1 Two problems

1.1 A colorful version of Gallai’s theorem

Let $F$ be a family of sets in $\mathbb{R}^d$. A matching in $F$ is a sub-family of pairwise disjoint sets. The matching number $\nu(F)$ is the largest size of a matching in $F$. A cover in $F$ is a set of points in $\mathbb{R}^d$ intersecting all the sets in $F$. The covering number $\tau(F)$ is the minimal size of a cover in $F$.

A well-known theorem of Gallai from the 1960’s is the following:

**Theorem 1.1** (Gallai). Let $F$ be a finite family of compact intervals in $\mathbb{R}$. Then $\tau(F) = \nu(F)$.

**Exercise 1.2.** Prove Gallai’s theorem (hint: construct an algorithm for finding a cover and a matching of the same size. Why is this enough?)

Another way to state the theorem: if $\tau(F) > k$ then there exists a matching in $F$ of size $k + 1$. Now, suppose that we have $k + 1$ finite families $F_1, \ldots, F_{k+1}$ of compact intervals in $\mathbb{R}$, with $\tau(F_i) > k$ for all $i$. Can we find a rainbow matching (that is a matching $\mathcal{M}$ with $\mathcal{M} \cap F_i = 1$ for all $i$)?

1.2 Fair division of a cake

Suppose that we have $k$ players with subjective preferences on a given cake (identified with the $[0, 1]$ interval). In any partition of the cake each player gives a list of pieces they prefer from the cake in that partition. Two conditions are satisfied:

1. Players are hungry (define)
2. Preference sets are closed (define).

Does there necessarily exist a partition of the cake and allocation of pieces such that every player receives one of his favorite pieces?
2 Sperner’s Lemma, Brouwer’s fixed-point theorem, and the KKM theorem

Sperner’s Lemma is an important result in combinatorial topology. It was originally proved by Sperner in 1928 to obtain a simple proof of Brouwer’s fixed-point theorem (1910).

**Theorem 2.1** (Brouwer’s Fixed Point Theorem, 1911). Any continuous map $f$ from a finite dimensional ball $B$ to itself has a fixed point, namely a point $x \in B$ such that $f(x) = x$.

BFPT has numerous applications in mathematics and economics as does Sperner’s Lemma.

**Definition 2.2.**

- The $n$-dimensional simplex is the convex hull of $n + 1$ affinely independent points in $\mathbb{R}^{n+1}$.

- The standard $n$-dimensional simplex is

$$
\Delta^n = \text{conv}\{e_1, \ldots, e_{n+1}\} \subset \mathbb{R}^{n+1},
$$

where $e_1, \ldots, e_{n+1}$ are the standard basis vector in $\mathbb{R}^{n+1}$.

- The convex hull of any nonempty subset of the $n + 1$ points that define an simplex is called a a face of the simplex (so a face of a simplex is also a simplex). If the subset defining a face is of size $k$, then the dimension of the face is $k - 1$.

- The 0-dimensional faces of a simplex are called vertices. The 1-dimensional faces of are called edges.

- A triangulation is a subdivision of a simplex (or more generally - a polytope) into simplices.

- If $x \in \Delta^{n+1}$, then $\text{supp}(x)$ is the minimal face of $\Delta^{n+1}$ containing $x$.

**Definition 2.3** (Sperner coloring). Let $T$ be a triangulation of a $n$-dimensional simplex $\Delta$. Let $\lambda : v(T) \to [n + 1]$ be a coloring of the vertices of $T$ with colors $[n + 1]$ such that:

- Every vertex of $\Delta$ gets a distinct color.

- For every $v \in T$ we have $\lambda(v) \in \lambda(V(\text{supp}(v)))$.

Then $\lambda$ is called a Sperner coloring of $T$.

Let $T$ be a triangulation of $\Delta^n$ and let $\lambda$ be a Sperner coloring of $V(T)$. A rainbow simplex is a simplex in $T$ whose vertices have all distinct colors.
Theorem 2.4 (Sperner’s lemma 1928). Let $\Delta$ be a $n$-dimensional simplex, let $T$ be a triangulation of $\Delta$, and let $\lambda : V(T) \to [n+1]$ be a Sperner coloring of $\Delta$. Then the number of $n$-dimensional rainbow simplices in $T$ is odd. In particular, there is at least one $n$-dimensional rainbow simplex.

Proof. By induction on $n$.

Base case: $n = 1$. Then $\Delta$ is a 1-dimensional simplex, namely a segment $[a, b]$. $T$ is a triangulation of $\Delta$, namely a subdivision of $[a, b]$ into smaller segments. We have two colors $\{1, 2\}$ and $a, b$ receive different colors. Now, going from $a$ to $b$, we must switch color an odd number of times so that we get a different color in $b$. Hence there is an odd number of subsegments (simplices in $T$) that receive two different colors.

Case 2: $n = 2$. Consider the face $12$ of $\Delta$. By induction, it has odd many rainbow 1-dimensional simplices of $T$, colored $12$. Define a graph $G$ as follows: the vertices of $G$ are the 2-dimensional simplices of $T$, and one additional vertex $v$ in the outer face. Two vertices are connected be an edge in $G$ if they share an edge of $T$ colored by $12$. By the handshake lemma, $G$ has even many vertices of odd degree, and since $\deg(v)$ is odd, it has odd many vertices corresponding to 2-dimensional simplices of $T$. Such simplices must have exactly one $12$ edge, and hence they must be rainbow.

General $n$. Consider the face $12\cdots n$ of $\Delta$. By induction, it has odd many rainbow $(n-1)$-dimensional simplices of $T$, colored $12\cdots n$. Define a graph $G$ as follows: the vertices of $G$ are the $n$-dimensional simplices of $T$, and one additional vertex $v$ in the outer face. Two vertices are connected be an edge in $G$ if they share an edge of $T$ colored by $12\cdots n$. By the handshake lemma, $G$ has even many vertices of odd degree, and since $\deg(v)$ is odd, it has odd many vertices corresponding to $n$-dimensional simplices of $T$. Such simplices must have exactly one $12\cdots n$ edge, and hence they must be rainbow.

Another proof by induction: Let $Q$ be the number of simplices in $T$ colored $(1, 1, 2)$ or $(1, 2, 2)$. Let $R$ be the number of rainbow simplices in $T$. Let $X$ be the number of $(1, 2)$ edges on the boundary of $\Delta$. Let $Y$ be the number of $(1, 2)$ edges in the interior of $\Delta$.

- For each simplex on $T$ colored $(1, 1, 2)$ of $(1, 2, 2)$ we get two $(1, 2)$ edges, while for each rainbow simplex we get one $(1, 2)$ edge.

- On the other hand, this way we count all of internal edges colored $(1, 2)$ twice, and all of the boundary edges colored $(1, 2)$ once. Thus, $2Q + R = 2Y + X$.

We know that $X$ is odd because the $[1, 2]$ boundary of $\Delta$ is colored in a Sperner coloring. So $R$ must be odd.

General Case: We have a Sperner coloring on $T$ by $n + 1$ colors. Let $R$ denote the number of rainbow simplices in $T$. Let $Q$ denote the number of $d$-dimensional simplices in $T$ that get all of the colors except $n + 1$, i.e. they are colored by all of the colors in $[n]$, so that exactly one of these colors is used twice and the others are used once.

Example: In the $d = 3$ case $Q$ counts the number of simplices colored as follows: $(1, 1, 2, 3), (1, 2, 2, 3), (1, 2, 3, 3)$.

Also, consider the $(n-1)$-dimensional faces that are colored by exactly the colors in $[n]$ (namely $"[n]-rainbow$ simplices”). Let $X$ be the number of such faces of the boundary of $\Delta$, and let $Y$ be the number of such faces in the interior of $\Delta$.
Again we count in two different ways:

- Every simplex of type \( R \) ([\( n + 1 \)]-rainbow simplex) contributes exactly one \((n-1)\)-face colored by \( \{1, 2, \ldots, n\} \). Every simplex of the type \( Q \) (colored by \( \{1, 2, \ldots, n\} \)) contributes exactly 2 \((n-1)\)-faces colored by \( \{1, 2, \ldots, n\} \).

- \((n-1)\)-dimensional faces that are colored by \( \{1, 2, \ldots, n\} \) and lie on the boundary of \( \Delta \) appear in one \( d \)-dimensional simplex of \( T \), while if it does not lie on the boundary it appears in 2 simplices of \( T \). Hence we get \( 2Q + R = X + 2Y \).

