The joint numerical range of commuting matrices

by

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Abstract. It is shown that for $n \leq 3$ the joint numerical range of a family of commuting $n \times n$ complex matrices is always convex; for $n \geq 4$ there are two commuting matrices whose joint numerical range is not convex.

1. Introduction. Let $M_{m,n}$ be the set of $m \times n$ complex matrices. For $A \in M_{m,n}$, $A^*$ (resp. $A^t$) stands for the conjugate transpose (resp. transpose) of $A$; for example, see [9, 10]. Denote by $\mathbb{C}^n$ (resp. $\mathbb{R}^n$) the set of column vectors with $n$ complex (resp. real) entries. Let $M_n = M_{n,n}$ and $M_n^m$ be the set of all $m$-tuples of $n \times n$ matrices. We identify $\mathbb{C}^n$ with $M_{n,1}$. For notational convenience, we will also say that $z \in \mathbb{C}^n$ for a complex row vector $z = (z_1, \ldots, z_n)$. The joint numerical range of $A = (A_1, \ldots, A_m) \in M_n^m$ is defined by

$$W(A) = \{(x^* A_1 x, \ldots, x^* A_m x) : x \in \mathbb{C}^n, x^* x = 1\} \subseteq \mathbb{C}^m.$$ 

When $m = 1$, it reduces to the classical numerical range $W(A_1)$ of $A_1 \in M_n$, which is a useful tool for studying matrices and operators; for example, see [10, Chapter 1]. The joint numerical range of $m$ matrices is useful in studying the behavior of the family of matrices $\{A_1, \ldots, A_m\} \subseteq M_n$, and has applications in many pure and applied areas. We refer the readers to the excellent survey [14] and the paper [15] on this subject.

When $m = 1$, the Toeplitz–Hausdorff theorem asserts that $W(A_1)$ is always convex. However, for $m \geq 2$, $W(A_1, \ldots, A_m)$ may fail to be convex; see [11]. Many researchers have studied matrices with certain commutative properties that have convex joint numerical ranges; e.g., see [3, 4, 5, 6, 11, 12, 13]. In particular, Dash [5, Proposition 2.4] proved that $W(A_1, \ldots, A_m)$ is always convex for any commuting family $\{A_1, \ldots, A_m\} \subseteq M_2$ and raised the ques-
tion on the same result for \{A_1, \ldots, A_m\} \subseteq M_n with n > 2. In [13], Müller gave a simple example, which was incorporated in [15] with some improvements, of a commuting family \{A_1, A_2, A_3\} \subseteq M_4 such that \(W(A_1, A_2, A_3)\) is not convex, and raised the question of whether \(W(A_1, A_2)\) is convex if \(A_1A_2 = A_2A_1\); see [13 Problem 2]. We will show that the answer is negative if \(A_1, A_2\) is a commuting pair of matrices (or infinite-dimensional operators) with dimension at least 4. However, for a commuting pair of matrices \(A_1, A_2 \in M_3\), \(W(A_1, A_2)\) is always convex. We can then deduce from the results that \(W(A_1, \ldots, A_m)\) is always convex for any commuting family \(\{A_1, \ldots, A_m\} \subseteq M_3\).

Our paper is organized as follows. In Section 2, we present some preliminary results including a short proof of the convexity of \(W(A_1, \ldots, A_m)\) for every commuting family \(\{A_1, \ldots, A_m\} \subseteq M_2\). In Section 3, we present examples of commuting matrices (or infinite-dimensional operators) \(A_1, A_2\) of dimension at least 4 such that \(W(A_1, A_2)\) is not convex. We then state our main result that \(W(A_1, A_2)\) is convex if \(A_1, A_2 \in M_3\) commute, and we deduce that \(W(A_1, \ldots, A_m)\) is convex for any commuting family \(\{A_1, \ldots, A_m\} \subseteq M_3\). The rather involved proof of the main theorem on the convexity of \(W(A_1, A_2)\) for a commuting pair \(A_1, A_2 \in M_3\) will be given in Section 4.

2. Preliminaries and commuting families in \(M_2\). Let 

\[\mathcal{H}_n = \{A \in M_n : A = A^*\}\]

be the real space of all \(n \times n\) Hermitian matrices and \(I_n\) be the \(n \times n\) identity matrix. We summarize some properties of joint numerical ranges which are useful for what follows. We refer the interested readers to [1], [8], [11].

**Proposition 2.1.** Let \(F = \{A_1, \ldots, A_m\} \subseteq M_n\). Suppose the complex space spanned by \(\{A_1, \ldots, A_m\}\) has a basis \(\{C_1, \ldots, C_s\}\). Let \(A_j = H_j + iG_j\), where \(H_j, G_j \in \mathcal{H}_n\) for \(j = 1, \ldots, m\). Then:

(a) \(W(A_1, \ldots, A_m) = W(U^*A_1U, \ldots, U^*A_mU)\) for any unitary \(U \in M_n\).
(b) \(W(A_1, \ldots, A_m) = W(A^*_1, \ldots, A^*_m)\).
(c) \(W(A_1, \ldots, A_m)\) is convex if and only if \(W(C_1, \ldots, C_s)\) is convex.
(d) The family \(F\) is commuting if and only if \(\{C_1, \ldots, C_s\}\) is commuting.
(e) \(W(A_1, \ldots, A_m) \subseteq \mathbb{C}^m\) can be identified with \(W(H_1, G_1, \ldots, H_m, G_m) \subseteq \mathbb{R}^{2m}\).
(f) For \(n = 2\) and \(H_1, \ldots, H_m \in \mathcal{H}_2\), \(W(H_1, \ldots, H_m)\) is convex if and only if \(\text{span} \{I_2, H_1, \ldots, H_m\} \neq \mathcal{H}_2\).
(g) Suppose \(n \geq 3\) and \(H_1, \ldots, H_m \in \mathcal{H}_n\). If \(\text{span} \{I_n, H_1, \ldots, H_m\}\) has dimension at most 4, then \(W(H_1, \ldots, H_m)\) is convex.

Note that (c) and (f) are given in [8] Corollary 2.4 and Example 1 and (g) is given in [1] Corollary 1. By (e), the study of convexity of \(W(A_1, \ldots, A_m)\)
can be reduced to $W(H_1, G_1, \ldots, H_m, G_m)$ for Hermitian matrices $H_1, G_1, \ldots, H_m, G_m$. However, it is clear that the commutativity of $A_1, \ldots, A_m$ does not imply the commutativity of $H_1, G_1, \ldots, H_m, G_m$. In fact, if $\{H_1, G_1, \ldots, H_m, G_m\}$ is a commuting family, then $\{A_1, \ldots, A_m\}$ is a commuting family of normal matrices, and $W(A_1, \ldots, A_m)$ will be polyhedral, i.e., a convex hull of finitely many points in $\mathbb{C}^m$; see [5, Theorem 2.5]. It is clear that $(\mu_1, \ldots, \mu_m) \in W(A_1, \ldots, A_m)$ if and only if $(1, \mu_1, \ldots, \mu_m) \in W(I_n, A_1, \ldots, A_m)$ for any $W(A_1, \ldots, A_m) \in M_{n}^{m}$. By Proposition 2.1 to study the convexity of $W(A_1, \ldots, A_m)$, one may focus on $W(C_1, \ldots, C_s)$ where $\{I_n, C_1, \ldots, C_s\}$ is a basis for the span of $\{I_n, A_1, \ldots, A_m\}$. It is well-known that if $\{A_1, \ldots, A_m\}$ is a commuting family of matrices then there is a unitary $U$ such that $U^* A_j U$ are in upper triangular form for all $j = 1, \ldots, m$; see [10]. Our proofs often use this property.

Denote by $\text{conv } S$ and $\partial S$ respectively the convex hull and the boundary of a set $S$ in $\mathbb{R}^m$ or $\mathbb{C}^m$. The next result describes the intersection of support planes of $\text{conv } W(A_1, \ldots, A_m)$ with $W(A_1, \ldots, A_m)$.

