Preservation of the joint essential matricial range

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Abstract

Let $\mathbf{A} = (A_1, \ldots, A_m)$ be an m-tuple of elements of a unital C*-algebra $\mathcal{A}$ and let M_q denote the set of $q \times q$ complex matrices. The *joint q-matricial range* $W^q(\mathbf{A})$ is the set of $(B_1, \ldots, B_m) \in M_q^m$ such that $B_j = \Phi(A_j)$ for some unital completely positive linear map $\Phi : \mathcal{A} \to M_q$. When $\mathcal{A} = \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on the Hilbert space $\mathcal{H}$, the *joint spatial q-matricial range* $W_s^q(\mathbf{A})$ of $\mathbf{A}$ is the set of $(B_1, \ldots, B_m) \in M_q^m$ for which there is a q-dimensional subspace V of $\mathcal{H}$ such that B_j is the compression of A_j to V for $j = 1, \ldots, m$. Suppose $\mathcal{K}(\mathcal{H})$ is the set of compact operators in $\mathcal{B}(\mathcal{H})$. The *joint essential spatial q-matricial range* is defined as

$$W_{ess}^q(\mathbf{A}) = \cap\{\mathbf{cl}(W_s^q(A_1 + K_1, \ldots, A_m + K_m)) : K_1, \ldots, K_m \in \mathcal{K}(\mathcal{H})\},$$

where $\mathbf{cl}(T)$ denotes the closure of the set T. Let π be the canonical surjection from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. We prove that $W_{ess}^q(\mathbf{A}) = W^q(\pi(\mathbf{A}))$, where $\pi(\mathbf{A}) = (\pi(A_1), \ldots, \pi(A_m))$. Furthermore, for any positive integer N, we prove that there are self-adjoint compact operators $K_1, \ldots, K_m$ such that

$$\mathbf{cl}\,(W_s^q(A_1 + K_1, \ldots, A_m + K_m)) = W_{ess}^q(\mathbf{A}) \quad \text{for all } q \in \{1, \ldots, N\}.$$

These results generalize those of Narcowich-Ward and Smith-Ward, obtained in the $m = 1$ case, and also generalize a result of Müller obtained in case $m \geq 1$ and $q = 1$. Furthermore, if $W_{ess}^1(\mathbf{A})$ is a simplex in $\mathbb{R}^m$, then we prove that there are self-adjoint compact operators $K_1, \ldots, K_m$ such that $\mathbf{cl}\,(W_s^q(A_1 + K_1, \ldots, A_m + K_m)) = W_{ess}^q(\mathbf{A})$ for all positive integers q.

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1 Introduction

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on an infinite dimensional Hilbert space $\mathcal{H}$ and let $\mathcal{K}(\mathcal{H})$ denote the set of compact operators in $\mathcal{B}(\mathcal{H})$. The *numerical range* of $A \in \mathcal{B}(\mathcal{H})$ is defined and denoted by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

It is a useful concept for studying matrices and operators. The Toeplitz-Hausdorff Theorem states that this set is always convex [6, 17], i.e. $tw_1 + (1-t)w_2 \in W(A)$ for all $w_1,\ w_2 \in W(A)$ and $0 \leq t \leq 1$. In general, $W(A)$ is not closed.

To study the joint behavior of multiple operators in $\mathcal{B}(\mathcal{H})$, researchers have considered the *joint numerical range* of an m-tuple $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$ defined by

$$W(\mathbf{A}) = \{(\langle A_1 x, x\rangle, \ldots, \langle A_m x, x\rangle) : x \in \mathcal{H}, \|x\| = 1\},$$

see [2, 9] and its references.

The *joint essential numerical range* of $\mathbf{A}$ is defined as

$$W_{ess}(\mathbf{A}) = \cap\{\mathbf{cl}(W(A_1 + K_1, \ldots, A_m + K_m)) : K_1, \ldots, K_m \in \mathcal{K}(\mathcal{H})\},$$

where $\mathbf{cl}(T)$ denotes the closure of the set T. These concepts were further extended to the joint spatial q-matricial range, and the joint essential spatial q-matricial range defined as follows. Let $\mathcal{V}_q$ denote the set of operators $X : \mathcal{K} \to \mathcal{H}$ such that $X^*X = I_{\mathcal{K}}$ for some q-dimensional subpace $\mathcal{K}$ of $\mathcal{H}$. The *joint spatial q-matricial range* is

$$W_s^q(\mathbf{A}) = \{(X^*A_1X, \ldots, X^*A_mX) : X \in \mathcal{V}_q\},$$

and the *joint essential q-matricial range* is

$$W_{ess}^q(\mathbf{A}) = \cap\{\mathbf{cl}(W_s^q(A_1 + K_1, \ldots, A_m + K_m)) : K_1, \ldots, K_m \text{ are compact operators}\},$$

respectively. Evidently, when $q = 1$, these concepts reduce to $W(\mathbf{A})$ and $W_{ess}(\mathbf{A})$.

Let $\mathcal{A}$ be a unital C^*-algebra. The *joint q-matricial range* of $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{A}^m$ is defined as

$$W^q(\mathbf{A}) = \{(\Phi(A_1), \ldots, \Phi(A_m)) : \Phi \text{ is a unital completely positive map from } \mathcal{A} \text{ to } M_q\}.$$

Let $\mathcal{S}$ denote the operator system [13] spanned by $\{I_{\mathcal{H}}, A_1, A_1^*, ..., A_m, A_m^*\}$. Then by Arveson's extension theorem,

$$W^q(\mathbf{A}) = \{(\Phi(A_1), \ldots, \Phi(A_m)) : \Phi \text{ is a unital completely positive map from } \mathcal{S} \text{ to } M_q\}.$$

Let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the canonical map from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and set $\pi(\mathbf{A}) = (\pi(A_1), \ldots, \pi(A_m))$ for $A_1, \ldots, A_m \in \mathcal{B}(\mathcal{H})$. Then it is clear that

$$\mathbf{cl}(W_s^q(\mathbf{A})) \subseteq W^q(\mathbf{A}) \qquad \text{and} \qquad W^q(\pi(\mathbf{A})) \subseteq W^q(\mathbf{A} + \mathbf{K}) \quad \text{for all} \quad \mathbf{K} \in \mathcal{K}(\mathcal{H})^m$$

In this paper, we will show that $W_{ess}^q(\mathbf{A}) = W^q(\pi(\mathbf{A}))$ and consequently, $W_{ess}^q(\mathbf{A})$ is C^*-convex [4]. See the definition of C^*-convex in Section 2.