Now note that on the boundary of \( \Delta \), the only \((n-1)\)-dimensional faces colored by all colors \( \{1, 2, \ldots, n\} \) can be of the \((n-1)\)-face of \( \Delta \) whose vertices are colored by \( \{1, 2, \ldots, n\} \) (because this is a Sperner Coloring). By induction, the number of such faces \( X \) is odd. So \( R \) must be odd too.

The KKM Theorem is a continuous version of the Sperner lemma that was proved by Knaster-Kuratowski-Mazurkiewicz in 1929.

**Theorem 2.5** (The KKM theorem, 1928). Let \( \Delta \) be an \( n \)-dimensional simplex on the vertex set \( \{v_1, v_2, \ldots, v_{n+1}\} \). Let \( A_1, A_2, \ldots, A_{n+1} \) be closed sets covering \( \Delta \) so that \( \sigma \subseteq \bigcup_{v_i \in \sigma} A_i \).

Then \( \bigcap_{i=1}^{n+1} A_i \neq \emptyset \).

**Proof.** Embed \( \Delta \) in \( \mathbb{R}^{n+1} \) in the standard way. For every \( i \in [n+1] \) define a function \( g_i : \Delta_n \to \mathbb{R} \) by

\[
g_i(x) = \text{dist}(x, A_i) = \inf \{|x - a| : a \in A_i\} = \min \{|x - a| : a \in A_i\}.
\]

Define \( f : \Delta \to \Delta \) by

\[
f(x) = f((x_1, x_2, \ldots, x_{n+1})) = \frac{(x_1 + g_1(x), \ldots, x_{n+1} + g_{n+1}(x))}{1 + \sum_{j=1}^{n+1} g_j(x)}.
\]

This is indeed a map to \( \Delta \), moreover, it is continuous. So by BFPT there exists a \( z \in \Delta \) such that \( f(z) = z \).

Let \( S(z) = \{i \in [n+1] : z_i > 0\} = V(\text{supp}(z)) \). By the conditions of the theorem,

\[
z \in \text{supp}(z) = \text{conv}\{e_i : i \in S(z)\} \subseteq \bigcup_{i \in S(z)} A_i.
\]

Thus there exists \( i_0 \in S(z) \) so that \( z \in A_{i_0} \), and therefore \( g_{i_0}(z) = \text{dist}(z, A_{i_0}) = 0 \).

Now, \( f(z) = z \) implies \( (f(z))_{i_0} = z_{i_0} \), and therefore

\[
z_{i_0} = \frac{z_{i_0} + g_{i_0}(z)}{1 + \sum_{j=1}^{n+1} g_j(z)} = \frac{z_{i_0}}{1 + \sum_{j=1}^{n+1} g_j(z)}.
\]

Note that \( z_{i_0} \neq 0 \) since \( i_0 \in S(z) \), and therefore we can divide by \( z_{i_0} \) to get

\[
\frac{1}{1 + \sum_{j=1}^{n+1} g_j(z)} = 1,
\]

implying \( \sum_{j=1}^{n+1} g_j(z) = 0 \). This entails \( g_j(z) = 0 \) for all \( j \in [n+1] \). Since \( A_j \) are closed, this implies \( z \in A_j \) for all \( j \).
Proposition 2.6. KKM is true also if all of the sets $A_i$ are open.

Proof. We can find closed sets $B_i$ satisfying KKM such that $b_i \subseteq A_i$ and $\bigcap_{i \in K} B_i \neq \emptyset$ if and only if $\bigcap_{i \in K} A_i \neq \emptyset$ for every $K \subseteq [n + 1]$. Then we apply the theorem with the sets $B_i$. 

Exercise 2.7. Sperner’s lemma, BFPT, and the KKM theorem are easily proved one from the other. Prove all six implications.

Exercise 2.8.
1. Prove the KKM theorem from the Borsuk-Ulam theorem.
2. Can you prove the opposite?

3 Warm-up: Proving Gallai’s theorem with KKM

Proof. Let $F$ be a family of intervals with $\tau(F) = k + 1$. We show that $\nu(F) \geq k + 1$. (This will give $\nu \geq \tau$, and we already know that $\nu \leq \tau$). Since $F$ is finite, by rescaling $\mathbb{R}$ we may assume that all of the intervals in $F$ are contained in the open segment $(0, 1)$.

Let $\Delta$ be the $k$-dimensional standard simplex in $\mathbb{R}^{k+1}$. Every point in $\Delta$ corresponds to a distribution of $k$ (not necessarily distinct) points $u_1(x), \ldots, u_k(x)$ on $[0, 1]$, where $u_i(x) = \sum_{j=1}^{i} x_j$.

Since $k$ or less points do not cover $F$ (because $\tau(F) > k$), for every $x \in \Delta$ there exists an interval $f \in F$ that does not contain any of the points $u_1(x), \ldots, u_k(x)$ corresponding to $x$. Thus $f \subset (u_{i-1}(x), u_i)$ for some $1 \leq i \leq k + 1$.

Define sets $A_1, \ldots, A_{k+1} \subseteq \Delta$ as follows:

$$A_i = \{x \in \Delta \mid \text{there exists } f \in F \text{ such that } f \subset (u_{i-1}(x), u_i(x))\}$$

Note that by the above, $\Delta \subseteq \bigcup_{i=1}^{k+1} A_i$.

Claim 3.1. $A_1, \ldots, A_{k+1}$ form a KKM cover.

Proof. First, since the intervals in $F$ are closed, the sets $A_i$ is open for all $i$. Indeed, if $f \in F$ witnesses the fact the $x \in A_i$ (that is $f \subset (u_{i-1}(x), u_i(x))$), then since $f$ is closed, for some small enough $\varepsilon$, every point $x' \in B_{\varepsilon}(x)$ satisfies $f \subset (u_{i-1}(x'), u_i(x'))$, and therefore $B_{\varepsilon}(x) \subset A_i$.

Second, let $\sigma$ be a face of $\Delta$ and let $x \in \sigma$. If $e_i \notin \sigma$ then $x_i = 0$, and thus $(u_{i-1}(x), u_i(x)) = \emptyset$, showing that no $f \in F$ satisfies $f \subset (u_{i-1}(x), u_i(x))$. Since $x \in \Delta \subseteq \bigcup_{i=1}^{k+1} A_i$, we must have $x \in A_i$ for some $i \in \sigma$, showing $\sigma \subset \bigcup_{i \in \sigma} A_i$. 

So by the KKM theorem there exists an $x \in \bigcap_{j=1}^{k+1} A_j$. Consider the distribution of $k$ points $u_1(x), \ldots, u_k(x)$ corresponding to $x$. We have $x \in A_i$, and thus there is an interval $f_i \in (u_{i-1}(x), u_i(x))$ for every $i \in [k + 1]$. The set of intervals $f_1, \ldots, f_{k+1}$ is a matching of size $k + 1$, showing $\nu(F) \geq k + 1$ as promised.
4 Colorful KKM

Theorem 4.1 (Gale, 1982). Let \( (A^i_j : i, j \in [n]) \) be \( n \) KKM covers of \( \Delta^{n-1} \). Then there exists a permutation \( \pi \in S_n \) so that \( \bigcap_{i \in \pi(j)} A^i_j \neq \emptyset \).

Proof. Let \( \{T_k\} \) be a sequence of barycentric subdivisions of \( \Delta_{n-1} \) (so the simplex diameters going to 0). For every vertex \( v \) the triangulation \( T_k \), assign a role \( r(v) \) to \( v \) according to the dimension of the face in \( T_{k-1} \) it subdivide: if \( v \) is the barycenter of a face of dimension \( m \) in \( T_{k-1} \), then let \( r(v) = m + 1 \). (For example if \( v \) subdivide an edge of \( T_{k-1} \) then \( r(v) = 2 \).

By the definition of the barycentric subdivision the roles of the vertex set of every simplex in \( T_k \) are distinct. So, every full dimensional simplex in \( T_k \) contains vertices of all \( n \) possible roles.

We construct a coloring function \( c : V(T_k) \to [n] \) as follows: For \( v \in V(T_k) \), choose \( j \) such that \( v \in A^r(v) \) and \( j \in \text{supp}(v) \). Such \( j \) exists because \( (A^r(v) : j \in [n]) \) is a KKM cover, and thus

\[
v \in \text{supp}(v) \subseteq \bigcup_{j \in \text{supp}(v)} A^r_j(v).
\]

Let \( c(v) = j \). Note that for every \( v \in V(T_k) \) we have \( c(v) \in \text{supp}(v) \), and thus \( c : V(T_k) \to [n] \) is a Sperner coloring.