**Proposition 2.2.** Let $B_1, \ldots, B_r \in \mathcal{H}_n$ be Hermitian matrices. For every unit vector $\mathbf{v} = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$, let

$$P_{\mathbf{v}} = \{ \mathbf{b} \in \mathbb{R}^r : \mathbf{b}^* \mathbf{v} \leq \lambda_1(\nu_1 B_1 + \cdots + \nu_r B_r) \},$$

where $\lambda_1(H)$ denotes the largest eigenvalue of $H \in \mathcal{H}_n$ and $\mathbf{b}^* \mathbf{v} = \sum_{i=1}^{r} b_i \nu_i$ for $\mathbf{b} = (b_1, \ldots, b_r) \in \mathbb{R}^r$. Then

$$\text{conv } W(B_1, \ldots, B_r) = \bigcap \{ P_{\mathbf{v}} : \mathbf{v} = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r, \mathbf{v}^* \mathbf{v} = 1 \}.$$

Consequently,

$$\partial P_{\mathbf{v}} \cap W(B_1, \ldots, B_r) = \{ (x^* B_1 x, \ldots, x^* B_r x) : x \in \mathbb{C}^n, x^* x = 1, B_{\mathbf{v}} x = \lambda_1(B_{\mathbf{v}}) x \},$$

where $B_{\mathbf{v}} = \sum_{j=1}^{r} \nu_j B_j$. Moreover, $\partial P_{\mathbf{v}} \cap W(B_1, \ldots, B_r)$ is convex if and only if

$$W(X^* B_1 X, \ldots, X^* B_r X)$$

is convex, where the columns of $X$ form an orthonormal basis for the null space of $B_{\mathbf{v}} - \lambda_1(B_{\mathbf{v}}) I_n$.

**Proof.** If $x \in \mathbb{C}^n$ is a unit vector and

$$\mathbf{b} = (x^* B_1 x, \ldots, x^* B_r x) \in W(B_1, \ldots, B_r),$$

then for any unit vector $\mathbf{v} = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$ we have

$$\mathbf{b}^* \mathbf{v} = x^* \left( \sum_{j=1}^{r} \nu_j B_j \right) x \leq \lambda_1 \left( \sum_{j=1}^{r} \nu_j B_j \right).$$

Thus, $W(B_1, \ldots, B_r) \subseteq P_{\mathbf{v}}$. As $P_{\mathbf{v}}$ is convex, $\text{conv } W(B_1, \ldots, B_r) \subseteq P_{\mathbf{v}}$. 


Conversely, suppose $b = (b_1, \ldots, b_r) \notin \text{conv } W(B_1, \ldots, B_r) \subseteq \mathbb{R}^r$. By the separation theorem, there exists a real unit vector $v = (v_1, \ldots, v_r) \in \mathbb{R}^r$ such that $\sum_{j=1}^r b_jv_j > \sum_{j=1}^r q_jv_j$ for all $(q_1, \ldots, q_r) \in W(B_1, \ldots, B_r)$, i.e., for every unit vector $x \in \mathbb{C}^n$,

$$\sum_{j=1}^r b_jv_j > \sum_{j=1}^r v_j(x^*B_jx) = x^*(\sum_{j=1}^r \nu_jB_j)x.$$ 

So, $\sum_{j=1}^r b_jv_j > \lambda_1(\sum_{j=1}^r \nu_jB_j)$.

The last two assertions are clear. □

The following result is proven in [5, Proposition 2.4]. Recently, it was also given in [2, Theorem 2.2]. We give a short proof here for completeness.

**Proposition 2.3.** For any commuting family $\mathcal{F} = \{A_1, \ldots, A_m\} \subseteq M_2$, $W(A_1, \ldots, A_m)$ is convex.

**Proof.** To avoid trivial considerations, suppose $\mathcal{F}$ contains a nonscalar matrix $X \in M_2$. Applying a unitary similarity, we may assume that all matrices in $\mathcal{F}$ are in upper triangular form. Let $X_0 = X - \frac{\text{tr}X}{2}I_2 = \begin{pmatrix} x_1 & x_2 \\ 0 & -x_1 \end{pmatrix}$. We claim that for every $Y \in \mathcal{F}$, $Y_0 = Y - \frac{\text{tr}Y}{2}I_2 = \begin{pmatrix} y_1 & y_2 \\ 0 & -y_1 \end{pmatrix}$ is a multiple of $X_0$; see [7, Theorem II] for an alternative proof. Thus every $A_j$ is a linear combination of $I_2$, $H_1 = (X_0 + X_0^*)/2$ and $H_2 = (X_0 - X_0^*)/(2i)$. By Proposition 2.1(f), $W(A_1, \ldots, A_m)$ is convex.

To prove our claim, note that $X_0$ commutes with $Y_0$, i.e., $x_1y_2 = x_2y_1$. Since $X$ is nonscalar, either $x_1 \neq 0$ or $x_2 \neq 0$.

If $x_1 = 0$, then $x_2 \neq 0$ and $x_2y_1 = 0$. Thus $y_1 = 0$ and $Y_0 = (y_2/x_2)X_0$. Our claim follows.

If $x_1 \neq 0$, then $x_1y_2 = x_2y_1$ implies $Y_0 = (y_1/x_1)X_0$. Again, our claim follows. □

**3. Convexity of commuting family of dimension at least 3.** In [13], the author gave an elegant example of a commuting family $\{A_1, A_2, A_3\} \subseteq M_4$ with nonconvex $W(A_1, A_2, A_3)$. The following example illustrates that $W(A_1, A_2)$ may not be convex for a commuting pair $A_1, A_2 \in M_4$.

**Example 3.1.** Let $A_1 = H_1 + iG_1$ and $A_2 = A_1 + A_1^2 - A_1^3 - 12I_4 = H_2 + iG_2$ with

$$H_1 = \text{diag}(2, 2, 1, 0),$$

$$G_1 = \begin{pmatrix} 1 & 0 & 2 - i & -i \\ 0 & 0 & -1 + i & 1 - i \\ 2 + i & -1 - i & 0 & 0 \\ i & 1 + i & 0 & 0 \end{pmatrix}.$$
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H_2 = \begin{pmatrix}
14 & -9 - 7i & 8 - 4i & -3i \\
-9 + 7i & 0 & 0 & 0 \\
8 + 4i & 0 & 10 & -2 - 4i \\
3i & 0 & -2 + 4i & -9 \\
\end{pmatrix},

G_2 = \begin{pmatrix}
6 & -2 - 2i & 12 - 4i & -4 - 6i \\
-2 + 2i & 0 & -3 + 7i & 5 - i \\
12 + 4i & -3 - 7i & 5 & 1 - 2i \\
-4 + 6i & 5 + i & 1 + 2i & 1 \\
\end{pmatrix}.

Then A_1A_2 = A_2A_1. Note that for the unit vector \( \nu = (1, 0, 0, 0) \), the matrix \( A_\nu = \nu_1H_1 + \nu_2G_1 + \nu_3H_2 + \nu_4G_1 = H_1 \) has the largest eigenvalue 2, and the null space of \( A_\nu - 2I_4 \) is spanned by the first two columns of \( I_4 \). Let \( X \in M_{4,2} \) be the matrix formed by these two columns. It is easy to check that

\[ \text{span} \{ X^*H_1X, X^*G_1X, X^*H_2X, X^*G_2X \} = H_2. \]

By Proposition 2.1(e,f), \( W(X^*A_1X, X^*A_2X) \) is not convex. Since

\[ W(X^*A_1X, X^*A_2X) = \{ (\mu_1, \mu_2) \in W(A_1, A_2) : \text{Re} \mu_1 = 2 \}, \]

\( W(A_1, A_2) \) is not convex.

**Remark 3.2.** For \( n > 4 \), one can extend the above example to \( \tilde{A}_1 = A_1 \oplus 0_N \) and \( \tilde{A}_2 = A_2 \oplus 0_N \), where \( 1 \leq N \leq \infty \). It is clear that \( \tilde{A}_1 \tilde{A}_2 = \tilde{A}_2 \tilde{A}_1 \) and \( W(\tilde{A}_1, \tilde{A}_2) \) is not convex.

For commuting \( A_1, A_2 \in M_3 \), we have the following.

**Theorem 3.3.** Suppose that \( A_1, A_2 \in M_3 \) commute. Then \( W(A_1, A_2) \) is convex.

The proof of the result is quite involved and technical. We will present it in the next section. From Theorem 3.3, we can deduce the following.

**Theorem 3.4.** Let \( \{ A_1, \ldots, A_m \} \subseteq M_3 \) be a commuting family of matrices. Then the complex linear span of \( \{ I_3, A_1, \ldots, A_m \} \) has dimension at most 3, and hence \( W(A_1, \ldots, A_m) \) is convex.

**Proof.** We may assume that \( A_1, \ldots, A_m \) are in upper triangular form, and \( \mathcal{F} = \{ I_3, A_1, \ldots, A_m \} \) is linearly independent. We are going to prove by contradiction that \( m \leq 2 \). In the following, we will use \( \text{diag} A \in \mathbb{C}^n \) as the vector of diagonal entries of \( A \in M_n \).