When $m = 1$, this reduces to a result of Narcowich and Ward [12]. Moreover, we study the preservation problem for $W_{ess}^q(\mathbf{A})$ and $W^q(\pi(\mathbf{A}))$, namely, we prove that for each N, there is an m-tuple of self-adjoint compact operator $\mathbf{K} = (K_1, \ldots, K_m)$ such that for $1 \le q \le N$,

$$\mathbf{cl}(W_s^q(\mathbf{A} + \mathbf{K})) = W_{ess}^q(\mathbf{A}) = W^q(\mathbf{A} + \mathbf{K}) = W^q(\pi(\mathbf{A})). \tag{1.1}$$

When $m = 1$, this reduces to a result of Smith and Ward [16]; when $N = 1$, this reduces to a result of Müller [11].

Let $\mathcal{S}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : A = A^*\}$. Note that every $A \in \mathcal{B}(\mathcal{H})$ has a unique Cartesian decomposition $A = A_1 + iA_2$ such that $A_1, A_2 \in \mathcal{S}(\mathcal{H})$. Thus, one can identify $W_s^q(A)$ with $W_s^q(A_1, A_2)$, and also identify $W_{ess}^q(A)$ with $W_{ess}^q(A_1, A_2)$. For this reason, we shall henceforth focus on $W_s^q(\mathbf{A})$ and $W_{ess}^q(\mathbf{A})$ for $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$.

We show that if $W_{ess}(\mathbf{A})$ is a simplex in $\mathbb{R}^m$, then there are compact operators $K_1, \ldots, K_m$ such that (1.1) holds for all positive integers q. This extends another result in [16].

In our discussion, we will always assume that $\mathcal{H}$ is infinite dimensional. In addition to the notations $\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H})$ introduced above, we will let $\mathcal{S}_{\mathcal{K}}(\mathcal{H})$ be the set of compact operators in $\mathcal{S}(\mathcal{H})$.

2 Some basic results

The following result extends [4, Theorem 2.1].

Theorem 2.1. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{A}^m$ be self-adjoint. Then $(B_1, \ldots, B_m) \in W^q(\mathbf{A})$ if and only if*

$$\|R_0 \otimes I_q + R_1 \otimes B_1 + \cdots + R_m \otimes B_m\| \leq \|R_0 \otimes I + R_1 \otimes A_1 + \cdots + R_m \otimes A_m\| \tag{2.1}$$

for all $R_0, \ldots, R_m \in M_q$.

Proof. Note that (2.1) is equivalent to the condition that the unital linear map sending $A_j \in \mathcal{A}$ to the Hermitian matrix $B_j \in M_q$ is completely contractive. Then the result follows from the fact that every unital completely contractive map from an operator system to M_q is completely positive; for example, see [13]. ∎

Given $\mathbf{B} = (B_1, ..., B_m) \in M_q^m$ and $L \in M_q$ we set

$$L^*\mathbf{B}L = (L^*B_1L, ..., L^*B_mL).$$

A subset $\mathcal{C} \subseteq M_q^m$ is *C^*-convex* if

$$\sum_{j=1}^{N} L_j^*\mathbf{B}_jL_j \in \mathcal{C}$$

for any $\mathbf{B}_1, \ldots, \mathbf{B}_N \in \mathcal{C}$ and $L_1, \ldots, L_N \in M_q$ satisfying $\sum_{j=1}^N L_j^*L_j = I_q$. It is well known that $W^q(\mathbf{A})$ is C^*-convex. We have the following result showing that $W_{ess}^q(\mathbf{A}) = W^q(\pi(\mathbf{A}))$. It will then follow that $W_{ess}^q(\mathbf{A})$ is also C^*-convex.

Theorem 2.2. *Let $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. Then*

$$W_{ess}^q(\mathbf{A}) = W^q(\pi(\mathbf{A})), \tag{2.2}$$

which consists of $\mathbf{B} = (B_1, \ldots, B_m) \in M_q^m$ satisfying:

$$\|R_0 \otimes I_q + R_1 \otimes B_1 + \cdots + R_m \otimes B_m\| \leq \|R_0 \otimes I + R_1 \otimes \pi(A_1) + \cdots + R_m \otimes \pi(A_m)\| \tag{2.3}$$

for all $R_0, \ldots, R_m \in M_q$. Consequently, $W_{ess}^q(\mathbf{A})$ is C^*- convex.

Proof. Let S be the finite dimensional operator system given by the span of $\{\pi(I), \pi(A_1), \ldots, \pi(A_m)\}$. Then by [7, Theorem 9.11], there exists a unital q-positive map $R : S \to \mathcal{B}(\mathcal{H})$ such that $R(\pi(A_j)) = A_j + K_j$ with $K_j \in \mathcal{S}_{\mathcal{K}}(\mathcal{H})$ for $1 \leq j \leq m$.

