NUMERICAL RANGE, DILATION, AND COMPLETELY
POSITIVE MAPS

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Abstract. A proof using the theory of completely positive maps is given to
the fact that if $A \in M_2$ or $A \in M_3$ has a reducing eigenvalue, then every
bounded linear operator $B$ with $W(B) \subseteq W(A)$ has a dilation of the form
$I \otimes A$. This gives a unified treatment for the different cases of the result
obtained by researchers using different techniques.

1. Introduction

Let $B(\mathcal{H})$ be the set of bounded linear operators acting on a Hilbert space \( \mathcal{H} \)
with inner product $\langle x, y \rangle$. If $\mathcal{H}$ has dimension $n$, we identify $B(\mathcal{H})$ with $M_n$ and
$\mathcal{H} = \mathbb{C}^n$ with $\langle x, y \rangle = y^*x$.

The numerical range of $A \in B(\mathcal{H})$ is defined and denoted by

$$W(A) = \{ \langle Ax, x \rangle : x \in \mathcal{H}, \langle x, x \rangle = 1 \}.$$

We say that an operator $B \in B(\mathcal{H})$ admits a dilation $A \in B(K)$ if there is a partial
isometry $X : \mathcal{H} \to K$ such that $X^*X = I_\mathcal{H}$ and $X^*AX = B$. If $B$ admits a dilation
$I_{\mathcal{K}_1} \otimes A \in B(\mathcal{K}_1 \otimes K)$ for some Hilbert space $\mathcal{K}_1$, we will simply say that $B$ admits a
dilation of the form $I \otimes A$.

There are interesting connections between the numerical range inclusion and
dilation relation between two operators. For example, the following is known; see
[7][8].

Theorem 1.1. Let $A \in M_3$ be a normal matrix. If $B \in B(\mathcal{H})$ satisfies $W(B) \subseteq W(A)$, then $B$ admits a dilation of the form $I \otimes A$.

Also, we have the following [1][2][5].

Theorem 1.2. Let $A \in M_2$. If $B \in B(\mathcal{H})$ satisfies $W(B) \subseteq W(A)$, then $B$ admits a dilation of the form $I \otimes A$.

The following theorem which generalizes Theorems 1.1 and 1.2 was proved in [6].

Theorem 1.3. Suppose $A \in M_3$ has a non-trivial reducing subspace. If $B \in B(\mathcal{H})$ satisfies $W(B) \subseteq W(A)$, then $B$ admits a dilation of the form $I \otimes A$.

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Furthermore, it was shown in [5] that the conclusion of Theorem 1.3 would fail if one considers a general matrix $A \in M_3$ or a normal matrix $A \in M_4$. The proofs in [1,2,5,8] used different techniques. In this note, we give a unified proof of the above results. In particular, we will give a proof of the theorems in Section 1 in an equivalent form involving unital positive and completely positive maps. We first introduce some background.

An operator system $S$ of $\mathcal{B}(\mathcal{H})$ is a self-adjoint subspace of $\mathcal{B}(\mathcal{H})$ which contains $I_{\mathcal{H}}$. A linear map $\Phi : S \to \mathcal{B}(\mathcal{K})$ is unital if $\Phi(I_{\mathcal{H}}) = I_{\mathcal{K}}$, $\Phi$ is positive if $\Phi(A)$ is positive semi-definite for every positive semi-definite $A \in S$, and $\Phi$ is said to be completely positive if $I_k \otimes \Phi : M_k(S) \to M_k(\mathcal{B}(\mathcal{K}))$ defined by $(T_{ij}) \mapsto (\Phi(T_{ij}))$ is positive for every $k \geq 1$.

Suppose $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Let $S$ be the operator system spanned by $\{I_{\mathcal{H}}, A, A^*\}$. Define a unital linear map $\Phi : S \to \mathcal{B}(\mathcal{K})$ by $\Phi(aI + bA + cA^*) = aI + bB + cB^*$. By [9] Lemma 4.1, $\Phi$ is positive if and only if $W(B) \subseteq W(A)$. On the other hand, Stinespring’s representation theorem [10] (see also the paragraphs after Theorems 4.1 and 4.6 in [9]) shows that $\Phi$ is completely positive if and only if $B$ has a dilation of the form $I \otimes A$. Therefore, Theorems 1.2 and 1.3 can be stated in the following form.

