Preservation of the joint essential matricial range

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Abstract

Let $A = (A_1, \ldots, A_m)$ be an $m$-tuple of elements of a unital C*-algebra $A$ and let $M_q$ denote the set of $q \times q$ complex matrices. The joint $q$-matricial range $W^q(A)$ is the set of $(B_1, \ldots, B_m) \in M_q^m$ such that $B_j = \Phi(A_j)$ for some unital completely positive linear map $\Phi : A \to M_q$. When $A = B(H)$, where $B(H)$ is the algebra of bounded linear operators on the Hilbert space $H$, the joint spatial $q$-matricial range $W^q_s(A)$ of $A$ is the set of $(B_1, \ldots, B_m) \in M_q^m$ for which there is a $q$-dimensional subspace $V$ of $H$ such that $B_j$ is the compression of $A_j$ to $V$ for $j = 1, \ldots, m$. Suppose $K(H)$ is the set of compact operators in $B(H)$. The joint essential spatial $q$-matricial range is defined as

$$W^q_{\text{ess}}(A) = \bigcap \{ \overline{\text{cl}(W^q_s(A_1 + K_1, \ldots, A_m + K_m))} : K_1, \ldots, K_m \in K(H) \},$$

where $\text{cl}(T)$ denotes the closure of the set $T$. Let $\pi$ be the canonical surjection from $B(H)$ to the Calkin algebra $B(H)/K(H)$. We prove that $W^q_{\text{ess}}(A) = W^q(\pi(A))$, where $\pi(A) = (\pi(A_1), \ldots, \pi(A_m))$. Furthermore, for any positive integer $N$, we prove that there are self-adjoint compact operators $K_1, \ldots, K_m$ such that

$$\overline{\text{cl}(W^q_s(A_1 + K_1, \ldots, A_m + K_m))} = W^q_{\text{ess}}(A) \quad \text{for all } q \in \{1, \ldots, N\}.$$ 

These results generalize those of Narcowich-Ward and Smith-Ward, obtained in the $m = 1$ case, and also generalize a result of Müller obtained in case $m \geq 1$ and $q = 1$. Furthermore, if $W^1_{\text{ess}}(A)$ is a simplex in $\mathbb{R}^m$, then we prove that there are self-adjoint compact operators $K_1, \ldots, K_m$ such that $\overline{\text{cl}(W^q_s(A_1 + K_1, \ldots, A_m + K_m))} = W^q_{\text{ess}}(A)$ for all positive integers $q$.

AMS Classification. 47A12, 47A13, 47A20

Keywords. Numerical range, joint essential matricial range, unital completely positive linear map, Calkin algebra.

1 Introduction

Let $B(H)$ be the algebra of bounded linear operators acting on an infinite dimensional Hilbert space $H$ and let $K(H)$ denote the set of compact operators in $B(H)$. The numerical range of $A \in B(H)$ is defined and denoted by

$$W(A) = \{ \langle Ax, x \rangle : x \in H, \|x\| = 1 \}.$$ 

It is a useful concept for studying matrices and operators. The Toeplitz-Hausdorff Theorem states that this set is always convex [6, 17], i.e. $tw_1 + (1-t)w_2 \in W(A)$ for all $w_1, w_2 \in W(A)$ and $0 \leq t \leq 1$. In general, $W(A)$ is not closed.
To study the joint behavior of multiple operators in $B(H)$, researchers have considered the joint numerical range of an $m$-tuple $A = (A_1, \ldots, A_m) \in B(H)^m$ defined by

$$W(A) = \{ (\langle A_1 x, x \rangle, \ldots, \langle A_m x, x \rangle) : x \in H, \|x\| = 1 \},$$

see [2, 9] and its references.

The joint essential numerical range of $A$ is defined as

$$W_{\text{ess}}(A) = \cap \{ \text{cl}(W(A_1 + K_1, \ldots, A_m + K_m)) : K_1, \ldots, K_m \in \mathcal{K}(H) \},$$

where $\text{cl}(T)$ denotes the closure of the set $T$. These concepts were further extended to the joint spatial $q$-matricial range, and the joint essential spatial $q$-matricial range defined as follows. Let $\mathcal{V}_q$ denote the set of operators $X : K \to H$ such that $X^*X = I_K$ for some $q$-dimensional subspace $K$ of $H$. The joint spatial $q$-matricial range is

$$W_s^q(A) = \{ (X^*A_1X, \ldots, X^*A_mX) : X \in \mathcal{V}_q \},$$

and the joint essential $q$-matricial range is

$$W_{\text{ess}}^q(A) = \cap \{ \text{cl}(W_s^q(A_1 + K_1, \ldots, A_m + K_m)) : K_1, \ldots, K_m \text{ are compact operators} \},$$

respectively. Evidently, when $q = 1$, these concepts reduce to $W(A)$ and $W_{\text{ess}}(A)$.

