Topological Methods in Combinatorics - KKM-Type Theorems

Lecture Notes

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1 Two problems

1.1 A colorful version of Gallai's theorem

Let F be a family of sets in $\mathbb{R}^d$. A *matching* in F is a sub-family of pairwise disjoint sets. The *matching number* $\nu(F)$ is the largest size of a matching in F. A *cover* in F is a set of points in $\mathbb{R}^d$ intersecting all the sets in F. The *covering number* $\tau(F)$ is the minimal size of a cover in F.

A well-known theorem of Gallai from the 1960's is the following:

Theorem 1.1 (Gallai). *Let $\mathcal{F}$ be a finite family of compact intervals in $\mathbb{R}$. Then $\tau(\mathcal{F}) = \nu(\mathcal{F})$.*

Exercise 1.2. *Prove Gallai's theorem (hint: construct an algorithm for finding a cover and a matching of the same size. Why is this enough?)*

Another way to state the theorem: if $\tau(\mathcal{F}) > k$ then there exists a matching in $\mathcal{F}$ of size $k+1$. Now, suppose that we have $k+1$ finite families $\mathcal{F}_1, \dots, \mathcal{F}_{k+1}$ of compact intervals in $\mathbb{R}$, with $\tau(\mathcal{F}_i) > k$ for all i. Can we find a *rainbow matching* (that is a matching $\mathcal{M}$ with $\mathcal{M} \cap F_i = 1$ for all i)?

1.2 Fair division of a cake

Suppose that we have k players with subjective preferences on a given cake (identified with the $[0,1]$ interval). In any partition of the cake each player gives a list of pieces they prefer from the cake in that partition. Two conditions are satisfied:

1. Players are hungry (define)
2. Preference sets are closed (define).

Does there necessarily exist a partition of the cake and allocation of pieces such that every player receives one of his favorite pieces?

2 Sperner's Lemma, Brouwer's fixed-point theorem, and the KKM theorem

Sperner's Lemma is an important result in combinatorical topology. It was originally proved by Sperner in 1928 to obtain a simple proof of Brouwer's fixed-point theorem (1910).

Theorem 2.1 (Brouwer's Fixed Point Theorem, 1911)**.** *Any continuous map f from a finite dimensional ball B to itself has a fixed point, namely a point $x \in B$ such that $f(x) = x$.*

BFPT has numerous applications in mathematics and economics as does Sperner's Lemma.

Definition 2.2.

- *The* n-dimensional simplex *is the convex hull of $n+1$ affinely independent points in $\mathbb{R}^{n+1}$.*
- *The* standard *n-dimensional simplex is*

$$\Delta^n = \mathrm{conv}\{e_1, \ldots, e_{n+1}\} \subset \mathbb{R}^{n+1},$$

  where $e_1, \ldots, e_{n+1}$ are the standard basis vector in $\mathbb{R}^{n+1}$.
- *The convex hull of any nonempty subset of the $n+1$ points that define an simplex is called a a* face *of the simplex (so a face of a simplex is also a simplex). If the subset defining a face is of size k, then the* dimension *of the face is $k-1$.*
- *The 0-dimensional faces of a simplex are called* vertices*. The 1-dimensional faces of are called* edges*.*
- *A* triangulation *is a subdivision of a simplex (or more generally - a polytope) into simplices.*
- *If $x \in \Delta^{n+1}$, then supp(x) is the minimal face of Δ^{n+1} containing x.*

Definition 2.3 (Sperner coloring)**.** *Let T be a triangulation of a n-dimensional simplex Δ. Let $\lambda : v(T) \to [n+1]$ be a coloring of the vertices of T with colors $[n+1]$ such that:*

- *Every vertex of Δ gets a distinct color.*
- *For every $v \in T$ we have $\lambda(v) \in \lambda(V(supp(v)))$.*

Then λ is called a Sperner coloring *of T.*

Let T be a triangulation of Δ^n and let λ be a Sperner coloring of $V(T)$. A *rainbow simplex* is a simplex in T whose vertices have all distinct colors.

Theorem 2.4 (Sperner's lemma 1928)**.** *Let Δ be a n-dimensional simplex, let T be a triangulation of Δ, and let $\lambda : V(T) \to [n+1]$ be a Sperner coloring of Δ. Then the number of n-dimensional rainbow simplices in T is odd. In particular, there is at least one n-dimensional rainbow simplex.*

Proof. By induction on n.

Base case: $n = 1$. Then Δ is a 1-dimensional simplex, namely a segment $[a, b]$. T is a triangulation of Δ, namely a subdivision of $[a, b]$ into smaller segments. We have two colors $\{1, 2\}$ and a, b receive different colors. Now, going from a to b, we must switch color an odd number of times so that we get a different color in b. Hence there is an odd number of subsegments (simplices in T) that receive two different colors.

Case 2: $n = 2$. Consider the face 12 of Δ. By induction, it has odd many rainbow 1-dimensional simplices of T, colored 12. Define a graph G as follows: the vertices of G are the 2-dimensional simplices of T, and one additional vertex v in the outer face. Two vertices are connected be an edge in G if they share an edge of T colored by 12. By the handshake lemma, G has even many vertices of odd degree, and since $deg(v)$ is odd, it has odd many vertices corresponding to 2-dimensional simplices of T. Such simplices must have exactly one 12 edge, and hence they must be rainbow.

General n. Consider the face $12 \cdots n$ of Δ. By induction, it has odd many rainbow $(n-1)$-dimensional simplices of T, colored $12 \cdots n$. Define a graph G as follows: the vertices of G are the n-dimensional simplices of T, and one additional vertex v in the outer face. Two vertices are connected be an edge in G if they share an edge of T colored by $12 \cdots n$. By the handshake lemma, G has even many vertices of odd degree, and since $deg(v)$ is odd, it has odd many vertices corresponding to n-dimensional simplices of T. Such simplices must have exactly one $12 \cdots n$ edge, and hence they must be rainbow.

Another proof by induction: Let Q be the number of simplices in T colored $(1, 1, 2)$ or $(1, 2, 2)$. Let R be the number of rainbow simplices in T. Let X be the number of $(1, 2)$ edges on the boundary of Δ. Let Y be the number of $(1, 2)$ edges in the interior of Δ.

- For each simplex on T colored $(1, 1, 2)$ of $(1, 2, 2)$ we get two $(1, 2)$ edges, while for each rainbow simplex we get one $(1, 2)$ edge.
- On the other hand, this way we count all of internal edges colored $(1, 2)$ twice, and all of the boundary edges colored $(1, 2)$ once. Thus, $2Q + R = 2Y + X$

We know that X is odd because the $[1, 2]$ boundary of Δ is colored in a Sperner coloring. So R must be odd.

General Case: We have a Sperner coloring on T by $n + 1$ colors. Let R denote the number of rainbow simplices in T. Let Q denote the number of d-dimensional simplices in T that get all of the colors except $n + 1$, i.e. they are colored by all of the colors in $[n]$, so that exactly one of these colors is used twice and the others are used once.

Example: In the $d = 3$ case Q counts the number of simplices colored as follows: $(1, 1, 2, 3), (1, 2, 2, 3), (1, 2, 3, 3)$.

Also, consider the $(n-1)$-dimensional faces that are colored by exactly the colors in $[n]$ (namely "$[n]$-rainbow simplices"). Let X be the number of such faces of the boundary of Δ, and let Y be the number of such faces in the interior of Δ.

Again we count in two different ways:

- Every simplex of type R ($[n+1]$-rainbow simplex) contributes exactly one $(n-1)$-face colored by $\{1, 2, \ldots, n\}$. Every simplex of the type Q (colored by $\{1, 2, \ldots, n\}$) contributes exactly 2 $(n-1)$-faces colored by $\{1, 2, \ldots, n\}$.

- $(n-1)$-dimensional faces that are colored by $\{1, 2, \ldots, n\}$ and lie on the boundary of Δ appear in one d-dimensional simplex of T, while if it does not lie on the boundary it appears in 2 simplices of T. Hence we get $2Q + R = X + 2Y$.

