The joint essential numerical range of operators: convexity and related results

by

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Abstract. Let $W(A)$ and $W_e(A)$ be the joint numerical range and the joint essential numerical range of an $m$-tuple of self-adjoint operators $A = (A_1, \ldots, A_m)$ acting on an infinite-dimensional Hilbert space. It is shown that $W_e(A)$ is always convex and admits many equivalent formulations. In particular, for any fixed $i \in \{1, \ldots, m\}$, $W_e(A)$ can be obtained as the intersection of all sets of the form

$$\text{cl}(W(A_1, \ldots, A_{i+1}, A_i + F, A_{i+1}, \ldots, A_m)),$$

where $F = F^*$ has finite rank. Moreover, the closure $\text{cl}(W(A))$ of $W(A)$ is always star-shaped with the elements in $W_e(A)$ as star centers. Although $\text{cl}(W(A))$ is usually not convex, an analog of the separation theorem is obtained, namely, for any element $d \notin \text{cl}(W(A))$, there is a linear functional $f$ such that $f(d) > \sup \{f(a) : a \in \text{cl}(W(A))\}$, where $A$ is obtained from $A$ by perturbing one of the components $A_i$ by a finite rank self-adjoint operator. Other results on $W(A)$ and $W_e(A)$ extending those on a single operator are obtained.

1. Introduction. Let $B(\mathcal{H})$ denote the algebra of bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. The numerical range of $A \in B(\mathcal{H})$ is defined as

$$W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H}, \langle x, x \rangle = 1\},$$

which is useful in studying operators; see [10, 11, 22, 24] and [25, Chapter 1]. Let $S(\mathcal{H})$ denote the set of self-adjoint operators in $B(\mathcal{H})$. Since every $A \in B(\mathcal{H})$ admits a decomposition $A = A_1 + iA_2$ with $A_1, A_2 \in S(\mathcal{H})$, we can identify $W(A)$ with

$$\{\langle (A_1x, x), (A_2x, x) \rangle : x \in \mathcal{H}, \langle x, x \rangle = 1\} \subseteq \mathbb{R}^2.$$
which has been studied by many researchers in order to understand the joint behavior of several operators \(A_1, \ldots, A_m\). One may see \cite{1, 5, 12, 14, 15, 16, 19, 23, 28, 31, 33, 35} and their references for the background and many applications of the joint numerical range.

Let \(\mathcal{F}(\mathcal{H})\) and \(\mathcal{K}(\mathcal{H})\) be the sets of finite rank and compact operators in \(\mathcal{B}(\mathcal{H})\). In the study of finite rank or compact perturbations of operators, researchers consider the joint essential numerical range of \(A \in \mathcal{S}(\mathcal{H})^m\) defined by

\[
W_e(A) = \bigcap \{\text{cl}(W(A + K)) : K = (K_1, \ldots, K_m) \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m \}.
\]

Here \(\text{cl}(S)\) denotes the closure of the set \(S\). For \(m = 2\), \(W_e(A)\) can be identified with the essential numerical range of \(A = A_1 + iA_2 \in \mathcal{B}(\mathcal{H})\), defined by

\[
W_e(A) = \bigcap \{\text{cl}(W(A + K)) : K \in \mathcal{K}(\mathcal{H}) \}.
\]

One may see \cite{2, 3, 6, 7, 13, 18, 20, 21, 26, 27, 30, 32, 36, 37} for many interesting results on \(W_e(A)\) and \(W_e(A)\).

In theoretical studies as well as applications, it is desirable to deal with \(A\) such that \(W(A)\) or \(\text{cl}(W(A))\) is convex. For example, if \(\text{cl}(W(A))\) is convex, one can apply the separation theorem to show that \(0 \notin \text{cl}(W(A))\) if and only if there exist \(r > 0\) and \(c = (c_1, \ldots, c_m) \in \mathbb{R}^m\) such that \((\sum_{i=1}^m c_i A_i) > rI_{\mathcal{H}}\). Unfortunately, \(\text{cl}(W(A))\) is not always convex. Here are some results concerning the convexity of \(W(A)\) and \(\text{cl}(W(A))\), and related to \(W_e(A)\) (for example, see \cite{5, 10, 11, 36, 21, 29, 31} and their references).