Suppose to the contrary that \( m > 2 \). Then \( \{ \text{diag} I_3, \text{diag} A_1, \ldots, \text{diag} A_m \} \) is linearly dependent. Therefore, \( \text{span} \mathcal{F} \) has a nonzero nilpotent. We may assume that \( A_1 \) is a nonzero nilpotent in \( \text{span} \mathcal{F} \) of the largest rank. Consider the following cases:
CASE 1: rank $A_1 = 2$. Then there is an invertible $S$ such that $S^{-1}A_1S = J$ is the upper triangular Jordan block. Then for every $2 \leq i \leq m$, $A_1A_i = A_iA_1$ implies that $S^{-1}A_iS = a_iI_3 + b_iJ + c_iJ^2$ for some $a_i, b_i, c_i \in \mathbb{C}$. Since \{I_3, A_1, \ldots, A_m\} is linearly independent, we have $m \leq 2$, a contradiction.

CASE 2: rank $A_1 = 1$. So, up to a nonzero multiple and a unitary similarity transform, we may assume that $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then for every $2 \leq i \leq m$, the condition $A_1A_i = A_iA_1$ implies that $A_i$ is in upper triangular form with the $(1,1)$-entry equal to the $(3,3)$-entry. We may then replace $A_i$ by $A_i - \alpha_iI_3 - \beta_iA_1$ for some $\alpha_i, \beta_i \in \mathbb{C}$ and assume that

$$A_i = \begin{pmatrix} 0 & b_i & 0 \\ 0 & a_i & c_i \\ 0 & 0 & 0 \end{pmatrix}$$

for some $a_i, b_i, c_i \in \mathbb{C}, i = 2, \ldots, m$.

If $a_i = 0$ for all $2 \leq i \leq m$, then span\{A_2, A_3\} would contain a nonzero nilpotent of rank 2, which contradicts the assumption that $A_1$ has the largest rank. Therefore, we may assume that $a_2 = 1$ and $a_3 = 0$. Then $A_2A_3 = A_3A_2$ implies that $b_3 = c_3 = 0$, which contradicts $F$ being linearly independent.

This shows that $m \leq 2$ and the convexity of $W(A_1, A_2)$ follows from Theorem 3.3. □

4. Proof of Theorem 3.3. We divide the proof into two subsections. We will always assume that $A_1 = H_1 + iG_1$ and $A_2 = H_2 + iG_2$, where $H_1, G_1, H_2, G_2$ are Hermitian. In view of Proposition 2.1(g), we always assume that span \{I_n, H_1, G_1, H_2, G_2\} has dimension 5 to avoid trivial considerations.

4.1. span \{I_3, A_1, A_2\} ⊆ M_3 does not contain a nonzero nilpotent. In this subsection, we assume that span \{I_3, A_1, A_2\} ⊆ M_3 does not contain a nonzero nilpotent. Without loss of generality, by applying unitary similarity transforms and taking linear combinations of $I_3, A_1, A_2$, one can assume that

$$A_1 = \begin{pmatrix} 1 & u & w_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & w_2 \\ 0 & 0 & v \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad u, v \geq 0, \quad w_1 + uv + w_2 = 0.$$

The reduction can be done as follows. Since $A_1$ and $A_2$ commute, we assume without loss of generality that both $A_1, A_2$ are in upper triangular form. Since span \{I_3, A_1, A_2\} does not contain a nonzero nilpotent matrix, \{diag I_3, \text{diag} A_1, \text{diag} A_2\} ⊆ C^3 is linearly independent. Replacing $A_j$ by $\alpha_j A_1 + \beta_j A_2 + \gamma_j I_3$ with suitable $\alpha_j, \beta_j, \gamma_j \in \mathbb{C}, j = 1, 2$, we may assume
are of the form in (4.1).

Therefore, \( A_1 = \begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix} \) and \( A_2 = \begin{pmatrix} 0 & b_1 & b_2 \\ 0 & 0 & b_3 \\ 0 & 0 & 1 \end{pmatrix} \).

Then we have

\[
A_1A_2 = \begin{pmatrix} 0 & b_1 & a_2 + b_2 + a_1b_3 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix} = A_2A_1 = \begin{pmatrix} 0 & 0 & a_3b_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Therefore, \( a_3 = b_1 = 0 = a_2 + b_2 + a_1b_3 \). Replacing \( A_j \) by \( DA_jD^{-1} \) with a diagonal unitary matrix \( D \), we may assume \( a_1, b_3 \geq 0 \), so that we get (4.1).

By Proposition 2.1, the convexity of \( W(A_1, A_2) \) is equivalent to the convexity of the numerical range of \((A_1, A_2)\) transformed into the form (4.1).

In the following, we will show that \( W(A_1, A_2) \) is convex if \( A_1, A_2 \in M_3 \) are of the form in (4.1).

**Proposition 4.1.** Let \( A_1, A_2 \in M_3 \) be of the form (4.1). If (0, 0) \( \notin \{(u, w_1), (v, w_2), (u, v)\} \), then \( W(A_1, A_2) \) is convex.

**Proof.** If \( u = w_1 = 0 \), then set \( (H_2, G_2) = (A_2 + A^*_2, i(A^*_2 - A_2))/2 \) and identify \( W(A_1, A_2) \) with \( W(A_1, H_2, G_2) \subseteq \mathbb{R}^3 \), which is convex by Proposition 2.1(g).

If \( v = w_2 = 0 \), then set \( (H_1, G_1) = (A_1 + A^*_1, i(A^*_1 - A_1))/2 \) and identify \( W(A_1, A_2) \) with \( W(H_1, G_1, A_2) \subseteq \mathbb{R}^3 \), which is convex.

If \( u = v = 0 \), then \( w_1 + w_2 = 0 \). By the previous argument, \( A_1 + A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \), hence \( W(A_1 + A_2, A_2) \) is convex, and so is \( W(A_1, A_2) \). \( \blacksquare \)

Next, we treat the case where (0, 0) \( \notin \{(u, w_1), (v, w_2), (u, v)\} \). First, we show that \( W(A_1, A_2) \) has convex boundary.

**Proposition 4.2.** Let \( A_1, A_2 \in M_3 \) be commuting matrices of the form (4.1) such that (0, 0) \( \notin \{(u, w_1), (v, w_2), (u, v)\} \). Then \( W(A_1, A_2) \) contains all of the boundary points of \( \text{conv} \, W(A_1, A_2) \).

**Proof.** Suppose \( A_1 \) and \( A_2 \) satisfy the hypothesis, and \( A_1 = H_1 + iG_1, \)
\( A_2 = H_2 + iG_2 \), where \( H_1, H_2, G_1, G_2 \in \mathcal{H}_3 \). For every unit vector \( \nu = (\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{R}^4 \), let

\[
P_\nu = \left\{(b_1, \ldots, b_4) \in \mathbb{R}^4 : \sum_{i=1}^{4} b_i\nu_i \leq \lambda_1(\nu_1H_1 + \nu_2G_1 + \nu_3H_2 + \nu_4G_2) \right\}.
\]

By Proposition 2.2, every boundary point of \( \text{conv} \, W(A_1, A_2) \) lies in \( \partial P_\nu \) for some \( \nu \in \mathbb{R}^4 \), and

\[
\partial P_\nu \cap \text{conv} \, W(A_1, A_2) = \text{conv}(\partial P_\nu \cap W(A_1, A_2)).
\]
We will show that $\partial P_\nu \cap \text{conv } W(A_1, A_2) \subseteq W(A_1, A_2)$.

**Case 1.** Suppose one of the following conditions holds:

(i) $uv = 0,$
(ii) $(w_1 - w_2)^2 = (uv)^2,$ or
(iii) $|w_1|\sqrt{1 + v^2} \neq |w_2|\sqrt{1 + u^2}.

In each of these cases, we will show that $\partial P_\nu \cap \text{conv } W(A_1, A_2)$ is a singleton lying in $W(A_1, A_2)$ for any unit vector $\nu$.

Let $\nu = (\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{R}^4$ be a unit vector. The matrix $B_\nu = \nu_1 H_1 + \nu_2 G_1 + \nu_3 H_2 + \nu_4 G_2$ has the form

$$
\begin{pmatrix}
v_1 & u(v_1 - iv_2) & w_1(v_1 - iv_2) + w_2(v_3 - iv_4) \\
u(v_1 + iv_2) & 0 & v(v_3 - iv_4)
\end{pmatrix}.
$$

Let $r$ be the multiplicity of $\lambda_1(B_\nu)$. Since $\{I_3, H_1, G_1, H_2, G_2\}$ is linearly independent, $r \leq 2$.

