The topological Tverberg problem beyond prime powers

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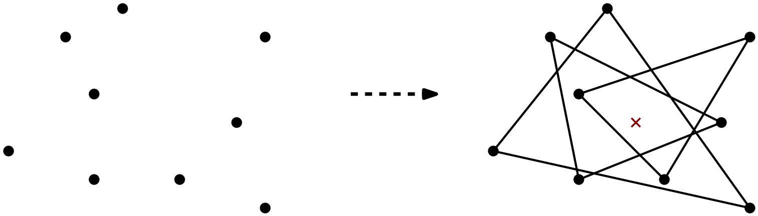
I. Prehistory

Birch 1959

Any $3q$ points in the plane can be partitioned into q triples that span triangles that share common point.



Birch 1959 — restatements

1. For any straight-line drawing of K_{3q} in the plane there is a point in the plane surrounded by q pairwise vertex-disjoint 3-cycles.

2. For any linear map from the 3-uniform hypergraph $H^{(3)}_{3q}$ on $3q$ vertices to the plane, there is a perfect matching that has a point in common.

Statement 2 is true for maps $K_{2q} = H^{(2)}_{2q} \to \mathbb{R}$. The point of intersection is a *median*.

Birch 1959 — restatements

1. For any straight-line drawing of K_{3q} in the plane there is a point in the plane surrounded by q pairwise vertex-disjoint 3-cycles.

2. For any linear map from the 3-uniform hypergraph $H_{3q}^{(3)}$ on $3q$ vertices to the plane, there is a perfect matching that has a point in the plane in common.

What about non-linear drawings?

What about higher dimensions? Say $H_{qd+q}^{(d+1)} \to \mathbb{R}^d$.

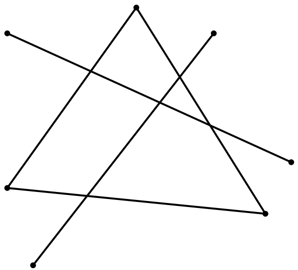
What about non-linear drawings in higher dimensions?

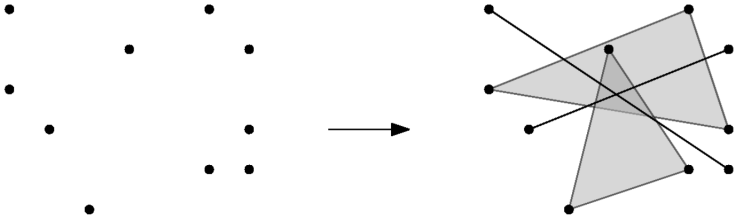
What about higher dimensions? Say $H^{(d+1)}_{qd+q} \to \mathbb{R}^d$.

This is true. One can even save d points:

Tverberg 1966

Any $(q-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into q sets whose convex hulls all share a common point.



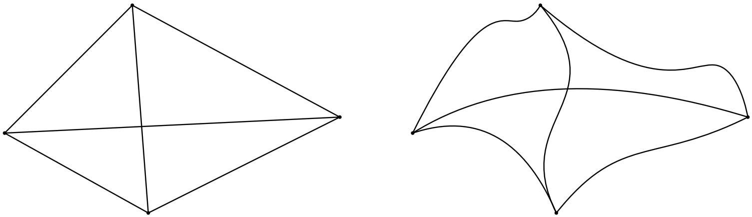


Bárány, Soberón, *Bull. Amer. Math. Soc.* (2018)

What about non-linear drawings?

What about non-linear drawings in higher dimensions?

Bárány asked Tverberg in a 1976 letter whether a continuous generalization of his theorem holds.

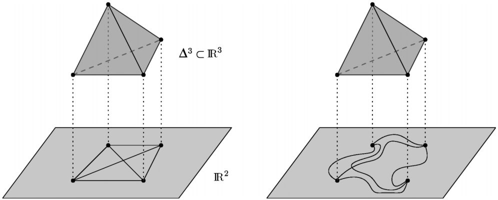


Tverberg 1966 — restatement

Any linear map $\Delta_{(q-1)(d+1)} \to \mathbb{R}^d$ identifies points from q pairwise disjoint faces.

Bárány 1976 (topological Tverberg conjecture)

Is this still true for continuous maps?



Bárány, Soberón, *Bull. Amer. Math. Soc.* (2018)

Intermission: Why would anyone care?

For one it is nice to know whether a theorem belongs to the field of geometry / linear algebra or topology. Is it rigid or flexible? Sometimes one encounters surprises, rigid–flexible dichotomies.

It is a natural generalization of non-embeddability, such as the characterization of non-planar graphs.

