Closedness of the $k$-numerical range

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ABSTRACT

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For every positive integer $k$ and $A \in \mathcal{B}(\mathcal{H})$, the $k$-numerical range of $A$ is the set

$$W_k(A) = \left\{ \sum_{j=1}^{k} \langle Ax_j, x_j \rangle : \{x_1, \ldots, x_k\} \text{ is an orthonormal set in } \mathcal{H} \right\}.$$  

In this note, we show that the closure of $W_k(A)$ can be written as the convex hull of sets involving the essential numerical range of $A$ and $W_\ell(A)$ for $1 \leq \ell \leq k$. We also show that if $W_k(A)$ is closed, then $W_\ell(A)$ is also closed for $1 \leq \ell \leq k$.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, the numerical range of $A$ is the set

$$W(A) = \{ \langle Ax, x \rangle : x \text{ is a unit vector in } \mathcal{H} \}.$$  

It is a bounded and convex subset of the complex plane $\mathbb{C}$, but is in general not closed. Let $\text{cl}(W(A))$ denote the closure of $W(A)$. In [1, Theorem 1], Lancaster proved that

$$\text{cl}(W(A)) = \text{conv}(W(A) \cup W_{\text{ess}}(A)),$$

where $W_{\text{ess}}(A)$ is the essential numerical range of $A$ and $\text{conv}(W(A) \cup W_{\text{ess}}(A))$ is the the convex hull of $W(A)$ and $W_{\text{ess}}(A)$. There are several equivalent definitions of $W_{\text{ess}}(A)$, see [2]. For our purpose, a point $\mu \in \mathbb{C}$ belongs to $W_{\text{ess}}(A)$ if and only if there is a weakly null sequence of unit vectors (or, a sequence of orthonormal vectors) $\{v_k\}$ in $\mathcal{H}$ such that $\langle Av_k, v_k \rangle \to \mu$. A consequence of Lancaster's theorem is that $W(A)$ is closed if and only if $W_{\text{ess}}(A) \subseteq W(A)$. This is an extension of an earlier result of Halmos [3, Problem 213],
who showed that if \( A \) is compact, \( W(A) \) is closed if and only if \( 0 \in W(A) \). Note that for any compact operator \( A \), \( W_{\text{ess}}(A) = \{0\} \).

There are different extensions of the notion of the numerical range. One of them is to define for each positive integer \( k \) the \( k \)-numerical range of \( A \in B(\mathcal{H}) \) by

\[
W_k(A) = \left\{ \sum_{j=1}^{k} \langle Ax_j, x_j \rangle : \{x_1, \ldots, x_k\} \text{ is an orthonormal set in } \mathcal{H} \right\}.
\]

When \( k = 1 \), \( W_k(A) \) reduces to the usual numerical range \( W(A) \). It is well-known that \( W_k(A) \) is always convex \([3, \text{Problem 211}]\). But just like \( W(A) \), \( W_k(A) \) is not always closed. In \([4]\) Li and Poon showed that, when \( A \in B(\mathcal{H}) \) is compact, \( W_k(A) \) is closed if and only if

\[
0 \in W_k(A) \quad \text{and} \quad W_\ell(A) \subseteq W_k(A) \quad \text{for all } \ell = 1, \ldots, k - 1.
\]

Actually their results are about the more general \( c \)-numerical range, but we shall confine our discussion to the \( k \)-numerical range.

In the next section, we give a description of \( \text{cl} (W_k(A)) \) when \( A \) is not necessarily compact. More precisely, we express the closure as the convex hull of sets involving \( W_{\text{ess}}(A) \) and \( W_\ell(A) \) for \( \ell \leq k \). Another question stemming from the condition \((*)\) is whether there are inclusion relations between the other \( W_\ell(A) \)'s when \( W_k(A) \) is closed. It turns out that if \( W_k(A) \) is closed, \( W_\ell(A) \) is also closed for \( \ell \leq k \). Consequently, \((*)\) can be written as

\[
\{0\} \subseteq W_1(A) \subseteq W_2(A) \subseteq \cdots \subseteq W_k(A).
\]

This is discussed in the last section.