Let $\mathbf{B} = (B_1, \ldots, B_m) \in W^q_{ess}(\mathbf{A})$. For $\varepsilon > 0$, there is

$$(\tilde{B}_1, \ldots, \tilde{B}_m) \in W^q_s(A_1 + K_1, \ldots, A_m + K_m)$$

such that $\|\tilde{B}_j - B_j\| < \varepsilon$ for all $1 \leq j \leq m$. Let $X \in \mathcal{V}_q$ be such that $X^*(A_j + K_j)X = \tilde{B}_j$ for all $1 \leq j \leq m$. Then the map $\Phi : S \to M_q$ given by $\Phi(T) = X^*R(T)X$ is a unital completely positive map satisfying $(\tilde{B}_1, \ldots, \tilde{B}_m) = (\Phi(\pi(A_1)), \ldots, \Phi(\pi(A_m))) \in W^q(\pi(\mathbf{A}))$. Since $W^q(\pi(\mathbf{A}))$ is close, $\mathbf{B} \in W^q(\pi(\mathbf{A}))$. Hence, $W^q_{ess}(\mathbf{A}) \subseteq W^q(\pi(\mathbf{A}))$.

For the reverse inclusion, if $\mathbf{B} = (B_1, \ldots, B_m) \in W^q(\pi(\mathbf{A}))$. Let $\phi : S \to M_q$ be a unital completely positive map such that $\phi(\pi(A_j)) = B_j$ for all $1 \leq j \leq m$. Let $\Phi = \phi \circ \pi$. Then Φ is a unital completely positive on the separable C^*-algebra $\mathcal{A}$ generated by $\{I, A_1, ..., A_m\}$, with $\Phi(\mathcal{A} \cap \mathcal{K}(\mathcal{H})) = \{0\}$. Hence, by [3, Theorem 2.5] (see also [1, Theorem 4]), we have that $\mathbf{B} = (B_1, \ldots, B_m) \in \mathbf{cl}(W^q_s(A_1, ..., A_m))$. Given $K_1, \ldots, K_m \in \mathcal{S}_{\mathcal{K}}(\mathcal{H})$, we also have that Φ vanishes on the intersection of the compact operators with the separable C^*-algebra generated by $\{I, A_1 + K_1, \ldots A_m + K_m\}$ with $\Phi(A_j + K_j) = \phi(\pi(A_j))$. Hence, by the same reasoning, $\mathbf{B} \in \mathbf{cl}(W^q_s(A_1 + K_1, \ldots, A_m + K_m))$.

This proves (2.2). Inequality (2.3) follows from Theorem 2.1 and the last statement is a consequence of the C^*-convexity of $W^q(\pi(\mathbf{A}))$. ∎

Theorem 2.3. *Let $\mathbf{A} = (A_1, \ldots, A_m)$ be an m-tuple of self-adjoint operators. The following conditions are equivalent.*

(a) *$W_{ess}(\mathbf{A}) = W(\pi(\mathbf{A}))$ has non-empty interior in $\mathbb{R}^{1\times m}$.*

(b) *For any positive integer q, $W^q_{ess}(\mathbf{A}) = W^q(\pi(\mathbf{A}))$ has non-empty interior in $\mathcal{H}^m_q$, where $\mathcal{H}_q$ is the real linear space of $q \times q$ Hermitian matrices.*

(c) *The set $\{I, \pi(A_1), \ldots, \pi(A_m)\}$ is linearly independent.*

Proof. (a) $\Rightarrow$ (b): Suppose $W_{ess}(\mathbf{A})$ has non-empty interior. We may assume that for some $r > 0$, $D_r = \{\mathbf{a} \in \mathbb{R}^{1\times m} : \|\mathbf{a}\|_\infty \leq r\} \subseteq W_{ess}(\mathbf{A})$. Let $q > 1$. We are going to show that $(B_1, \ldots, B_m) \in W^q_{ess}(\mathbf{A})$ for all $B_1, \ldots, B_m \in H_q$ with $\|B_i\| \leq \dfrac{r}{m}$, $1 \leq i \leq m$.

Let $|b| \leq r$. Then $(b, 0, \ldots, 0) \in W_{ess}(\mathbf{A})$. So there exists [10, Theorem 2.1] an orthonormal sequence of vectors $\{\mathbf{x}_n\}$ such that $\lim\limits_{n\to\infty} (\langle A_1\mathbf{x}_n, \mathbf{x}_n\rangle, \ldots, \langle A_m\mathbf{x}_n, \mathbf{x}_n\rangle) = (b, 0, \ldots, 0)$. For every $K \in \mathcal{K}(\mathcal{H})$, we have $\lim\limits_{n\to\infty} \langle K\mathbf{x}_n, \mathbf{x}_n\rangle = 0$. Let $S = \text{span}\ \{I, A_1, \ldots, A_m\}$. Using the quotient map π from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, we have $\phi(\pi(A)) = \lim\limits_{n\to\infty} \langle A\mathbf{x}_n, \mathbf{x}_n\rangle$ defines a unital completely positive map on $\pi(S)$ such that $\phi(\pi(A_1)) = b$ and $\phi(\pi(A_2)) = \cdots = \phi(\pi(A_m)) = 0$. ϕ can be extended to a unital completely positive map on $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Let $B = U^*\text{diag}(b_1, \ldots, b_q)U \in H_q$ with $(b_1, \ldots, b_q) \in D_r$. Then for each $1 \leq i \leq q$, there is a unital completely positive map ϕ_i on $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ such that $\phi_i(\pi(A_1)) = b_i$ and $\phi_i(\pi(A_2)) = \cdots =$

$\phi_i(\pi(A_m)) = 0$. Let $\Phi(\pi(A)) = U^* \left(\operatorname{diag}(\phi_1(\pi(A)), \ldots, \phi_q(\pi(A)))\right) U$. Then $\Phi : \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to M_q$ is a unital completely positive map such that $\Phi\left(\pi(A_1)\right) = B$ and $\Phi\left(\pi(A_2)\right) = \cdots = \Phi\left(\pi(A_m)\right) = 0$. Suppose $(B_1, \ldots, B_m) \in H_q^m$ with $\|B_i\| \leq \dfrac{r}{m}$ for all $1 \leq i \leq m$. From the above discussion, there exist unital completely positive maps $\Phi_i : \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to M_q$ such that $\Phi_i(\pi(A_j)) = \delta_{i\,j}\,(mB_j)$ for all $1 \leq i,\, j \leq m$. Let $\Phi = \dfrac{1}{m}\left(\sum_{i=1}^m \Phi_i\right)$. Then Φ is a unital completely positive map satisfying $\Phi(\pi(A_j)) = B_j$ for all $1 \leq j \leq m$.