Let $A$ be a unital $C^*$-algebra. The joint $q$-matricial range of $A = (A_1, \ldots, A_m) \in A^m$ is defined as

$$W^q(A) = \{ (\Phi(A_1), \ldots, \Phi(A_m)) : \Phi \text{ is a unital completely positive map from } A \text{ to } M_q \}.$$ 

Let $S$ denote the operator system [13] spanned by $\{I_H, A_1, A_1^*, \ldots, A_m, A_m^*\}$. Then by Arveson’s extension theorem,

$$W^q(A) = \{ (\Phi(A_1), \ldots, \Phi(A_m)) : \Phi \text{ is a unital completely positive map from } S \text{ to } M_q \}.$$ 

Let $\pi : B(H) \to B(H)/\mathcal{K}(H)$ be the canonical map from $B(H)$ to the Calkin algebra $B(H)/\mathcal{K}(H)$ and set $\pi(A) = (\pi(A_1), \ldots, \pi(A_m))$ for $A_1, \ldots, A_m \in B(H)$. Then it is clear that

$$\text{cl}(W_s^q(A)) \subseteq W^q(A) \quad \text{and} \quad W^q(\pi(A)) \subseteq W^q(A + K) \quad \text{for all} \quad K \in \mathcal{K}(H)^m.$$ 

In this paper, we will show that $W_{\text{ess}}^q(A) = W^q(\pi(A))$ and consequently, $W_{\text{ess}}^q(A)$ is $C^*$-convex [4]. See the definition of $C^*$-convex in Section 2.

When $m = 1$, this reduces to a result of Narcowich and Ward [12]. Moreover, we study the preservation problem for $W_{\text{ess}}^q(A)$ and $W^q(\pi(A))$, namely, we prove that for each $N$, there is an $m$-tuple of self-adjoint compact operator $K = (K_1, \ldots, K_m)$ such that for $1 \leq q \leq N$,

$$\text{cl}(W_s^q(A + K)) = W_{\text{ess}}^q(A) = W^q(A + K) = W^q(\pi(A)). \quad (1.1)$$

When $m = 1$, this reduces to a result of Smith and Ward [16]; when $N = 1$, this reduces to a result of Müller [11].
Let $S(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : A = A^*\}$. Note that every $A \in \mathcal{B}(\mathcal{H})$ has a unique Cartesian decomposition $A = A_1 + iA_2$ such that $A_1, A_2 \in S(\mathcal{H})$. Thus, one can identify $W^q(A)$ with $W^q(A_1, A_2)$, and also identify $W^q_{\text{ess}}(A)$ with $W^q_{\text{ess}}(A_1, A_2)$. For this reason, we shall henceforth focus on $W^q(A)$ and $W^q_{\text{ess}}(A)$ for $A \in S(\mathcal{H})^m$.

We show that if $W^q_{\text{ess}}(A)$ is a simplex in $\mathbb{R}^m$, then there are compact operators $K_1, \ldots, K_m$ such that (1.1) holds for all positive integers $q$. This extends another result in [16].

In our discussion, we will always assume that $\mathcal{H}$ is infinite dimensional. In addition to the notations $\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H})$ and $S(\mathcal{H})$ introduced above, we will let $\mathcal{S}_K(\mathcal{H})$ be the set of compact operators in $S(\mathcal{H})$.

2 Some basic results

The following result extends [4, Theorem 2.1].

Theorem 2.1. Let $\mathcal{A}$ be a unital C*-algebra and let $A = (A_1, \ldots, A_m) \in \mathcal{A}^m$ be self-adjoint. Then $(B_1, \ldots, B_m) \in W^q(A)$ if and only if

$$\|R_0 \otimes I_q + R_1 \otimes B_1 + \cdots + R_m \otimes B_m\| \leq \|R_0 \otimes I + R_1 \otimes A_1 + \cdots + R_m \otimes A_m\| \quad (2.1)$$

for all $R_0, \ldots, R_m \in M_q$.

Proof. Note that (2.1) is equivalent to the condition that the unital linear map sending $A_j \in \mathcal{A}$ to the Hermitian matrix $B_j \in M_q$ is completely contractive. Then the result follows from the fact that every unital completely contractive map from an operator system to $M_q$ is completely positive; for example, see [13].

Given $B = (B_1, \ldots, B_m) \in M_q^m$ and $L \in M_q$ we set

$$L^*BL = (L^*B_1L, \ldots, L^*B_mL).$$

A subset $\mathcal{C} \subseteq M_q^m$ is $C^*$-convex if

$$\sum_{j=1}^N L_j^*B_jL_j \in \mathcal{C}$$

for any $B_1, \ldots, B_N \in \mathcal{C}$ and $L_1, \ldots, L_N \in M_q$ satisfying $\sum_{j=1}^N L_j^*L_j = I_q$. It is well known that $W^q(A)$ is $C^*$-convex. We have the following result showing that $W^q_{\text{ess}}(A) = W^q(\pi(A))$. It will then follow that $W^q_{\text{ess}}(A)$ is also $C^*$-convex.