Topological generalizations of Tverberg's theorem are a testing ground for methods. The same / similar methods are used for

Fair divisions of necklaces

Chromatic numbers of Kneser hypergraphs

Measure partitions by fans

Inscribability problems (square peg problem)

Finding points in a certain skeleton of a polytope with a prescribed barycenter

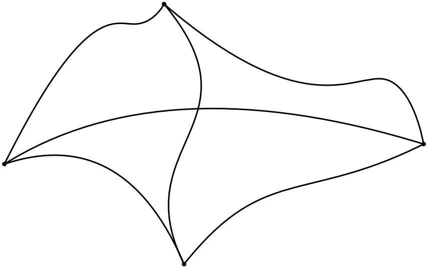
Accurately representing a partition of the vertices of a graph by an independent set

II. Enter Topology

The case $q = 2$

Bárány and Bajmoczy 1979

For any continuous $f\colon \Delta_{d+1} \to \mathbb{R}^d$ there are disjoint faces σ and τ with $f(\tau) \cap f(\sigma) \neq \emptyset$.



Use $f\colon \Delta_{d+1} \to \mathbb{R}^d$ to define a map $F\colon \Diamond_{d+2} \to \mathbb{R}^{d+1}$.

Fix two opposite facets of $\Diamond_{d+2}$ define $F(x) = (f(x), 1)$ on one facet and $F(x) = (-f(x), -1)$ on the other facet. Extend linearly and use the Borsuk–Ulam theorem, i.e., $F\colon S^{d+1} \to \mathbb{R}^{d+1}$ with $F(-x) = -F(x)$ for all x has a zero.

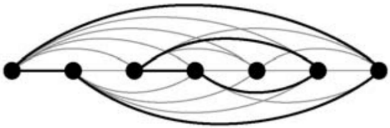
No facet of $\Diamond_{d+2}$ contains antipodal points.

Another way to think about this proof:

Given $f\colon \Delta_{d+1} \to \mathbb{R}^d$, assign the label 1 or 2 to every vertex. Check whether the face spanned by vertices with label 1 intersects the face spanned by vertices with label 2.

The facets of the cross-polytope are in one-to-one correspondence with 1–2-labellings.

The Borsuk–Ulam theorem guarantees that for some labelling the corresponding faces intersect.



Schöneborn, Ziegler, *JCTA* (2005)

For $q > 2$ we need

1. A generalized Borsuk–Ulam theorem that guarantees that certain symmetric maps have zeros.
2. The generalization for 1. must hold for the simplicial complex of valid vertex labellings with labels $1, 2, \ldots, q$.

Generalized Borsuk–Ulam theorem:

Dold 1983

Let G act on $\mathbb{R}^d$ such that the action on $\mathbb{R}^d \setminus \{0\}$ is free. Let X be a $(d-1)$-connected simplicial complex with an action by G. Then any G-equivariant map $X \to \mathbb{R}^d$ has a zero.

$G = \mathbb{Z}/2$ and $X = S^d$ is Borsuk–Ulam.

We want to use this for $G = \mathbb{Z}/q$ and $\mathbb{Z}/q$ acts on $\mathbb{R}^{q-1}$ by shifting the vertices of the regular $(q-1)$-simplex centered at zero. This is free if q is a prime.

The q-labelling complex:

The complex $(\Delta_n)^{*q}_{\Delta}$ has vertex set $[n+1] \times [q]$ and a set of vertices is a face if it does not contain two copies of the same vertex.

$(\Delta_n)^{*q}_{\Delta}$ is $(n-1)$-connected.

Bárány, Shlosman, Szűcs 1981

For q a prime any continuous map $\Delta_{(q-1)(d+1)} \to \mathbb{R}^d$ identifies points from q pairwise disjoint faces.

If $q = p^k$ for a prime p, use the group $G = (\mathbb{Z}/p)^k$ (and extend Dold's theorem to actions of such groups without fixed points) to get:

Özaydin 1987 and **Volovikov** 1996

For q a power of a prime any continuous map $\Delta_{(q-1)(d+1)} \to \mathbb{R}^d$ identifies points from q pairwise disjoint faces.

Özaydin showed that this topological machinery fails beyond prime powers.

Avvakumov, Karasev, Skopenkov 2019

If q is not a prime power and X is a complex with a free action by the symmetric group, then there is an equivariant map $X \to (\mathbb{R}^{q-1})^2$ without a zero.

F. building on deep work of **Mabillard, Wagner** 2015

For q not a prime power $d \geq 3q$, there is a continuous map $\Delta_{(q-1)(d+1)} \to \mathbb{R}^d$ that does not identify points from q pairwise disjoint faces.