**P1** \cite{31} \(W(A_1, \ldots, A_m)\) is convex if

(a) \(\text{span}\{I, A_1, \ldots, A_m\}\) has dimension at most 3, or

(b) \(\dim \mathcal{H} \geq 3\) and \(\text{span}\{I, A_1, \ldots, A_m\}\) has dimension at most 4.

**P2** \cite{31} For any \(A_1, A_2, A_3 \in \mathcal{S}(\mathcal{H})\) such that \(\text{span}\{I, A_1, A_2, A_3\}\) has dimension 4, there is always an \(A_4 \in \mathcal{S}(\mathcal{H})\) for which \(W(A_1, \ldots, A_4)\) is not convex.

**P3** \cite{31} If \(m \geq 4\) then there exists \(A \in \mathcal{S}(\mathcal{H})^m\) such that \(W(A)\) is non-convex.

**P4** For any positive integer \(m\) and any \(A \in \mathcal{S}(\mathcal{H})^m\), \(W_e(A)\) is a compact set contained in \(W(A)\). If \(\text{span}\{I, A_1, \ldots, A_m\}\) has dimension at most 4, then \(W_e(A)\) is convex.

**P5** \cite{36} For \(S \subseteq \mathbb{R}^m\), let \(\text{Ext}(S)\) be the set of all points in \(S\) that do not lie in the open line segment joining two distinct points in \(S\). Then \(\text{Ext}(\text{cl}(W(A))) \subseteq \text{Ext}(W(A)) \cup \text{Ext}(W_e(A))\).

We remark that (P1)–(P3) also hold if we replace \(W(A)\) by \(\text{cl}(W(A))\). In view of (P2) and (P3), if \(m > 3\), then for \(A \in \mathcal{S}(\mathcal{H})^m\) and \(K \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\) the set \(\text{cl}(W(A + K))\) is usually non-convex. Since \(W_e(A)\)
is the intersection of non-convex sets, one does not expect the set \( W_e(A) \) to be convex. This might be the reason why the convexity of \( W_e(A) \) is seldom discussed for \( m > 3 \). In fact, some researchers have studied different geometrical properties of \( W_e(A) \) under the assumption that \( W_e(A) \) is convex, and some have examined \( W_e(A) \) for different classes of operators without discussing their convexity; for example, see [6, 26, 27, 30, 32].

In this paper, we prove the rather unexpected result that \( W_e(A) \) is always convex. Moreover, it is shown that the closure \( \text{cl}(W(A)) \) of \( W(A) \) is always star-shaped with the elements in \( W_e(A) \) as star centers. Many results relating \( W_e(A) \) and \( W(A) \) are also obtained. Our paper is organized as follows.

In Section 2, we extend the results of [21] by establishing several equivalent formulations of the essential joint numerical range for \( A \in \mathcal{S}(\mathcal{H})^m \). One key obstacle for such an extension is the fact that \( W(A) \) may not be convex. To get around this problem, we show that \( \text{cl}(W(A)) \) is star-shaped. The star-shapedness of \( \text{cl}(W(A)) \) and the conditions equivalent to membership in \( W_e(A) \), given in Section 2, lead to our main result that \( W_e(A) \) is convex and its elements are star centers of the set \( \text{cl}(W(A)) \), which is presented in Section 3. With the convexity theorem, we obtain additional descriptions of \( W_e(A) \) in Section 4 in terms of the perturbations of one of the components of \( A \), and also in terms of linear combinations of the components of \( A \). For example, we show that \( W_e(A_1, \ldots, A_m) \) is equal to the sets

\[
\bigcap \{ \text{cl}(W(A_1, \ldots, A_{i-1}, A_i + F, A_{i+1}, \ldots, A_m) : F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H}) \}
\]