Now note that on the boundary of Δ, the only $(n-1)$-dimensional faces colored by all colors $\{1, 2, \ldots, n\}$ can be of the $(n-1)$-face of Δ whose vertices are colored by $\{1, 2, \ldots, n\}$ (because this is a Sperner Coloring). By induction, the number of such faces X is odd. So R must be odd too. $\square$

The KKM Theorem is a continuous version of the Sperner lemma that was proved by Knaster-Kuratowski-Mazurkiewicz in 1929.

Theorem 2.5 (The KKM theorem, 1928)**.** *Let Δ be an n-dimensional simplex on the vertex set $\{v_1, v_2, \ldots, v_{n+1}\}$. Let $A_1, A_{2, n+1}$ be closed sets covering Δ so that $\sigma \subseteq \bigcup_{v_i \in \sigma} A_i$. Then $\bigcap_{i=1}^{n+1} A_i \neq \emptyset$.*

Proof. Embed Δ in $\mathbb{R}^{n+1}$ in the standard way. For every $i \in [n+1]$ define a function $g_i : \Delta_n \to \mathbb{R}$ by

$$g_i(x) = dist(x, A_i) = inf\{|x - a| : a \in A_i\} = min\{|x - a| : a \in A_i\}.$$

Define $f : \Delta \to \Delta$ by

$$f(x) = f((x_1, x_2, \ldots, x_{n+1})) = \frac{(x_1 + g_1(x), \ldots, x_{n+1} + g_{n+1}(x))}{1 + \sum_{j=1}^{n+1} g_j(x)}.$$

This is indeed a map to Δ, moreover, it is continuous. So by BFPT there exists a $z \in \Delta$ such that $f(z) = z$.

Let $S(z) = \{i \in [n+1] : z_i > 0\} = V(\mathrm{supp}(z))$. By the conditions of the theorem,

$$z \in \mathrm{supp}(z) = \mathrm{conv}\{e_i \mid i \in S(z)\} \subseteq \bigcup_{i \in S(z)} A_i.$$

Thus there exists $i_0 \in S(z)$ so that $z \in A_{i_0}$, and therefore $g_{i_0}(z) = dist(z, A_{i_0}) = 0$.

Now, $f(z) = z$ implies $(f(z))_{i_0} = z_{i_0}$, and therefore

$$z_{i_0} = \frac{z_{i_0} + g_{i_0}(z)}{1 + \sum_{j=1}^{n+1} g_j(z)} = \frac{z_{i_0}}{1 + \sum_{j=1}^{n+1} g_j(z)}.$$

Note that $z_{i_0} \neq 0$ since $i_0 \in S(z)$, and therefore we can divide by z_{i_0} to get

$$\frac{1}{1 + \sum_{j=1}^{n+1} g_j(z)} = 1,$$

implying $\sum_{j=1}^{n+1} g_j(z) = 0$. This entails $g_j(z) = 0$ for all $j \in [n+1]$. Since A_j are closed, this implies $z \in A_j$ for all j. $\square$

Proposition 2.6. *KKM is true also if all of the sets A_i are open.*

Proof. We can find closed sets B_i satisfying KKM such that $b_i \subseteq A_i$ and $\bigcap_{i \in K} B_i \neq \emptyset$ if and only if $\bigcap_{i \in K} A_i \neq \emptyset$ for every $K \subseteq [n+1]$. Then we apply the theorem with the sets B_i □

Exercise 2.7. *Sperner's lemma, BFPT, and the KKM theorem are easily proved one from the other. Prove all six implications.*

Exercise 2.8.
1. Prove the KKM theorem from the Borsuk-Ulam theorem.
2. Can you prove the opposite?

3 Warm-up: Proving Gallai's theorem with KKM

Proof. Let F be a family of intervals with $\tau(F) = k+1$. We show that $\nu(F) \geqslant k+1$. (This will give $\nu \geqslant \tau$, and we already know that $\nu \leqslant \tau$). Since F is finite, by rescaling $\mathbb{R}$ we may assume that all of the intervals in F are contained in the open segment $(0,1)$.

Let Δ be the k-dimensional standard simplex in $\mathbb{R}^{k+1}$. Every point in Δ corresponds to a distribution of k (not necessarily distinct) points $u_1(x), \ldots, u_k(x)$ on $[0,1]$, where $u_i(x) = \sum_{j=1}^{i} x_j$.

Since k or less points do not cover F (because $\tau(F) > k$), for every $x \in \Delta$ there exists an interval $f \in F$ that does not contain any of the points $u_1(x), \ldots, u_k(x)$ corresponding to x. Thus $f \subset (u_{i-1}, u_i)$ for some $1 \leqslant i \leqslant k+1$.

Define sets $A_1, \ldots, A_{k+1} \subset \Delta$ as follows:

$$A_i = \{x \in \Delta \mid \text{ there exists } f \in \mathcal{F} \text{ such that } f \subset (u_{i-1}(x), u_i(x))\}$$

Note that by the above, $\Delta \subseteq \bigcup_{i=1}^{k+1} A_i$.

Claim 3.1. *$A_1, \ldots, A_{k+1}$ form a KKM cover.*

Proof. First, since the intervals in $\mathcal{F}$ are closed, the sets A_i is open for all i. Indeed, if $f \in F$ witnesses the fact the $x \in A_i$ (that is $f \subset (u_{i-1}(x), u_i(x))$), then since f is closed, for some small enough ε, every point $x' \in B_\varepsilon(x)$ satisfies $f \subset (u_{i-1}(x'), u_i(x'))$, and therefore $B_\varepsilon(x) \subset A_i$.

Second, let σ be a face of Δ and let $x \in \sigma$. If $e_i \notin \sigma$ then $x_i = 0$, and thus $(u_{i-1}(x), u_i(x)) = \emptyset$, showing that no $f \in F$ satisfies $f \subset (u_{i-1}(x), u_i(x))$. Since $x \in \Delta \subseteq \bigcup_{i=1}^{k+1} A_i$, we must have $x \in A_i$ for some $i \in \sigma$, showing $\sigma \subset \bigcup_{i \in \sigma} A_i$. □

So by the KKM theorem there exists an $x \in \bigcap_{j=1}^{k+1} A_j$. Consider the distribution of k points $u_1(x), \ldots, u_k(x)$ corresponding to x. We have $x \in A_i$, and thus there is an interval $f_i \in (u_{i-1}(x), u_i(x))$ for every $i \in [k+1]$. The set of intervals $f_1, \ldots, f_{k+1}$ is a matching of size $k+1$, showing $\nu(F) \geqslant k+1$ as promised. □

4 Colorful KKM

Theorem 4.1 (Gale, 1982)**.** *Let* $(A_i^j \mid i, j \in [n])$ *be* n *KKM covers of* Δ^{n-1}*. Then there exists a permutation* $\pi \in S_n$ *so that* $\bigcap A^i_{\pi(i)} \neq \emptyset$*.*

Proof. Let $\{T_k\}$ be a sequence of barycentric subdivisions of Δ_{n-1} (so the simplex diameters going to 0). For every vertex v the triangulation T_k, assign a role $r(v)$ to v according to the dimension of the face in T_{k-1} it subdivide: if v is the barycenter of a face of dimension m in T_{k-1}, then let $r(v) = m + 1$. (For example if v subdivide an edge of T_{k-1} then $r(v) = 2$.)

By the definition of the barycentric subdivision the roles of the vertex set of every simplex in T_k are distinct. So, every full dimensional simplex in T_k contains vertices of all n possible roles.

We construct a coloring function $c : V(T_k) \to [n]$ as follows: For $v \in V(T_k)$, choose j such that $v \in A_j^{r(v)}$ and $j \in \text{supp}(v)$. Such j exists because $(A_j^{r(v)} \mid j \in n)$ is a KKM cover, and thus

$$v \in \text{supp}(v) \subset \bigcup_{j \in \text{supp}(v)} A_j^{r(v)}.$$

Let $c(v) = j$. Note that for every $v \in V(T_k)$ we have $c(v) \in \text{supp}(v)$, and thus $c : V(T_k) \to [n]$ is a Sperner coloring.