Now apply Sperner’s lemma. We obtain a rainbow simplex \( \sigma_k \) in \( T_k \). That is, \( \sigma_k \) has the following property: there exists \( \pi_k \in S_n \) so that in \( V(\sigma_k) = \{v_1, \ldots, v_n\} \) with \( r(v_i) = i \) and \( c(v_i) = \pi_k(i) \) for all \( i \in [n] \). Since \( \Delta \) is compact, and the diameter of \( \sigma_k \) tends to 0 when \( k \) tends to infinity, the sequence \( \{\sigma_k\}_{k \geq 1} \) has a subsequence converging to a point \( x \in \Delta \), and since \( S_n \) is finite, this subsequence has an infinite subsequence \( \{\sigma_{k_j}\}_{j \geq 1} \) in which all the permutations \( \pi_{k_j} \) are the same permutation \( \pi \). By construction, this means that \( \text{dist}(x, A^j_{\pi(i)}) \leq \varepsilon \) for all \( i \in [n] \) and for every \( \varepsilon > 0 \). Since the sets \( A^i_j \) are closed we have \( x \in \bigcap_{i \in [n]} A^i_{\pi(i)} \), as needed. \( \square \)

5 Solution to the two problems

5.1 Colorful Gallai’s theorem

Theorem 5.1. Let \( F_1, \ldots, F_{k+1} \) be finite families of compact intervals in \( \mathbb{R} \), with \( \tau(F_i) > k \) for all \( i \). Then there exists a full rainbow matching.

Proof. Since \( F_i \) are finite and all the intervals are bounded, by rescaling \( \mathbb{R} \) we may assume that all of the intervals in \( \bigcup F_i \) are contained in the open segment \((0, 1)\). Let \( \Delta \) be the \( k \)-dimensional standard simplex in \( \mathbb{R}^{k+1} \). Every point in \( \Delta \) corresponds to a distribution of \( k \) (not necessarily distinct) points \( u_1(x), \ldots, u_k(x) \) on \((0, 1)\), where \( u_i(x) = \sum_{j=1}^{i} x_j \).

For \( j \in [k+1] \) define sets \( A^1_j, \ldots, A^j_{k+1} \subset \Delta \) as follows:

\[
A^j_i = \{x \in \Delta \mid \text{there exists } f \in F_j \text{ such that } f \subset (u_{i-1}(x), u_i(x))\}.
\]
Like in the proof of Gallai’s theorem, for every \( j \in [k+1] \), the collection \( (A^j_1, \ldots, A^j_{k+1}) \) forms a KKM cover of \( \Delta \). So by the colorful KKM theorem, there exists an \( x \in \bigcap_{i=1}^{k+1} A^i_{\pi(i)} \). Consider the distribution of \( k \) points \( u_1(x), \ldots, u_k(x) \) corresponding to \( x \). We have \( x \in A^i_{\pi(i)} \), and thus there is an interval \( f_i \in F_i \) with \( f_i \in (u_{\pi(i)-1}(x), u_{\pi(i)}(x)) \) for every \( i \in [k+1] \). Thus the set of intervals \( f_1, \ldots, f_{k+1} \) is a colorful matching of size \( k+1 \), as needed.

5.2 Fair division of a cake

**Theorem 5.2** (Stromquist 1980, Woodall 1980). If \( n \) hungry players have closed preference sets on a cake then there exists a fair division.

**Proof.** Let \( A^j_i \) be the set of partitions in which player \( j \) prefers piece \( i \).

**Claim 5.3.** The “hungry player” assumption implies that the KKM covering conditions hold.

**Proof.** Let \( x \in \sigma \) be a partition of the cake. If \( i \notin \sigma \), then \( x_i = 0 \), and since player \( j \) is hungry, he does not prefer piece \( i \) in the partition \( x \). Therefore \( x \in A^j_i \) for some \( i \in \sigma \), showing \( \sigma \subset \bigcup_{i \in \sigma} A^j_i \).

**Claim 5.4.** The “closed preference set” assumption implies that the sets \( A^j_i \) are closed.

**Proof.** This is immediate by definition. If a converging sequence of partitions \( \{x_t\}_{t \geq 1} \subset A^j_i \) then by definition of the closed preference set” assumption, the limit partition is also in \( A^j_i \).

Thus by the colorful KKM theorem there exists a partition \( x \) with \( x \in \bigcap_{i=1}^{n} A^j_{\pi(i)} \) for some \( \pi \in S_n \). In this partition every player prefer a distinct piece, as needed.

6 Another application: Piercing sets in the plane with lines

Let \( \mathcal{F} \) be a family of convex sets in the plane. We say that \( \mathcal{F} \) has property \( T(r) \) if every \( r \) or fewer sets in \( \mathcal{F} \) admit a line transversal, that is, there exists a line intersecting these sets. We say that \( \mathcal{F} \) is pierced by \( k \) lines if there are \( k \) lines in the plane whose union intersects all the sets in \( \mathcal{F} \). The line-piercing number of the family is the minimum \( k \) so that \( \mathcal{F} \) is pierced by \( k \) lines. Some known bounds:

- Santalo (1940): For any \( k \), \( T(k) \) property does not imply \( \mathcal{F} \) is pierced by one line.
- Eckhoff (1969): \( T(4) \) property implies \( \mathcal{F} \) is pierced by two lines.
- This implies: for \( k \geq 4 \), \( T(k) \) property implies \( \mathcal{F} \) is pierced by two lines.
- \( T(2) \) property does not imply \( \mathcal{F} \) is pierced by constant many lines (e.g., \( n \) points in general position).
- Eckhoff (1975): \( T(3) \) property does not imply \( \mathcal{F} \) is pierced by 2 lines.
- Eckhoff (1993): $T(3)$ property implies $\mathcal{F}$ is pierced by 4 lines.

Eckhoff conjectured in 1993 that the latter can be further improved, namely, that the $T(3)$ property implies that $\mathcal{F}$ is pierced by 3 lines. The following theorem is a generalization of this statement: it specializes to Eckhoff’s conjecture when all the families are the same.

**Theorem 6.1** (McGinnis-Zerbib 2021). Let $\mathcal{F}_1, \ldots, \mathcal{F}_6$ be families of compact connected sets in $\mathbb{R}^2$. If every three sets $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}$, $1 \leq i_1 < i_2 < i_3 \leq 6$, have a line transversal, then there exists $i \in [6]$ such that the line-piercing number of $\mathcal{F}_i$ is at most 3.

**Proof of Theorem 6.1** We may scale the plane so that every set in $\mathcal{F}_j$ is contained in the unit disk $D$ for each $j$. Denote by $U$ the unit circle. Let $f(t)$ be a parameterization of $U$ defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$.

A point $x = (x_1, \ldots, x_6) \in \Delta^5$ corresponds to 6 points on $U$ given by $f_i(x) = f(\sum_{j=1}^{i-1} x_j)$ for $1 \leq i \leq 6$. Let $l_1(x) = l_4(x) = [f_1(x), f_4(x)], l_2(x) = l_5(x) = [f_2(x), f_5(x)]$, and $l_3(x) = l_6(x) = [f_3(x), f_6(x)]$.

For $i = 1, \ldots, 6$ let $R^i_x$ be the interior of the region bounded by $l_{i-1}(x), l_i(x)$ and the arc on $U$ connecting $f_{i-1}(x)$ and $f_i(x)$. Notice that $R^i_x = \emptyset$ when $x_i = 0$. Also, it is possible that some of the regions $R^i_x$ intersect.

Set $1 \leq j \leq 6$ and let $A^j_i$ be the set of points $x \in \Delta^5$ so that $R^i_x$ contains a set $F \in \mathcal{F}_j$. Since the sets $F \in \mathcal{F}_j$ are closed, $A^j_i$ is open. If there is some $x \in \Delta^5$ for which $x \notin \bigcup_{i=1}^{6} A^j_i$, then since the sets in $\mathcal{F}_j$ are connected, every set in $\mathcal{F}_j$ must intersect $\bigcup_{i=1}^{6} l_i(x)$, and we are done. So we assume for contradiction that $\Delta^5 = \bigcup_{i=1}^{6} A^j_i$ for all $j$. Observe that if $x \in \text{conv}\{e_i : i \in I\}$ for some $I \subset [6]$ then $R^k_x = \emptyset$ for $k \notin I$, and therefore, $x \in \bigcup_{i \in I} A^j_i$ for all $j$. This shows that the conditions of the colorful KKM theorem hold.