**Theorem 1.4.** Suppose $A = A_0$ or $A_0 + [\mu]$ with $A_0 \in M_2$ and $B \in \mathcal{B}(\mathcal{H})$. Define a linear map $\Phi : S \to \mathcal{B}(\mathcal{H})$ by $$\Phi(aI + bA + cA^*) = aI + bB + cB^*$$ for any $a, b, c \in \mathbb{C}$.

Then $\Phi$ is positive if and only if $\Phi$ is completely positive.

2. Proofs

To prove Theorem 1.4 we need several lemmas, some of which are well known. In our discussion, we will let $E_{ij}$ be the basic matrix unit of appropriate size and write $P \geq 0$ if a matrix or operator $P$ is positive semi-definite.

**Lemma 2.1** ([9] Corollary 6.7). Let $S$ be an operator system. Then every positive linear map $\Phi : S \to \mathcal{B}(\mathcal{H})$ is completely positive for every Hilbert space $\mathcal{H}$ if and only if every positive linear map $\Psi : S \to M_n$ is completely positive for all positive integers $n$.

Recall that $f : \mathbb{R}^m \to \mathbb{R}^m$ is an affine map if it has the form $x \mapsto Rx + x_0$ for a real matrix $R \in M_m$ and $x_0 \in \mathbb{R}^m$. The affine map is invertible if $R$ is invertible, and the inverse of $f$ has the form $y \mapsto R^{-1}y - R^{-1}x_0$. One can extend the definition of affine map to an $m$-tuple of self-adjoint operators in $\mathcal{B}(\mathcal{H})$ by $$(A_1, \ldots, A_m) \mapsto (A_1, \ldots, A_m)(r_{ij}I_{\mathcal{H}}) + (B_0, \ldots, B_m)$$ for a real matrix $R = (r_{ij}) \in M_m$ and an $m$-tuple $(B_1, \ldots, B_m)$ of self-adjoint operators in $\mathcal{B}(\mathcal{H})$. We have the following result, which can be easily verified.

**Lemma 2.2.** Let $S$ be an operator system with a basis $\{I, A_1, \ldots, A_m\}$, and let $\Phi : S \to \mathcal{B}(\mathcal{H})$ be a unital linear map defined by $\Phi(A_j) = B_j \in \mathcal{B}(\mathcal{H})$ for $j = 1, \ldots, m$. Suppose $f$ is an invertible affine map such that $f(A_1, \ldots, A_m) = (\tilde{A}_1, \ldots, \tilde{A}_m)$ and $f(B_1, \ldots, B_m) = (\tilde{B}_1, \ldots, \tilde{B}_m)$. Then $\Phi$ is positive (respectively, completely positive) if and only if the unital map $\tilde{\Phi}$ defined by $\tilde{\Phi}(\tilde{A}_j) = \tilde{B}_j$ for $j = 1, \ldots, m$ is positive (respectively, completely positive).
Lemma 2.3. Let \( S = \text{span}\{E_{jj} : 1 \leq j \leq m\} \subseteq M_m \). A linear map \( \Phi : S \to M_n \) is completely positive if and only if \( \Phi(E_{jj}) \geq 0 \) for \( j = 1, \ldots, m \). As a result, every positive linear map \( \Phi : S \to M_n \) is completely positive.

Proof. If \( \Phi : S \to M_n \) is positive, then \( \Phi(E_{jj}) \geq 0 \) for all \( j = 1, \ldots, m \).

Suppose \( \Phi(E_{jj}) \geq 0 \) for all \( j = 1, \ldots, m \). Let \( C = (C_{ij}) \in M_k(S) \) be positive semi-definite for a positive integer \( k \). Then \( C = C_{11} \otimes E_{11} + \cdots + C_{mm} \otimes E_{mm} \geq 0 \), where \( C_{jj} \geq 0 \) for \( j = 1, \ldots, m \). Thus,

\[
(I_k \otimes \Phi)(C) = C_{11} \otimes \Phi(E_{11}) + \cdots + C_{mm} \otimes \Phi(E_{mm}) \geq 0.
\]

Hence, \( \Phi \) is completely positive. \( \square \)

Lemma 2.4. Let \( S = \text{span}\{E_{jj} : 1 \leq j \leq m\} \cup \{E_{12} + E_{21}\} \) in \( M_m \) with \( m \geq 2 \). Then every positive linear map \( \Phi : S \to M_n \) is completely positive.

Proof. Suppose \( \Phi : S \to M_n \) is a positive map. If \( m = 2 \), the result is due to Choi [3, Theorem 7]. The proof in [4] relies on a result of Calderon [3]. We give a short and direct proof using basic theory of completely positive maps for completeness as follows.