Now apply Sperner's lemma. We obtain a rainbow simplex σ_k in T_k. That is, σ_k has the following property: there exists $\pi_k \in S_n$ so that in $V(\sigma_k) = \{v_1, \dots, v_n\}$ with $r(v_i) = i$ and $c(v_i) = \pi_k(i)$ for all $i \in [n]$. Since Δ is compact, and the diameter of σ_k tends to 0 when k tends to infinity, the sequence $\{\sigma_k\}_{k \geqslant 1}$ has a subsequence converging to a point $x \in \Delta$, and since S_n is finite, this subsequence has an infinite subsequence $\{\sigma_{k_j}\}_{j \geqslant 1}$ in which all the permutations π_{k_j} are the same permutation π. By construction, this means that $\text{dist}(x, A^i_{\pi(i)}) \leqslant \varepsilon$ for all $i \in [n]$ and for every $\varepsilon > 0$. Since the sets A_i^j are closed we have $x \in \bigcap_{i \in [n]} A^i_{\pi(i)}$, as needed. □

5 Solution to the two problems

5.1 Colorful Gallai's theorem

Theorem 5.1. *Let* $\mathcal{F}_1, \dots, F_{k+1}$ *be finite families of compact intervals in* $\mathbb{R}$*, with* $\tau(\mathcal{F}_i) > k$ *for all* i*. Then there exists a full rainbow matching.*

Proof. Since F_i are finite and all the intervals are bounded, by rescaling $\mathbb{R}$ we may assume that all of the intervals in $\bigcup F_i$ are contained in the open segment $(0, 1)$. Let Δ be the k-dimensional standard simplex in $\mathbb{R}^{k+1}$. Every point in Δ corresponds to a distribution of k (not necessarily distinct) points $u_1(x), \dots, u_k(x)$ on $(0, 1)$, where $u_i(x) = \sum_{j=1}^{i} x_j$. For $j \in [k + 1]$ define sets $A_1^j, \dots, A_{k+1}^j \subset \Delta$ as follows:

$$A_i^j = \{x \in \Delta \mid \text{there exists } f \in F_j \text{ such that } f \subset (u_{i-1}(x), u_i(x))\}.$$

Like in the proof of Gallai's theorem, for every $j \in [k+1]$, the collection $(A_1^j, \ldots, A_{k+1}^j)$ forms a KKM cover of Δ. So by the colorful KKM theorem, there exists an $x \in \bigcap_{i=1}^{k+1} A_{\pi(i)}^i$. Consider the distribution of k points $u_1(x), \ldots, u_k(x)$ corresponding to x. We have $x \in A_{\pi(i)}^i$, and thus there is an interval $f_i \in F_i$ with $f_i \in (u_{\pi(i)-1}(x), u_{\pi(i)}(x))$ for every $i \in [k+1]$. Thus the set of intervals $f_i, \ldots, f_{k+1}$ is a colorful matching of size $k+1$, as needed. □

5.2 Fair division of a cake

Theorem 5.2 (Stromquist 1980, Woodall 1980)**.** *If n hungry players have closed preference sets on a cake then there exists a fair division.*

Proof. Let A_i^j be the set of partitions in which player j prefers piece i.

Claim 5.3. *The "hungry player" assumption implies that that the KKM covering conditions hold.*

Proof. Let $x \in \sigma$ be a partition of the cake. If $i \notin \sigma$, then $x_i = 0$, and since player j is hungry, he does not prefer piece i in the partition x. Therefore $x \in A_i^j$ for some $i \in \sigma$, showing $\sigma \subset \bigcup_{i \in \sigma} A_j^i$. □

Claim 5.4. *The "closed preference set" assumption implies that that the sets A_i^j are closed.*

Proof. This is immidiate by definition. If a converging sequence of partitions $\{x_t\}_{t \geqslant 1} \subset A_i^j$ then by definition of the closed preference set" assumption, the limit partition is also in A_i^j. □

Thus by the colorful KKM theorem there exists a partition x with $x \in \bigcap_{i=1}^{n} A_{\pi(i)}^i$ for some $\pi \in S_n$. In this partition every player prefer a distinct piece, as needed. □

6 Another application: Piercing sets in the plane with lines

Let $\mathcal{F}$ be a family of convex sets in the plane. We say that $\mathcal{F}$ has *property* $T(r)$ if every r or fewer sets in $\mathcal{F}$ admit a *line transversal*, that is, there exists a line intersecting these sets. We say that $\mathcal{F}$ is *pierced* by k lines if there are k lines in the plane whose union intersects all the sets in $\mathcal{F}$. The *line-piercing number* of the family is the minimum k so that $\mathcal{F}$ is pierced by k lines. Some known bounds:

- Santalo (1940): For any k, $T(k)$ property does not imply $\mathcal{F}$ is pierced by one line.
- Eckhoff (1969): $T(4)$ property implies $\mathcal{F}$ is pierced by two lines.
- This implies: for $k \geqslant 4$, $T(k)$ property implies $\mathcal{F}$ is pierced by two lines.
- $T(2)$ property does not imply $\mathcal{F}$ is pierced by constant many lines (e.g., n points in general position).
- Eckhoff (1975): $T(3)$ property does not imply $\mathcal{F}$ is pierced by 2 lines.

- Eckhoff (1993): $T(3)$ property implies $\mathcal{F}$ is pierced by 4 lines.

Eckhoff conjectured in 1993 that the latter can be further improved, namely, that the $T(3)$ property implies that $\mathcal{F}$ is pierced by 3 lines. The following theorem is a generalization of this statement: it specializes to Eckhoff's conjecture when all the families are the same.

Theorem 6.1 (McGinnis-Zerbib 2021)**.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_6$ be families of compact connected sets in $\mathbb{R}^2$. If every three sets $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}$, $1 \leqslant i_1 < i_2 < i_3 \leqslant 6$, have a line transversal, then there exists $i \in [6]$ such that the line-piercing number of $\mathcal{F}_i$ is at most* 3.

Proof of Theorem 6.1. We may scale the plane so that every set in $\mathcal{F}_j$ is contained in the unit disk D for each j. Denote by U the unit circle. Let $f(t)$ be a parameterization of U defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$.

A point $x = (x_1, \dots, x_6) \in \Delta^5$ corresponds to 6 points on U given by $f_i(x) = f(\sum_{j=1}^{i} x_j)$ for $1 \leqslant i \leqslant 6$. Let $l_1(x) = l_4(x) = [f_1(x), f_4(x)]$, $l_2(x) = l_5(x) = [f_2(x), f_5(x)]$, and $l_3(x) = l_6(x) = [f_3(x), f_6(x)]$.

For $i = 1, \dots, 6$ let R_x^i be the interior of the region bounded by $l_{i-1}(x)$, $l_i(x)$ and the arc on U connecting $f_{i-1}(x)$ and $f_i(x)$. Notice that $R_x^i = \emptyset$ when $x_i = 0$. Also, it is possible that some of the regions R_x^i intersect.

Set $1 \leqslant j \leqslant 6$ and let A_i^j be the set of points $x \in \Delta^5$ so that R_x^i contains a set $F \in \mathcal{F}_j$. Since the sets $F \in \mathcal{F}_j$ are closed, A_i^j is open. If there is some $x \in \Delta^5$ for which $x \notin \bigcup_{i=1}^{6} A_i^j$, then since the sets in $\mathcal{F}_j$ are connected, every set in $\mathcal{F}_j$ must intersect $\bigcup_{i=1}^{3} l_i(x)$, and we are done. So we assume for contradiction that $\Delta^5 = \bigcup_{i=1}^{6} A_i^j$ for all j. Observe that if $x \in \operatorname{conv}\{e_i : i \in I\}$ for some $I \subset [6]$ then $R_x^k = \emptyset$ for $k \notin I$, and therefore, $x \in \bigcup_{i \in I} A_i^j$ for all j. This shows that the conditions of the colorful KKM theorem hold.