### 2. Closure of \( W_k(A) \)

In this section we prove

**Theorem 2.1:** Let \( A \in B(\mathcal{H}) \) and \( k \) be a positive integer. Set \( W_0(A) = \{0\} \). Then

\[
\text{cl} (W_k(A)) = \text{conv} \bigcup_{\ell=0}^{k} [W_\ell(A) + (k-\ell)W_{\text{ess}}(A)].
\]

Consequently, \( W_k(A) \) is closed if and only if

\[
W_\ell(A) + (k-\ell)W_{\text{ess}}(A) \subseteq W_k(A), \quad \ell = 0, \ldots, k.
\]

**Proof:** To prove (1), note that when \( k = 1 \), it is just \([1, \text{Theorem 1}]\). So, assume that \( k > 1 \).

The inclusion can be deduced from \([5, \text{Theorem 3}]\) by putting \( \beta_{k-\ell,j} = (k-\ell)/k \) and \( \beta_{jk} = 0 \) for \( j \neq k - \ell \). We include a short proof for the sake of completeness. It suffices to show that \( W_\ell(A) + (k-\ell)W_{\text{ess}}(A) \subseteq \text{cl} (W_k(A)) \) for \( 0 \leq \ell \leq k \). Let

\[
\mu = \sum_{j=1}^{\ell} \langle Av_j, v_j \rangle + (k-\ell)\xi \in W_\ell(A) + (k-\ell)W_{\text{ess}}(A),
\]

for orthonormal vectors \( v_1, \ldots, v_\ell \) in \( \mathcal{H} \) and \( \xi \in W_{\text{ess}}(A) \). Choosing an orthonormal basis of \( \mathcal{H} \) with \( \{v_1, \ldots, v_\ell\} \) as the first \( \ell \) vectors, we can represent \( A \) as the matrix \( \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \),
where the $\ell \times \ell$ matrix $A_{11}$ has $(j,j)$ entry equal to $\langle Av_j, v_j \rangle$ for $j = 1, \ldots, \ell$. If $F$ is the finite rank operator represented as $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix}$, then $\xi \in W_{ess}(A-F)$ so that there is an orthonormal sequence of unit vectors \( \{u_1, u_2, \ldots \} \subseteq \{v_1, \ldots, v_\ell \}^\perp \) such that $\langle Au_j, u_j \rangle \to \xi$ [2, Theorem 5.1]. Thus,

$$
\mu = \sum_{j=1}^{\ell} \langle Av_j, v_j \rangle + \lim_{m \to \infty} \sum_{j=1}^{k-\ell} \langle Au_{m+j}, u_{m+j} \rangle \in \text{cl} (W_k(\mathcal{A})).
$$

‘$\subseteq$’ Let $\mu \in \text{cl} (W_k(\mathcal{A}))$. There are orthonormal sets of vectors $\{v_1^{(n)}, \ldots, v_k^{(n)} \}$ in $\mathcal{H}$ such that $\sum_{j=1}^{k} \langle Av_j^{(n)}, v_j^{(n)} \rangle \to \mu$. As the closed unit ball is weakly sequentially compact, by passing to subsequences, we may assume that for each $j$, $v_j^{(n)} \to v_j$ weakly and $\langle Av_j^{(n)}, v_j^{(n)} \rangle \to \mu_j$, for $v_j$ in the closed unit ball of $\mathcal{H}$ and $\mu_j \in \text{cl} (W(\mathcal{A}))$. There are three possibilities (see the proof of [6, Theorem 2.1] for detail),

(i) $v_j = 0$ and $\mu_j \in W_{ess}(\mathcal{A}),$

(ii) $\|v_j\| = 1$, $v_j^{(n)} \to v_j$ strongly and $\mu_j = \langle Av_j, v_j \rangle \in W(\mathcal{A}),$

(iii) $0 < \|v_j\| < 1$ and $\mu_j = \|v_j\|^2 \langle A(v_j/\|v_j\|), v_j/\|v_j\| \rangle + (1 - \|v_j\|^2) \xi_j$ for some $\xi_j \in W_{ess}(\mathcal{A})$ so that $\mu_j$ is a convex combination of points in $W(\mathcal{A})$ and $W_{ess}(\mathcal{A})$.

