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NUMERICAL RANGE, DILATION, AND COMPLETELY POSITIVE MAPS

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Abstract. A proof using the theory of completely positive maps is given to the fact that if $A \in M_2$ or $A \in M_3$ has a reducing eigenvalue, then every bounded linear operator B with $W(B) \subseteq W(A)$ has a dilation of the form $I \otimes A$. This gives a unified treatment for the different cases of the result obtained by researchers using different techniques.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators acting on a Hilbert space $\mathcal{H}$ with inner product $\langle \mathbf{x}, \mathbf{y} \rangle$. If $\mathcal{H}$ has dimension n, we identify $\mathcal{B}(\mathcal{H})$ with M_n and $\mathcal{H} = \mathbb{C}^n$ with $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* \mathbf{x}$.

The numerical range of $A \in \mathcal{B}(\mathcal{H})$ is defined and denoted by

$$W(A) = \{\langle A\mathbf{x}, \mathbf{x} \rangle : \mathbf{x} \in \mathcal{H}, \langle \mathbf{x}, \mathbf{x} \rangle = 1\}.$$

We say that an operator $B \in \mathcal{B}(\mathcal{H})$ admits a dilation $A \in \mathcal{B}(\mathcal{K})$ if there is a partial isometry $X : \mathcal{H} \to \mathcal{K}$ such that $X^*X = I_{\mathcal{H}}$ and $X^*AX = B$. If B admits a dilation $I_{\mathcal{K}_1} \otimes A \in \mathcal{B}(\mathcal{K}_1 \otimes \mathcal{K})$ for some Hilbert space $\mathcal{K}_1$, we will simply say that B admits a dilation of the form $I \otimes A$.

There are interesting connections between the numerical range inclusion and dilation relation between two operators. For example, the following is known; see [7, 8].

Theorem 1.1. *Let $A \in M_3$ be a normal matrix. If $B \in \mathcal{B}(\mathcal{H})$ satisfies $W(B) \subseteq W(A)$, then B admits a dilation of the form $I \otimes A$.*

Also, we have the following [1, 2, 5].

Theorem 1.2. *Let $A \in M_2$. If $B \in \mathcal{B}(\mathcal{H})$ satisfies $W(B) \subseteq W(A)$, then B admits a dilation of the form $I \otimes A$.*

The following theorem which generalizes Theorems 1.1 and 1.2 was proved in [6].

Theorem 1.3. *Suppose $A \in M_3$ has a non-trivial reducing subspace. If $B \in \mathcal{B}(\mathcal{H})$ satisfies $W(B) \subseteq W(A)$, then B admits a dilation of the form $I \otimes A$.*

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Furthermore, it was shown in [5] that the conclusion of Theorem 1.3 would fail if one considers a general matrix $A \in M_3$ or a normal matrix $A \in M_4$. The proofs in [1, 2, 5–8] used different techniques. In this note, we give a unified proof of the above results. In particular, we will give a proof of the theorems in Section 1 in an equivalent form involving unital positive and completely positive maps. We first introduce some background.

An *operator system* $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is a self-adjoint subspace of $\mathcal{B}(\mathcal{H})$ which contains $I_{\mathcal{H}}$. A linear map $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{K})$ is *unital* if $\Phi(I_{\mathcal{H}}) = I_{\mathcal{K}}$, Φ is *positive* if $\Phi(A)$ is positive semi-definite for every positive semi-definite $A \in \mathcal{S}$, and Φ is said to be *completely positive* if $I_k \otimes \Phi : M_k(\mathcal{S}) \to M_k(\mathcal{B}(\mathcal{K}))$ defined by $(T_{ij}) \mapsto (\Phi(T_{ij}))$ is positive for every $k \geq 1$.

Suppose $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Let $\mathcal{S}$ be the operator system spanned by $\{I_{\mathcal{H}}, A, A^*\}$. Define a unital linear map $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{K})$ by $\Phi(aI + bA + cA^*) = aI + bB + cB^*$. By [6, Lemma 4.1], Φ is positive if and only if $W(B) \subseteq W(A)$. On the other hand, Stinespring's representation theorem [10] (see also the paragraphs after Theorems 4.1 and 4.6 in [9]) shows that Φ is completely positive if and only if B has a dilation of the form $I \otimes A$. Therefore, Theorems 1.2 and 1.3 can be stated in the following form.