By Proposition 2.2, if $r = 1$, then $\partial P_\nu \cap W(A_1, A_2)$ is singleton and equals $\partial P_\nu \cap \text{conv } W(A_1, A_2)$.

We show that $r = 2$ is impossible under any one of the assumptions (i), (ii) or (iii). Assume to the contrary that $r = 2$. As $(0,0) \notin \{(u,v), (u,w_1), (v,w_2)\}$, we see that $\lambda_1(B_\nu) \neq 0$. Since the $(2,2)$-entry of $B_\nu$ is 0, we have $\lambda_1(B_\nu) = \lambda_2(B_\nu) > 0 \geq \lambda_3(B_\nu)$. Thus, there is a nonzero real vector $(a, b, c, d)$ such that

$$R = I_3 + aH_1 + bG_1 + cH_2 + dG_2 = zz^*$$

for some nonzero $z \in \mathbb{C}^3$, so that $R$ is a rank 1 positive semidefinite matrix.

Let $S$ be the set of all nonzero real vectors $(a, b, c, d)$ such that $R$ is a rank 1 positive semidefinite matrix. We are going to show that $S = \emptyset$, thus arriving at a contradiction. In such a case, we may assume that

$$z = (z_1, z_2, z_3) = (\sqrt{a + 1}e^{i\theta_1}, 1, \sqrt{c + 1}e^{i\theta_2})$$

with

$$\sqrt{a + 1}e^{i\theta_1} = z_1 = z_1\bar{z}_2 = \frac{u}{2}(a - i)b, \quad \sqrt{c + 1}e^{-i\theta_2} = \bar{z}_3 = z_2\bar{z}_3 = \frac{v}{2}(c - id),$$

and

$$uv(a - ib)(c - id) = \sqrt{(a + 1)(c + 1)}e^{i(\theta_1 - \theta_2)} = z_1\bar{z}_3 = \frac{w_1(a - ib) + w_2(c - id)}{2}.$$
The matrix $R$ given by \((4.2)\) then has the form
\[
(4.3) \quad R = \begin{pmatrix}
a + 1 & u(a - ib)/2 & uv(a - ib)(c - id)/4 \\
u(a + ib)/2 & 1 & v(c - id)/2 \\
uv(a + ib)/(c + id)/4 & v(c + id)/2 & c + 1
\end{pmatrix}.
\]

Since $R$ has rank 1, we have $(a, b) \neq (0, 0)$ and $(c, d) \neq (0, 0)$. If any one of the assumptions (i), (ii) or (iii) holds, we are going to derive a contradiction.

Suppose that (i) holds, i.e., $uv = 0$. Recall from (4.1) that $u$ and $v$ are nonnegative. Since $(u, v) \neq (0, 0)$, we assume $u = 0 < v$ or $v = 0 < u$. Let $u = 0$ and $v > 0$. Since $(u, w_1) \neq (0, 0)$, we may replace $(A_1, A_2)$ by $(D^* A_1 D, D^* A_2 D)$ for some suitable diagonal unitary matrix $D$ and assume that $-w_2 = w_1 > 0$. Suppose there is a real vector $(a, b, c, d)$ such that $R$ given by (4.2) is a rank 1 positive semidefinite matrix of the form (4.3). Since the $(1,2)$-entry is zero, we see that $a = -1$. Now, $-w_2 = w_1 > 0$ and the $(1,3)$-entry of $R$ is $w_1((-a) - (c - id)) = 0$. Thus $c = a = -1$ and $b = d$. As a result, the $(3,3)$-entry of $R$ is zero and so must be the $(2,3)$-entry. Hence, $v = 0$, which is a contradiction. Similarly, we can show that for $u > 0$ and $v = 0$, $S = \emptyset$.

Suppose now that $u, v > 0$. As the matrix $R$ in (4.3) is rank 1, we have $4(a + 1)/u^2 = (a^2 + b^2)$ and $4(c + 1)/v^2 = (c^2 + d^2)$. Therefore,
\[
(4.4) \quad a + ib \in \mathcal{E}_u := \{ x + iy : (x - 2/u^2)^2 + y^2 = 4(1/u^2 + 1/u^4) \},
\]
\[
(4.5) \quad c + id \in \mathcal{E}_v := \{ x + iy : (x - 2/v^2)^2 + y^2 = 4(1/v^2 + 1/v^4) \}.
\]
Since $w_1 + uv + w_2 = 0$, we may let $w_1 = -uv(1 - \xi)/2$, $w_2 = -uv(1 + \xi)/2$ for some $\xi \in \mathbb{C}$. As $\xi = (w_1 - w_2)/uv$, assumption (ii) holds, i.e., $(w_1 - w_2)^2 = (uv)^2$, if and only if $\xi = \pm 1$. Now the $(1,3)$-entry of $R$ becomes
\[
uv(a - ib)(c - id)/4 = [w_1(a - ib) + w_2(c - id)]/2
\]
\[
= -[uv(a - ib)(1 - \xi) + uv(c - id)(1 + \xi)]/4.
\]
Thus,
\[
(4.6) \quad (a - ib)(c - id) = (\xi - 1)(a - ib) - (\xi + 1)(c - id).
\]
If $\xi = 1$, then we have $(a - ib)(c - id) = -2(c - id)$ so that $a - ib = -2$. Thus, the $(1,1)$-entry of $R$ is $-1$, which is impossible. Similarly, if $\xi = -1$, then the $(3,3)$-entry of $R$ is $-1$, which is impossible.

Suppose $\xi \neq \pm 1$ and (iii) holds. Substituting $w_1 = -uv(1 - \xi)/2$, $w_2 = -uv(1 + \xi)/2$, we have
\[
(4.7) \quad |1 - \xi|\sqrt{1 + v^2} \neq |1 + \xi|\sqrt{1 + u^2}.
\]
Since $a - ib, c - id \neq 0$, (4.6) is equivalent to
\[
(4.8) \quad \frac{1 + \xi}{a - ib} + \frac{1 - \xi}{c - id} + 1 = 0.
\]
Note that \( \mu \in \mathbb{C} \) lies on a circle with center \( \mu_0 \geq 0 \) and radius \( r > \mu_0 \) if and only if
\[
0 = (\mu - \mu_0)(\bar{\mu} - \mu_0) - r^2 = \mu\bar{\mu} - (\mu_0\bar{\mu} + \mu_0\mu) + (\mu_0^2 - r^2).
\]
Dividing by \( \mu\bar{\mu}(\mu_0^2 - r^2) \), we have
\[
(\mu\bar{\mu})^{-1} - \left( \frac{\mu_0}{\mu_0^2 - r^2} \right) \mu^{-1} - \left( \frac{\mu_0}{\mu_0^2 - r^2} \bar{\mu}^{-1} \right) = -\frac{1}{\mu_0^2 - r^2},
\]
equivalently,
\[
\left( \mu^{-1} - \frac{\mu_0}{\mu_0^2 - r^2} \right) \left( \bar{\mu}^{-1} - \frac{\mu_0}{\mu_0^2 - r^2} \right) = \frac{\mu_0^2}{(\mu_0^2 - r^2)^2} - \frac{1}{\mu_0^2 - r^2} = \frac{r^2}{(\mu_0^2 - r^2)^2}.
\]
Applying this to the circles \( \mathcal{E}_u \) and \( \mathcal{E}_v \), we see that
\[
\mathcal{E}_u^{-1} = \{ 1/\mu : \mu \in \mathcal{E}_u \} = \{ -1/2 + 1/2\sqrt{1 + u^2} e^{i\theta} : t \in [0, 2\pi) \},
\]
\[
\mathcal{E}_v^{-1} = \{ 1/\mu : \mu \in \mathcal{E}_v \} = \{ -1/2 + 1/2\sqrt{1 + v^2} e^{i\theta} : t \in [0, 2\pi) \}.
\]
Since \( c - id \in \mathcal{E}_v \) is nonzero, \( (4.8) \) yields
\[
(4.9) \quad \frac{1}{c - id} = \frac{1}{\xi - 1} + \frac{\xi + 1}{(\xi - 1)(a - ib)} \in \hat{\mathcal{E}}_u \cap \mathcal{E}_v^{-1},
\]
where
\[
\hat{\mathcal{E}}_u = \left\{ \frac{1}{\xi - 1} + \frac{(\xi + 1)}{2(\xi - 1)} (-1 + \sqrt{1 + u^2} e^{i\theta}) : \theta \in [0, 2\pi) \right\}
\]
\[
= \left\{ -\frac{1}{2} + \frac{(\xi + 1)}{2(\xi - 1)} \sqrt{1 + u^2} e^{i\theta} : \theta \in [0, 2\pi) \right\}.
\]
By \( (4.7) \), \( \hat{\mathcal{E}}_u \cap \mathcal{E}_v^{-1} = \emptyset \), a contradiction to \( (4.9) \). Thus the proof in Case 1 is complete.

