Submultiplicativity of the numerical radius of commuting matrices of order two

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Denote by $w(T)$ the numerical radius of a matrix $T$. An elementary proof is given to the fact that $w(AB) \leq w(A)w(B)$ for a pair of commuting matrices of order two, and characterization is given for the matrix pairs that attain the quality.

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1. Introduction

Let $M_n$ be the set of $n \times n$ matrices. The numerical range and numerical radius of $A \in M_n$ are defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\} \quad \text{and} \quad w(A) = \max\{|\mu| : \mu \in W(A)\},$$

respectively. The numerical range and numerical radius are useful tools in studying matrices and operators. There are strong connection between the algebraic properties of a matrix $A$ and the geometric properties of $W(A)$. For example, $W(A) = \{\mu I\}$ if and only if $A = \mu I$; $W(A) \subseteq \mathbb{R}$ if and only if $A = A^*$; $W(A) \subseteq [0, \infty)$ if and only if $A$ is positive semi-definite.

The numerical radius is a norm on $M_n$, and has been used in the analysis of basic iterative solution methods [2]. Researchers have obtained interesting inequalities related to the numerical radius; for example, see [4–8]. We mention a few of them in the following. Let $\|A\|$ be the operator norm of $A$. It is known that

$$w(A) \leq \|A\| \leq 2w(A).$$

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While the spectral norm is submultiplicative, i.e., \( \|AB\| \leq \|A\|\|B\| \) for all \( A, B \in M_n \), the numerical radius is not. In general,

\[
w(AB) \leq \xi w(A)w(B) \quad \text{for all } A, B \in M_n
\]

if and only if \( \xi \geq 4 \); e.g., see [3]. Despite the fact that the numerical radius is not submultiplicative,

\[
w(A^m) \leq w(A)^m \quad \text{for all positive integers } m.
\]

For a normal matrix \( A \in M_n \), we have \( w(A) = \|A\| \). Thus, for a normal matrix \( A \) and any \( B \in M_n \),

\[
w(AB) \leq \|AB\| \leq \|A\|\|B\| = w(A)|B| \leq 2w(A)w(B),
\]

and also

\[
w(BA) \leq \|BA\| \leq \|B\|\|A\| = \|B\|w(A) \leq 2w(B)w(A).
\]

In case \( A, B \in M_n \) are normal matrices,

\[
w(AB) \leq \|AB\| \leq \|A\|\|B\| = w(A)w(B).
\]

Also, for any pairs of commuting matrices \( A, B \in M_n \),

\[
w(AB) \leq 2w(A)w(B).
\]

To see this, we may assume \( w(A) = w(B) = 1 \), and observe that

\[
4w(AB) = w((A + B)^2 - (A - B)^2) \leq w((A + B)^2) + w((A - B)^2)
\]

\[
\leq w(A + B)^2 + w(A - B)^2 \leq 8.
\]

The constant 2 is best (smallest) possible for matrices of order at least 4 because \( w(AB) = 2w(A)w(B) \) if \( A = E_{12} + E_{34} \) and \( B = E_{13} + E_{24} \), where \( E_{ij} \in M_n \) has 1 at the \((i, j)\) position and 0 elsewhere; see [3, Theorem 3.1].

In connection to the above discussion, there has been interested in studying the best (smallest) constant \( \xi > 0 \) such that

\[
w(AB) \leq \xi w(A)w(B)
\]

for all commuting matrices \( A, B \in M_n \) with \( n \leq 3 \). For \( n = 2 \), the best constant \( \xi \) is one; the existing proofs of the \( 2 \times 2 \) case depend on deep theory on analytic functions, von Neumann inequality, and functional calculus on operators with numerical radius equal to one, etc.; for example, see [6,7].

Researchers have been trying to find an elementary proof for this result in view of the fact that the numerical range of \( A \in M_2 \) is well understood, namely, \( W(A) \) is an elliptical disk with the eigenvalues \( \lambda_1, \lambda_2 \) as foci and the length of minor axis \( \sqrt{(tr A^*A) - |\lambda_1|^2 - |\lambda_2|^2} \); for example, see [10,11] and [8, Theorem 1.3.6].