Theorem 1.4. *Suppose $A = A_0$ or $A_0 \oplus [\mu]$ with $A_0 \in M_2$ and $B \in \mathcal{B}(\mathcal{H})$. Define a linear map $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ by*

$$\Phi(aI + bA + cA^*) = aI + bB + cB^* \quad \text{for any } a, b, c \in \mathbb{C}.$$

Then Φ is positive if and only if Φ is completely positive.

2. Proofs

To prove Theorem 1.4, we need several lemmas, some of which are well known. In our discussion, we will let E_{ij} be the basic matrix unit of appropriate size and write $P \geq 0$ if a matrix or operator P is positive semi-definite.

Lemma 2.1 ([9, Corollary 6.7])**.** *Let $\mathcal{S}$ be an operator system. Then every positive linear map $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is completely positive for every Hilbert space $\mathcal{H}$ if and only if every positive linear map $\Psi : \mathcal{S} \to M_n$ is completely positive for all positive integers n.*

Recall that $f : \mathbb{R}^m \to \mathbb{R}^m$ is an affine map if it has the form $\mathbf{x} \mapsto R\mathbf{x} + \mathbf{x}_0$ for a real matrix $R \in M_m$ and $\mathbf{x}_0 \in \mathbb{R}^m$. The affine map is invertible if R is invertible, and the inverse of f has the form $y \mapsto R^{-1}\mathbf{y} - R^{-1}\mathbf{x}_0$. One can extend the definition of affine map to an m-tuple of self-adjoint operators in $\mathcal{B}(\mathcal{H})$ by

$$(A_1, \ldots, A_m) \mapsto (A_1, \ldots, A_m)(r_{ij} I_{\mathcal{H}}) + (B_0, \ldots, B_m)$$

for a real matrix $R = (r_{ij}) \in M_m$ and an m-tuple $(B_1, \ldots, B_m)$ of self-adjoint operators in $\mathcal{B}(\mathcal{H})$. We have the following result, which can be easily verified.

Lemma 2.2. *Let $\mathcal{S}$ be an operator system with a basis $\{I, A_1, \ldots, A_m\}$, and let $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be a unital linear map defined by $\Phi(A_j) = B_j \in \mathcal{B}(\mathcal{H})$ for $j = 1, \ldots, m$. Suppose f is an invertible affine map such that $f(A_1, \ldots, A_m) = (\tilde{A}_1, \ldots, \tilde{A}_m)$ and $f(B_1, \ldots, B_m) = (\tilde{B}_1, \ldots, \tilde{B}_m)$. Then Φ is positive (respectively, completely positive) if and only if the unital map $\tilde{\Phi}$ defined by $\tilde{\Phi}(\tilde{A}_j) = \tilde{B}_j$ for $j = 1, \ldots, m$ is positive (respectively, completely positive).*

Lemma 2.3. *Let $\mathcal{S} = \operatorname{span}\{E_{jj} : 1 \le j \le m\} \subseteq M_m$. A linear map $\Phi : \mathcal{S} \to M_n$ is completely positive if and only if $\Phi(E_{jj}) \ge 0$ for $j = 1, \dots, m$. As a result, every positive linear map $\Phi : \mathcal{S} \to M_n$ is completely positive.*

Proof. If $\Phi : \mathcal{S} \to M_n$ is positive, then $\Phi(E_{jj}) \ge 0$ for all $j = 1, \dots, m$.

Suppose $\Phi(E_{jj}) \ge 0$ for all $j = 1, \dots, m$. Let $C = (C_{ij}) \in M_k(\mathcal{S})$ be positive semi-definite for a positive integer k. Then $C = C_{11} \otimes E_{11} + \cdots + C_{mm} \otimes E_{mm} \ge 0$, where $C_{jj} \ge 0$ for $j = 1, \dots, m$. Thus,

$$(I_k \otimes \Phi)(C) = C_{11} \otimes \Phi(E_{11}) + \cdots + C_{mm} \otimes \Phi(E_{mm}) \ge 0\,.$$

Hence, Φ is completely positive. □

Lemma 2.4. *Let $\mathcal{S} = \operatorname{span}(\{E_{jj} : 1 \le j \le m\} \cup \{E_{12} + E_{21}\})$ in M_m with $m \ge 2$. Then every positive linear map $\Phi : \mathcal{S} \to M_n$ is completely positive.*

Proof. Suppose $\Phi : \mathcal{S} \to M_n$ is a positive map. If $m = 2$, the result is due to Choi [4, Theorem 7]. The proof in [4] relies on a result of Calderon [3]. We give a short and direct proof using basic theory of completely positive maps for completeness as follows.