(b) $\Rightarrow$ (c): Suppose (c) is not true. Then there are real numbers $a_0, \ldots, a_m$ not all zero such that $a_0 I + a_1\pi(A_1) + \cdots + a_m\pi(A_m) = 0$. Then $a_1 B + \cdots + a_m B = -a_0 I_q$ for every $(B_1, \ldots, B_q) \in W^q(\pi(\mathbf{A}))$. So, $W^q(\pi(\mathbf{A}))$ has empty interior.

(c) $\Rightarrow$ (a): Suppose (a) is not true. Then the convex set $W(\pi(\mathbf{A}))$ has empty interior. So, it must lie in an affine space, say, $\{(x_1, \ldots, x_m) \in \mathbb{R}^m : a_0 + a_1 x_1 + \cdots + a_m x_m = 0\}$, where $(a_0, a_1, \ldots, a_m) \in \mathbb{R}^{m+1}$ is a unit vector. It then follows that $\langle (a_0 I + \cdots + a_m\pi(A_m))\, v, v\rangle = 0$ for any unit vector v. Thus, $a_0 I + \cdots + a_m\pi(A_m) = 0$, i.e., $\{I, \pi(A_1), \ldots, \pi(A_m)\}$ is linearly dependent.

■

3 Preservation problems

Narcowich and Ward [12] proved that given a single operator $A \in \mathcal{B}(\mathcal{H})$ one has $W^q(\pi(A)) = W^q_{ess}(A)$ for every q. Smith and Ward [16, Section 5] proved that given a single operator $A \in \mathcal{B}(\mathcal{H})$ and a positive integer N, there exists $K \in \mathcal{K}(\mathcal{H})$ such that $W^q(A+K) = W^q(\pi(A))$, for all $q = 1, \ldots, N$. Müller [11, Corollary 14] proves that given an m-tuple of operators $\mathbf{A} = (A_1, \ldots, A_m)$, there is an m-tuple of compact operators $\mathbf{K} = (K_1, \ldots, K_m)$ such that $\mathbf{cl}(W(\mathbf{A}+\mathbf{K})) = W_{ess}(\mathbf{A})$. The following result extends these results to joint (spatial, essential) q-matricial ranges of tuples of operators. Again, we can focus on tuples of self-adjoint operators.

Theorem 3.1. *Suppose $\mathbf{A} = (A_1, \ldots, A_m)$ is an m-tuple of self-adjoint operators. Then for any positive integer N, there is an m-tuple of compact self-adjoint operators $\mathbf{K} = (K_1, \ldots, K_m)$ such that*

$$\mathbf{cl}(W^q_s(\mathbf{A}+\mathbf{K})) = W^q(\mathbf{A}+\mathbf{K}) = W^q(\pi(\mathbf{A})) = W^q_{ess}(\mathbf{A}) \quad \textit{for all } q = 1, \ldots, N. \tag{3.1}$$

Proof. First, we show that there is $\mathbf{K}$ such that

$$W^N(\mathbf{A}+\mathbf{K}) = W^N(\pi(\mathbf{A})).$$

Given $\mathbf{A} = (A_1, ..., A_m)$, let S be the operator system of dimension not larger than $m+1$ spanned by $\pi(I), \pi(A_1), \ldots, \pi(A_m)$. This operator system is embedded into the Calkin algebra by the identity map.

By [7, Theorem 9.11], there is an N-positive lifting map of π, $R : S \to \mathcal{B}(\mathcal{H})$. Since it is an N-positive lifting, $R(\pi(A_i)) = A_i + K_i$ for some $(K_1, \ldots, K_m) \in \mathcal{S}_{\mathcal{K}}(\mathcal{H})$.

Since $A_i + K_i \mapsto \pi(A_i)$ is completely positive, it follows that

$$W^q(A_1+K_1,...,A_m+K_m) = W^q(\pi(A_1),...,\pi(A_m)) \quad \text{for all } q \le N.$$

To see this, let $(T_1,...,T_m)$ be in the left hand side. Then there is a completely positive map $\phi : S_0 \to M_q$ with $T_i = \phi(A_i+K_i) = \phi(R(\pi(A_i)))$. But then the map $\gamma(\pi(A_i)) = T_i$ is N-positive. Since $q \le N$, by [13, Theorem 6.1], γ is completely positive. Hence $(T_1,...,T_m)$ is in the right hand side. The proof of other containment is similar.

Now, using the notation $\mathbf{A} = (A_1,\dots,A_m), \mathbf{K} = (K_1,\dots,K_m)$ we note that

$$W^q_{ess}(\mathbf{A}) \subseteq \mathbf{cl}(W^q_s(\mathbf{A}+\mathbf{K})) \subseteq W^q(\mathbf{A}+\mathbf{K})$$