The purpose of this note is to provide such a proof. Our analysis is based on elementary theory in convex analysis, co-ordinate geometry, and inequalities. Using our approach, we readily give a characterization of commuting pairs of matrices \( A, B \in M_2 \) satisfying \( w(AB) = w(A)w(B) \), which was done in [3, Theorem 4.1] using yet another deep result of Ando [1] that a matrix \( A \) has numerical radius bounded by one if and only if \( A = (I - Z)^{1/2}C(I + Z)^{1/2} \) for some contractions \( C \) and \( Z \), where \( Z = Z^* \). Here is our main result.
Theorem 1. Let $A, B \in M_2$ be nonzero matrices such that $AB = BA$. Then $w(AB) \leq w(A)w(B)$. The equality holds if and only if one of the following holds.

(a) $A$ or $B$ is a scalar matrix, i.e. of the form $\mu I_2$ for some $\mu \in \mathbb{C}$.
(b) There is a unitary $U$ such that $U^*AU = \text{diag}(a_1, a_2)$ and $U^*BU = \text{diag}(b_1, b_2)$ with $|a_1| \geq |a_2|$ and $|b_1| \geq |b_2|$.

One can associate the conditions (a) and (b) in the theorem with the geometry of the numerical range of $A$ and $B$ as follows. Condition (a) means that $W(A)$ or $W(B)$ is a single point; condition (b) means that $W(A), W(B), W(AB)$ are line segments with three sets of end points, $\{a_1, a_2\}, \{b_1, b_2\}, \{a_1b_1, a_2b_2\}$, respectively, such that $|a_1| \geq |a_2|$ and $|b_1| \geq |b_2|$.

2. Proof of Theorem 1

Let $A, B \in M_2$ be commuting matrices. We may replace $(A, B)$ by $(A/w(A), B/w(B))$ and assume that $w(A) = w(B) = 1$. We need to show that $w(AB) \leq 1$.

Since $AB = BA$, there is a unitary matrix $U \in M_2$ such that both $U^*AU$ and $U^*BU$ are in triangular form; for example, see [9, Theorem 2.3.3]. We may replace $(A, B)$ by $(U^*AU, U^*BU)$ and assume that $A = \begin{pmatrix} a_1 & a_3 \\ 0 & a_2 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix}$ and $w(A) = w(B) = 1$. The result is clear if $A$ or $B$ is normal. So, we assume that $a_3, b_3 \neq 0$. Furthermore, comparing the $(1, 2)$ entries on both sides of $AB = BA$, we see that $a_1a_2 - a_3 = b_1b_2 - b_3$. Applying a diagonal unitary similarity to both $A$ and $B$, we may further assume that $\gamma = \frac{a_1 - a_2}{a_3} \geq 0$. Let $r = \frac{1}{\sqrt{\gamma^2 + 1}}$. We have $0 < r \leq 1$. Then $A = z_1I + s_1C$ and $B = z_2I + s_2C$ with

\[ z_1 = \frac{a_1 + a_2}{2}, \quad z_2 = \frac{b_1 + b_2}{2}, \quad s_1 = \frac{a_3}{2r}, \quad s_2 = \frac{b_3}{2r}, \quad \text{and} \quad C = \begin{pmatrix} \sqrt{1 - r^2} & 2r \\ -r \sqrt{1 - r^2} & 0 \end{pmatrix}. \]

Note that $W(C)$ is the elliptical disk with boundary

\[ \{ \cos \theta + ir \sin \theta : \theta \in [0, 2\pi] \}; \]

see [10] and [8, Theorem 1.3.6]. Replacing $(A, B)$ with $(e^{it_1}A, e^{it_2}B)$ for suitable $t_1, t_2 \in [0, 2\pi]$, if necessary, we may assume that $\text{Re} z_1, \text{Re} z_2 \geq 0$ and $s_1, s_2$ are real.