Taking any $\xi_j \in W_{ess}(\mathcal{A})$ in (ii), we can always write $\mu_j = \langle Av_j, v_j \rangle + (1 - \|v_j\|^2) \xi_j$. As in [4], consider the positive semidefinite operator $H = \sum_{j=1}^{k} (\cdot, v_j)v_j$. Let $d_1 \geq d_2 \geq \cdots \geq d_k \geq 0$ be the $k$ largest eigenvalues of $H$, and $\{u_1, \ldots, u_k \}$ an orthonormal set of corresponding eigenvectors. For each $j$,

$$
d_j = \langle Hu_j, u_j \rangle = \sum_{i=1}^{k} |(u_j, v_i)|^2 = \lim_{n \to \infty} \sum_{i=1}^{k} |\langle u_j, v_i^{(n)} \rangle|^2 \leq \|u_j\|^2 = 1.
$$

For each $\ell = 1, \ldots, k$, let $G_\ell = \sum_{j=1}^{\ell} (\cdot, u_j)u_j$. Then

$$
H = \sum_{j=1}^{k} d_j (\cdot, u_j)u_j = d_k G_k + (d_{k-1} - d_k) G_{k-1} + \cdots + (d_1 - d_2) G_1.
$$

Now, $\text{tr} A H = \sum_{j=1}^{k} \text{tr} A (\cdot, v_j)v_j = \sum_{j=1}^{k} \langle Av_j, v_j \rangle$, and similarly, $\text{tr} A G_\ell = \sum_{j=1}^{\ell} \langle Au_j, u_j \rangle$. In particular, $\text{tr} A G_\ell \in W_\ell(\mathcal{A})$. We have

$$
\sum_{j=1}^{k} \langle Av_j, v_j \rangle = \text{tr} A H = d_k \text{tr} A G_k + (d_{k-1} - d_k) \text{tr} A G_{k-1} + \cdots + (d_1 - d_2) \text{tr} A G_1.
$$

Observe that $d_1 + \cdots + d_k = \text{tr} H = \sum_{j=1}^{k} \text{tr} (\cdot, v_j)v_j = \sum_{j=1}^{k} \|v_j\|^2$. So,

$$
\sum_{j=1}^{k} (1 - \|v_j\|^2) \xi_j
$$
\[(k - d_1 - \cdots - d_k) \left( \frac{1 - \|v_1\|^2}{k - d_1 - \cdots - d_k} \xi_1 + \cdots + \frac{1 - \|v_k\|^2}{k - d_1 - \cdots - d_k} \xi_k \right).\]

As \(W_{\text{ess}}(A)\) is convex, the number above is equal to \((k - d_1 - \cdots - d_k)\xi\) for some \(\xi \in W_{\text{ess}}(A)\). We can write

\[
\mu = \sum_{j=1}^{k} \mu_j = \sum_{j=1}^{k} \langle Av_j, v_j \rangle + \sum_{j=1}^{k} (1 - \|v_j\|^2) \xi_j
\]

\[
= d_k \text{tr} G_k A + (d_{k-1} - d_k) \text{tr} G_{k-1} A + \cdots + (d_1 - d_2) \text{tr} G_1 A + (k - d_1 - \cdots - d_k) \xi
\]

\[
= d_k \text{tr} G_k A + (d_{k-1} - d_k)(\text{tr} G_{k-1} A + \xi) + \cdots + (d_1 - d_2)(\text{tr} G_1 A + (k - 1) \xi) + (1 - d_1)(k \xi),
\]

where, as observed above, \(\text{tr} AG_\ell \in W_\ell(A)\). The proof of (1) is complete. Statement (2) follows easily from (1).

The following example shows that one needs to check the condition in (2) for every \(\ell\) to conclude that \(W_k(A)\) is closed.

**Example 2.2:** Let \(\ell\) and \(k\) be nonnegative integers with \(\ell < k\), and \(A\) the compact operator \(A = -I_\ell \oplus I_k \oplus \text{diag}(1/2, 1/3, \ldots)\) acting on \(\mathcal{H} = \ell_2\). Then \(W_{\text{ess}}(A) = \{0\}\). We have

\[
W_j(A) + (k - j)W_{\text{ess}}(A) = \begin{cases} 
[-j, j] & \text{for } j = 0, 1, \ldots, \ell, \\
(-\ell, j] & \text{for } j = \ell + 1, \ldots, k,
\end{cases}
\]

and

\[
W_k(A) = (-\ell, k].
\]

In particular, \(W_k(A)\) is not closed. In this example we have \(W_j(A) + (k - j)W_{\text{ess}}(A) \subseteq W_k(A)\) for all \(j \in \{0, 1, \ldots, k\} \setminus \{\ell\}\).