Theorem 2.2. Let $A = (A_1, \ldots, A_m) \in S(\mathcal{H})^m$. Then

$$W^q_{\text{ess}}(A) = W^q(\pi(A)), \quad (2.2)$$

which consists of $B = (B_1, \ldots, B_m) \in M_q^m$ satisfying:

$$\|R_0 \otimes I_q + R_1 \otimes B_1 + \cdots + R_m \otimes B_m\| \leq \|R_0 \otimes I + R_1 \otimes \pi(A_1) + \cdots + R_m \otimes \pi(A_m)\| \quad (2.3)$$

for all $R_0, \ldots, R_m \in M_q$. Consequently, $W^q_{\text{ess}}(A)$ is $C^*$-convex.
Proof. Let $S$ be the finite dimensional operator system given by the span of $\{\pi(I), \pi(A_1), \ldots, \pi(A_m)\}$. Then by [7, Theorem 9.11], there exists a unital $q$-positive map $R : S \to \mathcal{B}(\mathcal{H})$ such that $R(\pi(A_j)) = A_j + K_j$ with $K_j \in \mathcal{S}_K(\mathcal{H})$ for $1 \leq j \leq m$.

Let $B = (B_1, \ldots, B_m) \in W^q_{\text{ess}}(A)$. For $\varepsilon > 0$, there is

$$(\tilde{B}_1, \ldots, \tilde{B}_m) \in W^q_s(A_1 + K_1, \ldots, A_m + K_m)$$

such that $\|\tilde{B}_j - B_j\| < \varepsilon$ for all $1 \leq j \leq m$. Let $X \in V_q$ be such that $X^*(A_j + K_j)X = \tilde{B}_j$ for all $1 \leq j \leq m$. Then the map $\Phi : S \to M_q$ given by $\Phi(T) = X^*R(T)X$ is a unital completely positive map satisfying $(\tilde{B}_1, \ldots, \tilde{B}_m) = (\Phi(\pi(A_1)), \ldots, \Phi(\pi(A_m))) \in W^q(\pi(A))$. Since $W^q(\pi(A))$ is close, $B \in W^q(\pi(A))$. Hence, $W^q_{\text{ess}}(A) \subseteq W^q(\pi(A))$.

For the reverse inclusion, if $B = (B_1, \ldots, B_m) \in W^q(\pi(A))$. Let $\phi : S \to M_q$ be a unital completely positive map such that $\phi(\pi(A_j)) = B_j$ for all $1 \leq j \leq m$. Let $\Phi = \phi \circ \pi$. Then $\Phi$ is a unital completely positive on the separable $C^*$-algebra $A$ generated by $\{I, A_1, \ldots, A_m\}$, with $\Phi(A \cap K(\mathcal{H})) = \{0\}$. Hence, by [3, Theorem 2.5] (see also [1, Theorem 4]), we have that $B = (B_1, \ldots, B_m) \in \text{cl}(W^q_s(A_1, \ldots, A_m))$. Given $K_1, \ldots, K_m \in \mathcal{S}_K(\mathcal{H})$, we also have that $\Phi$ vanishes on the intersection of the compact operators with the separable $C^*$-algebra generated by $\{I, A_1 + K_1, \ldots, A_m + K_m\}$ with $\Phi(A_j + K_j) = \phi(\pi(A_j))$. Hence, by the same reasoning, $B \in \text{cl}(W^q_s(A_1 + K_1, \ldots, A_m + K_m))$.

This proves (2.2). Inequality (2.3) follows from Theorem 2.1 and the last statement is a consequence of the $C^*$-convexity of $W^q(\pi(A))$. 

\begin{theorem}
Let $A = (A_1, \ldots, A_m)$ be an $m$-tuple of self-adjoint operators. The following conditions are equivalent.

(a) $W^q_{\text{ess}}(A) = W(\pi(A))$ has non-empty interior in $\mathbb{R}^{1 \times m}$.

(b) For any positive integer $q$, $W^q_{\text{ess}}(A) = W^q(\pi(A))$ has non-empty interior in $\mathcal{H}_q^m$, where $\mathcal{H}_q$ is the real linear space of $q \times q$ Hermitian matrices.

(c) The set $\{I, \pi(A_1), \ldots, \pi(A_m)\}$ is linearly independent.
\end{theorem}

\begin{proof}
(a) $\Rightarrow$ (b): Suppose $W^q_{\text{ess}}(A)$ has non-empty interior. We may assume that for some $r > 0$, $D_r = \{a \in \mathbb{R}^{1 \times m} : \|a\|_\infty \leq r\} \subseteq W^q_{\text{ess}}(A)$. Let $q > 1$. We are going to show that $(B_1, \ldots, B_m) \in W^q_{\text{ess}}(A)$ for all $B_1, \ldots, B_m \in H_q$ with $\|B_i\| \leq \frac{r}{m}$, $1 \leq i \leq m$.