Suppose $z_1 = \alpha_1 + i\alpha_2$ with $\alpha_1 \geq 0$ and the boundary of $W(A)$ touches the unit circle at the point $\cos \phi_1 + i \sin \phi_1$ with $\phi_1 \in [-\pi/2, \pi/2]$. Then $W(A)$ has boundary $\{ \alpha_1 + |s_1| \cos \theta + i(\alpha_2 + |s_1|r \sin \theta) : \theta \in [0, 2\pi] \}$.

We **claim** that the matrix $A$ is a convex combination of $A_0 = e^{i\phi_1}I$ and another matrix $A_1$ of the form $A_1 = i(1 - r^2) \sin \phi_1 I + \xi C$ for some $\xi \in \mathbb{R}$ such that $w(A_1) \leq 1$.

To prove our claim, we first determine $\theta_1 \in [-\pi/2, \pi/2]$ satisfying

\[ \cos \phi_1 + i \sin \phi_1 = (\alpha_1 + |s_1| \cos \theta_1) + i(\alpha_2 + |s_1|r \sin \theta_1). \]

Since the boundary of $W(A)$ touches the unit circle at the point $\cos \phi_1 + i \sin \phi_1$, using the parametric equation

\[ x + iy = (\alpha_1 + |s_1| \cos \theta) + i(\alpha_2 + |s_1|r \sin \theta), \]

(1)
of the boundary of $W(A)$, we see that the direction of the tangent at the intersection point $\cos \phi_1 + i \sin \phi_1$ is $-\sin \theta_1 + ir \cos \theta_1$, which agrees with $-\sin \phi_1 + i \cos \phi_1$, the direction of the tangent line of the unit circle at the same point. As a result, we have

$$(\cos \theta_1, \sin \theta_1) = \frac{(\cos \phi_1, r \sin \phi_1)}{\sqrt{\cos^2 \phi_1 + r^2 \sin^2 \phi_1}}.$$ 

Furthermore, since $\cos \phi_1 + i \sin \phi_1 = (\alpha_1 + |s_1| \cos \theta_1) + i(\alpha_2 + |s_1| r \sin \theta_1)$, we have

$$\alpha_1 = \cos \phi_1 - \frac{|s_1| \cos \phi_1}{\sqrt{\cos^2 \phi_1 + r^2 \sin^2 \phi_1}} \geq 0 \quad \text{and} \quad \alpha_2 = \sin \phi_1 - \frac{|s_1| r \sin \phi_1}{\sqrt{\cos^2 \phi_1 + r^2 \sin^2 \phi_1}}.$$ 

**Assertion.** If $\hat{s}_1 = \sqrt{\cos^2 \phi_1 + r^2 \sin^2 \phi_1}$, then $|s_1| \leq \hat{s}_1$.

If $\cos \phi_1 > 0$, then $\alpha_1 = \left(1 - \frac{|s_1|}{\hat{s}_1}\right) \cos \phi_1 \geq 0$, and hence $|s_1| \leq \hat{s}_1$.

If $\cos \phi_1 = 0$, then $\sin \phi_1 = \pm 1$, $\hat{s}_1 = r$ and $(\alpha_1, \alpha_2) = (0, \sin \phi_1(1 - |s_1|r))$ so that the parametric equation of the boundary of $W(A)$ in (1) becomes

$$x + iy = |s_1| \cos \theta + i(\sin \phi_1(1 - |s_1|r) + |s_1|r \sin \theta).$$