**Case 2.** Suppose conditions (i), (ii) and (iii) in Case 1 do not hold. Then \( |w_1|\sqrt{1 + v^2} = |w_2|\sqrt{1 + u^2} \). If \( m \in \mathbb{N} \), then \( B_m = A_1 + E_{13}/m \) and \( C_m = A_2 - E_{13}/m \) are commuting matrices in \( M_3 \) with \((1, 3)\)-entries \( w_1+1/m \) and \( w_2 - 1/m \), respectively. We are going to show that
\[
(4.10) \quad |w_1 + 1/m|\sqrt{1 + v^2} = |w_2 - 1/m|\sqrt{1 + u^2}
\]
for at most one \( m \).

Note that \( (4.10) \) holds if and only if
\[
(4.11) \quad (mw_1 + 1)(m\bar{w}_1 + 1)(1 + v^2) = (mw_2 - 1)(m\bar{w}_2 - 1)(1 + u^2)
\]
\[
\iff 2((\text{Re } w_1)(1 + v^2) + (\text{Re } w_2)(1 + u^2))m + (v^2 - u^2) = 0.
\]
If \( (4.11) \) holds for more than one \( m \), then \( v^2 = u^2 \) and \( \text{Re } w_1 = -\text{Re } w_2 \). Then it follows from \( u, v \geq 0 \) and \( w_1 + uv + w_2 = 0 \) in \( (4.1) \) that \( uv = 0 \) and (i) holds, a contradiction.
So there exists $m_0$ such that $|w_1 + 1/m|\sqrt{1 + v^2} \neq |w_2 - 1/m|\sqrt{1 + u^2}$ for all $m \geq m_0$. By Case 1, $\partial \text{conv} W(B_m, C_m) \subseteq W(B_m, C_m)$. Now, every boundary point $(\mu_1, \mu_2) \in \text{conv} W(A_1, A_2)$ is the limit of a sequence of points $\{(\mu_1(m), \mu_2(m)) : m \geq m_0\}$ with $(\mu_1(m), \mu_2(m)) \in \partial(\text{conv} W(B_m, C_m)) \subseteq W(B_m, C_m)$. Note that $W(B_m, C_m) \rightarrow W(A_1, A_2)$ as $m \rightarrow \infty$ in the Hausdorff metric on compact subsets of $\mathbb{R}^2$. We have $(\mu_1, \mu_2) \in W(A_1, A_2)$. Hence, $\partial(\text{conv} W(A_1, A_2)) \subseteq W(A_1, A_2)$. This finishes the proof in Case 2, and thus also the proof of Proposition 4.2. \hfill \blacksquare

Let $\mu_1 \in W(A_1)$ and $W(\mu_1, A_2) = \{\mu : (\mu_1, \mu) \in W(A_1, A_2)\}$. Now, we know that $W(A_1, A_2)$ has convex boundary if $A_1, A_2 \in M_3$ commute. Therefore, to prove that $W(A_1, A_2)$ is convex, we only need to show that $W(\mu_1, A_2)$ is simply connected for every $\mu_1 \in W(A_1)$.

To prove the latter property, we will show that

$$W(\mu_1, A_2) = \{\mu : (\mu_1, \mu) \in \text{conv} W(A_1, A_2)\}.$$ 

To this end, using linear combinations, unitary similarity and transposition of matrices, we find that the matrices $A_1, A_2$ in (4.1) can be transformed as

\begin{equation}
A_1 = E_{11} + a E_{12}, \quad A_2 = \begin{pmatrix} -a & 0 & 0 \\ 1 & 1 & \xi \\ b & 0 & 0 \end{pmatrix} \quad \text{where} \quad a > 0, \ b \geq 0, \ \xi \in \mathbb{C}.
\end{equation}

To prove this, observe that if $w_1 = 0$, then $w_2 = -uv$ and we can replace $A_2$ with

$$I_3 - (A_1 + A_2) = \begin{pmatrix} 0 & -u & uv \\ 0 & 1 & -v \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -u \\ 1 \\ 0 \end{pmatrix}(0 \ 1 \ -v).$$

If $w_2 = 0$, replace $(A_1, A_2)$ with $(TA_2^T, TA_1^T)$, where $T = E_{13} + E_{22} + E_{31}$. We have

$$TA_2^T = \begin{pmatrix} 1 & v & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad TA_1^T = \begin{pmatrix} 0 & 0 & -uv \\ 0 & 0 & u \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we can proceed as in the above case for $w_1 = 0$.

Suppose $w_1, w_2 \neq 0$. Let $a = \sqrt{u^2 + |w_1|^2}$ and $U = (1) \oplus \frac{1}{a}(\frac{u}{w_1} w_1 - u)$ be unitary. Then

$$U^* A_1 U = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U^* A_2 U = \gamma \begin{pmatrix} 0 & -a & -ac \\ 0 & 1 & c \\ 0 & b & bc \end{pmatrix},$$

where $\gamma = -\frac{(\bar{w}_1 w_2)}{a^2}$, $b = (u - v\bar{w}_1)/w_2$ and $c = -u/\bar{w}_1$. Let $b = |b| e^{i\theta}$.
and $D = \text{diag}(1, 1, e^{i\theta})$. Replace $(A_1, A_2)$ with $(D^*U^*A_1UD, \frac{1}{\gamma}D^*U^*A_2UD)$. Direct calculation gives

$$DU^*A_1UD^* = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D^*A_2UD^* = \begin{pmatrix} -a \\ 1 \\ \frac{1}{|b|} \end{pmatrix}(0 1 \xi),$$

where $\xi = ce^{i\theta}$. If $\xi = 0 = b$, then $A_1 + A_2 = \text{diag}(1, 1, 0)$ is Hermitian. By Proposition 2.1(g), $W(A_1, A_1 + A_2)$ is convex and hence $W(A_1, A_2)$ is also convex. So, we now assume that $(b, \xi) \neq (0, 0)$.

Recall that a set $S$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ is star-shaped with star center $s_0 \in S$ if $ts_0 + (1 - t)s \in S$ for all $t \in [0, 1]$ and $s \in S$. We have the following.

**Proposition 4.3.** Suppose that $A_1$ and $A_2$ are as in (4.1). For every $\mu_1 \in W(A_1)$, the set

$$W(\mu_1, A_2) = \{\mu : (\mu_1, \mu) \in W(A_1, A_2)\}$$

is star-shaped. Consequently,

$$W(\mu_1, A_2) = \{\mu : (\mu_1, \mu) \in \text{conv}W(A_1, A_2)\},$$

and $W(A_1, A_2)$ is convex.

**Proof.** Without loss of generality, we may assume that $A_1$ and $A_2$ are of the form (4.12). Suppose $\mu_1 \in W(A_1)$. We are going to show that $W(\mu_1, A_2)$ is star-shaped with star center $1 - \mu_1$.