Suppose \( \Phi \) is positive. Then \( \Phi(E_{11}), \Phi(E_{22}) \geq 0 \), and for any real number \( d \),

\[
(1) \quad \Phi(E_{11} + d(E_{12} + E_{21}) + d^2 E_{22}) = \Phi(E_{11}) + d \Phi(E_{12} + E_{21}) + d^2 \Phi(E_{22}) \geq 0.
\]

Let \( C = (C_{ij}) \in M_k(S) \) be positive semi-definite for a positive integer \( k \). Then

\[
C = C_{11} \otimes E_{11} + C_{22} \otimes E_{22} + C_{12} \otimes E_{12} + C_{21} \otimes E_{21}
\]

such that \( C_{21} = C_{12} = C_{21}^* \) so that \( C_{12} = C_{21} \) is Hermitian and

\[
Q = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}
\]

is positive semi-definite. We need to show that

\[
(2) \quad (I_k \otimes \Phi)(C) = C_{11} \otimes \Phi(E_{11}) + C_{12} \otimes \Phi(E_{12} + E_{21}) + C_{22} \otimes \Phi(E_{22}) \geq 0.
\]

We focus on the case when \( C_{11} \) is invertible. The general case can be derived by continuity argument. We may replace \( C_{ij} = C_{11}^{-1/2} C_{ij} C_{11}^{-1/2} \) for \( i, j \in \{1, 2\} \) in (2) and assume that \( C_{11} = I \). Because \( C_{12} = C_{21}^* \), we may further replace \( C_{ij} \) by \( U^* C_{ij} U \) in (2) for \( i, j \in \{1, 2\} \) and assume that \( C_{11} = I \) and \( C_{12} = C_{21} = D = \text{diag}(d_1, \ldots, d_k) \) with \( d_1, \ldots, d_k \in \mathbb{R} \). Since \( Q \geq 0 \), we have

\[
\tilde{C}_{22} = C_{22} - C_{21} C_{11}^{-1} C_{12} = C_{22} - D^2 \geq 0.
\]

As \( \phi(E_{22}) \) and \( \tilde{C}_{22} \) are positive semi-definite, it follows from (1) that

\[
C_{11} \otimes \Phi(E_{11}) + C_{12} \otimes \Phi(E_{12} + E_{21}) + C_{22} \otimes \Phi(E_{22})
\]

\[
= I \otimes \Phi(E_{11}) + D \otimes \Phi(E_{12} + E_{21}) + D^2 \otimes \phi(E_{22}) + \tilde{C}_{22} \otimes \Phi(E_{22}) \geq 0
\]

as asserted.

Suppose \( m > 2 \) and \( k \) is a positive integer. Let \( C = (C_{ij}) \in M_k(S) \) be positive semi-definite. Then

\[
C = C_{11} \otimes E_{11} + C_{22} \otimes E_{22} + C_{12} \otimes E_{12} + C_{21} \otimes E_{21} + \sum_{j=3}^{m} C_{jj} \otimes E_{jj},
\]
where \(\sum_{1 \leq r,s \leq 2}(C_{rs} \otimes E_{rs}) \geq 0\) and \(C_{jj} \geq 0\) for all \(3 \leq j \leq m\). We need to show that

\[
(I_k \otimes \Phi)(C) = \sum_{1 \leq r,s \leq 2} C_{rs} \otimes \Phi(E_{rs}) + \sum_{j=3}^m C_{jj} \otimes \Phi(E_{jj})
\]

is positive semi-definite. Since \(\sum_{j=3}^m C_{jj} \otimes \Phi(E_{jj}) \geq 0\), it suffices to prove that

\[
\sum_{1 \leq r,s \leq 2} C_{rs} \otimes \Phi(E_{rs}) \geq 0,
\]

which is true because the restriction of \(\Phi\) to \(\{E_{12}, E_{21}, E_{12} + E_{21}\}\) is positive and is completely positive by the result when \(m = 2\). The asserted result follows. \(\square\)

Now, we are ready to present the following.

**Proof of Theorem 1.4** By Lemma 2.1, it suffices to show that if \(\Phi: S \rightarrow M_n\) is positive, then it is completely positive, where \(S = \text{span}\{I, A, A^*\}\) satisfies the assumption of the theorem. We assume that \(A\) is not a scalar matrix to avoid trivial considerations.