Thus, by the colorful KKM theorem, there exists some permutation $\pi \in S_6$ and a point $p = (p_1, \ldots, p_6) \in \bigcap_{i=1}^{6} A^{\pi(i)}$. Therefore, each of the open regions $R^i_p$ contains a set $S_i \in \mathcal{F}_{\pi(i)}, i = 1, \ldots, 6$, and in particular $R^i_p \neq \emptyset$ and thus $p_i \neq 0$ for all $i$. We claim that at least one of the triples $\{S_1, S_3, S_5\}$ or $\{S_2, S_4, S_6\}$ is not a tight triple. To see this, note that the regions $R^1_p, R^2_p, R^3_p$ are pairwise disjoint or the regions $R^2_p, R^4_p, R^6_p$ are pairwise disjoint (depending on the orientation of the triangle bounded by the lines $l_1, l_2, l_3$). Without loss of generality, we assume $R^1_p, R^3_p, R^5_p$ are pairwise disjoint, and in this case, the three sets $S_1, S_3, S_5$ is not a tight triple. This is a contradiction. 

By a similar method, one can also prove:

**Theorem 6.2** (McGinnis-Zerbib, 2021). Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ be finite families of compact, connected sets in the plane such that any collection of four sets, one from each $\mathcal{F}_i$, has a line transversal. Then for some $i \in [4]$, $\mathcal{F}_i$ has line piercing number at most 2.

When all the families are the same this specializes to Eckhoff’s result the the $T(4)$ property implies that $\mathcal{F}$ is pierced by 2 lines.

**Exercise 6.3.** Prove Theorem 6.2.
7 Generalizing Gallai’s theorem: Piercing $d$-intervals

7.1 Theorems on piercing $d$-interval families

A $d$-interval is a union of at most $d$ intervals on $\mathbb{R}$.

**Theorem 7.1** (Tardos-Kaiser 1995). If $F$ a finite family of compact $d$-intervals, then $\tau(F) \leq (d^2 - d + 1)\nu(F)$.

For $d = 2$ this is tight. For larger $d$ it is known to be tight up to $\log d$ factor: Matoušek showed that there are families of $d$-intervals with $\tau(F) = \Omega(d^2 \log d)\nu(F)$.

The following a colorful version of the Tardos-Kaiser theorem. It specializes to the Tardos-Kaiser theorem when all the families are the same:

**Theorem 7.2** (Frick-Zerbib 2019). Let $F_1, \ldots, F_{k+1}$ be $k+1$ finite families of compact $d$-intervals. If $\tau(F_i) > k$ for all $i$, then there exists a rainbow matching of size at least $\frac{d^2 - d + 1}{k+1}$.

All known proofs for Theorems 7.1 and 7.2 are topological. Alon showed via elementary methods a slightly worse bound: $\tau \leq 2d^2$.

7.2 Notions from hypergraph theory

A hypergraph $H$ is a family $E(H)$ of subsets, called edges, of a ground set $V(H)$ of vertices. A hypergraph $H$ is $r$-uniform if all its edges are of size $r$. It is $r$-partite if there exists a partition $V_1 \cup \cdots \cup V_r$ of $V(H)$ such that $|e \cap V_i| = 1$ for every edge $e \in H$ and every $1 \leq i \leq r$. The sets $V_i$ are called the vertex sides of $H$. Note that an $r$-partite hypergraph in particular $r$-uniform.

A graph is a 2-uniform hypergraph and a 2-partite graph is also called bipartite.

Let $H = (V, E)$ be a hypergraph. A matching in $H$ is a set of disjoint edges. The matching number $\nu(H)$ is the maximum size of a matching in $H$. A cover of $H$ is a set of vertices intersecting all edges. The covering number $\tau(H)$ is the minimum size of a cover in $H$.

We can also regard a matching as a function $f : E(H) \to \{0, 1\}$ satisfying the condition that adjacent edges do not get both 1, namely, for all $v \in V$, $\sum_{e \in e} f(e) \leq 1$. Then the matching number is

$$\nu(H) = \max \{ \sum_{e \in E} f(e) \mid f \text{ is a matching} \}.$$ 

Now we can consider a fractional relaxation of this notion: A fractional matching in a hypergraph $H$ is a function $f : E(H) \to [0, 1]$, satisfying the condition for all $v \in V$, $\sum_{e \in e} f(e) \leq 1$. The fractional matching number is defined as

$$\nu^*(H) = \max \{ \sum_{e \in E} f(e) \mid f \text{ is a functional matching} \}.$$
Similarly, a cover can be viewed as a function \( g : V(H) \to \{0, 1\} \), satisfying the condition that for all \( e \in E(H) \), \( \sum_{v \in e} g(v) \geq 1 \), and the covering number is

\[
\tau(H) = \min \{ \sum_{v \in V} g(v) \mid g \text{ is a cover} \}.
\]

Now, a fractional cover of \( H \) is a function \( g : V(H) \to [0, 1] \), satisfying the condition that for all \( e \in E(H) \), \( \sum_{v \in e} g(v) \geq 1 \). The fractional covering number \( \tau^*(H) \) is defined by

\[
\tau^*(H) = \min \{ \sum_{v \in V} g(v) \mid g \text{ is a fractional cover} \}.
\]

**Exercise 7.3.** Prove the following:

1. By linear programming duality, show that \( \nu^*(H) = \tau^*(H) \) for all \( H \).

2. If \( r \) is the maximum size of an edge in a hypergraph \( H \) then \( \nu(H) \leq \nu^*(H) = \tau^*(H) \leq \tau(H) \leq r \nu(H) \).

**Example 7.4.** In \( K_3 \), \( \nu = 1 \), \( \nu^* = \tau^* = 3/2 \), \( \tau^* = 2 \).

A perfect fractional matching (PFM) is a fractional matching \( f : E(H) \to [0, 1] \) such that for all \( v \in V \), \( \sum_{e \ni v} f(e) = 1 \). Not every hypergraph has a PFM. For example, \( K_3 \) has a PFM, but a path on 3 vertices does not. A hypergraph is called balanced if it has a PFM.

The following is a trivial consequence of Exercise 7.3.

**Proposition 7.5.** If \( |e| \leq r \) for all \( e \in E(H) \), then \( \nu(H) \geq \frac{\nu^*(H)}{r} \).

**Proof.** We have \( \frac{\nu^*(H)}{r} \leq \frac{\tau(H)}{r} \leq \frac{r \nu(H)}{r} = \nu(H) \).

A theorem of Füredi shows the bound in Proposition 7.5 can be slightly improved:

**Theorem 7.6** (Füredi). If \( |e| \leq r \) for all \( e \in E(H) \), then \( \nu(H) \geq \frac{\nu^*(H)}{r-1+\frac{1}{r}} \). Moreover, if \( H \) is \( r \)-partite, then \( \nu(H) \geq \frac{\nu^*(H)}{r-1} \).

We will need also:

**Proposition 7.7.** If \( H \) is a balanced hypergraph with maximal edge size \( r \), then \( \nu^*(H) \geq \frac{|V(H)|}{r} \).

**Proof.** Let \( f : E(H) \to [0, 1] \) be a perfect fractional matching. Then \( \nu^*(H) \geq \sum_{e \in E} f(e) \). Now,

\[
|V| = \sum_{v \in V} 1 = \sum_{v \in V} \sum_{e \ni v} f(e) = \sum_{e \in E} \sum_{v \in e} f(e) \leq r \sum_{e \in E} f(e) \leq r \nu^*(H).
\]
7.3 The KKMS Theorem

For a face $\sigma$ of the standard $k$-simplex $\Delta^k$, let $s(k) = \{i \in [k+1] \mid e_i \text{ is a vertex in } \sigma\}$.

**Definition 7.8.** Let $\Delta$ be the standard $k$-dimensional simplex on vertex set $e_1, \ldots, e_{k+1}$. We say that faces $\sigma_1, \ldots, \sigma_m$ are balanced if the hypergraph with vertex set $V(H) = [k+1]$ and edge set $E(H) = \{s(\sigma_1), \ldots, s(\sigma_m)\}$ is balanced.

**Example 7.9.** In $\Delta^2$ the faces 12, 23, 13 are balanced. Also, the faces 1, 23 are balanced.

**Theorem 7.10** (The KKMS theorem, Shapley 1973). Let $\Delta$ be the $n$-dimensional standard simplex and let $A_\tau$ be a closed (open) set for every nonempty face of $\Delta$, such that for every face $\sigma$, $\sigma \subseteq S_\tau \subseteq \sigma$ $A_\tau$. Then there exists balanced faces $\sigma_1, \ldots, \sigma_{k+1}$, such that $\sum_{i=1}^{k+1} A_{\sigma_i} \neq \emptyset$.