Let $\nu \in \mathbb{C}^3$ be a unit vector such that $\nu^*A_1\nu = \mu_1$. By replacing $\nu$ with $\tilde{\nu} = e^{i\theta}\nu$ for some $\theta \in \mathbb{R}$, we may assume that the first entry of $\nu$ is nonnegative. Let

$$S = \{(p_1, p_2e^{i\theta}, p_3e^{i\phi})^t : \theta, \phi \in [0, 2\pi), p_1, p_2, p_3 \geq 0, p_1^2 + p_2^2 + p_3^2 = 1\}.$$ 

If $\nu = (0, p_2e^{i\theta}, p_3e^{i\phi})^t \in S$, we have $\mu_1 = \nu^*A_1\nu = 0$. Moreover

$$\nu^*A_2\nu \in W\left(\begin{pmatrix} 1 & \xi \\ b & b\xi \end{pmatrix}\right) \subseteq W(0, A_2).$$

Since $W\left(\begin{pmatrix} 1 & \xi \\ b & b\xi \end{pmatrix}\right)$ is convex, and it contains the point $\{1\}$, we can see that $t + (1 - t)\nu^*A_2\nu \in W(0, A_2)$ for all $t \in [0, 1]$. Now assume $\nu \in S$ with $\nu^*A_1\nu = \mu_1$ and $p_1 > 0$. Then

$$\mu_1 = p_1^2 + ap_1p_2e^{i\theta}, \quad \text{i.e.,} \quad p_2e^{i\theta} = \frac{\mu_1 - p_1^2}{ap_1},$$

and

$$1 - p_3^2 = p_1^2 + p_2^2 = p_1^2 + \left|\frac{\mu_1 - p_1^2}{ap_1}\right|^2 = \frac{a^2p_1^4 + |\mu_1 - p_1|^2}{a^2p_1^2} = \frac{(a^2 + 1)p_1^4 + |\mu_1|^2 - 2(\text{Re}\, \mu_1)p_1^2}{a^2p_1^2}.$$
Therefore, we have
\[
- a^2 p_1^2 p_3^2 = (a^2 + 1)p_1^4 - (2 \Re \mu_1 + a^2)p_1^2 + |\mu_1|^2 \leq 0.
\]
By the above calculation, \( \nu \in S \) with positive first entry and \( \nu^* A_1 \nu = \mu_1 \) if and only if
\[
\nu = (p_1, (\mu_1/p_1 - p_1)/a, p_3e^{i\phi})
\]
for \( p_1 > 0 \) satisfying (4.13), \( \phi \in [0, 2\pi) \) and \( p_3 = \sqrt{1 - p_1^2 - |(\mu_1/p_1 - p_1)/a|^2} \). Now
\[
\nu^* A_2 \nu = (p_1 (\mu_1/p_1 - p_1)/a, p_3e^{-i\phi}) \begin{pmatrix} 0 & -a & -a\xi \\ 0 & 1 & \xi \\ 0 & b & b\xi \end{pmatrix} \begin{pmatrix} p_1 \\ (\mu_1/p_1 - p_1)/a \\ p_3e^{i\phi} \end{pmatrix}
\]
\[
= (-ap_1 + (\mu_1/p_1 - p_1)/a + b p_3e^{-i\phi})((\mu_1/p_1 - p_1)/a + \xi p_3e^{i\phi})
\]
\[
= p_1^2 - \mu_1 + |\mu_1/p_1 - p_1|^2/a^2 + b\xi p_3^2
\]
\[
+ p_3\{(-ap_1 + (\mu_1/p_1 - p_1)/a)\xi e^{i\phi} + (\mu_1/p_1 - p_1)(b/a) e^{-i\phi}\}
\]
\[
= 1 - \mu_1 + (b\xi - 1)p_3^2
\]
\[
+ p_3\{(-ap_1 + (\mu_1/p_1 - p_1)/a)\xi e^{i\phi} + (\mu_1/p_1 - p_1)(b/a) e^{-i\phi}\}.
\]
For a fixed \( p_1 > 0 \), if we let \( \phi \) vary in \([0, 2\pi)\), we see that \( \nu^* A_2 \nu \) generates all the points of an ellipse denoted by \( \mathcal{E}(p_1) \). Hence, \( \mathcal{E}(p_1) \subseteq W(\mu_1, A_2) \). For a fixed \( \mu_1 \in W(A_1) \), let \( p_u \) and \( p_e \) be the largest and smallest nonnegative values of \( p_1 \) respectively for which the inequality
\[
(a^2 + 1)p_1^4 - (2 \Re \mu_1 + a^2)p_1^2 + |\mu_1|^2 \leq 0
\]
in (4.13) is satisfied. Then
\[
W(\mu_1, A_2) = \bigcup_{p \in [p_e, p_u]} \mathcal{E}(p).
\]
Here we denote \( \mathcal{E}(0) = W\left(\begin{pmatrix} 1 & 0 \\ b & b\xi \end{pmatrix}\right) \). Next we show that every point inside the ellipse \( \mathcal{E}(p) \) also lies in \( W(\mu_1, A_2) \). As \( \mu_1 \in W(A_1) = W(A_0) \) with \( A_0 = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \), there is a unit vector \( \tilde{\nu} = (\tilde{\rho}, \nu_2) \in \mathbb{C}^2 \) with \( \tilde{\rho} \geq 0 \) such that \( \tilde{\nu}^* A_0 \tilde{\nu} = \mu_1 \). Thus, with \( \nu = (\tilde{\rho}, \nu_2, 0) \in \mathbb{C}^3 \) we have \( \nu^* A_1 \nu = \mu_1 \). The corresponding ellipse \( \mathcal{E}(\tilde{\rho}) = \{1 - \mu_1\} \) is a singleton as \( p_3 = 0 \). For every \( p_1 \in [p_e, p_u] \), we may let \( p_1 \) change continuously to \( \tilde{\rho} \). Recall that \( \nu = (p_1, (\mu_1/p_1 - p_1)/a, p_3e^{i\phi}) \). As the entries of \( \nu \) are continuous functions in \( p_1 > 0 \), the ellipse \( \mathcal{E}(p_1) \) will deform continuously to the singleton \( \mathcal{E}(\tilde{\rho}) \) in the set \( W(\mu_1, A_2) \). Hence, by continuity all the points inside the ellipse \( \mathcal{E}(p_1) \) also lie in \( W(\mu_1, A_2) \), i.e.,
\[
W(\mu_1, A_2) = \bigcup_{p \in [p_e, p_u]} \mathcal{E}(p) = \bigcup_{p \in [p_e, p_u]} \mathcal{E}(p),
\]
where \( \mathcal{E}(p) \) is the elliptical disk with \( \mathcal{E}(p) \) as boundary.
We will show that $\bigcup_{p \in [\ell, \mu]} \overline{E}(p)$ is star-shaped with star center $1 - \mu_1$. Expressing $p_3$ as a function of $p_1$ from (4.13), we see that $p_3$ attains the maximum value

$$\hat{p}_3 = \sqrt{1 - p_1^2 - |(\mu_1/p_1 - 1)/a|^2} = \sqrt{a^2 + 2(\text{Re} \mu_1 - \sqrt{1 + a^2|\mu_1|})}$$

when $p_1 = \hat{p} = \sqrt{|\mu_1|/\sqrt{1 + a^2}}$. In general, for each choice of $p_3 \in [0, \hat{p}_3]$, there are $p_1^- \in [\ell, \hat{p}]$ and $p_1^+ \in [\hat{p}, \mu_3]$ satisfying the equality in (4.13). For every $0 < r < 1$ and $p_3 \in [0, \hat{p}_3]$, set $\hat{p}_3 = rp_3$ and let $\hat{p}_1^- \in [\ell, \hat{p}]$ and $\hat{p}_1^+ \in [\hat{p}, \mu_3]$ satisfy (4.13) for $p_3$. With some intricate arguments presented in the Appendix, we will show that

(I) If $|\xi|^2(1 + a^2) \geq b^2$, then $\overline{E}(p_1^-) \subseteq \overline{E}(p_1^+)$, and for every $\mu_2 \in \overline{E}(p_1^+)$, $(1 - r^2)(1 - \mu_1) + r^2 \mu_2 \in \overline{E}(\hat{p}_1^+)$. 

(II) If $|\xi|^2(1 + a^2) \leq b^2$, then $\overline{E}(p_1^+) \subseteq \overline{E}(p_1^-)$, and for every $\mu_2 \in \overline{E}(p_1^-)$, $(1 - r^2)(1 - \mu_1) + r^2 \mu_2 \in \overline{E}(\hat{p}_1^-)$.

Once (I) and (II) are proved, by (4.14) we see that $W(\mu_1, A_2)$ is star-shaped with $1 - \mu_1$ as a star center, i.e., for any $\mu_2 \in W(\mu_1, A_2)$ and $t \in [0, 1]$,

$$t\mu_2 + (1 - t)(1 - \mu_1) \in W(\mu_1, A_2).$$

Let $S = \{\mu : (\mu_1, \mu) \in \text{conv } W(A_1, A_2)\}$. We have $W(\mu_1, A_2) \subseteq S$. Note that $S \subseteq \mathbb{C}$ is convex and compact. By Proposition 4.2

$$\partial S \subseteq \{\mu : (\mu_1, \mu) \in \partial (\text{conv } W(A_1, A_2))\} \subseteq \{\mu : (\mu_1, \mu) \in W(A_1, A_2)\} = W(\mu_1, A_2).$$

The star-shapedness of $W(\mu_1, A_2)$ implies that this set is simply connected. Therefore, $S \subseteq W(\mu_1, A_2)$. Hence, $S = W(\mu_1, A_2)$.