Suppose Φ is positive. Then $\Phi(E_{11}), \Phi(E_{22}) \ge 0$, and for any real number d,

$$\Phi(E_{11} + d(E_{12} + E_{21}) + d^2 E_{22}) = \Phi(E_{11}) + d\Phi(E_{12} + E_{21}) + d^2\Phi(E_{22}) \ge 0. \tag{1}$$

Let $C = (C_{ij}) \in M_k(\mathcal{S})$ be positive semi-definite for a positive integer k. Then

$$C = C_{11} \otimes E_{11} + C_{22} \otimes E_{22} + C_{12} \otimes E_{12} + C_{21} \otimes E_{21}$$

such that $C_{21} = C_{12} = C_{21}^*$ so that $C_{12} = C_{21}$ is Hermitian and

$$Q = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

is positive semi-definite. We need to show that

$$(I_k \otimes \Phi)(C) = C_{11} \otimes \Phi(E_{11}) + C_{12} \otimes \Phi(E_{12} + E_{21}) + C_{22} \otimes \Phi(E_{22}) \ge 0. \tag{2}$$

We focus on the case when C_{11} is invertible. The general case can be derived by continuity argument. We may replace $C_{ij} = C_{11}^{-1/2} C_{ij} C_{11}^{-1/2}$ for $i, j \in \{1, 2\}$ in (2) and assume that $C_{11} = I$. Because $C_{12} = C_{12}^*$, we may further replace C_{ij} by $U^* C_{ij} U$ in (2) for $i, j \in \{1, 2\}$ and assume that $C_{11} = I$ and $C_{12} = C_{21} = D = \operatorname{diag}(d_1, \dots, d_k)$ with $d_1, \dots, d_k \in \mathbb{R}$. Since $Q \ge 0$, we have

$$\tilde{C}_{22} = C_{22} - C_{21} C_{11}^{-1} C_{12} = C_{22} - D^2 \ge 0.$$

As $\phi(E_{22})$ and $\tilde{C}_{22}$ are positive semi-definite, it follows from (1) that

$$\begin{aligned} & C_{11} \otimes \Phi(E_{11}) + C_{12} \otimes \Phi(E_{12} + E_{21}) + C_{22} \otimes \Phi(E_{22}) \\ = \;& I \otimes \Phi(E_{11}) + D \otimes \Phi(E_{12} + E_{21}) + D^2 \otimes \phi(E_{22}) + \tilde{C}_{22} \otimes \Phi(E_{22}) \ge 0 \end{aligned}$$

as asserted.

Suppose $m > 2$ and k is a positive integer. Let $C = (C_{ij}) \in M_k(\mathcal{S})$ be positive semi-definite. Then

$$C = C_{11} \otimes E_{11} + C_{22} \otimes E_{22} + C_{12} \otimes E_{12} + C_{21} \otimes E_{21} + \sum_{j=3}^{m} C_{jj} \otimes E_{jj},$$

where $\sum_{1\le r,s\le 2}(C_{rs}\otimes E_{rs})\ge 0$ and $C_{jj}\ge 0$ for all $3\le j\le m$. We need to show that

$$(I_k\otimes\Phi)(C)=\sum_{1\le r,s\le 2}C_{rs}\otimes\Phi(E_{rs})+\sum_{j=3}^{m}C_{jj}\otimes\Phi(E_{jj})$$

is positive semi-definite. Since $\sum_{j=3}^{m}C_{jj}\otimes\Phi(E_{jj})\ge 0$, it suffices to prove that

$$\sum_{1\le r,s\le 2}C_{rs}\otimes\Phi(E_{rs})\ge 0,$$

which is true because the restriction of Φ to $\{E_{12},E_{21},E_{12}+E_{21}\}$ is positive and is completely positive by the result when $m=2$. The asserted result follows. $\square$

Now, we are ready to present the following.

Proof of Theorem 1.4. By Lemma 2.1, it suffices to show that if $\Phi:\mathcal{S}\to M_n$ is positive, then it is completely positive, where $\mathcal{S}=\operatorname{span}\{I,A,A^*\}$ satisfies the assumption of the theorem. We assume that A is not a scalar matrix to avoid trivial considerations.

We will use the fact that the conclusion will not change if we replace A by $\alpha I+\beta U^*AU$ for any unitary matrix U and $\alpha,\beta\in\mathbb{C}$ with $\beta\ne 0$. Furthermore, a unital linear map $\Phi:\mathcal{S}\to M_n$ is positive if and only if $B=\Phi(A)$ satisfies $W(B)\subseteq W(A)$ [6, Lemma 4.1]. Applying an affine transform to $A=A_1+iA_2$ with $(A_1,A_2)=(A_1^*,A_2^*)$ will always mean applying a (real) affine transform to (A_1,A_2).