Thus, by the colorful KKM theorem, there exists some permutation $\pi \in S_6$ and a point $p = (p_1, \dots, p_6) \in \bigcap_{i=1}^{6} A_i^{\pi(i)}$. Therefore, each of the open regions R_p^i contains a set $S_i \in \mathcal{F}_{\pi(i)}$, $i = 1, \dots, 6$, and in particular $R_p^i \neq \emptyset$ and thus $p_i \neq 0$ for all i. We claim that at least one of the triples $\{S_1, S_3, S_5\}$ or $\{S_2, S_4, S_6\}$ is not a tight triple. To see this, note that the regions R_p^1, R_p^3, R_p^5 are pairwise disjoint or the regions R_p^2, R_p^4, R_p^6 are pairwise disjoint (depending on the orientation of the triangle bounded by the lines l_1, l_2, l_3). Without loss of generality, we assume R_p^1, R_p^3, R_p^5 are pairwise disjoint, and in this case, the three sets S_1, S_3, S_5 is not a tight triple. This is a contradiction. □

By a similar method, one can also prove:

Theorem 6.2 (McGinnis-Zerbib, 2021)**.** *Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ be finite families of compact, connected sets in the plane such that any collection of four sets, one from each $\mathcal{F}_i$, has a line transversal. Then for some $i \in [4]$, $\mathcal{F}_i$ has line piercing number at most 2.*

When all the families are the same this specializes to Eckhoff's result the the $T(4)$ property implies that $\mathcal{F}$ is pierced by 2 lines.

Exercise 6.3. *Prove Theorem 6.2.*

7 Generalizing Gallai's theorem: Piercing d-intervals

7.1 Theorems on piercing d-interval families

A d-interval is a union of at most d intervals on $\mathbb{R}$.

Theorem 7.1 (Tardos-Kaiser 1995). *If $\mathcal{F}$ a finite family of compact d-intervals, then $\tau(\mathcal{F}) \leqslant (d^2 - d + 1)\nu(\mathcal{F})$.*

For $d = 2$ this is tight. For larger d it is known to be tight up to $\log d$ factor: Matoušek showed that there are families of d-intervals with $\tau(\mathcal{F}) = \Omega(\frac{d^2}{\log d})\nu(\mathcal{F})$

The following a colorful version of the Tardos-Kaiser theorem. It specializes to the Tardos-Kaiser theorem when all the families are the same:

Theorem 7.2 (Frick-Zerbib 2019). *Let $\mathcal{F}_1, \ldots, \mathcal{F}_{k+1}$ be $k+1$ finite families of compact d-intervals. If $\tau(\mathcal{F}_i) > k$ for all i, then there exists a rainbow matching of size at least $\frac{k+1}{d^2-d+1}$.*

All known proofs for Theorems 7.1 and 7.2 are topological. Alon showed via elementary methods a slightly worse bound: $\tau \leqslant 2d^2$.

7.2 Notions from hypergraph theory

A *hypergraph* H is a family $E(H)$ of subsets, called *edges*, of a ground set $V(H)$ of *vertices*. A hypergraph H is r*-uniform* if all its edges are of size r. It is r*-partite* if there exists a partition $V_i \cup \cdots \cup V_r$ of $V(H)$ such that $|e \cap V_i| = 1$ for every edge $e \in H$ and every $1 \leqslant i \leqslant r$. The sets V_i are called the *vertex sides* of H. Note that an r-partite hypergraph is in particular r-uniform. A *graph* is a 2-uniform hypergraph and a 2-partite graph is also called *bipartite*.

Let $H = (V, E)$ be a hypergraph. A *matching* in a H is a set of disjoint edges. The *matching number* $\nu(H)$ is the maximum size of a matching in H. A *cover* of H is a set of vertices intersecting all edges. The *covering number* $\tau(H)$ is the minimum size of a cover in H.

We can also regard a matching as a function $f : E(H) \to \{0, 1\}$ satisfying the condition that adjacent edges do not get both 1, namely, for all $v \in V$, $\sum_{v \in e} f(e) \leqslant 1$. Then the matching number is

$$\nu(H) = \max\Big\{\sum_{e \in E} f(e) \mid f \text{ is a matching}\Big\}.$$

Now we can consider a fractional relaxation of this notion: A *fractional matching* in a hypergraph H is a function $f : E(H) \to [0, 1]$, satisfying the condition for all $v \in V$, $\sum_{v \in e} f(e) \leqslant 1$. The *fractional matching number* is defined as

$$\nu^*(H) = \max\Big\{\sum_{e \in E} f(e) \mid f \text{ is a functional matching}\Big\}.$$

Similarly, a cover can be viewed as a function $g : V(H) \to \{0,1\}$, satisfying the condition that for all $e \in E(H)$, $\sum_{v\in e} g(v) \geqslant 1$, and the covering number is

$$\tau(H) = \min\{\sum_{v\in V} g(v) \mid g \text{ is a cover}\}.$$

Now, a *fractional cover* of H is a function $g : V(H) \to [0,1]$, satisfying the condition that for all $e \in E(H)$, $\sum_{v\in e} g(v) \geqslant 1$. The *fractional covering number* $\tau^*(H)$ is defined by

$$\tau^*(H) = \min\{\sum_{v\in V} g(v) \mid g \text{ is a fractional cover}\}.$$

Exercise 7.3. *Prove the following:*

1. *By linear programming duality, show that* $\nu^*(H) = \tau^*(H)$ *for all* H.
2. *If* r *is the maximum size of an edge in a hypergraph* H *then*

$$\nu(H) \leqslant \nu^*(H) = \tau^*(H) \leqslant \tau(H) \leqslant r\nu(H).$$

Example 7.4. *In* K_3, $\nu = 1$, $\nu^* = \tau^* = 3/2$, $\tau^* = 2$.

A *perfect fractional matching* (PFM) is a fractional matching $f : E(H) \to [0,1]$ such that for all $v \in V$, $\sum_{v\in e} f(e) = 1$. Not every hypergraph has a PFM. For example, K_3 has a PFM, but a path on 3 vertices does not. A hypergraph is called *balanced* if it has a PFM.

The following is a trivial consequence of Exercise 7.3.

Proposition 7.5. *If* $|e| \leqslant r$ *for all* $e \in E(H)$, *then* $\nu(H) \geqslant \frac{\nu^*(H)}{r}$.

Proof. We have $\frac{\nu^*(H)}{r} \leqslant \frac{\tau(H)}{r} \leqslant \frac{r\nu(H)}{r} = \nu(H)$. □

A theorem of Füredi shows the bound in Proposition 7.5 can be slightly improved:

Theorem 7.6 (Füredi)**.** *If* $|e| \leqslant r$ *for all* $e \in E(H)$, *then* $\nu(H) \geqslant \frac{\nu^*(H)}{r-1+\frac{1}{r}}$. *Morover, if* H *is* r*-partite, then* $\nu(H) \geqslant \frac{\nu^*(H)}{r-1}$.

We will need also:

Proposition 7.7. *If* H *is a balanced hypergraph with maximal edge size* r, *then* $\nu^*(H) \geqslant \frac{|V(H)|}{r}$.

Proof. Let $f : E(H) \to [0,1]$ be a perfect fractional matching. Then $\nu^*(H) \geqslant \sum_{e\in E} f(e)$. Now,

$$|V| = \sum_{v\in V} 1 = \sum_{v\in V}\sum_{e\ni v} f(e) = \sum_{e\in E}\sum_{v\in e} f(e) \leqslant r\sum_{e\in E} f(e) \leqslant r\nu^*(H).$$

□

7.3 The KKMS Theorem

For a face σ of the standard k-simplex Δ^k, let $s(k) = \{i \in [k+1] \mid e_i \text{ is a vertex in } \sigma\}$.