Blagojević, F., Ziegler 2019

The same is true for $\Delta_{(q-1)(d+1)+c} \to \mathbb{R}^d$ for d large.

Avvakumov, Karasev, Skopenkov 2019

The same is true for $\Delta_n \to \mathbb{R}^d$ for $n = q(d+1) - q\lceil \frac{d+2}{q+1} \rceil - 2$.

III. Extremal constructions to rectify Topology's shortcomings

But of course given an integer $q \geq 2$, let $r \geq q$ be a prime power, then any map $\Delta_{(r-1)(d+1)} \to \mathbb{R}^d$ identifies points from q pairwise disjoint faces.

…even though if we tried to apply the usual topological machinery directly for q, it would fail.

We can save $r - q$ additional points (by agreeing in advance to delete the at most $r - q$ faces containing the last $r - q$ vertices). This was the best upper bound. For $d \geq 2$ and q with at least two distinct prime divisors I am unaware of a non-trivial upper bound.

In particular, if $q+1$ is a prime power then any map $\Delta_{q(d+1)-1} \to \mathbb{R}^d$ identifies points from q pairwise disjoint faces.

Thus a non-linear, higher-dimensional Birch theorem is true if q or $q+1$ is a prime power.

Maybe Tverberg's theorem does not generalize to continuous maps for q with at least two distinct prime divisors, but Birch's theorem that allows for a few more points does?

F., Soberón 2020

Let $q \geq 2$ be an integer. Then any continuous map $\Delta_{q(d+1)-1} \to \mathbb{R}^d$ identifies points from q pairwise disjoint faces.

Observe that a generic map $\Delta_{q(d+1)-1} \to \mathbb{R}^d$ does not identify points from $q+1$ pairwise disjoint faces.

This was conjectured to be the exact bound by **Blagojević, F., Ziegler**.

Part 1: A special case

Suppose $2q + 1$ is a prime.

Given a drawing of K_{3q} in the plane, produce a degenerate drawing of K_{6q} by drawing everything twice.

Recall $6q = (2q + 1 - 1)(2 + 1)$.

Vučić, Živaljević 1993

For p a prime any map $\Delta_{(p-1)(d+1)} \to \mathbb{R}^d$ identifies p points from pairwise disjoint faces such that if vertex 1 is in face j, vertex 2 is in face j or $j + 1$, if vertex 3 is in face j, vertex 4 is in face j or $j + 1$, ...

The label complex is a join of circles, and thus highly connected.

Vučić, Živaljević 1993

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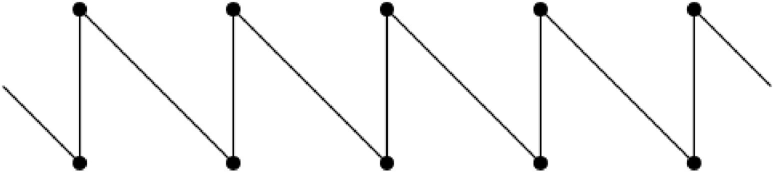

F., Soberón 2020

Let $q \geq 2$ be an integer. Then any continuous map $\Delta_{q(d+1)-1} \to \mathbb{R}^d$ identifies points from q pairwise disjoint faces.

This simple argument, combined with the earlier reduction, proves this result for q that are prime powers, or such that $q+1$ is a prime power, or where $2q+1$ is a prime.

2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21
22	23	24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39	40	41
42	43	44	45	46	47	48	49	50	51

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Part 2: The key idea

Recall: To prove the topological Tverberg theorem for q a prime power, we label every vertex with one number in $1, 2, \ldots, q$ and then ask the question whether the q induced faces have a point in common. This fails for q with at least two distinct prime divisors.

Key Idea

Instead of labeling each vertex with a number in $1, 2, \ldots, q$, label it with a subset of $1, 2, \ldots, p$ taken from a simplicial complex Σ on $[p]$, where p is a large prime.

We always want to find a subset of q labels such that no two appear at the same vertex.

To apply Dold's theorem the label complex Σ needs to be $\mathbb{Z}/p$-symmetric and highly connected, namely $(\frac{p}{q} - c)$-connected for arbitrarily large p.

To get q pairwise disjoint faces, Σ needs to have an independent set of size q.

symmetric — highly connected — large independent set

Given $q \geq 2$, find an integer $a \gg cqd$
such that $p = aq + 1$ is a prime.

Construct a label complex Σ on $[p]$ that is
– $\mathbb{Z}/p$-symmetric
– sufficiently dense to be $(\frac{p}{q} - c)$-connected
– sufficiently sparse to have an independent set of size q

The last two properties can just barely be balanced.