Since $w(A) = 1$ and $\sin \phi_1 = \pm 1$, for all $\theta \in [0, 2\pi)$, we have

$$0 \leq 1 - \left[|s_1| (|\cos \theta|)^2 + (\sin \phi_1(1 - |s_1|r) + |s_1|r \sin \theta)^2\right]$$

$$= 1 - \left[|s_1|^2(1 - \sin^2 \theta) + (\pm(1 - |s_1|r) + |s_1|r \sin \theta)^2\right]$$

$$= 1 - \left[|s_1|^2(1 - (\pm 1 \mp (1 \mp \sin \theta)^2) + (1 - |s_1|r(1 \mp \sin \theta))^2\right]$$

$$= 1 - \left[|s_1|^2(2(1 \mp \sin \theta) - (1 \mp \sin \theta)^2) + 1 - 2|s_1|r(1 \mp \sin \theta) + |s_1|^2 r^2(1 \mp \sin \theta)^2\right]$$

$$= 2|s_1|(r - |s_1|)(1 \mp \sin \theta) + (1 - r^2)|s_1|^2(1 \mp \sin \theta)^2.$$ 

Therefore, $(r - |s_1|) \geq 0$, which gives $|s_1| \leq r = \hat{s}_1$.

Now, we show that our claim holds with

$$A_0 = e^{i\phi_1} I \quad \text{and} \quad A_1 = i(1 - r^2) \sin \phi_1 I + \nu_1 \hat{s}_1 C,$$ 

(2)

where $\nu_1 = 1$ if $s_1 \geq 0$ and $\nu_1 = -1$ if $s_1 < 0$.

Note that $W(A_1)$ is the elliptical disk with boundary $\{ \hat{s}_1 \cos \theta + i[(1 - r^2) \sin \phi_1 + \hat{s}_1 r \sin \theta] : \theta \in [0, 2\pi)\}$, and for every $\theta \in [0, 2\pi]$, we have

$$(\hat{s}_1 \cos \theta)^2 + ((1 - r^2) \sin \phi_1 + \hat{s}_1 r \sin \theta)^2$$

$$= \hat{s}_1^2(1 - \sin^2 \theta) + (1 - r^2)^2 \sin^2 \phi_1 + \hat{s}_1^2 r^2 \sin^2 \theta + 2\hat{s}_1 r(1 - r^2) \sin \phi_1 \sin \theta$$

$$= \hat{s}_1^2 + (1 - r^2)^2 \sin^2 \phi_1 + (1 - r^2)^2 \sin^2 \phi_1 - (1 - r^2) (\hat{s}_1^2 \sin^2 \theta - 2\hat{s}_1 r \sin \phi_1 \sin \theta + r^2 \sin^2 \phi_1)$$

$$= (\cos^2 \phi_1 + r^2 \sin^2 \phi_1) + (1 - r^2)^2 \sin^2 \phi_1 - (1 - r^2)(\hat{s}_1 \sin \theta - r \sin \phi_1)^2$$

$$\leq 1.$$ 

Therefore, $w(A_1) \leq 1$. By the Assertion, $|s_1| \leq \hat{s}_1$. Hence $A = \left(1 - \frac{|s_1|}{\hat{s}_1}\right) A_0 + \frac{|s_1|}{\hat{s}_1} A_1$ is a convex combination of $A_0$ and $A_1$. 

Similarly, if $W(B)$ touches the unit circle at $e^{i\phi_2}$ with $\phi_2 \in [-\pi/2, \pi/2]$, then $B$ is a convex combination of
\[
B_0 = e^{i\phi_2}I \quad \text{and} \quad B_1 = i(1 - r^2) \sin \phi_2 I + \nu_2 \hat{s}_2 C
\]
with $\hat{s}_2 = \sqrt{\cos^2 \phi_2 + r^2 \sin^2 \phi_2}$ and $\nu_2 \in \{1, -1\}$. Let $U = \left( \frac{-r}{\sqrt{1 - r^2}} \frac{\sqrt{1 - r^2}}{r} \right)$. Then $U^*CU = -C$. If $\nu_2 = -1$, we may replace $(A, B)$ by $(U^*AU, U^*BU)$ so that $(\nu_1, \nu_2)$ will change to $(-\nu_1, -\nu_2)$. So, we may further assume that $\nu_2 = 1$.