**Exercise 7.11.** Prove that the KKMS theorem implies the KKM theorem.

7.4 Proof of the KKMS theorem

In this section we will prove the KKMS theorem. We first need a few preperations.

**Proposition 7.12.** Faces $\sigma_1, \ldots, \sigma_m$ of $\Delta$ are balanced if and only if $b_\Delta \in \text{conv}\{b_{\sigma_1}, \ldots, b_{\sigma_m}\}$.

**Proof.** Let $s_i = s(\sigma_i) = \{j : e_j \in \sigma_i\}$. Let $\chi^i$ be the characteristic vector of $S_i$, that is

$$\chi^i_j = \begin{cases} 1 & j \in s_i \\ 0 & \text{otherwise} \end{cases}.$$ 

Now, the faces $\sigma_1, \ldots, \sigma_m$ are balanced if and only if the hypergraph $([k+1], \{s_1, \ldots, s_m\})$ has a PFM, which means that there exist weights $\alpha_1, \ldots, \alpha_m \in [0,1]$ such that

$$\alpha_1 \chi_1 + \alpha_2 \chi_2 + \cdots + \alpha_m \chi_m = (1, \ldots, 1) = \chi_\Delta.$$

This is equivalent to

$$\frac{\alpha_1}{k+1} \chi_1 + \frac{\alpha_2}{k+1} \chi_2 + \cdots + \frac{\alpha_m}{k+1} \chi_m = \left(\frac{1}{k+1}, \ldots, \frac{1}{k+1}\right) = b_\Delta,$$

which we can write as

$$\frac{\alpha_1}{k+1} |s_1| b_{\sigma_1} + \frac{\alpha_2}{k+1} |s_2| b_{\sigma_2} + \cdots + \frac{\alpha_m}{k+1} |s_m| b_{\sigma_m} = b_\Delta.$$

To see that this is a convex combination note that

$$\sum_{i=1}^{m} \frac{\alpha_i |s_i|}{k+1} = \frac{1}{k+1} \sum_{i=1}^{m} \alpha_i |s_i| = \frac{1}{k+1} \sum_{i=1}^{m} \sum_{j \in s_i} \alpha_i = \frac{1}{k+1} \sum_{j=1}^{k+1} \sum_{i \in s_j} \alpha_i = \frac{1}{k+1} \sum_{j=1}^{k+1} 1 = 1,$$

because the function $s_i \mapsto \alpha_i$ is a PFM.


Exercise 7.13. Prove that if \( f : \Delta \to \Delta \) is continuous and homotopic to identity on the boundary of \( \Delta \), then \( f \) is surjective.

We first prove a “discrete version” of the KKMS theorem:

**Theorem 7.14.** Let \( T \) be a triangulation of the \( n \)-dimensional simplex \( \Delta \). Let

\[
\lambda : V(T) \to \{ \sigma : \sigma \neq \emptyset \text{ is a face of } \Delta \}
\]

be a labeling function such that \( \lambda(v) \subseteq \operatorname{supp}(v) \). Then there exists a simplex \( \tau \in T \) whose vertex labelings are balanced.

**Proof.** Define a map \( f : V(T) \to \Delta \) by \( v \mapsto b_{\lambda(v)} \). Extend \( f \) linearly to a map \( F : \Delta \to \Delta \). That is, if \( x \) is in a simplex \( \sigma = \text{conv}\{v_1, \ldots, v_{n+1}\} \) of \( T \), and \( x = \alpha_1 v_1 + \cdots + \alpha_{n+1} v_{n+1} \) (here \( \alpha_i \) are the coefficients in the convex combination that gives \( x \)) then \( F(x) = \alpha_1 f(v_1) + \cdots + \alpha_{n+1} f(v_{n+1}) \).

By definition, \( F \) is continuous. Moreover, if \( v \in V(T) \) lies in a face \( \sigma \) of \( \Delta \), then \( \lambda(v) \in \sigma \) and thus \( f(v) \in \sigma \) and thus \( F(\sigma) = \sigma \). So \( F \) is homotopic to the identity map on the boundary of \( \Delta \). By the exercise, \( F \) is surjective. Therefore there exists a point \( p \in \Delta \), such that \( F(p) = b_\Delta \).

Let \( \tau = \text{conv}\{v_1, \ldots, v_{n+1}\} \) be a simplex in \( T \) containing \( p \). Then by definition, there exist \( \alpha_1, \ldots, \alpha_{n+1} \) with \( \sum \alpha_i = 1 \), \( \alpha_i \geq 0 \) such that

\[
b_\Delta = F(p) = \alpha_1 f(v_1) + \cdots + \alpha_{n+1} f(v_{n+1}) = \alpha_1 b_{\lambda(v_1)} + \cdots + \alpha_{n+1} b_{\lambda(v_{n+1})}
\]

Thus \( b_\Delta \in \text{conv}\{b_{\lambda(v_1)}, \ldots, b_{\lambda(v_{n+1})}\} \). By the proposition, \( \lambda(v_1), \ldots, \lambda(v_{n+1}) \) are balanced. \( \square \)

Exercise 7.15. Prove the KKMS theorem from the previous theorem (similarly to the way KKM is proved from Sperner).

### 7.5 Proof of the Tardos-Kaiser theorem

**Theorem 7.16** \((d = 2): \) Tardos 1995, \( d \geq 2 \): Kaiser 1997. If \( \mathcal{F} \) is a finite family of compact \( d \)-intervals then \( \tau(\mathcal{F}) \leq (d^2 - d + 1) \nu(\mathcal{F}) \).

**Proof.** Since \( \mathcal{F} \) is finite and the \( d \)-interval are compact, we can assume that all the sets in \( \mathcal{F} \) are contained in \((0, 1)\). Suppose \( \tau(\mathcal{F}) = k + 1 \). We will show that \( \nu(\mathcal{F}) \geq \frac{k+1}{d^2-d+1} \).

This will imply \( \frac{\tau(\mathcal{F})}{\nu(\mathcal{F})} \leq \frac{k+1}{d^2-d+1} = d^2 - d + 1 \). Let \( \Delta \) be the standard \( k \)-dimensional simplex.

Every point \( x = (x_1, \ldots, x_{k+1}) \) corresponds to a distribution of \( k \) points \( u_1(x), \ldots, u_k(x) \) on \((0, 1)\), where \( u_i(x) = \sum_{j=1}^{i} x_j \).

Define \( u_0(x) = 0 \), \( u_{k+1}(x) = 1 \). For every face \( \sigma \) of \( \Delta \) define a set \( A_\sigma \) as follows: \( x = (x_1, \ldots, x_{k+1}) \in A_\sigma \) if and only if there exists a \( d \)-interval \( f \in \mathcal{F} \) such that following two conditions hold:

(a) \( f \subseteq \bigcup_{i \in \sigma} (u_{i-1}(x), u_i(x)) \), and
(b) for every \( i \in \sigma \), \( f \cap (u_{i-1}(x), u_i(x)) \neq \emptyset \).

Note that since \( \tau(\mathcal{F}) = k + 1 \), the points \( u_1(x), \ldots, u_k(x) \) do not cover \( \mathcal{F} \). So there is a \( d \)-interval in \( \mathcal{F} \) that is not covered, showing that \( \Delta^k \subseteq \bigcup_{\sigma \subseteq \Delta^k} A_\sigma \).