Let $|b| \leq r$. Then $(b, 0, \ldots, 0) \in W^q_{\text{ess}}(A)$. So there exists [10, Theorem 2.1] an orthonormal sequence of vectors $\{x_n\}$ such that $\lim_{n \to \infty} \langle A_1x_n, x_n \rangle, \ldots, \langle A_mx_n, x_n \rangle = (b, 0, \ldots, 0)$. For every $K \in K(\mathcal{H})$, we have $\lim_{n \to \infty} \langle Kx_n, x_n \rangle = 0$. Let $S = \text{span} \{I, A_1, \ldots, A_m\}$. Using the quotient map $\pi$ from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{B}(\mathcal{H})/K(\mathcal{H})$, we have $\phi(\pi(A)) = \lim_{n \to \infty} \langle Ax_n, x_n \rangle$ defines a unital completely positive map on $\pi(S)$ such that $\phi(\pi(A_1)) = b$ and $\phi(\pi(A_2)) = \cdots = \phi(\pi(A_m)) = 0$. $\phi$ can be extended to a unital completely positive map on $\mathcal{B}(\mathcal{H})/K(\mathcal{H})$. Let $B = U^*\text{diag}(b_1, \ldots, b_q)U \in H_q$ with $(b_1, \ldots, b_q) \in D_r$. Then for each $1 \leq i \leq q$, there is a unital completely positive map $\phi_i$ on $\mathcal{B}(\mathcal{H})/K(\mathcal{H})$ such that $\phi_i(\pi(A_1)) = b_i$ and $\phi_i(\pi(A_2)) = \cdots = $
\( \phi_i(\pi(A_m)) = 0 \). Let \( \Phi(\pi(A)) = U^* (\text{diag}(\phi_1(\pi(A)), \ldots, \phi_q(\pi(A)))) U \). Then \( \Phi : B(H)/K(H) \to M_q \) is a unital completely positive map such that \( \Phi(\pi(A_1)) = B \) and \( \Phi(\pi(A_2)) = \cdots = \Phi(\pi(A_m)) = 0 \). Suppose \( (B_1, \ldots, B_m) \in H_q^m \) with \( \|B_i\| \leq \frac{r}{m} \) for all \( 1 \leq i \leq m \). From the above discussion, there exist unital completely positive maps \( \Phi_i : B(H)/K(H) \to M_q \) such that \( \Phi_i(\pi(A_j)) = \delta_{ij} (mB_j) \) for all \( 1 \leq i, j \leq m \). Let \( \Phi = \frac{1}{m} (\sum_{i=1}^m \Phi_i) \). Then \( \Phi \) is a unital completely positive map satisfying \( \Phi(\pi(A_j)) = B_j \) for all \( 1 \leq j \leq m \).

(b) \( \Rightarrow \) (c): Suppose (c) is not true. Then there are real numbers \( a_0, \ldots, a_m \) not all zero such that \( a_0 I + a_1 \pi(A_1) + \cdots + a_m \pi(A_m) = 0 \). Then \( a_1 B + \cdots + a_m B = -a_0 I_q \) for every \( (B_1, \ldots, B_q) \in W^q(\pi(A)) \). So, \( W^q(\pi(A)) \) has empty interior.

(c) \( \Rightarrow \) (a): Suppose (a) is not true. Then the convex set \( W(\pi(A)) \) has empty interior. So, it must lie in an affine space, say, \( \{(x_1, \ldots, x_m) \in \mathbb{R}^m : a_0 + a_1 x_1 + \cdots + a_m x_m = 0\} \), where \( (a_0, a_1, \ldots, a_m) \in \mathbb{R}^{m+1} \) is a unit vector. It then follows that \( \langle (a_0 I + \cdots + a_m \pi(A_m)) v, v \rangle = 0 \) for any unit vector \( v \). Thus, \( a_0 I + \cdots + a_m \pi(A_m) = 0 \), i.e., \( \{I, \pi(A_1), \ldots, \pi(A_m)\} \) is linearly dependent.

\[ \text{3 Preservation problems} \]

Narcowich and Ward [12] proved that given a single operator \( A \in B(H) \) one has \( W^q(\pi(A)) = W^q_{\text{ess}}(A) \) for every \( q \). Smith and Ward [16, Section 5] proved that given a single operator \( A \in B(H) \) and a positive integer \( N \), there exists \( K \in K(H) \) such that \( W^q(A + K) = W^q(\pi(A)) \), for all \( q = 1, \ldots, N \). Müller [11, Corollary 14] proves that given an \( m \)-tuple of operators \( A = (A_1, \ldots, A_m) \), there is an \( m \)-tuple of compact operators \( K = (K_1, \ldots, K_m) \) such that \( \text{cl}(W(A + K)) = W_{\text{ess}}(A) \). The following result extends these results to joint (spatial, essential) \( q \)-matricial ranges of tuples of operators. Again, we can focus on tuples of self-adjoint operators.

**Theorem 3.1.** Suppose \( A = (A_1, \ldots, A_m) \) is an \( m \)-tuple of self-adjoint operators. Then for any positive integer \( N \), there is an \( m \)-tuple of compact self-adjoint operators \( K = (K_1, \ldots, K_m) \) such that

\[
\text{cl}(W^q_s(A + K)) = W^q(A + K) = W^q(\pi(A)) = W^q_{\text{ess}}(A) \quad \text{for all } q = 1, \ldots, N.
\]

**Proof.** First, we show that there is \( K \) such that

\[
W^N(A + K) = W^N(\pi(A)).
\]

Given \( A = (A_1, \ldots, A_m) \), let \( S \) be the operator system of dimension not larger than \( m + 1 \) spanned by \( \pi(I), \pi(A_1), \ldots, \pi(A_m) \). This operator system is embedded into the Calkin algebra by the identity map.

By [7, Theorem 9.11], there is an \( N \)-positive lifting map of \( \pi, R : S \to B(H) \). Since it is an \( N \)-positive lifting, \( R(\pi(A_i)) = A_i + K_i \) for some \( (K_1, \ldots, K_m) \in S_K(H) \).