Actually, if \(W_k(A)\) is closed, then for every \(\lambda \in W_{\text{ess}}(A)\), \(k\lambda \in W_k(A)\) so that there are orthonormal vectors \(v_1, \ldots, v_k\) such that \(k\lambda = \langle Av_1, v_1 \rangle + \cdots + \langle Av_k, v_k \rangle\). Then \(\lambda = (1/k)(\langle Av_1, v_1 \rangle + \cdots + \langle Av_k, v_k \rangle)\) \(\in W(A)\). By Lancaster [1, Corollary 1], \(W(A)\) is closed. In the next section, we prove that indeed \(W_\ell(A)\) is closed for all \(1 \leq \ell \leq k\).

### 3. Closedness of \(W_k(A)\) and \(W_{k+1}(A)\)

The main result of this section is the following.

**Theorem 3.1:** Let \(A \in \mathcal{B}(\mathcal{H})\) and \(k \geq 1\). If \(W_{k+1}(A)\) is closed, then so is \(W_k(A)\).

Note that the converse is not true. Let \(\{e_j\}_{j=1}^\infty\) be the standard orthonormal basis of \(\ell_2\). If \(A = \langle \cdot, e_1 \rangle e_1 - \sum_{j=2}^\infty 2^{-j} \langle \cdot, e_j \rangle e_j\), then \(W(A) = [-1/4, 1]\) is closed while \(W_2 = [-3/8, 1]\) is not closed.
**Proof:** First consider the simpler situation when \( A \in \mathcal{B}(\mathcal{H}) \) is self-adjoint. The set \( W_k(A) \) is a line segment on \( \mathbb{R} \), which can be computed as follows. Let

\[
\lambda_\ell(A) = \inf_{W \subseteq \mathcal{H}} \sup_{\dim W = \ell - 1} \{ \langle Ax, x \rangle : x \in W^\perp \text{ and } \|x\| = 1 \}.
\]

Then \( \lambda_1(A) \geq \lambda_2(A) \geq \cdots \). Denote by \( \sigma(A) \) the spectrum of \( A \). Then \( \lambda_1(A) = \sup \sigma(A) \) and one of the following holds.

1. There is an orthonormal set \( \{v_1, v_2, \ldots\} \subseteq \mathcal{H} \) such that \( Av_j = \lambda_j(A)v_j \) for \( j = 1, 2, \ldots \).
2. There exists an orthonormal set \( \{v_1, \ldots, v_\ell\} \subseteq \mathcal{H} \), \( \ell \geq 0 \) satisfying \( Av_j = \lambda_j(A)v_j \) for \( j = 1, \ldots, \ell \), and for all \( j > \ell \) we have \( \lambda_j(A) = \lambda_{\ell + 1}(A) = \max W_{\text{ess}}(A) \), which is a limit point of \( \sigma(A) \). Here, if \( \ell = 0 \), we have \( \lambda_j(A) = \max W_{\text{ess}}(A) \) for all \( j \geq 1 \).

Let \( a_k = -\sum_{j=1}^k \lambda_j(A) \) and \( b_k = \sum_{j=1}^k \lambda_j(A) \). Then \( (a_k, b_k) \subseteq W_k(A) \subseteq [a_k, b_k] \). If \( W_{k+1}(A) \) is closed, then \( W_{k+1}(A) = [a_{k+1}, b_{k+1}] \) and there is an orthonormal set of vectors \( \{v_1, \ldots, v_{k+1}\} \) such that \( Av_j = \lambda_j(A)v_j \). It follows that the right hand endpoint \( b_k = \sum_{j=1}^k \lambda_j(A) = \sum_{j=1}^k \langle Av_j, v_j \rangle \) of \( \text{cl} (W_k(A)) \) lies in \( W_k(A) \). Similarly, the left hand endpoint \( a_k = -\sum_{j=1}^k \lambda_j(A) \) of \( \text{cl} (W_k(A)) \) also lies in \( W_k(A) \). Thus, \( W_k(A) = [a_k, b_k] \) is closed.

Now, we turn to the case of a general operator \( A \in \mathcal{B}(\mathcal{H}) \). If for some \( \mu \in \mathbb{C} \) and \( t \in [0, 2\pi) \), \( e^{it}(A - \mu I) \) is self-adjoint, then the result follows from the discussion above. So, assume that it is not the case. We prove the contra-positive, i.e. if \( W_k(A) \) is not closed, then \( W_{k+1}(A) \) is also not closed.