We will use the fact that the conclusion will not change if we replace \(A\) by \(\alpha I + \beta U^*AU\) for any unitary matrix \(U\) and \(\alpha, \beta \in \mathbb{C}\) with \(\beta \neq 0\). Furthermore, a unital linear map \(\Phi: S \rightarrow M_n\) is positive if and only if \(B = \Phi(A)\) satisfies \(W(B) \subset W(A)\) [6, Lemma 4.1]. Applying an affine transform to \(A = A_1 + iA_2\) with \((A_1, A_2) = (A_1^*, A_2^*)\) will always mean applying a (real) affine transform to \((A_1, A_2)\).

First, suppose \(A\) is normal. If \(A \in M_2\) has eigenvalues \(a_1, a_2\), we may assume that \(A = \text{diag}(A_1, A_2)\). By Lemma 2.2, we may apply an invertible affine map to \(A\) and assume that \((a_1, a_2) = (1, 0)\). Then \(S = \text{span}\{E_{11}, E_{22}\}\). By Lemma 2.3, every unital positive linear map \(\Phi: S \rightarrow M_n\) is completely positive.

Suppose \(A \in M_3\) is normal. By Lemma 2.2, we can apply an invertible affine map and assume that \((1) A = \text{diag}(1, r, 0)\) with \(r \in [0, 1]\) if the three eigenvalues of \(A\) are collinear or \((2) A = \text{diag}(1, i, 0)\) otherwise.

Suppose \((1)\) holds and \(\Phi: S \rightarrow M_n\) is a unital positive linear map with \(\Phi(A) = B\). Then \(\Phi\) is positive if and only if \(x^*Bx \in W(A) = [0, 1]\) for all unit vector \(x \in \mathbb{C}^n\); i.e., \(B\) is a positive semi-definite contraction. One can extend \(\Phi\) to a map from \(\hat{S} = \text{span}\{E_{11}, E_{22}, E_{33}\} \subset M_3\) to \(M_n\) with \(\Phi(E_{11}) = B, \Phi(E_{22}) = 0\). By Lemma 2.3, \(\Phi\) is completely positive on \(\hat{S}\), and therefore we can restrict the map on \(S\).

If \((2)\) holds, then \(S = \text{span}\{E_{11}, E_{22}, E_{33}\}\). By Lemma 2.3, every unital positive linear map \(\Phi: S \rightarrow M_n\) is completely positive.

Next, we consider the case when \(A\) is not normal.

Suppose \(A \in M_2\). We may replace \(A\) by \(U^* \left( \begin{smallmatrix} \alpha & 2b \\ 0 & -\alpha \end{smallmatrix} \right) U\) for a suitable unitary \(U \in M_2\) and assume that \(A = \left( \begin{smallmatrix} \alpha & 2b \\ 0 & -\alpha \end{smallmatrix} \right)\) with \(\alpha \geq 0\) and \(b > 0\). Then \(W(A)\) is an elliptical disk with major axis \([-r, r]\) and minor axis \([i[-b, b]]\) with \(r = \sqrt{\alpha^2 + b^2}\). We may further apply an affine transform

\[
A = A_1 + iA_2 \mapsto \frac{1}{r} A_1 + \frac{i}{b} A_2.
\]
Then $A$ is unitarily similar to the symmetric matrix $C = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$, where $W(C)$ is the unit disk centered at the origin. So, we may assume that $S = \text{span}\{I_2, C, C^*\} = \text{span}\{E_{11}, E_{22}, E_{12} + E_{21}\}$ is the set of symmetric matrices in $M_2$. By Lemma 2.4, every positive linear map $\Phi : S \rightarrow M_n$ is completely positive.

Finally, suppose $A = A_0 \oplus [\mu]$. If $\mu \in W(A_0)$, then $W(A) = W(A_0)$ and the result follows from the previous case. So we can assume that $\mu \not\in W(A_0)$. We may apply an affine transform to $A_0$ as in the preceding case so that $W(A_0)$ is the unit disk centered at the origin and $A_0 \in M_2$ is nilpotent with norm 2. Applying the same affine transform to $A$ yields $A = A_0 \oplus [\hat{\mu}]$. Now, replacing $A$ by $e^{it}(U^*AU - \mu I)$ for a suitable $t \in \mathbb{R}$, we may assume that $A = (rI_2 + C) \oplus [0]$, where $r = |\hat{\mu}| > 1$, and $C = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$. So, $A = \begin{pmatrix} r + i & 1 \\ 1 & r - i \end{pmatrix} \oplus [0]$ with $r > 1$.