Since \( A_i + K_i \mapsto \pi(A_i) \) is completely positive, it follows that
To see this, let \((T_1, \ldots, T_m)\) be in the left hand side. Then there is a completely positive map 
\(\phi : S_0 \to M_q\) with \(T_i = \phi(A_i + K_i) = \phi(R(\pi(A_i)))\). But then the map \(\gamma(\pi(A_i)) = T_i\) is \(N\)-positive. Since \(q \leq N\), by [13, Theorem 6.1], \(\gamma\) is completely positive. Hence \((T_1, \ldots, T_m)\) is in the right hand side. The proof of other containment is similar.

Now, using the notation \(A = (A_1, \ldots, A_m), K = (K_1, \ldots, K_m)\) we note that
\[ W^q_{\text{ess}}(A) \subseteq \text{cl}(W^q_s(A + K)) \subseteq W^q(A + K) \]
is always true. By Theorem 2.2 and the above discussion,
\[ W^q(A + K) = W^q(\pi(A)) = W^q_{\text{ess}}(A), \]
so that (3.1) is true.

\[ \square \]

**Remark 3.2.** The still open Smith-Ward problem asks: given \(A \in \mathcal{B}(\mathcal{H})\) does there exist a compact operator \(K\) such that for every \(q \in \mathbb{N}\), \(W^q(A + K) = W^q(\pi(A))\)? Let \(A = A_1 + iA_2\) be the Cartesian decomposition of \(A\). It is known [14] that this problem is equivalent to asking if the operator system \(\mathcal{S}_\pi := \text{span}\{\pi(I), \pi(A_1), \pi(A_2)\} \subseteq \mathcal{B}(\mathcal{H})/K(\mathcal{H})\) has a completely positive lifting to \(\mathcal{B}(\mathcal{H})\), i.e., if there exists a unital completely positive map \(\Phi : \mathcal{S}_\pi \to \mathcal{B}(\mathcal{H})\) such that \(\pi(\Phi(x)) = x\) for all \(x \in \mathcal{S}_\pi\). The analogue of this problem for tuples of operators is known to be false. In [14, Theorem 3.3] an example is given of an operator \(A \in \mathcal{B}(\mathcal{H})\) such that the span of \(\{\pi(I), \pi(A), \pi(A^*), \pi(A^*A), \pi(AA^*)\}\) does not have a completely positive lifting to \(\mathcal{B}(\mathcal{H})\). This implies that for the tuple \(A = (A_1, A_2, A^*A, AA^*)\), there does not exist four compact operators \(K = (K_1, K_2, K_3, K_4)\) such that \(W^q(A + K) = W^q(\pi(A))\) for every \(q\). This example shows that the finite range on \(q\) in the above theorem is necessary. It also shows that the analogue of the Smith-Ward problem is false for four or more self-adjoint operators.

In [16, Theorem 5.1], the authors showed that if \(A \in \mathcal{B}(\mathcal{H})\) is such that \(W_{\text{ess}}(A)\) is a line segment, then there is a compact operator \(K\) such that \(W^q_s(A + K) = W^q_{\text{ess}}(A)\) for all \(q = 1, 2, \ldots\). We can extend the result to the situation when \(W_{\text{ess}}(A) = W(\pi(A))\) is a simplex in \(\mathbb{R}^{1 \times m}\). To achieve this, we need the following result from [2, Theorem 1.1] and a lemma.

**Theorem 3.3.** Let \(T = (T_1, \ldots, T_m) \in \mathcal{S}(\mathcal{H})^m\), and \(D = (D_1, \ldots, D_m) \in M^m_{m+1}\) be an \(m\) tuple of diagonal matrices such that \(W(D)\) is a simplex in \(\mathbb{R}^m\). Then \(W(T) \subseteq W(D)\) if and only if there is a Hilbert space \(\hat{\mathcal{H}}\) and \(X : \mathcal{H} \to \hat{\mathcal{H}} \otimes \mathbb{C}^{m+1}\) satisfying \(X^*X = I_{\hat{\mathcal{H}}}\) and \(T_j = X^*(I_{\hat{\mathcal{H}}} \otimes D_j)X\) for \(j = 1, \ldots, m\).

**Lemma 3.4.** Let \(A = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m\). Suppose \(B_i = (B_{i1}, \ldots, B_{im}) \in W^p_{\text{ess}}(A)\) for \(1 \leq i \leq N\). Then for all \(\varepsilon > 0\) and \(K_1, \ldots, K_m \in \mathcal{S}_K(\mathcal{H})\), there exists \(B = (B_1, \ldots, B_m) \in W^{Np}(A_1 + K_1, \ldots, A_m + K_m)\) such that \(B_j = \bigoplus_{i=1}^N \tilde{B}_{ij}\) with \(\|B_{ij} - \tilde{B}_{ij}\| < \varepsilon\) for all \(1 \leq i \leq N\) and \(1 \leq j \leq m\).
Proof. Without loss of generality, we may assume that $K_1 = \cdots = K_m = 0$. We are going to prove by induction on $N$. For $N = 1$, the result follows from definition.