Under the assumption, there is an extreme point \( \mu \) of \( \text{cl} (W_k(A)) \) that does not belong to \( W_k(A) \). Replacing \( A \) by \( e^{it}(A - \mu I) \) for a suitable \( t \in [0, 2\pi) \), we may assume that \( \mu = 0 \) and \( W_k(A) \) lies on the left half of the complex plane and the right support line of \( \text{cl} (W_k(A)) \) is the imaginary axis \( L = \{iy : y \in \mathbb{R}\} \). Considering \( A^* \) instead of \( A \) if necessary, we may further assume that \( 0 \) is the upper endpoint of the line segment \( \text{cl} (W_k(A)) \cap L \).

By Theorem 2.1, \( 0 \in W_\ell(A) + (k - \ell)W_{\text{ess}}(A) \) for some \( \ell \in \{0, \ldots, k\} \). That is,

\[
0 = \sum_{j=1}^\ell \langle Av_j, v_j \rangle + (k - \ell)(h + ig),
\]

for orthonormal vectors \( v_1, \ldots, v_\ell \) and \( h + ig \in W_{\text{ess}}(A) \). We shall assume that

\[
\ell \text{ is the largest integer such that } 0 \in W_\ell(A) + (k - \ell)W_{\text{ess}}(A).
\]

As \( 0 \notin W_k(A) \), \( \ell < k \). If we write \( A = H + iG \) for self-adjoint \( H \) and \( G \), then

\[
0 = \left( \sum_{j=1}^\ell \langle Hv_j, v_j \rangle + (k - \ell)h \right) + i \left( \sum_{j=1}^\ell \langle Gv_j, v_j \rangle + (k - \ell)g \right).
\]

If \( \ell = 0 \), then \( 0 = k(h + ig) \), or \( h + ig = 0 \). So, \( 0 = (k + 1)(h + ig) \in \text{cl} (W_{k+1}(A)) \). We will show that \( 0 \notin W_{k+1}(A) \) to conclude that \( W_{k+1}(A) \) is not closed. Suppose on the
contrary that

\[ 0 = \sum_{j=1}^{k+1} \langle Au_j, u_j \rangle = \sum_{j=1}^{k+1} \langle Hu_j, u_j \rangle + i \sum_{j=1}^{k+1} \langle Gu_j, u_j \rangle \in W_{k+1}(A) \]

for orthonormal vectors \( u_1, \ldots, u_{k+1} \). As \( W_k(A) \) lies on the left half of the complex plane, the sum of any \( k \) terms of \( \langle Hu_1, u_1 \rangle, \ldots, \langle Hu_{k+1}, u_{k+1} \rangle \) is less than or equal to zero. The sum of all \( k + 1 \) of them is zero implies that the sum of any \( k \) terms is indeed zero. We must have \( \langle Hu_j, u_j \rangle = 0 \) for all \( j \). Thus, the sum of any \( k \) terms of \( \langle Au_1, u_1 \rangle, \ldots, \langle Au_{k+1}, u_{k+1} \rangle \) belongs to \( \text{cl}(W_k(A)) \cap L \). As 0 is the upper endpoint of this line segment, the sum of any \( k \) terms of \( \langle Gu_1, u_1 \rangle, \ldots, \langle Gu_{k+1}, u_{k+1} \rangle \) is less than or equal to zero. An argument as above yields that \( \langle Gu_j, u_j \rangle = 0 \) for all \( j \). In particular, we have \( 0 = \sum_{j=1}^{k} \langle Au_j, u_j \rangle \in W_k(A) \), a contradiction.

So, assume in the rest of the proof that \( \ell > 0 \). Since \( 0 = \max \text{cl}(W_k(H)) \), we must have \( \sum_{j=1}^{\ell} \langle Hv_j, v_j \rangle = \max W_{\ell}(H) \). Otherwise we can find orthonormal vectors \( u_1, \ldots, u_{\ell} \) such that \( \sum_{j=1}^{\ell} \langle Hu_j, u_j \rangle > \sum_{j=1}^{\ell} \langle Hv_j, v_j \rangle \) to get the point

\[ \sum_{j=1}^{\ell} \langle Au_j, u_j \rangle + (k - \ell)(h + ig) \in \text{cl}(W_k(A)) \]

with real part \( \sum_{j=1}^{\ell} \langle Hv_j, v_j \rangle + (k - \ell)h > 0 \). Thus, \( \sum_{j=1}^{\ell} \langle Hv_j, v_j \rangle = \sum_{j=1}^{\ell} \lambda_j(H) \). Without loss of generality, we can assume that \( \langle Hv_j, v_j \rangle = \lambda_j(H) \) for each \( 1 \leq j \leq \ell \), so that they are the \( \ell \) largest eigenvalues of \( H \), counting multiplicities, with each \( v_j \) as an eigenvector corresponding to \( \lambda_j(H) \). Again, it follows from \( 0 = \max \text{cl}(W_k(H)) \) that

\[ \lambda_1(H) \geq \cdots \geq \lambda_{\ell}(H) \geq h \quad \text{and} \quad \lambda_j(H) = h \quad \text{for} \quad j > \ell. \]