Now, we can show that $W(A_1, A_2)$ is convex as follows. Suppose $(x_1, y_1), (x_2, y_2) \in W(A_1, A_2)$, $t \in [0, 1]$ and $(\mu_1, \mu_2) = t(x_1, y_1) + (1 - t)(x_2, y_2)$. Then $(\mu_1, \mu_2) \in \text{conv } W(A_1, A_2)$. We have $\mu_2 \in \{\mu : (\mu_1, \mu) \in \text{conv } W(A_1, A_2)\} = W(\mu_1, A_2)$. Thus, $(\mu_1, \mu_2) \in W(A_1, A_2)$. So, $W(A_1, A_2)$ is convex.

4.2. span $\{I_3, A_1, A_2\} \subseteq M_3$ contains a nonzero nilpotent. Here we present the proof of Theorem 3.3 when span $\{I_3, A_1, A_2\}$ contains a nonzero nilpotent matrix. We may assume that $\{I_3, A_1, A_2\}$ is linearly independent and $A_1$ is nilpotent.

Similar to the case considered in Subsection 4.1, we can apply linear combinations and unitary similarity transforms to change $A_1, A_2$ to a simpler form. First, we show that one may assume that $A_1$ is rank 1. Suppose $A_1$ is rank 2. Then there is an invertible $S$ such that $S^{-1}A_1S = J$ is the upper triangular Jordan block. Then $A_1A_2 = A_2A_1$ implies that $S^{-1}A_2S = aI_3 + bJ + cJ^2$. We may replace $A_2$ by $A_2 - aI_3 - bA_1$. Then $A_2$ is a rank 1
nilpotent. We may then interchange the roles of $A_1$ and $A_2$. Now, $A_1$ is a rank 1 nilpotent matrix in span \{I_3, A_1, A_2\}. So, up to a nonzero multiple and a unitary similarity transform, we may assume that $A_1 = E_{13}$, where as before \{$E_{ij} : i, j = 1, 2, 3$\} is the standard basis of $M_3$. The condition $A_1 A_2 = A_2 A_1$ implies that $A_2$ is in upper triangular form with the $(1, 1)$-entry equal to the $(3, 3)$-entry. We may then replace $A_2$ by $A_2 - \gamma_1 I_3 - \gamma_2 A_1$ and assume that

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & b & 0 \\ a & c & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

If necessary, we may also replace $(A_1, A_2)$ with $(DA_1^t D, DA_2^t D)$, where $D = E_{13} + E_{22} + E_{31}$, and assume that $|b| \geq |c|$.

If $b = 0$, then we may assume that $A_2 = E_{22}$. By Proposition 2.1(g,e),

$$W(A_1, A_2) \cong W\left(\frac{(E_{13} + E_{31})}{2}, i\frac{(E_{13} - E_{31})}{2}, E_{22}\right)$$

is convex.

If $b \neq 0$, let $\zeta = |a/b|$ and $\xi = |c/b|$. Suppose $a/b = \zeta e^{i\theta}$ and $c/b = \xi e^{i\phi}$, $\theta, \phi \in [0, 2\pi)$. Let $U = \text{diag}(1, e^{i\theta}, e^{i(2\theta - \phi)})$. Replacing $(A_1, A_2)$ with $(e^{i(\phi - 2\theta)}U^* A_1 U, e^{-i\theta}U^* A_2 U/b)$, we have $(A_1, A_2) = (E_{13}, \zeta E_{22} + E_{12} + \xi E_{23})$, where $\zeta \geq 0$ and $\xi \in [0, 1]$.

Let $P_m = E_{11}/m$ and $Q_m = (E_{22} - E_{32})/m$ for $m \in \mathbb{N}$. Then

$$A_1 + P_m = \begin{pmatrix} 1/m & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 + Q_m = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \zeta + 1/m & \xi \\ 0 & -1/m & 0 \end{pmatrix}$$

commute. Moreover,

$$aI_3 + b(A_1 + P_m) + c(A_2 + Q_m) = \begin{pmatrix} a + b/m & c & b \\ 0 & a + c(\zeta + 1/m) & c\xi \\ 0 & -c/m & a \end{pmatrix}$$

is nilpotent if and only if

$$a + b/m = 0, \quad 2a + c(\zeta + 1/m) = 0 \quad \text{and} \quad a^2 + ac(\zeta + 1/m) + c^2\xi/m = 0.$$ 

From the last two equations, if $\zeta + 1/m \neq 0$, then

$$\frac{a}{c} = \frac{-(\zeta + 1/m)}{2}, \quad 0 = \left(\frac{a}{c}\right)^2 + \frac{a}{c}\left(\zeta + \frac{1}{m}\right) + \frac{\xi}{m} = \frac{\xi}{m} - \frac{1}{4}\left(\zeta + \frac{1}{m}\right)^2,$$

which can be true for at most two choices of $m$. Hence, except for finitely many values of $m$, the linear span of the set \{$I_3, A_1 + P_m, A_2 + Q_m$\} contains
no nonzero nilpotent and \( W(A_1 + P_m, A_2 + Q_m) \) is convex by Proposition 4.3 in Subsection 4.1.

Suppose \( L \) is the line segment joining \((x^* A_1, x^* A_2), (y^* A_1 y, y^* A_2 y) \in W(A_1, A_2)\). Let \( L_m \) be the line segment joining \((x^* (A_1 + P_m), x^* (A_2 + Q_m)) \) and \((y^* (A_1 + P_m) y, y^* (A_2 + Q_m) y) \). Clearly, the endpoints of the line segments \( L_m \) converge to those of \( L \). Thus, \( L_m \to L \) in the Hausdorff metric as \( m \to \infty \). Note that \( L \subseteq W(A_1 + P_m, A_2 + Q_m) \) because \( W(A_1 + P_m, A_2 + Q_m) \) is convex by Proposition 4.3. Since \( W(A_1 + B_m, A_2 + Q_m) \to W(A_1, A_2) \) in the Hausdorff metric as \( m \to \infty \), we infer that \( L_m \to L \) as \( m \to \infty \), so that \( L \subseteq W(A_1, A_2) \), and therefore \( W(A_1, A_2) \) is convex. 

Appendix: Proof of (I) and (II). We use the notation introduced in Section 4.2. For every \( q \in [p_\ell, p_u] \), let

\[
C_q = \begin{pmatrix}
0 & \xi((\bar{\mu}/q - q)/a - aq) \\
b(\mu q - q)/a & 0
\end{pmatrix}.
\]

If \( q \in [p_\ell, p_u] \) and \( q_3^2 = 1 - q^2 - |(\mu q - q)/a|^2 \), then

\[
\mathcal{E}(q) = 1 - \mu_1 + (b\xi - 1)q_3^2 + 2q_3 W(C_q).
\]

It is clear that \( W(C_{p_1}) \subseteq W(C_{p_1}^+) \) if and only if \( \mathcal{E}(p_1^-) \subseteq \mathcal{E}(p_1^+) \). For every \( 0 < r < 1 \) and \( \mu_2 \in \mathcal{E}(p_1^+) \), we have

\[
(1 - r^2)(1 - \mu_1) + r^2 \mu_2 \in 1 - \mu_1 + (b\xi - 1)(rq_3)^2 + 2(rq_3) W(rC_{p_1}^+).
\]

Let \( p_3 = rq_3 \). Thus, to prove (I), it suffices to show that

\[
W(rC_{p_1}^-) \subseteq W(rC_{p_1}^+) \subseteq W(C_{p_1}^+),
\]

By Proposition 2.2, the inclusions (A.1) are equivalent to

\[
r\lambda_1(e^{i\theta} C_{p_1}^- + e^{-i\theta} C_{p_1}^*) \leq r\lambda_1(e^{i\theta} C_{p_1}^+ + e^{-i\theta} C_{p_1}^*) \leq \lambda_1(e^{i\theta} C_{p_1}^+ + e^{-i\theta} C_{p_1}^*),
\]

for every \( \theta \in [0, 2\pi] \).