Suppose the result holds for some $N = k \geq 1$. Then there is $X_1 : K_1 \to \mathcal{H}$ with $X_1^*X_1 = I_{K_1}$ for some $kp$-dimensional subspace $K_1$ of $\mathcal{H}$ such that $X_1^*A_jX_1 = \bigoplus_{i=1}^{k} \tilde{B}_{ij}$ for $j = 1, \ldots, m$ and $\|B_{ij} - \tilde{B}_{ij}\| < \varepsilon$ for all $1 \leq i \leq k$ and $1 \leq j \leq m$.

Extend the operator $X_1$ to an unitary operator $U : \mathcal{H} \to \mathcal{H}$ so that $U|_{K_1} = X_1$. Let $\mathcal{L}$ be the subspace spanned by

$$\{K_1, U^*A_1U(K_1), \ldots, U^*A_mU(K_1)\}.$$ 

Then $\mathcal{L}$ has dimension at most $kp(m + 1)$. Define $Y = U|_{\mathcal{L}^\perp}$. Then $Y^*Y = I_{\mathcal{L}^\perp}$. Furthermore, as $X_1 = U|_{K_1}$, $Y = U|_{\mathcal{L}^\perp}$ and $K_1 \subseteq \mathcal{L}$, we have $Y^*X_1 = 0$. Also, for every $1 \leq j \leq m$ and $v \in K_1$, $U^*A_jUv \in \mathcal{L}$. Hence, for every $u \in \mathcal{L}^\perp$,

$$\langle u, Y^*A_jX_1v \rangle = \langle Yu, A_jX_1v \rangle = \langle Lu, A_jUv \rangle = \langle u, U^*A_jUv \rangle = 0.$$ 

Thus, $Y^*A_jX_1 = 0$ for all $j = 1, \ldots, m$.

Since $Y^*A_jY - A_j \in S_\mathcal{K}(\mathcal{H})$ for all $1 \leq j \leq m$, there exists $X_2 : K_2 \to \mathcal{L}^\perp$ with $X_2^*X_2 = I_{K_2}$ for some $p$-dimensional subspace $K_2$ of $\mathcal{L}^\perp$ such that $X_2^*(Y^*A_jY)X_2 = \tilde{B}_{(k+1)j}$ satisfies

$$\|B_{(k+1)j} - \tilde{B}_{(k+1)j}\| < \varepsilon$$ 

for all $1 \leq j \leq m$.

Observe that $K_1$ and $K_2$ are two mutually orthogonal subspaces of $\mathcal{H}$. Furthermore, $X_1 : K_1 \to \mathcal{H}$ and $YX_2 : K_2 \to \mathcal{H}$ satisfy

$$X_1^*X_1 = I_{K_1}, \quad (YX_2)^*(YX_2) = I_{K_2}, \quad (YX_2)^*X_1 = 0 \quad \text{and} \quad (YX_2)^*A_jX_1 = 0$$

for $j = 1, \ldots, m$. Then the operator $Z = X_1 \oplus (YX_2) : K_1 \oplus K_2 \to \mathcal{H}$ satisfies $Z^*Z = I_{K_1 \oplus K_2}$ and $Z^*A_jZ = \bigoplus_{i=1}^{k+1} \tilde{B}_{ij}$ for $j = 1, \ldots, m$.

Now we are ready to present the other main result for this section.

Theorem 3.5. Suppose $A = (A_1, \ldots, A_m)$ is an $m$-tuple of self-adjoint operators acting on $\mathcal{H}$. If $W_{\text{ess}}(A)$ is a simplex $S$ in $\mathbb{R}^{1 \times m}$, i.e., $W_{\text{ess}}(A)$ is a polyhedron with $m + 1$ vertices, then there is an $m$-tuple of self-adjoint compact operators $K = (K_1, \ldots, K_m)$ such that for all positive integer $q$,

$$\text{cl}(W_q^\mathcal{K}(A + K)) = W_q(A + K) = W_q^{\text{ess}}(A) = W_q^{\text{ess}}(\pi(A)). \quad (3.2)$$

Proof. By Theorem 3.1, there is an $m$-tuple of compact operator $K$ such that

$$\text{cl}(W(A + K)) = W(A + K) = W(\pi(A)) = W_{\text{ess}}(A).$$

We will show that for any positive integer $q$, we have

$$\text{cl}(W_q^\mathcal{K}(A + K)) = W_q(A + K) = W_q^{\text{ess}}(\pi(A)) = W_q^{\text{ess}}(A).$$
Let $D = (D_1, \ldots, D_m)$ such that $D_i = \text{diag}(v_{i1}, \ldots, v_{i,m+1})$ for $i = 1, \ldots, m$, where

$$v_1 = (v_{11}, \ldots, v_{m1}), \ldots, v_{m+1} = (v_{1,m+1}, \ldots, v_{m,m+1}) \in \mathbb{R}^{1 \times m}$$

are the vertices of the simplex $S$. Then $S = W(D_1, \ldots, D_m) = W_{\text{ess}}(A)$.