**Claim 7.17.** \( \{A_\sigma\} \) is a KKMS cover.

**Proof.** The set \( A_\sigma \) are all open (since the \( d \)-intervals are closed). We want to show \( \tau \subseteq \bigcup_{\sigma \subseteq \tau} A_\sigma \), for every face \( \tau \). Let \( x \in \tau \). For \( i \notin \tau \) we have \( x_i = 0 \) and therefore \( u_i(x) = u_{i-1}(x) \). This implies that the segment \( (u_{i-1}, u_i) = \emptyset \), and thus cannot contain satisfy condition (b). So \( x \notin A_\sigma \) when \( \sigma \) contains a vertex \( i \notin \tau \). Since \( x \in \Delta^k \subseteq \bigcup_{\sigma \subseteq \Delta^k} A_\sigma \), we conclude \( x \) must be in some \( A_\sigma \) with \( \sigma \subseteq \tau \), showing \( \tau \subseteq \bigcup_{\sigma \subseteq \tau} A_\sigma \). \( \square \)

By the KKMS theorem there exist balanced faces \( \sigma_1, \ldots, \sigma_m \) of \( \Delta \) such that \( \bigcap_{i=1}^m A_{\sigma_i} \neq \emptyset \). Let \( x \in \bigcap_{i=1}^m A_{\sigma_i} \), and let \( u_1, \ldots, u_k \) be the corresponding distribution on \((0,1)\). Let \( s_i = s(\sigma_i) \). The fact that \( \sigma_1, \ldots, \sigma_{k+1} \) are balanced implies that the hypergraph

\[
H = ([k+1], \{s_1, \ldots, s_{k+1}\})
\]

has a PFM. Note that \( |s_i| \leq d \) for all \( i \), since every \( f \in \mathcal{F} \) has at most \( d \) non-empty interval components. Thus by Proposition 7.7 \( \nu^*(H) \geq \frac{|V(H)|}{d} \geq \frac{k+1}{d} \). By Füredi’s theorem \( \nu(H) \geq \frac{\nu^*(H)}{d-1+\frac{1}{d}} \), and thus

\[
\nu(H) \geq \frac{\nu^*(H)}{d-1+\frac{1}{d}} \geq \frac{k+1}{d^2 - d + 1}.
\]

Therefore, there is a matching \( M = \{s_{i_1}, \ldots, s_{i_\nu(H)}\} \) in \( H \) of size at least \( \frac{k+1}{d^2 - d + 1} \). Note that the fact that this is a matching implies that the sets \( U_{i_t} = \bigcup_{j \in s_{i_t}} (u_{j-1}, u_j) \) are disjoint.

For every \( 1 \leq t \leq \nu(H) \) let \( f_{i_t} \in \mathcal{F} \) be a \( d \)-interval witnessing the fact that \( x \in A_{\sigma_{i_t}} \). Then by definition \( f_{i_t} \subset U_{i_t} \), and therefore the set \( \{f_{i_1}, \ldots, f_{i_{\nu(H)}}\} \) is a matching in \( \mathcal{F} \) of size at least \( \nu(H) \), as needed. \( \square \)

### 7.6 The colorful KKMS theorem

**Theorem 7.18** (Colorful KKMS, Shih-Lee 1993). Let \( (A_\sigma^i), i \in [k+1] \) be \( k + 1 \) KKMS covers of \( \Delta^k \). Then there exists balanced faces \( \sigma_1, \ldots, \sigma_{k+1} \) such that \( \bigcap_{i=1}^{k+1} A_{\sigma_i} \neq \emptyset \).

We will prove a more general theorem later on.

**Exercise 7.19.** Prove Theorem 7.2 using the colorful KKMS theorem.
8 Separated $d$-intervals and fair-division of multiple cakes

8.1 Theorems on piercing separated $d$-intervals

A separated $d$-interval is a union of $d$ intervals, one on each of $d$ separated copies of $\mathbb{R}$.

**Theorem 8.1** (Tardos-Kaiser). If $\mathcal{F}$ is a finite set of compact separated $d$-intervals, then $\tau(\mathcal{F}) \leq (d^2 - d)\nu(\mathcal{F})$.

**Theorem 8.2** (Frick-Zerbib). Let $\mathcal{F}_i$, $i \in [kd+1]$, be $kd+1$ hypergraphs of separated $d$-intervals. If $\tau(\mathcal{F}_i) > kd$ for all $i$, then there exists a rainbow matching of size at least $\frac{k+1}{d-1}$.

If all the families $\mathcal{F}_i$ are the same family $\mathcal{F}$, with $\tau(\mathcal{F}) = [kd+1]$ then we get Theorem 8.1. Indeed, in that case $\frac{\tau(\mathcal{F})}{\nu(\mathcal{F})} \leq (kd+1)\frac{k+1}{k+1} \leq d(d-1)$.

8.2 A theorem on fair division of multiple cakes

Suppose that there are $k$ cakes and $p = k(n-1)+1$ hungry players with closed preference sets. In every partition of the $k$ cakes into $n$ pieces each, each player chooses his favorite $k$-tuples of pieces (a $k$-tuple of pieces is a choice of one piece in each cake).

**Theorem 8.3** (Nyman-Su-Zerbib). There exists a division of the $k$ cakes where at least $\lceil \frac{p}{k(k-1)} \rceil$ players prefer pairwise disjoint $k$-tuple of pieces.

**Remark 8.4.** The theorem can be stated also for other values of $p$ (with slightly different bound on the number of satisfied players), but for simplicity we will only give the proof only for the case $p = k(n-1)+1$.

8.3 Komiya’s theorem

Komiya’s theorem is a far-reaching polytopal generalization of the KKMS theorem. Before stating it, let are give a reformulation of the KKMS theorem:

**Theorem 8.5.** Let $P = \Delta$ be the $k$-dimensional simplex. For every non-empty face $\sigma$ of $P$ let $A_\sigma \subset P$ be a closed set and let $y_\sigma$ be the barycenter of $\sigma$. If for every face $\tau$ of $P$ we have $\tau \subset \bigcup_{\sigma \subset \tau} A_\sigma$, then there exist faces $\sigma_1, \ldots, \sigma_{k+1}$ of $P$ such that $\bigcap_{i=1}^{k+1} A_{\sigma_i} \neq \emptyset$ and $y_\Delta \in \text{conv}\{y_{\sigma_1}, \ldots, y_{\sigma_{k+1}}\}$.

Now Komiya’s theorem states that we can replace $\Delta$ by any $k$-dimensional polytope $P$, and we can replace the barycenters by any points $y_\sigma \in \sigma$, and the theorem will still be correct.

**Theorem 8.6** (Komiya, 1994). Let $P$ be a $k$-dimensional polytope. For every non-empty face $\sigma$ of $P$ let $A_\sigma \subset P$ be a closed set and let $y_\sigma$ be the point in $\sigma$. If for every face $\tau$ of $P$ we have $\tau \subset \bigcup_{\sigma \subset \tau} A_\sigma$, then there exist faces $\sigma_1, \ldots, \sigma_{k+1}$ of $P$ such that $\bigcap_{i=1}^{k+1} A_{\sigma_i} \neq \emptyset$ and $y_P \in \text{conv}\{y_{\sigma_1}, \ldots, y_{\sigma_{k+1}}\}$.
8.4 Proof of Komiya’s theorem

First, we prove a “discrete version” of Komiya’s theorem. Given a triangulation $T$ of $P$, a Komiya labeling of $T$ is a map $f: V(T) \to \{ \sigma \mid \sigma \text{ a non-empty face of } P \}$ such that $f(v) \subseteq \text{supp}(v)$.

**Theorem 8.7.** Let $T$ be a triangulation of $P$, and let $f$ be a Komiya labeling of $T$. For every nonempty face $\sigma$ of $P$ choose a point $y_\sigma \in \sigma$. Then there is a face $\tau$ of $T$ such that $y_\tau \in \text{conv}\{y_{f(v)} \mid v \text{ vertex of } \tau\}$.

**Proof.** Let $g : V(T) \to P$ be the map $v \mapsto y_{f(v)}$, and let $G : P \to P$ be a linear extension of $g$. Then $G$ is a continuous map. Note that for every face $\sigma$ of $P$, we have that $G(\sigma) \subset \sigma$: indeed, if $x \in \sigma$ then $G(x) = \text{conv}\{g(v_1), \ldots, g(v_m)\}$ for some $v_1, \ldots, v_m \in \sigma$, and by the Komiya labeling condition, $g(v_i) \in \sigma$ for all $i$, and thus $G(x) \in \sigma$ as well. This implies that $G$ is homotopic to the identity on $\partial P$, and thus $G$ is surjective. Therefore, there exists a point $x \in P$ such that $G(x) = y_\tau$. Let $\tau$ be a full dimension face of $T$ containing $x$. By definition

$$y_\tau = G(x) \in G(\tau) = \text{conv}\{g(v) \mid v \text{ is a vertex of } \tau\} = \text{conv}\{y_{f(v)} \mid v \text{ is a vertex of } \tau\}.$$ 

$\Box$

**Proof of Komiya’s theorem.** Let $\varepsilon > 0$, and let $T$ be a triangulation of $P$ such that every face of $T$ has diameter at most $\varepsilon$. Given a Komiya cover $(A_\sigma)$ we define a Komiya labeling on $T$ in the following way: For a vertex $v$ of $T$, label $v$ by a face $\sigma \subset \text{supp}(v)$ such that $v \in A_\sigma$. Such a face $\sigma$ exists since $v \in \text{supp}(v) \subset \bigcup_{\sigma \subset \text{supp}(v)} A_\sigma$. Thus by Theorem 8.7 there is a full dimensional face $\tau$ of $T$ whose vertices are labeled by faces $\sigma_1, \ldots, \sigma_{k+1}$ of $P$ such that $y_\tau \in \text{conv}\{y_{\sigma_1}, \ldots, y_{\sigma_{k+1}}\}$. In particular, the $\varepsilon$-neighborhoods of the sets $A_{\sigma_i}$, $i \in [k+1]$, intersect. Now let $\varepsilon$ tend to zero. As there are only finitely many collections of faces of $P$, one collection $\sigma_1, \ldots, \sigma_{k+1}$ must appear infinitely many times. By compactness of $P$ the sets $A_{\sigma_i}$, $i \in [k+1]$, then all intersect since they are closed. $\Box$

Note that Komiya’s theorem is true also if all the sets $A_\sigma$ are open, by the same argument as before.