First, suppose A is normal. If $A\in M_2$ has eigenvalues a_1,a_2, we may assume that $A=\operatorname{diag}(a_1,a_2)$. By Lemma 2.2, we may apply an invertible affine map to A and assume that $(a_1,a_2)=(1,0)$. Then $\mathcal{S}=\operatorname{span}\{E_{11},E_{22}\}$. By Lemma 2.3, every unital positive linear map $\Phi:\mathcal{S}\to M_n$ is completely positive.

Suppose $A\in M_3$ is normal. By Lemma 2.2, we can apply an invertible affine map and assume that (1) $A=\operatorname{diag}(1,r,0)$ with $r\in[0,1]$ if the three eigenvalues of A are collinear or (2) $A=\operatorname{diag}(1,i,0)$ otherwise.

Suppose (1) holds and $\Phi:\mathcal{S}\to M_n$ is a unital positive linear map with $\Phi(A)=B$. Then Φ is positive if and only if $\mathbf{x}^*B\mathbf{x}\in W(A)=[0,1]$ for all unit vector $\mathbf{x}\in\mathbb{C}^n$; i.e., B is a positive semi-definite contraction. One can extend Φ to a map from $\hat{\mathcal{S}}=\operatorname{span}\{E_{11},E_{22},E_{33}\}\subseteq M_3$ to M_n with $\Phi(E_{11})=B,\Phi(E_{22})=0,\Phi(E_{33})=I-B$. By Lemma 2.3, Φ is completely positive on $\tilde{\mathcal{S}}$, and therefore so must the restriction map on $\mathcal{S}$.

If (2) holds, then $\mathcal{S}=\operatorname{span}\{E_{11},E_{22},E_{33}\}$. By Lemma 2.3, every unital positive linear map $\Phi:\mathcal{S}\to M_n$ is completely positive.

Next, we consider the case when A is not normal.

Suppose $A\in M_2$. We may replace A by $U^*\left(e^{it}(A-(\operatorname{tr}A)I/2)\right)U$ for a suitable unitary $U\in M_2$ and assume that $A=\begin{pmatrix}\alpha & 2b\\ 0 & -\alpha\end{pmatrix}$ with $\alpha\ge 0$ and $b>0$. Then $W(A)$ is an elliptical disk with major axis $[-r,r]$ and minor axis $i[-b,b]$ with $r=\sqrt{\alpha^2+b^2}$. We may further apply an affine transform

$$A=A_1+iA_2\mapsto\frac{1}{r}A_1+\frac{i}{b}A_2.$$

Then A is unitarily similar to the symmetric matrix $C = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$, where $W(C)$ is the unit disk centered at the origin. So, we may assume that $\mathcal{S} = \operatorname{span}\{I_2, C, C^*\} = \operatorname{span}\{E_{11}, E_{22}, E_{12} + E_{21}\}$ is the set of symmetric matrices in M_2. By Lemma 2.4, every positive linear map $\Phi : \mathcal{S} \to M_n$ is completely positive.

Finally, suppose $A = A_0 \oplus [\mu]$. If $\mu \in W(A_0)$, then $W(A) = W(A_0)$ and the result follows from the previous case. So we can assume that $\mu \notin W(A_0)$. We may apply an affine transform to A_0 as in the preceding case so that $W(A_0)$ is the unit disk centered at the origin and $A_0 \in M_2$ is nilpotent with norm 2. Applying the same affine transform to A yields $A = A_0 \oplus [\hat{\mu}]$. Now, replacing A by $e^{it}(U^*AU - \hat{\mu}I)$ for a suitable $t \in \mathbb{R}$, we may assume that $A = (rI_2 + C) \oplus [0]$, where $r = |\hat{\mu}| > 1$, and $C = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$. So, $A = \begin{pmatrix} r+i & 1 \\ 1 & r-i \end{pmatrix} \oplus [0]$ with $r > 1$.

Suppose $B = H + iG$ with $H = H^*$ and $G = G^*$. We will construct P with $0 \le P \le I$ and a unital *positive* linear map $\Psi : \operatorname{span}\{E_{11}, E_{22}, E_{33}, E_{12} + E_{21}\} \to M_n$ with

$$\begin{aligned} &\Psi(E_{11} + E_{22}) = P, \qquad \Psi(E_{12} + E_{21}) = H - rP, \\ &\Psi(E_{11} - E_{22}) = G, \qquad \Psi(E_{33}) = I - P. \end{aligned} \tag{3}$$

By Lemma 2.4, Ψ is completely positive. One easily checks that Φ is the restriction of Ψ on $\mathcal{S}$ and is also unital completely positive.