Definition 7.8. *Let Δ be the standard k-dimensional simplex on vertex set $e_1, \ldots, e_{k+1}$. We say that faces $\sigma_1, \ldots, \sigma_m$ are* balanced *if the hypergraph with vertex set $V(H) = [k+1]$ and edge set $E(H) = \{s(\sigma_1), \ldots, s(\sigma_m)\}$ is balanced.*

Example 7.9. *In Δ^2 the faces $12, 23, 13$ are balanced. Also, the faces $1, 23$ are balanced.*

Theorem 7.10 (The KKMS theorem, Shapley 1973)**.** *Let Δ be the n-dimensional standard simplex and let A_τ be a closed (open) set for every nonempty face of Δ, such that for every face σ, $\sigma \subseteq \bigcup_{\tau \subseteq \sigma} A_\tau$. Then there exists balanced faces $\sigma_1, \ldots, \sigma_{k+1}$, such that $\bigcap_{i=1}^{k+1} A_{\sigma_i} \neq \emptyset$.*

Exercise 7.11. *Prove that the KKMS theorem implies the KKM theorem.*

7.4 Proof of the KKMS theorem

In this section we will prove the KKMS theorem. We first need a few preperations.

Proposition 7.12. *Faces $\sigma_1, \ldots, \sigma_m$ of Δ are balanced if and only if $b_\Delta \in \operatorname{conv}\{b_{\sigma_1}, \ldots, b_{\sigma_m}\}$.*

Proof. Let $s_i = s(\sigma_i) = \{j : e_j \in \sigma_i\}$. Let χ^i be the characteristic vector of S_i, that is

$$\chi^i_j = \begin{cases} 1 & j \in s_i \\ 0 & \text{otherwise.} \end{cases}$$

Now, the faces $\sigma_1, \ldots, \sigma_m$ are balanced if and only if the hypergraph $([k+1], \{s_1, \ldots, s_m\})$ has a PFM, which means that there exist weights $\alpha_1, \ldots, \alpha_m \in [0,1]$ such that

$$\alpha_1 \chi_1 + \alpha_2 \chi_2 + \cdots + \alpha_m \chi_m = (1, \ldots, 1) = \chi_\Delta.$$

This is equivalent to

$$\frac{\alpha_1}{k+1}\chi_1 + \frac{\alpha_2}{k+1}\chi_2 + \cdots + \frac{\alpha_m}{k+1}\chi_m = \left(\frac{1}{k+1}, \ldots, \frac{1}{k+1}\right) = b_\Delta,$$

which we can write as

$$\frac{\alpha_1 |s_1|}{k+1} b_{\sigma_1} + \frac{\alpha_2 |s_2|}{k+1} b_{\sigma_2} + \cdots + \frac{\alpha_m |s_m|}{k+1} b_{\sigma_{k+1}} = b_\Delta.$$

To see that this is a convex combination note that

$$\sum_{i=1}^{m} \frac{\alpha_i |s_i|}{k+1} = \frac{1}{k+1} \sum_{i=1}^{m} \alpha_i |s_i| = \frac{1}{k+1} \sum_{i=1}^{m} \sum_{j \in s_i} \alpha_i = \frac{1}{k+1} \sum_{j=1}^{k+1} \sum_{j \in s_i} \alpha_i = \frac{1}{k+1} \sum_{j=1}^{k+1} 1 = 1,$$

because the function $s_i \mapsto \alpha_i$ is a PFM. ◻

Exercise 7.13. *Prove that if $f : \Delta \to \Delta$ is continuous and homotopic to identity on the boundary of Δ, then f is surjective.*

We first prove a "discrete version" of the KKMS theorem:

Theorem 7.14. *Let T be a triangulation of the n-dimensional simplex Δ. Let*

$$\lambda : V(T) \to \{\sigma : \sigma \neq \emptyset \text{ is a face of } \Delta\}$$

be a labeling function such that $\lambda(v) \subseteq supp(v)$. Then there exists a simplex $\tau \in T$ whose vertex labelings are balanced.

Proof. Define a map $f : V(T) \to \Delta$ by $v \mapsto b_{\lambda(v)}$. Extend f linearly to a map $F : \Delta \to \Delta$. That is, if x is in a simplex $\sigma = \text{conv}\{v_1, \ldots, v_{n+1}\}$ of T, and $x = \alpha_1 v_1 + \cdots + \alpha_{n_1} v_{n+1}$ (here α_i are the coefficients in the convex combination that gives x) then $F(x) = \alpha_1 f(v_1) + \cdots + \alpha_{n+1} f(v_{n+1})$.

By definition, F is continuous. Moreover, if $v \in V(T)$ lies in a face σ of Δ, then $\lambda(v) \in \sigma$ and thus $f(v) \in \sigma$ and thus $F(\sigma) = \sigma$. So F is homotopic to the identity map on the boundary of Δ. By the exercise, F is surjective. Therefore there exists a point $p \in \Delta$, such that $F(p) = b_\Delta$.

Let $\tau = \text{conv}\{v_1, \ldots, v_{n+1}\}$ be a simplex in T containing p. Then by definition, there exist $\alpha_1, \ldots, \alpha_{n+1}$ with $\sum \alpha_i = 1$, $\alpha_i \geqslant 0$ such that

$$b_\Delta = F(p) = \alpha_1 f(v_1) + \cdots + \alpha_{n+1} f(v_{n+1}) = \alpha_1 b_{\lambda(v_1)} + \cdots + \alpha_{n+1} b_{\lambda(v_{n+1})}$$

Thus $b_\Delta \in \text{conv}\{b_{\lambda(v_1)}, \ldots, b_{\lambda(v_{n+1})}\}$. By the proposition, $\lambda(v_1), \ldots, \lambda(v_{n+1})$ are balanced. □

Exercise 7.15. *Prove the KKMS theorem from the previous theorem (similarly to the way KKM is proved from Sperner).*

7.5 Proof of the Tardos-Kaiser theorem

Theorem 7.16 ($d = 2$: Tardos 1995, $d \geqslant 2$: Kaiser 1997)**.** *If $\mathcal{F}$ is a finite family of compact d-intervals then $\tau(F) \leqslant (d^2 - d + 1)\nu(F)$.*

Proof. Since $\mathcal{F}$ is finite and the d-interval are compact, we can assume that all the sets in $\mathcal{F}$ are contained in $(0, 1)$. Suppose $\tau(\mathcal{F}) = k + 1$. We will show that $\nu(\mathcal{F}) \geqslant \frac{k+1}{d^2-d+1}$. This will imply $\frac{\tau(\mathcal{F})}{\nu(\mathcal{F})} \leqslant \frac{k+1}{\frac{k+1}{d^2-d+1}} = d^2 - d + 1$. Let Δ be the standard k-dimensional simplex. Every point $x = (x_1, \ldots, x_{k+1})$ corresponds to a distribution of k points $u_1(x), \ldots, u_k(x)$ on $(0, 1)$, where $u_i(x) = \sum_{j=1}^{i} x_j$.

Define $u_0(x) = 0$, $u_{k+1}(x) = 1$. For every face σ of Δ define a set A_σ as follows: $x = (x_1, \ldots, x_{k+1}) \in A_\sigma$ if and only if there exists a d-interval $f \in \mathcal{F}$ such that following two conditions hold:

(a) $f \subseteq \bigcup_{i \in \sigma} (u_{i-1}(x), u_i(x))$, and

(b) for every $i \in \sigma$, $f \cap (u_{i-1}(x), u_i(x)) \neq \emptyset$.

Note that since $\tau(\mathcal{F}) = k+1$, the points $u_1(x), \ldots, u_k(x)$ do not cover $\mathcal{F}$. So there is a d-interval in $\mathcal{F}$ that is not covered, showing that $\Delta^k \subseteq \bigcup_{\sigma \subset \Delta^k} A_\sigma$.