8.5 A colorful extension of Komiya’s theorem

For a face $\sigma$ of $P$ and a point $y_\sigma \in P$ we denote by $C_\sigma$ the cone of $\sigma$, that is, the union of all rays emanating from $y_\sigma$ that intersect $\sigma$.

**Theorem 8.8** (The colorful Komiya theorem, Frick-Zerbib 2019). Let $P$ be a $k$-dimensional polytope, and let $y_\sigma$ be a point in $P$. Suppose for every nonempty proper face $\sigma$ of $P$ we are given $k+1$ points $y^{(1)}_\sigma, \ldots, y^{(k+1)}_\sigma \in C_\sigma$ and $k+1$ closed sets $A^{(1)}_\sigma, \ldots, A^{(k+1)}_\sigma \subset P$. If $\sigma \subset \bigcup_{\tau \subset \sigma} A^{(j)}_\tau$ for every face $\sigma$ of $P$ and every $j \in [k+1]$, then there exist faces $\sigma_1, \ldots, \sigma_{k+1}$ of $P$ such that $y_\sigma \in \text{conv}\{y^{(1)}_{\sigma_1}, \ldots, y^{(k+1)}_{\sigma_{k+1}}\}$ and $\bigcap_{i=1}^{k+1} A^{(i)}_{\sigma_i} \neq \emptyset$. 

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Proof. Let $\varepsilon > 0$, and let $T$ be a triangulation of $P$ such that every face of $T$ has diameter at most $\varepsilon$. We will also assume that the chosen points $y_1^{(i)}, \ldots, y_{k+1}^{(i)}$ are contained in $\sigma$. This assumption does not restrict the generality of our proof since $y_P \in \text{conv}\{x_1, \ldots, x_{k+1}\}$ for vectors $x_1, \ldots, x_{k+1} \in \mathbb{R}^n$ if and only if $y_P \in \text{conv}\{\alpha_1 x_1, \ldots, \alpha_{k+1} x_{k+1}\}$ with arbitrary coefficients $\alpha_i > 0$.

Denote by $T'$ the barycentric subdivision of $T$. For $v \in V(T')$ let $r(v)$ be the dimension of the face in $T$ that $v$ subdivide, plus 1. We now define a Komiya labeling of $T'$: Let $v \in V(T')$. By the conditions of the theorem, $v$ is contained in a set $A_v^{(r)}$ where $r \in \text{supp}(v)$. We label $v$ by $r$. Thus by Theorem 8.7 there exists a full dimensional face $\tau$ of $T'$ whose vertices are labeled by faces $\sigma_1, \ldots, \sigma_{k+1}$ of $P$ such that $y_P \in \text{conv}\{y_{\sigma_1}^{(1)}, \ldots, y_{\sigma_{k+1}}^{(k+1)}\}$. In particular, the $\varepsilon$-neighborhoods of the sets $A_{\sigma_i}^{(i)}$, $i \in [k+1]$, intersect. Now use a limiting argument as before.

By taking $P = \Delta$ and $y_\sigma$ to be the barycenter of $\sigma$, we get a proof of Theorem 7.18.

### 8.6 Colorful $d$-intervals: Proof of Theorem 8.2

Let $\mathcal{F}_i$ be a family of separated $d$-intervals for all $i \in [kd+1]$. For $f \in \mathcal{F}$ let $f^t$ be the $t$-th interval component of $f$ on $\ell_t$. For a point $\vec{x} = (x_1, \ldots, x_{k+1}) \in \Delta_k$ let $p_\vec{x}(j) = \sum_{t=1}^{k+1} x_t \in [0,1]$. Since $\mathcal{F}$ is finite, by rescaling the $d$ copies $\mathbb{R}$ we may assume that for every $f \in \bigcup_{i \in [kd+1]} \mathcal{F}_i$, $f^t$ is a non-empty subset of $(0,1)$ on $\ell_t$. Let $P = (\Delta_k)^d$, and note that $\dim P = kd$.

Every point $\vec{X} = \vec{x}^1 \times \cdots \times \vec{x}^d \in P$ corresponds to a distribution of $kd$ points, $k$ points on each of the lines $\ell_1, \ldots, \ell_d$ as follows: on line $\ell_t$ the $k$ points are $p_\vec{x}^t(1), \ldots, p_\vec{x}^t(k)$. Since $r(\mathcal{F}_i) \geq kd + 1$, these $kd$ points do not cover $\mathcal{F}_i$. So there exists $f \in \mathcal{F}_i$ that is not covered. This means that $f^t \subset (p_\vec{x}^t(j_t - 1), p_\vec{x}^t(j_t))$ on $\ell_t$ for all $t \in [d]$, for some choice of $j_1, \ldots, j_d$.

We define a $kd + 1$ Komiya covers of $P$ as follows. Every face of $P$ corresponds to a tuple $T = (T_1, \ldots, T_d)$, with $T_i \subset [k+1]$ for all $i \in [d]$. In our setting $A_{T_i}$ is non-empty only if $T = (j_1, \ldots, j_d) \subset [k+1]^d$ (that is, all the $T_i$’s are singletons). For a $d$-tuple $T = (j_1, \ldots, j_d) \subset [k+1]^d$ let $A_T$ consist of all $\vec{X} = \vec{x}^1 \times \cdots \times \vec{x}^d \in P$ for which there exists $f \in \mathcal{F}_i$ satisfying $f^t \subset (p_\vec{x}^t(j_t - 1), p_\vec{x}^t(j_t))$ on $\ell_t$ for all $t \in [d]$.

By the same argument as before, the sets $A_T$ are open and satisfy the covering condition of the colorful Komiya theorem. Thus, by the colorful Komiya theorem, there exists a set $\mathcal{T} = \{T_1, \ldots, T_{kd+1}\}$ of $d$-tuples in $[k+1]^d$, such that the barycenters of the corresponding faces contain the point $b_P = (\frac{1}{k+1}, \ldots, \frac{1}{k+1}) \times \cdots \times (\frac{1}{k+1}, \ldots, \frac{1}{k+1}) \in P$ in their convex hull, and such that $\bigcap_{i \in [kd+1]} A_{T_i} \neq \emptyset$. Then the $d$-partite hypergraph $H = (\bigcup_{i=1}^d V_i, \mathcal{T})$, where $V_i = \{k+1\}$ for all $i$, has a perfect fractional matching, and hence by Proposition 7.7 we have $\nu^*(H) \geq k + 1$. By Füredi’s Theorem, this implies $\nu(H) \geq \nu^*(H) \geq \frac{k+1}{d-1}$. Now, by the same argument as before, taking $\vec{X} \in \bigcap_{i \in [kd+1]} A_T$, we obtain a matching in $\mathcal{F}$ of the same size as a maximal matching in $H$, concluding the proof of the theorem.
8.7 Multiple cakes: Proof of Theorem 8.3

Let $P = (Δ^{n-1})^k$ of dimension $k(n-1)$. Every point $\vec{x} = \vec{x}^1 \times \cdots \times \vec{x}^d$ in $P$ corresponds to a partition of the $k$ cakes into $n$ pieces each as follows: piece $j$ in cake $t$ is the piece $(p_{\vec{x}^1}(j-1), p_{\vec{x}^t}(j))$.

We define a $(n-1)+1$ Komia covers of $P$ as follows. As before, every face of $P$ corresponds to a tuple $T = (T_1, \ldots, T_k)$, with $T_i \subseteq [n]$ for all $i \in [k]$. In our setting $A_i^j$ is non-empty only if $T = (j_1, \ldots, j_k) \subseteq [n]^d$ (that is, all the $T_i$’s are singletons, so when the face is a vertex of $P$). For a $k$-tuple $T = (j_1, \ldots, j_k) \subseteq [n]^d$ let $A_T^j$ consist of all partitions $\vec{X} = \vec{x}^1 \times \cdots \times \vec{x}^d \in P$ of the cakes in which player $i$ prefer the $k$-tuple of pieces $(j_1, \ldots, j_k)$.