By the above analysis, $AB$ is a convex combination of $A_0B_0, A_0B_1, A_1B_0$ and $A_1B_1$. Since $w(e^{it}T) = w(T)$ for all $t \in \mathbb{R}$ and $T \in M_n$, the first three matrices have numerical radius 1. We will prove that
\[
w(A_1B_1) < 1.
\]
It will then follow that $w(AB) \leq 1$, where the equality holds only when $A = A_0$ or $B = B_0$.

For simplicity of notation, let $w_1 = \sin \phi_1$ and $w_2 = \sin \phi_2$. Then
\[
\hat{s}_i = \sqrt{1 - (1 - r^2)w_i^2} \quad \text{for} \quad i = 1, 2.
\]

Recall from (2) and (3) that $A_1 = i(1 - r^2)w_1I + \nu_1 \hat{s}_1 C$ and $B_1 = i(1 - r^2)w_2I + \hat{s}_2 C$ because $\nu_2 = 1$. Since $C^2 = (1 - r^2)I_2$, we have
\[
A_1B_1 = (1 - r^2)(uI_2 + ivC),
\]
where
\[
u = w_1 \hat{s}_2 + \nu_1 w_2 \hat{s}_1.
\]
If $r = 1$, then $A_1B_1 = 0$. Assume that $0 < r < 1$. We need to show that
\[
\frac{1}{1 - r^2} w(A_1B_1) = w(uI + ivC) < \frac{1}{(1 - r^2)}.
\]
Because $W(uI + ivC)$ is an elliptical disk with boundary $\{u + iv(\cos \theta + ir \sin \theta) : \theta \in [0, 2\pi]\}$, it suffices to show that
\[
f(\theta) = |u + iv(\cos \theta + ir \sin \theta)|^2 < \frac{1}{(1 - r^2)^2} \quad \text{for all} \quad \theta \in [0, 2\pi].
\]
Note that
\[
f(\theta) = (u - rv \sin \theta)^2 + (v \cos \theta)^2
\]
\[
= u^2 - 2ruv \sin \theta + r^2v^2 \sin^2 \theta + v^2(1 - \sin^2 \theta)
\]
\[
= \frac{u^2}{1 - r^2} + v^2 - \left( \sqrt{1 - r^2} v \sin \theta + \frac{ru}{\sqrt{1 - r^2}} \right)^2
\]
\[
\leq \frac{u^2}{1 - r^2} + v^2
\]
\[
= \frac{1}{(1 - r^2)} \left[ u^2 + (1 - r^2)v^2 \right]
\]
\[
= \frac{1}{(1-r^2)} \left[ (\nu_1 \hat{s}_1 \hat{s}_2 - w_1 w_2 (1-r^2))^2 + (1-r^2)(w_1 \hat{s}_2 + \nu_1 w_2 \hat{s}_1)^2 \right]
\]

\[
= \frac{1}{(1-r^2)} \left[ \hat{s}_1^2 \hat{s}_2^2 + w_1^2 w_2^2 (1-r^2)^2 + (1-r^2)(w_1^2 \hat{s}_2^2 + w_2^2 \hat{s}_1^2) \right]
\]

because \( \nu_1 = \pm 1 \)

\[
= \frac{1}{(1-r^2)} \left[ (\hat{s}_1^2 + (1-r^2)w_1^2)(\hat{s}_2^2 + (1-r^2)w_2^2) \right]
\]

\[
= \frac{1}{(1-r^2)} \ 	ext{by (5)}
\]

\[
< \frac{1}{(1-r^2)^2}
\]

because \( 0 < r < 1 \).

Consequently, we have \( w(A_1 B_1) < 1 \) as asserted in (4). Moreover, by the comment after (4), if \( w(AB) = w(A)w(B) \), then \( A = A_0 \) or \( B = B_0 \). Conversely, if \( A = A_0 \) or \( B = B_0 \), then we clearly have \( W(AB) = w(A)w(B) \). The proof of the theorem is complete. \( \Box \)

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