Claim 7.17. *$\{A_\sigma\}$ is a KKMS cover.*

Proof. The set A_σ are all open (since the d-intervals are closed). We want to show $\tau \subseteq \bigcup_{\sigma \subseteq \tau} A_\sigma$, for every face τ. Let $x \in \tau$. For $i \notin \tau$ we have $x_i = 0$ and therefore $u_i(x) = u_{i-1}(x)$. This implies that the segment $(u_{i-1}, u_i) = \emptyset$, and thus cannot contain satisfy condition (b). So $x \notin A_\sigma$ when σ contains a vertex $i \notin \tau$. Since $x \in \Delta^k \subseteq \bigcup_{\sigma \subset \Delta^k} A_\sigma$, we conclude x must be in some A_σ with $\sigma \subseteq \tau$, showing $\tau \subseteq \bigcup_{\sigma \subseteq \tau} A_\sigma$. □

By the KKMS theorem there exist balanced faces $\sigma_1, \ldots, \sigma_m$ of Δ such that $\bigcap_{i=1}^m A_{\sigma_i} \neq \emptyset$. Let $x \in \bigcap_{i=1}^m A_{\sigma_i}$, and let $u_1, \ldots, u_k$ be the corresponding distribution on $(0,1)$. Let $s_i = s(\sigma_i)$. The fact that $\sigma_1, \ldots, \sigma_{k+1}$ are balanced implies that the hypergraph

$$H = ([k+1], \{s_1, \ldots, s_{k+1}\})$$

has a PFM. Note that $|s_i| \leqslant d$ for all i, since every $f \in \mathcal{F}$ has at most d non-empty interval components. Thus by Proposition 7.7, $\nu^*(H) \geqslant \frac{|V(H)|}{d} \geqslant \frac{k+1}{d}$. By Füredi's theorem $\nu(H) \geqslant \frac{\nu^*(H)}{d-1+\frac{1}{d}}$, and thus

$$\nu(H) \geqslant \frac{\nu^*(H)}{d-1+\frac{1}{d}} \geqslant \frac{k+1}{d^2-d+1}.$$

Therefore, there is a matching $M = \{s_{i_1}, \ldots, s_{i_{\nu(H)}}\}$ in H of size at least $\frac{k+1}{d^2-d+1}$. Note that the fact that this is a matching implies that the sets $U_{i_t} = \bigcup_{j \in s_{i_t}} (u_{j-1}, u_j)$ are disjoint.

For every $1 \leqslant t \leqslant \nu(H)$ let $f_{i_t} \in \mathcal{F}$ be a d-interval witnessing the fact that $x \in A_{\sigma_{i_t}}$. Then by definition $f_{i_t} \subset U_{i_t}$, and therefore the set $\{f_{i_1}, \ldots, f_{i_{\nu(H)}}\}$ is a matching in $\mathcal{F}$ of size at least $\nu(H)$, as needed. □

7.6 The colorful KKMS theorem

Theorem 7.18 (Colorful KKMS, Shih-Lee 1993)**.** *Let $(A^i_\sigma), i \in [k+1]$ be $k+1$ KKMS covers of Δ^k. Then there exists balanced faces $\sigma_1, \ldots, \sigma_{k+1}$ such that $\bigcap_{i=1}^{k+1} A^i_{\sigma_i} \neq \emptyset$.*

We will prove a more general theorem later on.

Exercise 7.19. *Prove Theorem 7.2 using the colorful KKMS theorem.*

8 Separated d-intervals and fair-division of multiple cakes

8.1 Theorems on piercing separated d-intervals

A *separated d-interval* is a union of d intervals, one on each of d separated copies of $\mathbb{R}$.

Theorem 8.1 (Tardos-Kaiser). *If $\mathcal{F}$ is a finite set of compact separated d-intervals, then $\tau(\mathcal{F}) \leqslant (d^2 - d)\nu(\mathcal{F})$.*

Theorem 8.2 (Frick-Zerbib). *Let $\mathcal{F}_i$, $i \in [kd+1]$, be $kd+1$ hypergraphs of separated d-intervals. If $\tau(\mathcal{F}_i) > kd$ for all i, then there exists a rainbow matching of size at least $\frac{k+1}{d-1}$.*

If all the families $\mathcal{F}_i$ are the same family $\mathcal{F}$, with $\tau(\mathcal{F}) = \lfloor kd+1 \rfloor$ then we get Theorem 8.1. Indeed, in that case $\tau(\mathcal{F})/\nu(\mathcal{F}) \leqslant (kd+1)\frac{d-1}{k+1} \leqslant d(d-1)$.

8.2 A theorem on fair division of multiple cakes

Suppose that there are k cakes and $p = k(n-1)+1$ hungry players with closed preference sets. In every partition of the k cakes into n pieces each, each player chooses his favorite k-tuples of pieces (a k-tuple of pieces is a choice of one piece in each cake).

Theorem 8.3 (Nyman-Su-Zerbib). *There exists a division of the k cakes where at least $\lceil \frac{p}{k(k-1)} \rceil$ players prefer pairwise disjoint k-tuple of pieces.*

Remark 8.4. *The theorem can be stated also for other values of p (with slightly different bound on the number of satisfied players), but for simplicity we will only give the proof only for the case $p = k(n-1)+1$.*

8.3 Komiya's theorem

Komiya's theorem is a far-reaching polytopal generalization of the KKMS theorem. Before stating it, let are give a reformulation of the KKMS theorem:

Theorem 8.5. *Let $P = \Delta$ be the k-dimensional simplex. For every non-empty face σ of P let $A_\sigma \subset P$ be a closed set and let y_σ be the barycenter of σ. If for every face τ of P we have $\tau \subset \bigcup_{\sigma \subset \tau} A_\sigma$, then there exist faces $\sigma_1, \ldots, \sigma_{k+1}$ of P such that $\bigcap_{i=1}^{k+1} A_{\sigma_i} \neq \emptyset$ and $y_\Delta \in \operatorname{conv}\{y_{\sigma_1}, \ldots, y_{\sigma_{k+1}}\}$.*

Now Komiya's theorem states that we can replace Δ by any k-dimensional polytope P, and we can replace the barycenters by any points $y_\sigma \in \sigma$, and the theorem will still be correct.

Theorem 8.6 (Komiya, 1994). *Let P be a k-dimensional polytope. For every non-empty face σ of P let $A_\sigma \subset P$ be a closed set and let y_σ be the point in σ. If for every face τ of P we have $\tau \subset \bigcup_{\sigma \subset \tau} A_\sigma$, then there exist faces $\sigma_1, \ldots, \sigma_{k+1}$ of P such that $\bigcap_{i=1}^{k+1} A_{\sigma_i} \neq \emptyset$ and $y_P \in \operatorname{conv}\{y_{\sigma_1}, \ldots, y_{\sigma_{k+1}}\}$.*

8.4 Proof of Komiya's theorem

First, we prove a "discrete version" of Komiya's theorem. Given a triangulation T of P, a *Komiya labeling* of T is a map $f\colon V(T) \longrightarrow \{\sigma \mid \sigma \text{ a non-empty face of } P\}$ such that $f(v) \subseteq \operatorname{supp}(v)$.

Theorem 8.7. *Let T be a triangulation of P, and let f be a Komiya labeling of T. For every nonempty face σ of P choose a point $y_\sigma \in \sigma$. Then there is a face τ of T such that $y_P \in \operatorname{conv}\{y_{f(v)} \mid v \text{ vertex of } \tau\}$.*

Proof. Let $g : V(T) \to P$ be the map $v \mapsto y_{f(v)}$, and let $G : P \to P$ be a linear extension of g. Then G is a continuous map. Note that for every face σ of P, we have that $G(\sigma) \subset \sigma$: indeed, if $x \in \sigma$ then $G(x) = \operatorname{conv}\{g(v_1), \dots, g(v_m)\}$ for some $v_1, \dots, v_m \in \sigma$, and by the Komiya labeling condition, $g(v_i) \in \sigma$ for all i, and thus $G(x) \in \sigma$ as well. This implies that G is homotopic to the identity on ∂P, and thus G is surjective. Therefore, there exists a point $x \in P$ such that $G(x) = y_P$. Let τ be a full dimension face of T containing x. By definition

$$y_P = G(x) \in G(\tau) = \operatorname{conv}\{g(v) \mid v \text{ is a vertex of } \tau\} = \operatorname{conv}\{y_{f(v)} \mid v \text{ is a vertex of } \tau\}.$$