Like before, the hungry player condition implies that the Komia covering conditions hold, and the closed preference assumption implies that the sets $A_T^j$ are closed. Thus, by colorful Komia here exists a set $T = \{T_1, \ldots, T_k\}$ of $k$-tuples in $[n]^k$, such that the barycenters of the corresponding faces contain the point $b_T$ in their convex hull, and such that there exists a partition $\vec{X} \in \bigcap_{i \in [k]} A_T^i$. Then the $k$-partite hypergraph $H = (\bigcup_{i=1}^k V_i, T_i)$, where $V_i = [n]$ for all $i$, has a perfect fractional matching, and hence by Proposition 7.7 we have $\nu^*(H) \geq \frac{|V(H)|}{k} = \frac{n}{k} = n$. Since $H$ is $k$-partite, by Füredi’s Theorem, this implies $\nu(H) \geq \frac{\nu^*(H)}{k-1} \geq \frac{n}{k-1}$.

Thus in the partition $\vec{X}$ there are $\frac{n}{k-1}$ pairwise disjoint $k$-tuple of pieces that are chosen by distinct players. Since $p = k(n-1) + 1$, we have $n = \frac{p+k-1}{k}$, and therefore the number of satisfied players is at least $\frac{n}{k-1} = \frac{p+k-1}{k(k-1)} > \frac{p}{k(k-1)}$.

Exercise 8.9. Use the colorful KKMS theorem to derive a fair division theorem on a single cake with multiple piece selection for every player.

8.8 Another application of colorful Komia: the colorful Carathéodory theorem

For a point $x \neq 0$ in $\mathbb{R}^k$ let $H(x) = \{y \in \mathbb{R}^k : \langle x, y \rangle = 0\}$ be the hyperplane perpendicular to $x$ and let $H^+(x) = \{y \in \mathbb{R}^k : \langle x, y \rangle \geq 0\}$ be the closed halfspace with boundary $H(x)$ containing $x$.

Theorem 8.10 (Colorful Carathéodory theorem, Bárány). Let $X_1, \ldots, X_{k+1}$ be finite subsets of $\mathbb{R}^k$ with $0 \in \text{conv } X_i$ for every $i \in [k+1]$. Then there are $x_1 \in X_1, \ldots, x_{k+1} \in X_{k+1}$ such that $0 \in \text{conv } \{x_1, \ldots, x_{k+1}\}$.

Proof. We will assume that 0 is not contained in any of the sets $X_1, \ldots, X_{k+1}$, for otherwise we are done. Let $P \subseteq \mathbb{R}^k$ be a polytope containing 0 in its interior, such that if points $x$ and $y$ belong to the same face of $P$ then $\langle x, y \rangle \geq 0$. For example, a sufficiently fine subdivision of any polytope that contains 0 in its interior (slightly perturbed to be a strictly convex polytope) satisfies this condition. We can assume that any ray emanating from the origin intersects each $X_i$ in at most one point by arbitrarily deleting any additional points from $X_i$. This will not affect the property that 0 $\in \text{conv } X_i$. Furthermore, we can choose
Let $y_P = 0$. Let $i \in [k + 1]$. For each nonempty, proper face $\sigma$ of $P$ choose points $y^{(i)}_\sigma$ and sets $A^{(i)}_\sigma$ in the following way:

- If there exists $x \in C_\sigma \cap X_i$: let $y^{(i)}_\sigma = x$ and $A^{(i)}_\sigma = \{y \in P : \langle y, x \rangle \geq 0\} = P \cap H^+(x)$.
- Otherwise: let $y^{(i)}_\sigma$ be some point in $\sigma$ and $A^{(i)}_\sigma = \sigma$.

Suppose the theorem is not true. Then in particular, we can slightly perturb the vertices of $P$ and those points $y^{(i)}_\sigma$ that were chosen arbitrarily in $\sigma$, to make sure that for any collection of points $y^{(1)}_{\sigma_1}, \ldots, y^{(k+1)}_{\sigma_{k+1}}$ and any subset $S$ of this collection of size at most $k$, $0 \notin \text{conv } S$ (that is, 0 does not lie on any hyperplane defined by the points $y^{(i)}_\sigma$).

Let us now check that with these definitions the conditions of the colorful Komiya theorem hold. Clearly, all the sets $A^{(i)}_\sigma$ are closed. The fact that $P$ is covered by the sets $A^{(i)}_\sigma$ for every fixed $i$ follows from the condition $0 \notin \text{conv } X_i$. Indeed, this condition implies that for every $p \in P$ there exists a point $x \in X_i$ with $\langle p, x \rangle \geq 0$, and therefore, for the face $\sigma$ of $P$ for which $x \in C_\sigma$, we have $p \in A^{(i)}_\sigma$.

Now fix a proper face $\sigma$ of $P$. We claim that $\sigma \subset A^{(i)}_\sigma$ for every $i$. Indeed, either $X_i \cap C_\sigma = \emptyset$ in which case $A^{(i)}_\sigma = \sigma$, or otherwise, pick $x \in X_i \cap C_\sigma$ and let $\lambda > 0$ such that $\lambda x \in \sigma$; then for every $p \in \sigma$ we have $\langle p, \lambda x \rangle \geq 0$ by our assumption on $P$, and thus $\langle p, x \rangle \geq 0$, or equivalently $p \in A^{(i)}_\sigma$.

Thus by colorful Komiya there exist faces $\sigma_1, \ldots, \sigma_{k+1}$ of $P$ such that $\bigcap_{i=1}^{k+1} A^{(i)}_{\sigma_i} \neq \emptyset$ and $0 \in \text{conv } \{y^{(1)}_{\sigma_1}, \ldots, y^{(k+1)}_{\sigma_{k+1}}\}$.

We claim that $\bigcap_{i=1}^{k+1} A^{(i)}_{\sigma_i} = \{0\}$. Indeed, suppose that $0 \neq x_0 \in \bigcap_{i=1}^{k+1} A^{(i)}_{\sigma_i}$. Fix $i \in [k + 1]$. If $y^{(i)}_{\sigma_i} \in C_{\sigma_i} \cap X_i$, then since $x_0 \in A^{(i)}_{\sigma_i} = \{y \in P : \langle y, y^{(i)}_{\sigma_i} \rangle \geq 0\}$, we have $\langle x_0, y^{(i)}_{\sigma_i} \rangle \geq 0$, and therefore $y^{(i)}_{\sigma_i} \in H^+(x_0)$ by definition. Otherwise $x_0 \in A^{(i)}_{\sigma_i} = \sigma_i$ and $y^{(i)}_{\sigma_i} \in \sigma_i$, so by our choice of $P$ we have again that $\langle x_0, y^{(i)}_{\sigma_i} \rangle \geq 0$, and therefore $y^{(i)}_{\sigma_i} \in H^+(x_0)$. Thus all the points $y^{(1)}_{\sigma_1}, \ldots, y^{(k+1)}_{\sigma_{k+1}}$ are in $H^+(x_0)$. But since $0 \in \text{conv } \{y^{(1)}_{\sigma_1}, \ldots, y^{(k+1)}_{\sigma_{k+1}}\}$, this implies that the convex hull of the points in $\{y^{(1)}_{\sigma_1}, \ldots, y^{(k+1)}_{\sigma_{k+1}}\} \cap H(x_0)$ contains the origin. Now, the dimension of $H(x_0)$ is $k - 1$, and thus by Carathéodory’s theorem there exists a set $S$ of at most $k$ of the points in $y^{(1)}_{\sigma_1}, \ldots, y^{(k+1)}_{\sigma_{k+1}}$ with $0 \in \text{conv } S$, in contradiction to our general position assumption.

We have shown that $\bigcap_{i=1}^{k+1} A^{(i)}_{\sigma_i} = \{0\}$, and thus in particular, $A^{(i)}_{\sigma_i} \neq \sigma_i$ for all $i$. By our definitions, this implies $y^{(i)}_{\sigma_i} \in X_i$ for all $i$, concluding the proof of the theorem. \qed

**Remark 8.11.** Note that we could have avoided the usage of Carathéodory’s theorem in the proof of Theorem 8.10 by taking a more restrictive assumption on the polytope $P$, namely, that $\langle x, y \rangle > 0$ whenever the points $x$ and $y$ belong to the same face of $P$. Therefore, in particular, Theorem 8.10 specializes to Carathéodory’s theorem in the case where all the sets $X_i$ are the same.