Given $f\colon \Delta_{n-1} \to \mathbb{R}^d$ for $n = q(d+1)$, add a dummy vertex and label each of the $n+1$ vertices with labels in Σ.

Since $\Sigma^{*(n+1)}$ is at least $(p-1)(d+1)$-connected, we get p faces whose images under f all intersect, and such that each vertex is contained in faces whose indices form a set in Σ.

The dummy vertex cannot obstruct all independent sets of size q, since p is a prime.

Thus there are q pairwise disjoint faces of Δ_{n-1} whose images under f all intersect.

Constructing the label complex Σ.

First attempt: Let Σ be the simplicial complex on $[p]$ consisting of all q-stable sets, that is, σ is a face of Σ if any two distinct elements of σ are at least q apart in cyclic order.

Σ is $\mathbb{Z}/p$-symmetric and q consecutive vertices form an independent set.

But Σ is at most around $(\frac{p}{2q})$-connected.

For example, for $q = 4$ and $p = 9q + 1 = 37$, this Σ has the isolated face $\{1, 8, 15, 22, 29, 34\}$.

Refined attempt:

Only retain those q-stable sets that can be extended to have size $a - 1$.

For example, for $q = 4$ and $p = 9q + 1 = 37$, the complex Σ no longer contains the isolated face $\{1, 8, 15, 22, 29, 34\}$.

Only some gaps of length five are allowed as in $\{1, 6, 11, 16, 21, 26, 30, 34\}$

or one large gap as in $\{1, 5, 9, 13, 17, 21, 25, 29\}$ with a gap of length nine.

This complex is $(\frac{p}{q} - 4)$-connected.

First break symmetry, and disregard the condition that gaps have size q when wrapping around from large to small numbers. It is easy to see that such complexes are highly connected.

Cover Σ with q such complexes on a shorter vertex set. Then apply the Mayer–Vietoris sequence ad infinitum.

F., Soberón 2020

Let $q \geq 2$ be an integer. Then for any sufficiently large prime p the maximal size of an independent set in any $\mathbb{Z}/p$-invariant complex on $[p]$ that is $(\frac{p}{q} - o(p))$-connected is q.

Otherwise we could prove that any $q(d+1)$ points in $\mathbb{R}^d$ can be split into $q+1$ sets whose convex hulls all share a point. This is wrong.

IV. Getting even with Topology

An immediate consequence of Dold's theorem is:

If q is a prime and $n \geq (q-1)d$, then any map $(\Delta_n)^{*q}_{\Delta} = [q]^{*(n+1)} \to \mathbb{R}^d$ maps an entire $\mathbb{Z}/q$-orbit to one point.

This is the Borsuk–Ulam theorem for $q = 2$.

This is false if q is not a prime, no matter how large n is.

There is a natural projection $\pi\colon (\Delta_n)^{*q} \to \Delta_n$.

Consequence of previous result:

If q is a prime and $n \geq (q-1)d$, then any map $(\Delta_n)^{*q} \to \mathbb{R}^d$ identifies q points $x_1, \ldots, x_q$ with $\pi(x_1) = \cdots = \pi(x_q)$.

This is still precisely the Borsuk–Ulam theorem for $q = 2$.

If q is a prime and $n \geq (q-1)d$, then any map $(\Delta_n)^{*q} \to \mathbb{R}^d$ identifies q points $x_1, \dots, x_q$ with $\pi(x_1) = \cdots = \pi(x_q)$.

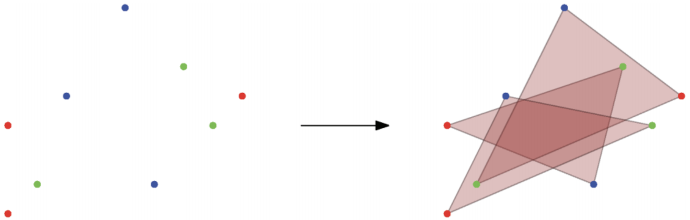
F., Soberón 2020

If $q \geq 2$ is an integer and $n \geq q(d+1)-1$, then any map $(\Delta_n)^{*(q+1)} \to \mathbb{R}^d$ identifies q points $x_1, \dots, x_q$ with $\pi(x_1) = \cdots = \pi(x_q)$.

V. Future directions

Bárány–Larman conjecture 1992

Let X be a set of $q(d+1)$ points in $\mathbb{R}^d$ partitioned into $d+1$ colors of size q. Then there is a rainbow partition of X into q sets whose convex hulls all share a point.



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Bárány–Larman 1992

True for $d=2$.

Blagojević, Matschke, Ziegler 2015

The continuous generalization is true for any d if $q+1$ is a prime.