There may be \( j \leq \ell \) such that \( \lambda_j(H) = h \). So, let \( r \) be the smallest integer such that \( \lambda_j(H) = h \) if \( j > r \). We have \( 0 \leq r \leq \ell \).

Consider the action of \( G \) on the eigenspaces of \( H \). If \( r > 0 \), let \( \mathcal{H}_1 = \text{span}\{v_1, \ldots, v_r\} \) be the direct sum of the eigenspaces of \( H \) corresponding to \( \lambda_1(H), \ldots, \lambda_r(H) \). If \( r = 0 \), let \( \mathcal{H}_1 = \{0\} \). On the finite dimensional subspace \( \mathcal{H}_1 \), we have

\[ \sum_{j=1}^{r} \langle Hw_j, w_j \rangle = \sum_{j=1}^{r} \langle Hv_j, v_j \rangle \quad \text{and} \quad \sum_{j=1}^{r} \langle Gw_j, w_j \rangle = \sum_{j=1}^{r} \langle Gv_j, v_j \rangle \]

for any orthonormal basis \( \{w_1, \ldots, w_r\} \) of \( \mathcal{H}_1 \).

Let \( \mathcal{H}_2 \) be the eigenspace of \( H \) corresponding to \( h \). Then \( \mathcal{H}_2 \) has dimension at least \( \ell - r \) and may even be infinite dimensional. Also, let \( \tilde{G} \) be the compression of \( G \) onto \( \mathcal{H}_2 \). It follows from the fact \( 0 = \sum_{j=1}^{\ell} \langle Gv_j, v_j \rangle + (k - \ell)g \) is the largest imaginary part of points
in $\text{cl}(W_k(A)) \cap L$ that
\[ \sum_{j=r+1}^{\ell} \langle Gv_j, v_j \rangle = \sum_{j=1}^{\ell-r} \lambda_j(\hat{G}). \]

Another observation is that if $u_1, \ldots, u_{\ell-r}$ are orthonormal vectors in $\mathcal{H}_2$ satisfying
\[ \sum_{j=1}^{\ell-r} \langle Gu_j, u_j \rangle = \sum_{j=r+1}^{\ell} \langle Gv_j, v_j \rangle = \sum_{j=1}^{\ell-r} \lambda_j(\hat{G}) \]

and $w$ is a unit vector in $\mathcal{H}_2$ orthogonal to $u_1, \ldots, u_{\ell-r}$, then $\langle Gw, w \rangle < g$. This is because if $\langle Gw, w \rangle > g$, then
\[ \sum_{j=1}^{r} \langle Av_j, v_j \rangle + \sum_{j=1}^{\ell-r} \langle Au_j, u_j \rangle + \langle Aw, w \rangle + (k - \ell - 1)(h + ig) \]

will be a point in $\text{cl}(W_k(A)) \cap L$ with imaginary part
\[ \sum_{j=1}^{\ell} \langle Gv_j, v_j \rangle + (Gw, w) + (k - \ell - 1)g > 0; \]

and if $\langle Gw, w \rangle = g$, $\langle Aw, w \rangle = h + ig$ so that the sum in (4) is zero, contradicting (3).