is always true. By Theorem 2.2 and the above discussion,

$$W^q(\mathbf{A}+\mathbf{K}) = W^q(\pi(\mathbf{A})) = W^q_{ess}(\mathbf{A}),$$

so that (3.1) is true. ■

Remark 3.2. The still open Smith-Ward problem asks: given $A \in \mathcal{B}(\mathcal{H})$ does there exist a compact operator K such that for every $q \in \mathbb{N}$, $W^q(A+K) = W^q(\pi(A))$? Let $A = A_1 + iA_2$ be the Cartesian decomposition of A. It is known [14] that this problem is equivalent to asking if the operator system $\mathcal{S}_\pi := \text{span}\{\pi(I), \pi(A_1), \pi(A_2)\} \subseteq \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ has a completely positive lifting to $\mathcal{B}(\mathcal{H})$, i.e., if there exists a unital completely positive map $\Phi : \mathcal{S}_\pi \to \mathcal{B}(\mathcal{H})$ such that $\pi(\Phi(x)) = x$ for all $x \in \mathcal{S}_\pi$. The analogue of this problem for tuples of operators is known to be false. In [14, Theorem 3.3] an example is given of an operator $A \in \mathcal{B}(\mathcal{H})$ such that the span of $\{\pi(I), \pi(A), \pi(A^*), \pi(A^*A), \pi(AA^*)\}$ does not have a completely positive lifting to $\mathcal{B}(\mathcal{H})$. This implies that for the tuple $\mathbf{A} = (A_1, A_2, A^*A, AA^*)$, there does not exist four compact operators $\mathbf{K} = (K_1, K_2, K_3, K_4)$ such that $W^q(\mathbf{A}+\mathbf{K}) = W^q(\pi(\mathbf{A}))$ for every q. This example shows that the finite range on q in the above theorem is necessary. It also shows that the analogue of the Smith-Ward problem is false for four or more self-adjoint operators.

In [16, Theorem 5.1], the authors showed that if $A \in \mathcal{B}(\mathcal{H})$ is such that $W_{ess}(A)$ is a line segment, then there is a compact operator K such that $W^q_s(A+K) = W^q_{ess}(A)$ for all $q = 1, 2, \dots$. We can extend the result to the situation when $W_{ess}(\mathbf{A}) = W(\pi(\mathbf{A}))$ is a simplex in $\mathbb{R}^{1\times m}$. To achieve this, we need the following result from [2, Theorem 1.1] and a lemma.

Theorem 3.3. *Let $\mathbf{T} = (T_1,\dots,T_m) \in \mathcal{S}(\mathcal{H})^m$, and $\mathbf{D} = (D_1,\dots,D_m) \in M^m_{m+1}$ be an m tuple of diagonal matrices such that $W(\mathbf{D})$ is a simplex in $\mathbb{R}^m$. Then $W(\mathbf{T}) \subseteq W(\mathbf{D})$ if and only if there is a Hilbert space $\tilde{\mathcal{H}}$ and $X : \mathcal{H} \to \tilde{\mathcal{H}} \otimes \mathbb{C}^{m+1}$ satisfying $X^*X = I_{\mathcal{H}}$ and $T_j = X^*(I_{\tilde{\mathcal{H}}} \otimes D_j)X$ for $j = 1,\dots,m$.*

Lemma 3.4. *Let $\mathbf{A} = (A_1,\dots,A_m) \in \mathcal{S}(\mathcal{H})^m$. Suppose $\mathbf{B}_i = (B_{i1},\dots,B_{im}) \in W^p_{ess}(\mathbf{A})$ for $1 \le i \le N$. Then for all $\varepsilon > 0$ and $K_1,\dots,K_m \in \mathcal{S}_{\mathcal{K}}(\mathcal{H})$, there exists $\tilde{\mathbf{B}} = (\tilde{B}_1,\dots,\tilde{B}_m) \in W^{Np}(A_1+K_1,\dots,A_m+K_m)$ such that $\tilde{B}_j = \oplus_{i=1}^N \tilde{B}_{ij}$ with $\|B_{ij} - \tilde{B}_{ij}\| < \varepsilon$ for all $1 \le i \le N$ and $1 \le j \le m$.*

Proof. Without loss of generality, we may assume that $K_1 = \cdots = K_m = 0$. We are going to prove by induction on N. For $N = 1$, the result follows from definition.

Suppose the result holds for some $N = k \geq 1$. Then there is $X_1 : \mathcal{K}_1 \to \mathcal{H}$ with $X_1^* X_1 = I_{\mathcal{K}_1}$ for some kp-dimensional subspace $\mathcal{K}_1$ of $\mathcal{H}$ such that $X_1^* A_j X_1 = \oplus_{i=1}^k \tilde{B}_{ij}$ for $j = 1, \ldots, m$ and $\|B_{ij} - \tilde{B}_{ij}\| < \varepsilon$ for all $1 \leq i \leq k$ and $1 \leq j \leq m$.

Extend the operator X_1 to an unitary operator $U : \mathcal{H} \to \mathcal{H}$ so that $U|_{\mathcal{K}_1} = X_1$. Let $\mathcal{L}$ be the subspace spanned by

$$\{\mathcal{K}_1,\, U^* A_1 U(\mathcal{K}_1),\, \ldots,\, U^* A_m U(\mathcal{K}_1)\}\,.$$

Then $\mathcal{L}$ has dimension at most $kp(m+1)$. Define $Y = U|_{\mathcal{L}^\perp}$. Then $Y^* Y = I_{\mathcal{L}^\perp}$. Furthermore, as $X_1 = U|_{\mathcal{K}_1}$, $Y = U|_{\mathcal{L}^\perp}$ and $\mathcal{K}_1 \subseteq \mathcal{L}$, we have $Y^* X_1 = 0$. Also, for every $1 \leq j \leq m$ and $v \in \mathcal{K}_1$, $U^* A_j U v \in \mathcal{L}$. Hence, for every $u \in \mathcal{L}^\perp$,

$$\langle u, Y^* A_j X_1 v\rangle = \langle Y u, A_j X_1 v\rangle = \langle U u, A_j U v\rangle = \langle u, U^* A_j U v\rangle = 0\,.$$

Thus, $Y^* A_j X_1 = 0$ for all $j = 1, \ldots, m$.