Note that

\[
\lambda_1(e^{i\theta} C_q + e^{-i\theta} C_q^*) = \sqrt{\det(e^{i\theta} C_q + e^{-i\theta} C_q^*)}.
\]

Hence, it suffices to show that for every \( \theta \in [0, 2\pi] \),

\[
r^2|\det(e^{i\theta} C_{p_1}^- + e^{-i\theta} C_{p_1}^*)| \leq r^2|\det(e^{i\theta} C_{p_1}^+ + e^{-i\theta} C_{p_1}^*)| \leq |\det(e^{i\theta} C_{p_1}^+ + e^{-i\theta} C_{p_1}^*)|.
\]

For every \( q \in [p_\ell, p_u] \) and \( q_3^2 = 1 - q^2 - |(\mu_1 q - q)/a|^2 \), we have

\[
|\det(e^{i\theta} C_{p_1} + e^{-i\theta} C_{p_1}^*)|
= |e^{i\theta} \xi(-aq + (\bar{\mu}_1 q - q)/a) + e^{-i\theta} b(\mu_1 q - q)/a|^2
\]
Now the first inequality in (A.2) follows from

\begin{align*}
&= |\xi|^2 (\bar{\mu}_1 / q - q) / a - aq|^2 + b^2 |(\bar{\mu}_1 / q - q) / a|^2
+ 2 \Re(e^{2i\theta} \xi b(-aq + (\bar{\mu}_1 / q - q) / a)(\mu_1 / q - q) / a)
&= |\xi|^2 (|\bar{\mu}_1 / q - q) / a|^2 + a^2 q^2 - 2 \Re(\bar{\mu}_1 - q^2) + b^2 |(\bar{\mu}_1 / q - q) / a|^2
+ 2 \Re(e^{2i\theta} \xi b(-aq + (\bar{\mu}_1 / q - q) / a)(\mu_1 / q - q) / a)
&= (|\xi|^2 (1 + a^2) - b^2) q^2 + (|\xi|^2 + b^2) (1 - q_3^2)
- 2 \Re(|\xi|^2 \bar{\mu}_1 + e^{2i\theta} \xi b(1 - \mu_1 - q_3^2)).
\end{align*}

As

\begin{align*}
1 - (p_1^-)^2 - |(\mu_1/p_1^- - p_1^-) / a|^2 = 1 - (p_1^+)^2 - |(\mu_1/p_1^+ - p_1^+) / a|^2 = p_3^2,
\end{align*}

the first inequality in (A.2) follows from $|\xi|^2 (1 + a^2) - b^2 \geq 0$ and $p_1^+ \geq p_1^-$. Now

\begin{align*}
\det |e^{i\theta} C_{p_1^+} + e^{-i\theta} C_{p_1^-}^*| - r^2 |\det(e^{i\theta} C_{p_1^+} + e^{-i\theta} C_{p_1^-}^*)|
&= (|\xi|^2 (1 + a^2) - b^2)(p_1^+)^2 - r^2(p_1^+)^2) + (1 - r^2)(|\xi|^2 + b^2)
- 2(1 - r^2) \Re(|\xi|^2 \bar{\mu}_1 + e^{2i\theta} \xi b(1 - \mu_1))
\geq (|\xi|^2 (1 + a^2) - b^2)(p_1^+)^2 + (|\xi|^2 + b^2) - 2(|\xi|^2 \Re \bar{\mu}_1 + |\xi b(1 - \bar{\mu}_1)|)
- r^2((|\xi|^2 (1 + a^2) - b^2)(p_1^+)^2 + (|\xi|^2 + b^2) - 2(|\xi|^2 \Re \bar{\mu}_1 + |\xi b(1 - \bar{\mu}_1)|)).
\end{align*}

For every $y \in [0, p_3^2]$, let

\begin{align*}
(q_y^+)^2 &= \frac{2 \Re \mu_1 + a^2 (1 - y) + \sqrt{(2 \Re \mu_1 + a^2 (1 - y))^2 - 4(a^2 + 1) |\mu_1|^2}}{2(1 + a^2)}.
\end{align*}

It is not hard to see that $q_y^+ \in [\hat{p}, p_u]$ satisfies the left-hand equality of (4.13) with $p_3 = \sqrt{\bar{y}}$, i.e.,

\begin{align*}
-a^2(q_y^+)^2 y = (a^2 + 1)(q_y^+)^4 - (2 \Re \mu_1 + a^2)(q_y^+)^2 + |\mu_1|^2.
\end{align*}

Define the function $M : [0, p_3^2] \rightarrow \mathbb{R}$ by

\begin{align*}
M(y) &= (|\xi|^2 (1 + a^2) - b^2)(q_y^+)^2 + (|\xi|^2 + b^2)
- 2(|\xi|^2 \Re \bar{\mu}_1 + |\xi b(1 - \bar{\mu}_1)|).
\end{align*}

For $y = 0$, we have $(1 + a^2)(q_0^+)^4 - (2 \Re \mu_1 + a^2)(q_0^+)^2 + |\mu_1|^2 = 0$ and

\begin{align*}
M(0) &= \frac{|1 - \mu_1|^2 |\xi|^2}{1 - (q_0^+)^2} - 2b|\xi| |1 - \bar{\mu}_1| + b^2 (1 - (q_0^+)^2) \geq 0.
\end{align*}

We will show that $M$ is concave so that

\begin{align*}
|\det(e^{i\theta} C_{p_1^+} + e^{-i\theta} C_{p_1^-}^*)| - r^2 |\det(e^{i\theta} C_{p_1^+} + e^{-i\theta} C_{p_1^-}^*)| \\
\geq M(r^2 p_3^2) - r^2 M(p_3^2) \geq (1 - r^2) M(0) \geq 0.
\end{align*}

Noting that $|\xi|^2 (1 + a^2) - b^2 \geq 0$, we have
\[
\frac{d^2 M}{dy^2} = (|\xi|^2(1 + a^2) - b^2)((q_y^+)^2)''
\]
\[
= \frac{|\xi|^2(1 + a^2) - b^2}{2(a^2 + 1)} \left( \sqrt{2 \operatorname{Re} \mu_1 + a^2(1 - y)} \right)^2 - 4(1 + a^2)|\mu_1|^2 \right)'' 
\]
\[
= \frac{- (|\xi|^2(1 + a^2) - b^2) (4a^4 (a^2 + 1)) |\mu_1|^2}{2(a^2 + 1)((2 \operatorname{Re} \mu_1 + a^2 (1 - y^2))^2 - 4(a^2 + 1)|\mu_1|^2)^{3/2} \leq 0.}
\]

Hence \( M \) is concave.

The proof of (II) is similar, and we just give a sketch. It suffices to show that for every \( \theta \in [0, 2\pi) \),

\[
(A.3) \quad r^2 |\det(e^{i\theta} C_{p_1^+} + e^{-i\theta} C_{p_1^+}^*)| \leq r^2 |\det(e^{i\theta} C_{p_1^-} + e^{-i\theta} C_{p_1^-}^*)| 
\]
\[
\leq |\det(e^{i\theta} C_{p_1^-} + e^{-i\theta} C_{p_1^-}^*)|. 
\]

Recall that
\[
|\det(e^{i\theta} C_q + e^{-i\theta} C_q^*)| = (|\xi|^2(1 + a^2) - b^2)q^2 + (|\xi|^2 + b^2)(1 - q^2_3) 
\]
\[
- 2 \operatorname{Re}(|\xi|^2 \bar{\mu}_1 + e^{2i\theta} \xi b(1 - \mu_1 - q^2_3)).
\]

So, the first inequality in (A.3) follows from \( |\xi|^2(1 + a^2) \leq b^2 \) and \( p_1^- \leq p_1^+ \). The second inequality will follow from the concavity of
\[
\tilde{M}(y) = (|\xi|^2(1 + a^2) - b^2)(q^-_y)^2 + (|\xi|^2 + b^2) - 2(|\xi|^2 \operatorname{Re} \bar{\mu}_1 + |\xi b(1 - \bar{\mu}_1)|),
\]
where
\[
(q^-_y)^2 = \frac{2 \operatorname{Re} \mu_1 + a^2(1 - y) - \sqrt{(2 \operatorname{Re} \mu_1 + a^2(1 - y))^2 - 4(a^2 + 1)|\mu_1|^2}}{2(a^2 + 1)}.
\]

Since \( |\xi|^2(1 + a^2) - b^2 \leq 0 \), we have
\[
\frac{d^2 \tilde{M}}{dy^2} = (|\xi|^2(1 + a^2) - b^2)((q^-_y)^2)''
\]
\[
= \frac{|\xi|^2(1 + a^2) - b^2}{2(a^2 + 1)} \left( - \sqrt{(2 \operatorname{Re} \mu_1 + a^2(1 - y))^2 - 4(1 + a^2)|\mu_1|^2} \right)'' \leq 0.
\]

Thus (II) holds. \( \blacksquare \)

**Remark A.1.** It is worth pointing out that our proofs use some continuity arguments and a simple idea of homotopy (in deforming ellipses inside the numerical range of a certain matrix). In particular, intricate linear-algebraic arguments are used. It would be nice if a less computational proof could be found.

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