Now consider $W_{k+1}(A)$. Note that
\[ \sum_{j=1}^{k+1} \lambda_j(H) = \sum_{j=1}^{k} \lambda_j(H) + \lambda_{k+1}(H) = h. \]

So, $\hat{L} = \{ h + iy : y \in \mathbb{R} \}$ is the right support line of $\text{cl}(W_{k+1}(A))$. Let $h + i\hat{g} \in \text{cl}(W_{k+1}(A)) \cap \hat{L}$ have maximum imaginary part. Then $\hat{g} \geq g$ as by Theorem 2.1,
\[ h + ig = \sum_{j=1}^{\ell} \langle Av_j, v_j \rangle + (k - \ell + 1)(h + ig) \in \text{cl}(W_{k+1}(A)). \]

We shall show that $h + i\hat{g}$ does not belong to $W_{k+1}(A)$ and therefore $W_{k+1}(A)$ is not closed. Assume the contrary that $h + i\hat{g} \in W_{k+1}(A)$. Then $h + i\hat{g} = \sum_{j=1}^{k+1} \langle Aw_j, w_j \rangle$ for orthonormal vectors $w_1, \ldots, w_{k+1}$. Recall that $r$ is the smallest integer such that $\lambda_j(H) = h$ if $j > r$. We have
\[ \sum_{j=1}^{k+1} \langle Hw_j, w_j \rangle = \sum_{j=1}^{k+1} \lambda_j(H) = \sum_{j=1}^{r} \langle Hv_j, v_j \rangle + (k + 1 - r)h. \]

Therefore,
\[ \mathcal{H}_1 = \text{span} \{ v_1, \ldots, v_r \} \subseteq \text{span} \{ w_1, \ldots, w_{k+1} \} = \mathcal{H}_3 \quad \text{and} \quad \mathcal{H}_1^\perp \cap \mathcal{H}_3 \subseteq \mathcal{H}_2, \]

where $\mathcal{H}_2$ is the eigenspace of $H$ corresponding to $h$. Without loss of generality, we may assume that $w_j = v_j$ for $j = 1, \ldots, r$. Therefore, $w_{r+1}, \ldots, w_{k+1} \in \mathcal{H}_2$. As $h + i\hat{g}$ has the
maximum imaginary part, we must have

$$\sum_{j=r+1}^{k+1} \langle Gw_j, w_j \rangle = \sum_{j=1}^{k-r+1} \lambda_j(\hat{G}),$$

where $\lambda_j(\hat{G})$ are the $k-r+1$ largest eigenvalues of the compression of $G$ onto $\mathcal{H}_2$. Again we can assume that

$$\sum_{j=r+1}^{\ell} \langle Gw_j, w_j \rangle = \sum_{j=1}^{\ell-r} \lambda_j(\hat{G}).$$

Therefore,

$$\langle Gw_{\ell+1}, w_{\ell+1} \rangle, \ldots, \langle Gw_{k+1}, w_{k+1} \rangle < g.$$

Hence

$$\hat{g} = \sum_{j=1}^{k+1} \langle Gw_j, w_j \rangle = \sum_{j=1}^{\ell} \langle Gw_j, w_j \rangle + \sum_{j=\ell+1}^{k+1} \langle Gw_j, w_j \rangle < \sum_{j=1}^{\ell} \langle Gw_j, w_j \rangle + (k - \ell + 1)g = g,$$

which is a contradiction. 

Combining Theorems 2.1 and 3.1, we get the following criterion for the closedness of $W_k(A)$.

**Corollary 3.2:** The $k$-numerical range $W_k(A)$ is closed if and only if

$$kW_{\text{ess}}(A) \subseteq W_1(A) + (k - 1)W_{\text{ess}}(A) \subseteq \cdots \subseteq W_{k-1}(A) + W_{\text{ess}}(A) \subseteq W_k(A).$$

In particular, if $A$ is compact, then $W_k(A)$ is closed if and only if

$$\{0\} \subseteq W_1(A) \subseteq \cdots \subseteq W_{k-1}(A) \subseteq W_k(A).$$

**Proof:** By Theorem 2.1, $W_k(A)$ is closed if and only if

$$W_j(A) + (k - j)W_{\text{ess}}(A) \subseteq W_k(A) \quad \text{for } j = 0, \ldots, k.$$

The implication ‘$\subseteq$’ is clear.

For the converse, if $W_k(A)$ is closed, then by Theorem 3.1, $W_{k-1}(A)$ is also closed and hence $W_{k-2}(A) + W_{\text{ess}}(A) \subseteq W_{k-1}(A)$. It follows that

$$W_{k-2}(A) + 2W_{\text{ess}}(A) \subseteq W_{k-1}(A) + W_{\text{ess}}(A).$$

The other inclusions can be obtained similarly.

**Acknowledgments**

Li is an affiliate member of the Institute for Quantum Computing, University of Waterloo.
Disclosure statement
No potential conflict of interest was reported by the author(s).

Funding
The research of Li was partially supported by the Simons Foundation Grant 351047.

References