□

Proof of Komiya's theorem. Let $\varepsilon > 0$, and let T be a triangulation of P such that every face of T has diameter at most ε. Given a Komiya cover (A_σ) we define a Komiya labeling on T in the following way: For a vertex v of T, label v by a face $\sigma \subset \operatorname{supp}(v)$ such that $v \in A_\sigma$. Such a face σ exists since $v \in \operatorname{supp}(v) \subset \bigcup_{\sigma \subset \operatorname{supp}(v)} A_\sigma$. Thus by Theorem 8.7 there is a full dimensional face τ of T whose vertices are labeled by faces $\sigma_1, \dots, \sigma_{k+1}$ of P such that $y_P \in \operatorname{conv}\{y_{\sigma_1}, \dots, y_{\sigma_{k+1}}\}$. In particular, the ε-neighborhoods of the sets A_{σ_i}, $i \in [k+1]$, intersect. Now let ε tend to zero. As there are only finitely many collections of faces of P, one collection $\sigma_1, \dots, \sigma_{k+1}$ must appear infinitely many times. By compactness of P the sets A_{σ_i}, $i \in [k+1]$, then all intersect since they are closed. □

Note that Komiya's theorem is true also if all the sets A_σ are open, by the same argument as before.

8.5 A colorful extension of Komiya's theorem

For a face σ of P and a point $y_P \in P$ we denote by C_σ the *cone of* σ, that is, the union of all rays emanating from y_P that intersect σ.

Theorem 8.8 (The colorful Komiya theorem, Frick-Zerbib 2019)**.** *Let P be a k-dimensional polytope, and let y_P be a point in P. Suppose for every nonempty proper face σ of P we are given $k+1$ points $y_\sigma^{(1)}, \dots, y_\sigma^{(k+1)} \in C_\sigma$ and $k+1$ closed sets $A_\sigma^{(1)}, \dots, A_\sigma^{(k+1)} \subset P$. If $\sigma \subset \bigcup_{\tau \subset \sigma} A_\tau^{(j)}$ for every face σ of P and every $j \in [k+1]$, then there exist faces $\sigma_1, \dots, \sigma_{k+1}$ of P such that $y_P \in \operatorname{conv}\{y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}\}$ and $\bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)} \neq \emptyset$.*

Proof. Let $\varepsilon > 0$, and let T be a triangulation of P such that every face of T has diameter at most ε. We will also assume that the chosen points $y_\sigma^{(1)}, \dots, y_\sigma^{(k+1)}$ are contained in σ. This assumption does not restrict the generality of our proof since $y_P \in \operatorname{conv}\{x_1, \dots, x_{k+1}\}$ for vectors $x_1, \dots, x_{k+1} \in \mathbb{R}^n$ if and only if $y_P \in \operatorname{conv}\{\alpha_1 x_1, \dots, \alpha_{k+1} x_{k+1}\}$ with arbitrary coefficients $\alpha_i > 0$.

Denote by T' the barycentric subdivision of T. For $v \in V(T')$ let $r(v)$ be the dimension of the face in T that v subdivide, plus 1. We now define a Komiya labeling of T': Let $v \in V(T')$. By the conditions of the theorem, v is contained in a set $A_\tau^{r(v)}$ where $\tau \subset \operatorname{supp}(v)$. We label v by τ. Thus by Theorem 8.7 there exists a full dimensional face τ of T' whose vertices are labeled by faces $\sigma_1, \dots, \sigma_{k+1}$ of P such that $y_P \in \operatorname{conv}\{y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}\}$. In particular, the ε-neighborhoods of the sets $A_{\sigma_i}^{(i)}$, $i \in [k+1]$, intersect. Now use a limiting argument as before. $\square$

By taking $P = \Delta$ and y_σ to be the barycenter of σ, we get a proof of Theorem 7.18.

8.6 Colorful d-intervals: Proof of Theorem 8.2

Let $\mathcal{F}_i$ be a family of separated d-intervals for all $i \in [kd+1]$. For $f \in \mathcal{F}$ let f^t be the t-th interval component of f on ℓ_t. For a point $\vec{x} = (x_1, \dots, x_{k+1}) \in \Delta_k$ let $p_{\vec{x}}(j) = \sum_{t=1}^{j} x_t \in [0,1]$. Since $\mathcal{F}$ is finite, by rescaling the d copies $\mathbb{R}$ we may assume that for every $f \in \bigcup_{i\in[kd+1]} \mathcal{F}_i$, f^t is a non-empty subset of $(0,1)$ on ℓ_t. Let $P = (\Delta_k)^d$, and note that $\dim P = kd$.

Every point $\vec{X} = \vec{x}^1 \times \cdots \times \vec{x}^d \in P$ corresponds to a distribution of kd points, k points on each of the lines $\ell_1, \dots, \ell_d$ as follows: on line ℓ_t the k points are $p_{\vec{x}^t}(1), \dots, p_{\vec{x}^t}(k)$. Since $\tau(\mathcal{F}_i) \geqslant kd+1$, these kd points do not cover $\mathcal{F}_i$. So there exists $f \in \mathcal{F}_i$ that is not covered. This means that $f^t \subset (p_{\vec{x}^t}(j_t - 1), p_{\vec{x}^t}(j_t))$ on ℓ_t for all $t \in [d]$, for some choice of $j_i, \dots, j_d$.

We define a $kd+1$ Komiya covers of P as follows. Every face of P corresponds to a tuple $T = (T_1, \dots, T_d)$, with $T_i \subset [k+1]$ for all $i \in [d]$. In our setting A_T^i is non-empty only if $T = (j_i, \dots, j_d) \subset [k+1]^d$ (that is, all the T_i's are singletons). For a d-tuple $T = (j_1, \dots, j_d) \subset [k+1]^d$ let A_T^i consist of all $\vec{X} = \vec{x}^1 \times \cdots \times \vec{x}^d \in P$ for which there exists $f \in \mathcal{F}_i$ satisfying $f^t \subset (p_{\vec{x}^t}(j_t - 1), p_{\vec{x}^t}(j_t))$ on ℓ_t for all $t \in [d]$.

By the same argument as before, the sets A_T^i are open and satisfy the covering condition of the colorful Komiya theorem. Thus, by the colorful Komiya theorem, there exists a set $\mathcal{T} = \{T_1, \dots, T_{kd+1}\}$ of d-tuples in $[k+1]^d$, such that the barycenters of the corresponding faces contain the point $b_P = (\frac{1}{k+1}, \dots, \frac{1}{k+1}) \times \cdots \times (\frac{1}{k+1}, \dots, \frac{1}{k+1}) \in P$ in their convex hull, and such that $\bigcap_{i\in[kd+1]} A_{T_i}^i \neq \emptyset$. Then the d-partite hypergraph $H = (\bigcup_{i=1}^d V_i, \mathcal{T})$, where $V_i = [k+1]$ for all i, has a perfect fractional matching, and hence by Proposition 7.7 we have $\nu^*(H) \geqslant k+1$. By Füredi's Theorem, this implies $\nu(H) \geqslant \frac{\nu^*(H)}{d-1} \geqslant \frac{k+1}{d-1}$.

Now, by the same argument as before, taking $\vec{X} \in \bigcap_{i\in[kd+1]} A_{T_i}^i$ we obtain a matching in $\mathcal{F}$ of the same size as a maximal matching in H, concluding the proof of the theorem.

8.7 Multiple cakes: Proof of Theorem 8.3

Let $P = (\Delta^{n-1})^k$ of dimension $k(n-1)$. Every point $\vec{X} = \vec{x}^1 \times \cdots \times \vec{x}^d$ in P corresponds to a partition of the k cakes into n pieces each as follows: piece j in cake t is the piece $(p_{\vec{x}^t}(j-1), p_{\vec{x}^t}(j))$.