Suppose $q > 1$ is a positive integer. Let $B = (B_1, \ldots, B_m) \in W^q(A + K)$. Then for any unital completely positive map $\phi : M_k \to M_1$, $(\phi(B_1), \ldots, \phi(B_m)) \in \text{cl}(W(A + K))$, which equals to the simplex $S$. Thus, $W(B_1, \ldots, B_m) \subseteq W(D_1, \ldots, D_m)$. By Theorem 3.3, $(B_1, \ldots, B_m)$ admits a joint dilation to $(I \otimes D_1, \ldots, I \otimes D_m)$. Hence, $W^q(B_1, \ldots, B_m) \subseteq W^q(D_1, \ldots, D_m)$.

Suppose $K = (K_1, \ldots, K_m) \in \mathcal{S}(\mathcal{H})^m$. Note that $v_1, \ldots, v_{m+1} \in W_{\text{ess}}(A_1 + K_1, \ldots, A_m + K_m)$. Applying Lemma 3.4 with $p = 1$, $N = (m+1)q$ and $B_i = v_j$ if $i \equiv j \pmod{m+1}$ for $1 \leq i \leq N$ and $1 \leq j \leq m+1$, we can get a sequence $D^{(k)} = (D_1^{(k)}, \ldots, D_m^{(k)}) \in W_{\text{ess}}(A_1 + K_1, \ldots, A_m + K_m)$ such that

$$\{D^{(1)}_j, D^{(2)}_j, D^{(3)}_j, \ldots\} \to I_q \otimes D_j.$$

Since $W^q(B_1, \ldots, B_k) \subseteq W^q(D_1, \ldots, D_m)$, for any $R_0, \ldots, R_m \in M_q$ we have

$$\|R_0 \otimes I + R_1 \otimes B_1 + \cdots + R_m \otimes B_m\| \leq \|R_0 \otimes I + R_1 \otimes D_1 + \cdots + R_m \otimes D_m\| = \|R_0 \otimes I + R_1 \otimes I_{m+1} \otimes I_q \otimes D_1 + \cdots + R_m \otimes I_{m+1} \otimes I_q \otimes D_m\| = \lim_{k \to \infty} \|R_0 \otimes I + R_1 \otimes I_{m+1} \otimes I + R_1 \otimes I_{m+1} \otimes (A_1 + K_1) + \cdots + R_m \otimes I_{m+1} \otimes (A_m + K_m)\| = \|R_0 \otimes I + R_1 \otimes (A_1 + K_1) + \cdots + R_m \otimes (A_m + K_m)\|.$$

By Theorem 2.1, $B \in W^q(A + K)$. Because this is true for all compact $K$, we see that $B \in W^q_{\text{ess}}(A)$. Hence, we have

$$\text{cl}(W^q(A + K)) \subseteq W^q(A + K) \subseteq W^q_{\text{ess}}(A).$$

Since $W^q(\pi(A)) = W^q_{\text{ess}}(A + K) \subseteq \text{cl}(W^q(A + K))$, we see that (3.2) holds.

4 Related results

By Theorem 3.5, we have the following extension of [15, 1.22.5].

**Proposition 4.1.** Let $A = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. Suppose $W_{\text{ess}}(A)$ is a subset of a simplex $S$ in $\mathbb{R}^m$ with vertices $v^{(1)}, \ldots, v^{(m+1)}$ such that $v^{(k)} = (v^{(k)}_1, \ldots, v^{(k)}_m)$ for $k = 1, \ldots, m+1$. Then there is an $m$-tuple of self-adjoint compact operators $K = (K_1, \ldots, K_m)$ such that for any $R_0, \ldots, R_m \in M_q$,

$$\|R_0 \otimes I + R_1 \otimes (A_1 + K_1) + \cdots + R_m \otimes (A_m + K_m)\| \leq \max\{|R_0 + v^{(k)}_1 R_1 + \cdots + v^{(k)}_m R_m| : 1 \leq k \leq m+1\}. \quad (4.3)$$

In fact, $K$ can be chosen such that the equality holds in (4.3) for any choice of $R_0, \ldots, R_m \in M_q$. 


Proof. Let $D_1, \ldots, D_m \in M_{m+1}$ with $D_j = \text{diag}(v_j^{(1)}, \ldots, v_j^{(m+1)})$ so that $W(D_1, \ldots, D_m) = S$. Let $\mathcal{H} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathbb{C}^{m+1})$. Then $A_j$ is a compression of $\tilde{A}_j = A_j \oplus (I_H \otimes D_j) \in \mathcal{B}(\mathcal{H})$ for all $j = 1, \ldots, m$. Evidently, $W^q_{\text{ess}}(\tilde{A}_1, \ldots, \tilde{A}_m) = W^q(D_1, \ldots, D_m)$. By Theorem 3.5, there are self-adjoint compact operators $\tilde{K}_1, \ldots, \tilde{K}_m \in \mathcal{B}(\mathcal{H})$ such that $\text{cl} \left( W(\tilde{A}_1 + \tilde{K}_1, \ldots, \tilde{A}_m + \tilde{K}_m) \right) = W_{\text{ess}}(\tilde{A}_1, \ldots, \tilde{A}_m)$. Moreover, for any positive integer $q$, we have

$$W^q(\tilde{A}_1 + \tilde{K}_1, \ldots, \tilde{A}_m + \tilde{K}_m) = W^q_{\text{ess}}(\tilde{A}_1, \ldots, \tilde{A}_m).$$