Since $Y^* A_j Y - A_j \in \mathcal{S}_{\mathcal{K}}(\mathcal{H})$ for all $1 \leq j \leq m$, there exists $X_2 : \mathcal{K}_2 \to \mathcal{L}^\perp$ with $X_2^* X_2 = I_{\mathcal{K}_2}$ for some p-dimensional subspace $\mathcal{K}_2$ of $\mathcal{L}^\perp$ such that $X_2^*(Y^* A_j Y) X_2 = \tilde{B}_{(k+1)j}$ satisfies

$$\|B_{(k+1)j} - \tilde{B}_{(k+1)j}\| < \varepsilon \text{ for all } 1 \leq j \leq m.$$

Observe that $\mathcal{K}_1$ and $\mathcal{K}_2$ are two mutually orthogonal subspaces of $\mathcal{H}$. Furthermore, $X_1 : \mathcal{K}_1 \to \mathcal{H}$ and $Y X_2 : \mathcal{K}_2 \to \mathcal{H}$ satisfy

$$X_1^* X_1 = I_{\mathcal{K}_1}, \quad (Y X_2)^*(Y X_2) = I_{\mathcal{K}_2}, \quad (Y X_2)^* X_1 = 0 \quad \text{and} \quad (Y X_2)^* A_j X_1 = 0$$

for $j = 1, \ldots, m$. Then the operator $Z = X_1 \oplus (Y X_2) : \mathcal{K}_1 \oplus \mathcal{K}_2 \to \mathcal{H}$ satisfies $Z^* Z = I_{\mathcal{K}_1 \oplus \mathcal{K}_2}$ and $Z^* A_j Z = \oplus_{i=1}^{k+1} \tilde{B}_{ij}$ for $j = 1, \ldots, m$ ∎

Now we are ready to present the other main result for this section.

Theorem 3.5. *Suppose $\mathbf{A} = (A_1, \ldots, A_m)$ is an m-tuple of self-adjoint operators acting on $\mathcal{H}$. If $W_{ess}(\mathbf{A})$ is a simplex $\mathbf{S}$ in $\mathbb{R}^{1\times m}$, i.e., $W_{ess}(\mathbf{A})$ is a polyhedron with $m+1$ vertices, then there is an m-tuple of self-adjoint compact operators $\mathbf{K} = (K_1, \ldots, K_m)$ such that for all positive integer q,*

$$\mathbf{cl}(W_s^q(\mathbf{A} + \mathbf{K})) = W^q(\mathbf{A} + \mathbf{K}) = W_{ess}^q(\mathbf{A}) = W^q(\pi(\mathbf{A})). \tag{3.2}$$

Proof. By Theorem 3.1, there is an m-tuple of compact operator $\mathbf{K}$ such that

$$\mathbf{cl}(W(\mathbf{A} + \mathbf{K})) = W(\mathbf{A} + \mathbf{K}) = W(\pi(\mathbf{A})) = W_{ess}(\mathbf{A}).$$

We will show that for any positive integer q, we have

$$\mathbf{cl}(W_s^q(\mathbf{A} + \mathbf{K})) = W^q(\mathbf{A} + \mathbf{K}) = W^q(\pi(\mathbf{A})) = W_{ess}^q(\mathbf{A}).$$

Let $\mathbf{D} = (D_1, \dots, D_m)$ such that $D_i = \mathrm{diag}(v_{i1}, \dots, v_{i,m+1})$ for $i = 1, \dots, m$, where

$$\mathbf{v}_1 = (v_{11}, \dots, v_{m1}), \dots, \mathbf{v}_{m+1} = (v_{1\,m+1}, \dots, v_{m\,m+1}) \in \mathbb{R}^{1\times m}$$

are the vertices of the simplex $\mathbf{S}$. Then $\mathbf{S} = W(D_1, \dots, D_m) = W_{ess}(\mathbf{A})$.

Suppose $q > 1$ is a positive integer. Let $\mathbf{B} = (B_1, \dots, B_m) \in W^q(\mathbf{A}+\mathbf{K})$. Then for any unital completely positive map $\phi : M_k \to M_1$, $(\phi(B_1), \dots, \phi(B_m)) \in \mathbf{cl}(W(\mathbf{A} + \mathbf{K}))$, which equals to the simplex $\mathbf{S}$. Thus, $W(B_1, \dots, B_m) \subseteq W(D_1, \dots, D_m)$. By Theorem 3.3, $(B_1, \dots, B_m)$ admits a joint dilation to $(I \otimes D_1, \dots, I \otimes D_m)$. Hence, $W^q(B_1, \dots, B_m) \subseteq W^q(D_1, \dots, D_m)$.

Suppose $\mathbf{K} = (K_1, \dots, K_m) \in \mathcal{S}(\mathcal{H})^m$. Note that $\mathbf{v}_1, \dots, \mathbf{v}_{m+1} \in W_{ess}(A_1 + K_1, \dots, A_m + K_m)$. Applying Lemma 3.4 with $p = 1$, $N = (m + 1)q$ and $\mathbf{B}_i = \mathbf{v}_j$ if $i \equiv j \pmod{m + 1}$ for $1 \le i \le N$ and $1 \le j \le m + 1$, we can get a sequence $\mathbf{D}^{(k)} = \left(D_1^{(k)}, \dots D_m^{(k)}\right) \in W_s^N(A_1 + K_1, \dots, A_m + K_m)$ such that

$$\{D_j^{(1)}, D_j^{(2)}, D_j^{(3)}, \dots, \} \to I_q \otimes D_j.$$

Since $W^q(B_1, \dots, B_k) \subseteq W^q(D_1, \dots, D_m)$, for any $R_0, \dots, R_m \in M_q$ we have

$$\begin{aligned}
& \|R_0 \otimes I_q + R_1 \otimes B_1 + \cdots + R_m \otimes B_m\| \\
\le\ & \|R_0 \otimes I_{m+1} + R_1 \otimes D_1 + \cdots + R_m \otimes D_m\| \\
=\ & \|R_0 \otimes I_{m+1} \otimes I_q \otimes I_{m+1} + R_1 \otimes I_{m+1} \otimes I_q \otimes D_1 + \cdots + R_m \otimes I_{m+1} \otimes I_q \otimes D_m\| \\
=\ & \lim_{k\to\infty} \|R_0 \otimes I_{m+1} \otimes I_N + R_1 \otimes I_{m+1} \otimes D_1^{(k)} + \cdots + R_m \otimes I_{m+1} \otimes D_m^{(k)}\| \\
\le\ & \|R_0 \otimes I_{m+1} \otimes I + R_1 \otimes I_{m+1} \otimes (A_1 + K_1) + \cdots + R_m \otimes I_{m+1} \otimes (A_m + K_m)\| \\
=\ & \|R_0 \otimes I + R_1 \otimes (A_1 + K_1) + \cdots + R_m \otimes (A_m + K_m)\|.
\end{aligned}$$