We define a $k(n-1)+1$ Komiya covers of P as follows. As before, every face of P corresponds to a tuple $T = (T_1, \dots, T_k)$, with $T_i \subset [n]$ for all $i \in [k]$. In our setting A^i_T is non-empty only if $T = (j_i, \dots, j_k) \subset [n]^d$ (that is, all the T_i's are singletons, so when the face is a vertex of P). For a k-tuple $T = (j_1, \dots, j_k) \subset [n]^d$ let A^i_T consist of all partitions $\vec{X} = \vec{x}^1 \times \cdots \times \vec{x}^d \in P$ of the cakes in which player i prefer the k-tuple of pieces $(j_1, \dots, j_k)$.

Like before, the hungry player condition implies that the Komiya covering conditions hold, and the closed preference assumption implies that the sets A^i_T are closed. Thus, by colorful Komiya here exists a set $\mathcal{T} = \{T_1, \dots, T_{k(n-1)+1}\}$ of k-tuples in $[n]^k$, such that the barycenters of the corresponding faces contain the point b_P in their convex hull, and such that there exists a partition $\vec{X} \in \bigcap_{i \in [k(n-1)+1]} A^i_{T_i}$. Then the k-partite hypergraph $H = (\bigcup_{i=1}^d V_i, \mathcal{T})$, where $V_i = [n]$ for all i, has a perfect fractional matching, and hence by Proposition 7.7 we have $\nu^*(H) \geqslant \frac{|V(H)|}{k} = \frac{nk}{k} = n$. Since H is k-partite, by Füredi's Theorem, this implies $\nu(H) \geqslant \frac{\nu^*(H)}{k-1} \geqslant \frac{n}{k-1}$.

Thus in the partition $\vec{X}$ there are $\frac{n}{k-1}$ pairwise disjoint k-tuple of pieces that are chosen by distinct players. Since $p = k(n-1)+1$, we have $n = \frac{p+k-1}{k}$, and therefore the number of satisfied players is at least $\frac{n}{k-1} = \frac{p+k-1}{k(k-1)} > \frac{p}{k(k-1)}$.

Exercise 8.9. *Use the colorful KKMS theorem to derive a fair division theorem on a single cake with multiple piece selection for every player.*

8.8 Another application of colorful Komiya: the colorful Carathéodory theorem

For a point $x \neq 0$ in $\mathbb{R}^k$ let $H(x) = \{y \in \mathbb{R}^k : \langle x, y \rangle = 0\}$ be the hyperplane perpendicular to x and let $H^+(x) = \{y \in \mathbb{R}^k : \langle x, y \rangle \geqslant 0\}$ be the closed halfspace with boundary $H(x)$ containing x.

Theorem 8.10 (Colorful Carathéodory theorem, Bárány)**.** *Let $X_1, \dots, X_{k+1}$ be finite subsets of $\mathbb{R}^k$ with $0 \in \operatorname{conv} X_i$ for every $i \in [k+1]$. Then there are $x_1 \in X_1, \dots, x_{k+1} \in X_{k+1}$ such that $0 \in \operatorname{conv}\{x_1, \dots, x_{k+1}\}$.*

Proof. We will assume that 0 is not contained in any of the sets $X_1, ..., X_{k+1}$, for otherwise we are done. Let $P \subset \mathbb{R}^k$ be a polytope containing 0 in its interior, such that if points x and y belong to the same face of P then $\langle x, y \rangle \geqslant 0$. For example, a sufficiently fine subdivision of any polytope that contains 0 in its interior (slightly perturbed to be a strictly convex polytope) satisfies this condition. We can assume that any ray emanating from the origin intersects each X_i in at most one point by arbitrarily deleting any additional points from X_i. This will not affect the property that $0 \in \operatorname{conv} X_i$. Furthermore, we can choose

P in such a way that for each face σ and $i \in [k+1]$ the intersection $C_\sigma \cap X_i$ contains at most one point.

Let $y_P = 0$. Let $i \in [k+1]$. For each nonempty, proper face σ of P choose points $y_\sigma^{(i)}$ and sets $A_\sigma^{(i)}$ in the following way:

- If there exists $x \in C_\sigma \cap X_i$: let $y_\sigma^{(i)} = x$ and $A_\sigma^{(i)} = \{y \in P : \langle y, x\rangle \geqslant 0\} = P \cap H^+(x)$.
- Otherwise: let $y_\sigma^{(i)}$ be some point in σ and $A_\sigma^{(i)} = \sigma$.

Suppose the theorem is not true. Then in particular, we can slightly perturb the vertices of P and those points $y_\sigma^{(i)}$ that were chosen arbitrarily in σ, to make sure that for any collection of points $y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}$ and any subset S of this collection of size at most k, $0 \notin \operatorname{conv} S$ (that is, 0 does not lie on any hyperplane defined by the points $y_\sigma^{(i)}$).

Let us now check that with these definitions the conditions of the colorful Komiya theorem hold. Clearly, all the sets $A_\sigma^{(i)}$ are closed. The fact that P is covered by the sets $A_\sigma^{(i)}$ for every fixed i follows from the condition $0 \in \operatorname{conv} X_i$. Indeed, this condition implies that for every $p \in P$ there exists a point $x \in X_i$ with $\langle p, x\rangle \geqslant 0$, and therefore, for the face σ of P for which $x \in C_\sigma$ we have $p \in A_\sigma^{(i)}$.

Now fix a proper face σ of P. We claim that $\sigma \subset A_\sigma^{(i)}$ for every i. Indeed, either $X_i \cap C_\sigma = \emptyset$ in which case $A_\sigma^{(i)} = \sigma$, or otherwise, pick $x \in X_i \cap C_\sigma$ and let $\lambda > 0$ such that $\lambda x \in \sigma$; then for every $p \in \sigma$ we have $\langle p, \lambda x\rangle \geqslant 0$ by our assumption on P, and thus $\langle p, x\rangle \geqslant 0$, or equivalently $p \in A_\sigma^{(i)}$.

Thus by colorful Komiya there exist faces $\sigma_1, \dots, \sigma_{k+1}$ of P such that $\bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)} \neq \emptyset$ and $0 \in \operatorname{conv}\{y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}\}$.

We claim that $\bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)} = \{0\}$. Indeed, suppose that $0 \neq x_0 \in \bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)}$. Fix $i \in [k+1]$. If $y_{\sigma_i}^{(i)} \in C_{\sigma_i} \cap X_i$, then since $x_0 \in A_{\sigma_i}^{(i)} = \{y \in P : \langle y, y_{\sigma_i}^{(i)}\rangle \geqslant 0\}$, we have $\langle x_0, y_{\sigma_i}^{(i)}\rangle \geqslant 0$, and therefore $y_{\sigma_i}^{(i)} \in H^+(x_0)$ by definition. Otherwise $x_0 \in A_{\sigma_i}^{(i)} = \sigma_i$ and $y_{\sigma_i}^{(i)} \in \sigma_i$, so by our choice of P we have again that $\langle x_0, y_{\sigma_i}^{(i)}\rangle \geqslant 0$, and therefore $y_{\sigma_i}^{(i)} \in H^+(x_0)$. Thus all the points $y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}$ are in $H^+(x_0)$. But since $0 \in \operatorname{conv}\{y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}\}$, this implies that the convex hull of the points in $\{y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}\} \cap H(x_0)$ contains the origin. Now, the dimension of $H(x_0)$ is $k-1$, and thus by Carathéodory's theorem there exists a set S of at most k of the points in $y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}$ with $0 \in \operatorname{conv} S$, in contradiction to our general position assumption.

We have shown that $\bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)} = \{0\}$, and thus in particular, $A_{\sigma_i}^{(i)} \neq \sigma_i$ for all i. By our definitions, this implies $y_{\sigma_i}^{(i)} \in X_i$ for all i, concluding the proof of the theorem. □

Remark 8.11. *Note that we could have avoided the usage of Carathéodory's theorem in the proof of Theorem 8.10 by taking a more restrictive assumption on the polytope P, namely, that $\langle x, y\rangle > 0$ whenever the points x and y belong to the same face of P. Therefore, in particular, Theorem 8.10 specializes to Carathéodory's theorem in the case where all the sets X_i are the same.*