Suppose $\tilde{K}_j$ has operator matrix $\begin{pmatrix} K_1 & * \\ * & * \end{pmatrix}$ with $K_1 \in \mathcal{B}(\mathcal{H})$ for $j = 1, \ldots, m$. Then for any $R_0, \ldots, R_m \in M_q$, we have

$$\| R_0 \otimes I + R_1 \otimes (A_1 + K_1) + \cdots + R_m \otimes (A_m + K_m) \|$$

$$\leq \| R_0 \otimes I + R_1 \otimes (\tilde{A}_1 + \tilde{K}_1) + \cdots + R_m \otimes (\tilde{A}_m + \tilde{K}_m) \|$$

$$= \| R_0 \otimes I + R_1 \otimes D_1 + \cdots + R_m \otimes D_m \|$$

$$= \max\{ \| R_0 + v_{1k}R_1 + \cdots + v_{mk}R_m \| : 1 \leq k \leq m + 1 \}. $$

For the last assertion, let $K$ be chosen such that (4.3) holds. Suppose $A_j + K_j$ has operator matrix $\begin{pmatrix} B_j & * \\ * & C_j \end{pmatrix}$ with $B_j \in M_{m+1}$ for $j = 1, \ldots, m$. There are finite rank operators $F_1, \ldots, F_m$ such that $A_j + K_j + F_j$ has operator matrix $D_j \oplus C_j$ for $j = 1, \ldots, m$. Then for any $R_0, \ldots, R_m \in M_q$, we have

$$\| R_0 \otimes I + R_1 \otimes C_1 + \cdots + R_m \otimes C_m \|$$

$$\leq \| R_0 \otimes I + R_1 \otimes (A_1 + K_1) + \cdots + R_m \otimes (A_m + K_m) \|$$

$$\leq \| R_0 \otimes I + R_1 \otimes D_1 + \cdots + R_m \otimes D_m \|.$$ 

Hence, if $\tilde{K}_j = K_j + F_j$ for $j = 1, \ldots, m$, then

$$\| R_0 \otimes I + R_1 \otimes (A_1 + \tilde{K}_1) + \cdots + R_m \otimes (A_m + \tilde{K}_m) \|$$

$$= \max\{ \| R_0 \otimes I + R_1 \otimes D_1 + \cdots + R_m \otimes D_m \|, \| R_0 \otimes I + R_1 \otimes C_1 + \cdots + R_m \otimes C_m \| \}$$

$$= \| R_0 \otimes I + R_1 \otimes D_1 + \cdots + R_m \otimes D_m \|$$

$$= \max\{ \| R_0 + v_{1k}R_1 + \cdots + v_{mk}R_m \| : 1 \leq k \leq m + 1 \}. $$

The last assertion follows.

Motivated by quantum error correction, researchers consider the joint rank $(p, q)$-matricial range of $A$, denoted by $\Lambda_{p,q}(A)$, consisting of $m$-tuples of $k \times k$ matrices $B = (B_1, \ldots, B_m)$ such that for some unitary $U$, $I_p \otimes B_j$ is the leading principal submatrix of $U^*A_jU$ for $j = 1, \ldots, m$; see [8] and its references. One can define the $\Lambda_{p,q}^{\text{ess}}(A)$ as

$$\Lambda_{p,q}^{\text{ess}}(A) = \cap\{ \text{cl}(\Lambda_{p,q}(A + K)) : K \in \mathcal{S}_K(\mathcal{H})^m \},$$

and deduce the following results from those in the previous sections.
Theorem 4.2. Let $A \in S(\mathcal{H})^m$.

1. For any positive integer $N$, there is $K \in S_K(\mathcal{H})^m$ such that
   
   $$ \Lambda_{p,q}(A + K) = \Lambda_{p,q}^{\text{ess}}(A), $$
   
   which is the same as
   
   $$ W_q^q(A) = W^q(\pi(A)) = W^q(A + K) = W^q_s(A + K) $$
   
   for all $q \in \{1, \ldots, N\}$.

2. Suppose $W_{\text{ess}}(A)$ is a simplex in $\mathbb{R}^m$. Then there is $K \in S_K(\mathcal{H})^m$ such that
   
   $$ \Lambda_{p,q}(A + K) = \Lambda_{p,q}^{\text{ess}}(A), $$
   
   which is the same as
   
   $$ W_q^q(A) = W^q_s(A + K) = W^q(\pi(A)) = W^q(A + K) $$
   
   for all positive integer $q$.

Proof. By the result in [8],

$$ \Lambda_{p,q}^{\text{ess}}(A) = W_q^q(A) \subseteq \Lambda_{p,q}(A + K) \subseteq W^q(A + K). $$

The results follow readily from Theorem 3.1 and Theorem 3.5. ☐

Acknowledgment

Li is an affiliate member of the Institute for Quantum Computing, University of Waterloo, and an honorary professor of the Shanghai University; his research was supported by the USA NSF DMS 1331021, the Simons Foundation Grant 351047, and NNSF of China Grant 11571220. Paulsen is a professor of Pure Mathematics and the Institute for Quantum Computing, University of Waterloo, his research is partially supported by NSERC 03784. Poon would like to express his gratitude to the generous support from the Institute for Quantum Computing during his visit in September to October, 2017.

References


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