Suppose $B = H + iG$ with $H = H^*$ and $G = G^*$. We will construct $P$ with $0 \leq P \leq I$ and a unital positive linear map $\Psi : \text{span}\{E_{11}, E_{22}, E_{33}, E_{12} + E_{21}\} \rightarrow M_n$ with

$$
\Psi(E_{11} + E_{22}) = P, \quad \Psi(E_{12} + E_{21}) = H - rP,
$$

$$
\Psi(E_{11} - E_{22}) = G, \quad \Psi(E_{33}) = I - P.
$$

By Lemma 2.4, $\Psi$ is completely positive. One easily checks that $\Phi$ is the restriction of $\Psi$ on $S$ and is also unital completely positive.

Let $t_0 \in (0, \pi)$ be such that $\cos(t_0) = -1/r$. For any unit vector $x \in \mathbb{C}^n$, we have

$$
\langle Bx, x \rangle \in W(B) \subseteq W(A)
$$

$$
= \{a + ib : |b|\sqrt{r^2 - 1} \leq a \text{ and } (a - r) \cos t + b \sin t \leq 1 \text{ for all } t \in [-t_0, t_0] \}.
$$

Equivalently,

$$
\pm \sqrt{r^2 - 1}G \leq H
$$

and

$$
\cos t(H - rI) + \sin tG \leq I \text{ for all } t \in [-t_0, t_0].
$$

Now, the desired map $\Psi$ satisfying (3) is positive if and only if

$$
\Psi \left( \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix} \oplus [0] \right) = \frac{1}{2} \left( \cos^2 \theta (P + G) + 2 \cos \theta \sin \theta (H - rP) + \sin^2 \theta (P - G) \right)
$$

is positive semi-definite for all $\theta \in \mathbb{R}$; equivalently,

$$
P \geq \cos t(H - rP) + \sin tG \text{ for all } t \in [-\pi, \pi].
$$

By (3), $H \geq 0$, and there exists a contraction $C = C^* \in M_n$ such that $G = \frac{1}{\sqrt{r^2 - 1}}H^{1/2}CH^{1/2}$. First, we show that for $Q = \frac{I}{r+1} + \frac{C^2}{r^2-1}$,

$$
Q \geq \cos t(I - rQ) + \sin t \frac{C}{\sqrt{r^2 - 1}} = \cos t \left( \frac{I}{r+1} - \frac{rC^2}{r^2 - 1} \right) + \sin t \frac{C}{\sqrt{r^2 - 1}}.
$$
Suppose second author visiting the Shenzhen Institute for Quantum Science and Engineering. Part of this research was done while the two authors were visiting the Institute University of Waterloo, and is an honorary professor of the Shanghai University.

To see this, apply a unitary similarity to $C$ and assume that $C = \text{diag}(c_1, \ldots, c_n)$ with $c_j \in [-1, 1]$. Then by Cauchy-Schwarz inequality and the fact that $c_j^2 \in [0, 1]$, we have
\[
\cos t \left( \frac{I}{r + 1} - \frac{rc_j^2}{r^2 - 1} \right) + \sin t \frac{c_j}{\sqrt{r^2 - 1}} \leq \sqrt{\left( \frac{1}{r + 1} - \frac{rc_j^2}{r^2 - 1} \right)^2 + \frac{c_j^2}{r^2 - 1}} 
\]
for each $j = 1, \ldots, n$. Hence (7) holds, and for $K = \frac{H}{r+1} + \frac{H^{1/2}C^2H^{1/2}}{r^2-1} \geq 0$, we have
\[
(8) \quad \cos tH + \sin tG \leq (1 + r \cos t) K \quad \text{for all } t \in [-\pi, \pi].
\]
Suppose $V$ is unitary such that $K = V^* \text{diag}(d_1, \ldots, d_n)V$ with $d_1 \geq \cdots \geq d_n \geq 0$. Let
\[
P = \min\{K, I\} = V^* \text{diag}(p_1, \ldots, p_n)V \quad \text{with } p_j = \min\{k_j, 1\} \quad \text{for } j = 1, \ldots, n.
\]
Then for $|t| \leq t_0$, it follows from (5) and (8) that
\[
\cos tH + \sin tG \leq (1 + r \cos t) \min\{I, K\} \leq (1 + r \cos t)P.
\]
For $|t| > t_0$, we have $1 + r \cos t < 0$. Together with (8), we also have
\[
\cos tH + \sin tG \leq (1 + r \cos t)K \leq (1 + r \cos t)P.
\]
Thus,
\[
(9) \quad \cos tH + \sin tG \leq (1 + r \cos t)P \quad \text{for all } t \in [-\pi, \pi].
\]
Hence, (6) holds, and the result follows.$\square$

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