By Theorem 2.1, $\mathbf{B} \in W^q(\mathbf{A} + \mathbf{K})$. Because this is true for all compact $\mathbf{K}$, we see that $\mathbf{B} \in W^q_{ess}(\mathbf{A})$. Hence, we have

$$\mathbf{cl}(W_s^q(\mathbf{A} + \mathbf{K})) \subseteq W^q(\mathbf{A} + \mathbf{K}) \subseteq W^q_{ess}(\mathbf{A}).$$

Since $W^q(\pi(\mathbf{A})) = W^q_{ess}(\mathbf{A} + \mathbf{K}) \subseteq \mathbf{cl}(W_s^q(\mathbf{A} + \mathbf{K}))$, we see that (3.2) holds. ∎

4 Related results

By Theorem 3.5, we have the following extension of [15, 1.22.5].

Proposition 4.1. *Let $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$. Suppose $W_{ess}(\mathbf{A})$ is a subset of a simplex $\mathbf{S}$ in $\mathbb{R}^m$ with vertices $v^{(1)}, \dots, v^{(m+1)}$ such that $v^{(k)} = (v_1^{(k)}, \dots, v_m^{(k)})$ for $k = 1, \dots, m + 1$. Then there is an m-tuple of self-adjoint compact operators $\mathbf{K} = (K_1, \dots, K_m)$ such that for any $R_0, \dots, R_m \in M_q$,*

$$\begin{aligned}
&\|R_0 \otimes I + R_1 \otimes (A_1 + K_1) + \cdots + R_m \otimes (A_m + K_m)\| \\
&\le \max\{\|R_0 + v_1^{(k)} R_1 + \cdots + v_m^{(k)} R_m\| : 1 \le k \le m + 1\}. \qquad (4.3)
\end{aligned}$$

In fact, $\mathbf{K}$ can be chosen such that the equality holds in (4.3) for any choice of $R_0, \dots, R_m \in M_q$.

Proof. Let $D_1, \ldots, D_m \in M_{m+1}$ with $D_j = \mathrm{diag}(v_j^{(1)}, \ldots, v_j^{(m+1)})$ so that $W(D_1, \ldots, D_m) = \mathbf{S}$. Let $\tilde{\mathcal{H}} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathbb{C}^{m+1})$. Then A_j is a compression of $\tilde{A}_j = A_j \oplus (I_{\mathcal{H}} \otimes D_j) \in \mathcal{B}(\tilde{\mathcal{H}})$ for all $j = 1, \ldots, m$. Evidently, $W^q_{ess}(\tilde{A}_1, \ldots, \tilde{A}_m) = W^q(D_1, \ldots, D_m)$. By Theorem 3.5, there are self-adjoint compact operators $\tilde{K}_1, \ldots, \tilde{K}_m \in \mathcal{B}(\tilde{\mathcal{H}})$ such that $\mathbf{cl}\Big(W(\tilde{A}_1 + \tilde{K}_1, \ldots, \tilde{A}_m + \tilde{K}_m)\Big) = W_{ess}(\tilde{A}_1, \ldots, \tilde{A}_m)$. Moreover, for any positive integer q, we have

$$W^q(\tilde{A}_1 + \tilde{K}_1, \ldots, \tilde{A}_m + \tilde{K}_m) = W^q_{ess}(\tilde{A}_1, \ldots, \tilde{A}_m).$$

Suppose $\tilde{K}_j$ has operator matrix $\begin{pmatrix} K_1 & * \\ * & * \end{pmatrix}$ with $K_1 \in \mathcal{B}(\mathcal{H})$ for $j = 1, \ldots, m$. Then for any $R_0, \ldots, R_m \in M_q$, we have

$$\begin{aligned} & \|R_0 \otimes I + R_1 \otimes (A_1 + K_1) + \cdots + R_m \otimes (A_m + K_m)\| \\ \leq\ & \|R_0 \otimes I + R_1 \otimes (\tilde{A}_1 + \tilde{K}_1) + \cdots + R_m \otimes (\tilde{A}_m + \tilde{K}_m)\| \\ =\ & \|R_0 \otimes I + R_1 \otimes D_1 + \cdots + R_m \otimes D_m\| \\ =\ & \max\{\|R_0 + v_{1k}R_1 + \cdots + v_{mk}R_m\| : 1 \leq k \leq m+1\}. \end{aligned}$$

For the last assertion, let $\mathbf{K}$ be chosen such that (4.3) holds. Suppose $A_j + K_j$ has operator matrix $\begin{pmatrix} B_j & * \\ * & C_j \end{pmatrix}$ with $B_j \in M_{m+1}$ for $j = 1, \ldots, m$. There are finite rank operators $F_1, \ldots, F_m$ such that $A_j + K_j + F_j$ has operator matrix $D_j \oplus C_j$ for $j = 1, \ldots, m$. Then for any $R_0, \ldots, R_m \in M_q$, we have

$$\begin{aligned} & \|R_0 \otimes I + R_1 \otimes C_1 + \cdots + R_m \otimes C_m\| \\ \leq\ & \|R_0 \otimes I + R_1 \otimes (A_1 + K_1) + \cdots + R_m \otimes (A_m + K_m)\| \\ \leq\ & \|R_0 \otimes I + R_1 \otimes D_1 + \cdots + R_m \otimes D_m\|. \end{aligned}$$