Let $t_0 \in (0, \pi)$ be such that $\cos(t_0) = -1/r$. For any unit vector $\mathbf{x} \in \mathbb{C}^n$, we have

$$\begin{aligned} \langle B\mathbf{x}, \mathbf{x}\rangle &\in W(B) \subseteq W(A) \\ &= \{a + ib : |b|\sqrt{r^2 - 1} \le a \quad \text{and} \\ &\qquad\qquad (a - r)\cos t + b \sin t \le 1 \quad \text{for all } t \in [-t_0, t_0]\}. \end{aligned}$$

Equivalently,

$$\pm\sqrt{r^2 - 1}G \le H \tag{4}$$

and

$$\cos t(H - rI) + \sin t\, G \le I \quad \text{for all } t \in [-t_0,\ t_0]. \tag{5}$$

Now, the desired map Ψ satisfying (3) is positive if and only if

$$\begin{aligned} &\Psi\left(\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \oplus [0]\right) \\ &= \frac{1}{2}\left(\cos^2\theta(P + G) + 2\cos\theta\sin\theta(H - rP) + \sin^2\theta(P - G)\right) \end{aligned}$$

is positive semi-definite for all $\theta \in \mathbb{R}$; equivalently,

$$P \ge \cos t(H - rP) + \sin t\, G \quad \text{for all } t \in [-\pi, \pi]. \tag{6}$$

By (4), $H \ge 0$, and there exists a contraction $C = C^* \in M_n$ such that $G = \frac{1}{\sqrt{r^2-1}} H^{1/2} C H^{1/2}$. First, we show that for $Q = \frac{I}{r+1} + \frac{C^2}{r^2-1}$,

$$Q \ge \cos t\,(I - rQ) + \sin t \frac{C}{\sqrt{r^2 - 1}} = \cos t\left(\frac{I}{r+1} - \frac{rC^2}{r^2 - 1}\right) + \sin t \frac{C}{\sqrt{r^2 - 1}}. \tag{7}$$

To see this, apply a unitary similarity to C and assume that $C = \operatorname{diag}(c_1, \dots, c_n)$ with $c_j \in [-1, 1]$. Then by Cauchy-Schwarz inquality and the fact that $c_j^2 \in [0, 1]$, we have

$$\cos t \left(\frac{I}{r+1} - \frac{rc_j^2}{r^2-1} \right) + \sin t \frac{c_j}{\sqrt{r^2-1}} \leq \sqrt{\left(\frac{1}{r+1} - \frac{rc_j^2}{r^2-1} \right)^2 + \frac{c_j^2}{r^2-1}}$$

$$\leq \sqrt{\left(\frac{1}{r+1} - \frac{rc_j^2}{r^2-1} \right)^2 + \frac{c_j^2}{r^2-1} + \frac{c_j^2(1-c_j^2)}{r^2-1}} = \left(\frac{1}{r+1} + \frac{c_j^2}{r^2-1} \right)$$

for each $j = 1, \dots, n$. Hence (7) holds, and for $K = \frac{H}{r+1} + \frac{H^{1/2}C^2H^{1/2}}{r^2-1} \geq 0$, we have

$$\cos tH + \sin tG \leq (1 + r\cos t)K \quad \text{for all } t \in [-\pi,\ \pi]. \tag{8}$$

Suppose V is unitary such that $K = V^* \operatorname{diag}(d_1, \dots, d_n)V$ with $d_1 \geq \cdots \geq d_n \geq 0$. Let

$$P = \min\{K, I\} = V^* \operatorname{diag}(p_1, \dots, p_n)V \quad \text{with } p_j = \min\{k_j, 1\} \quad \text{for } j = 1, \dots, n.$$

Then for $|t| \leq t_0$, it follows from (5) and (8) that

$$\cos tH + \sin tG \leq (1 + r\cos t)\min\{I, K\} \leq (1 + r\cos t)P.$$

For $\pi \geq |t| > t_0$, we have $1 + r\cos t < 0$. Together with (8), we also have

$$\cos tH + \sin tG \leq (1 + r\cos t)K \leq (1 + r\cos t)P.$$

Thus,

$$\cos tH + \sin tG \leq (1 + r\cos t)P \quad \text{for all } t \in [-\pi, \pi]. \tag{9}$$

Hence, (6) holds, and the result follows. □

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