Hence, if $\hat{K}_j = K_j + F_j$ for $j = 1, \ldots, m$, then

$$\begin{aligned} & \|R_0 \otimes I + R_1 \otimes (A_1 + \hat{K}_1) + \cdots + R_m \otimes (A_m + \hat{K}_m)\| \\ =\ & \max\{\|R_0 \otimes I + R_1 \otimes D_1 + \cdots + R_m \otimes D_m\|, \|R_0 \otimes I + R_1 \otimes C_1 + \cdots + R_m \otimes C_m\|\} \\ =\ & \|R_0 \otimes I + R_1 \otimes D_1 + \cdots + R_m \otimes D_m\| \\ =\ & \max\{\|R_0 + v_{1k}R_1 + \cdots + v_{mk}R_m\| : 1 \leq k \leq m+1\}. \end{aligned}$$

The last assertion follows. ∎

Motivated by quantum error correction, researchers consider the joint rank (p, q)-matricial range of $\mathbf{A}$, denoted by $\Lambda_{p,q}(\mathbf{A})$, consisting of m-tuples of $k \times k$ matrices $\mathbf{B} = (B_1, \ldots, B_m)$ such that for some unitary U, $I_p \otimes B_j$ is the leading principal submatrix of $U^* A_j U$ for $j = 1, \ldots, m$; see [8] and its references. One can define the $\Lambda^{ess}_{p,q}(\mathbf{A})$ as

$$\Lambda^{ess}_{p,q}(\mathbf{A}) = \cap\{\mathbf{cl}(\Lambda_{p,q}(\mathbf{A} + \mathbf{K})) : \mathbf{K} \in \mathcal{S}_{\mathcal{K}}(\mathcal{H})^m\},$$

and deduce the following results from those in the previous sections.

Theorem 4.2. *Let* $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$.

(1) *For any positive integer* N*, there is* $\mathbf{K} \in \mathcal{S}_{\mathcal{K}}(\mathcal{H})^m$ *such that*

$$\Lambda_{p,q}(\mathbf{A}+\mathbf{K}) = \Lambda_{p,q}^{ess}(\mathbf{A}),$$

which is the same as

$$W_{ess}^q(\mathbf{A}) = W^q(\pi(A)) = W^q(\mathbf{A}+\mathbf{K}) = W_s^q(\mathbf{A}+\mathbf{K}) \quad \text{for all } q \in \{1,\ldots,N\}.$$

(2) *Suppose* $W_{ess}(\mathbf{A})$ *is a simplex in* $\mathbb{R}^m$*. Then there is* $\mathbf{K} \in \mathcal{S}_{\mathcal{K}}(\mathcal{H})^m$ *such that*

$$\Lambda_{p,q}(\mathbf{A}+\mathbf{K}) = \Lambda_{p,q}^{ess}(\mathbf{A}),$$

which is the same as

$$W_{ess}^q(\mathbf{A}) = W_s^q(\mathbf{A}+\mathbf{K}) = W^q(\pi(A)) = W^q(\mathbf{A}+\mathbf{K})$$

for all positive integer q.

Proof. By the result in [8],

$$\Lambda_{p,q}^{ess}(\mathbf{A}) = W_{ess}^q(\mathbf{A}) \subseteq \Lambda_{p,q}(\mathbf{A}+\mathbf{K}) \subseteq W_s^q(\mathbf{A}+\mathbf{K}).$$

The results follow readily from Theorem 3.1 and Theorem 3.5. ∎

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References

[1] W.B. Arveson. Notes on extensions of C^*-algebras, Duke Math. J. 44(1977), no. 2, 329-355.

[2] P. Binding, D.R. Farenick and C.K. Li, A dilation and norm in several variable operator theory, Canad. J. Math. 47 (1995), 449-461.

[3] J. Bunce and N. Salinas, Completely positive maps on C^*-algebras and the left matricial spectra of an operator, Duke Math. J. 43(1976), no. 4, 747-774.

[4] D.R. Farenick, C^*-convexity and matricial ranges, Canad. J. Math. 44 (1992), 280-297.

[5] D.R. Farenick, Matrical extensions of the numerical range: a brief survey, Linear and Multilinear Algebra 34 (1993), 197-211.

[6] F. Hausdorff, Das Wertvorrat einer Bilinearform, Math. Zeit. 3 (1919), 314-316.

[7] A.S. Kavruk, Nuclearity related properties in operator systems, J. Operator Theory 71 (2014), no. 1, 95-156.

[8] P.S. Lau, C.K. Li, Y.T. Poon, and N.S. Sze, Convexity and Star-shapedness of Matricial Range, J. Func. Anal., 275 (2018), 2497-2515.

[9] C.K. Li and Y.T. Poon, Convexity of the joint numerical range, SIAM J. Matrix Anal. Appl. 21 (1999), 668-678.

[10] C.K. Li and Y.T. Poon, The joint essential numerical range of operators: Convexity and related results, Studia Math. 194 (2009), 91-104.

[11] V. Müller, The joint essential numerical range, compact perturbations, and the Olsen problem, Studia Math. 197 (2010), 275-290.

[12] F. Narcowich and J. D. Ward, A characterization of essential matrix ranges, Bulletin of the London Mathematical Society 14 (1982), 107-111.

[13] V.I. Paulsen, Completely Bounded Maps and Operator Algebras, Cambridge University Press, Cambridge, 2002.

[14] V.I. Paulsen, Preservation of essential matrix ranges by compact perturbations, J. Operator Theory 8 (1982), 299-317.

[15] S. Sakai, C^*-algebras and W^*-algebras, Springer Verlag, Berlin, 1971.

[16] R. R. Smith and J. D. Ward, Matrix Ranges for Hilbert Space Operators, American Journal of Mathematics 102 (1980), 1031-1081.

[17] O. Toeplitz, Das algebraische Analogon zu einem Satze von Fejer, Math. Zeit. 2 (1918), 187-197.

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