and

\[
\\{(a_1, \ldots, a_m) : \sum_{j=1}^m c_j a_j \in W_e \left( \sum_{j=1}^m c_j A_j \right) \text{ for all } (c_1, \ldots, c_m) \in \Omega \}\n\]

where \( \Omega = \{(c_1, \ldots, c_m) \in \mathbb{R}^m : \sum_{j=1}^m c_j^2 = 1 \} \). Also, we obtain an analog of the separation theorem for the not necessarily convex set \( \text{cl}(W(A)) \), namely, for any element \( d \notin \text{cl}(W(A)) \), there is a linear functional \( f \) such that \( f(d) > \sup \{ f(a) : a \in \text{cl}(W(A)) \} \), where \( \tilde{A} \) is obtained from \( A \) by perturbing one of the components \( A_j \) by a finite rank self-adjoint operator. In Section 5, we present additional results on \( W(A) \) and \( W_e(A) \). For instance, \( W_e(A) = \text{cl}(W(A)) \) if and only if the extreme points of \( W(A) \) are contained in \( W_e(A) \); the convex hull of \( \text{cl}(W(A)) \) can always be realized as the joint essential numerical range of \( \tilde{A}_1, \ldots, \tilde{A}_m \) for linear operators \( \tilde{A}_1, \ldots, \tilde{A}_m \) acting on a separable Hilbert space.

In our discussion, we always assume that \( \mathcal{H} \) is infinite-dimensional. For any vector \( x \in \mathcal{H} \) and \( A = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m \), we will use the notation

\[
\langle Ax, x \rangle = (\langle A_1 x, x \rangle, \ldots, \langle A_m x, x \rangle).
\]
Furthermore, $\mathbb{R}^m$ will be used to denote the inner product space of $1 \times m$ real vectors with the usual inner product $\langle x, y \rangle$.

2. Equivalent conditions for $W_e(A)$. Following [21, Theorem 5.1] and its corollary on a single operator $A \in \mathcal{B}(\mathcal{H})$, we obtain several conditions equivalent to membership in $W_e(A)$.

**Theorem 2.1.** Let $A = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. The following conditions are equivalent for a real vector $a = (a_1, \ldots, a_m)$:

1. $a \in W_e(A) = \bigcap \{ \text{cl}(W(A + K)) : K \in \mathcal{K}(\mathcal{H})^m \cap S(\mathcal{H})^m \}$.
2. $a \in \bigcap \{ \text{cl}(W(A + F)) : F \in \mathcal{F}(\mathcal{H})^m \cap S(\mathcal{H})^m \}$.
3. There is an orthonormal sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ of vectors such that
   \[ \lim_{n \to \infty} \langle Ax_n, x_n \rangle = a. \]
4. There is a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ of unit vectors converging weakly to $0$ in $\mathcal{H}$ such that
   \[ \lim_{n \to \infty} \langle Ax_n, x_n \rangle = a. \]
5. There is an infinite-dimensional projection $P \in \mathcal{S}(\mathcal{H})$ such that
   \[ P(A_j - a_j I)P \in \mathcal{K}(\mathcal{H}) \quad \text{for } j = 1, \ldots, k. \]

Most of the argument in [21] can be applied here except for one crucial step, where the convexity of $W(A)$ for $m = 2$ is needed. Since $W(A)$ may not be convex for $m > 3$, we need the following auxiliary result to overcome the obstacle. As a byproduct, it shows that $\text{cl}(W(A))$ is star-shaped.

**Theorem 2.2.** Let $A$ satisfy the hypothesis of Theorem 2.1, and let $W_3(A)$ be the set of real vectors $a$ satisfying condition (3) of Theorem 2.1. Then $W_3(A)$ is non-empty and closed. Moreover, each element $a \in W_3(A)$ is a star center of $\text{cl}(W(A))$, i.e., for any $b \in \text{cl}(W(A))$ we have $(1-t)a + tb \in \text{cl}(W(A))$ for all $0 \leq t \leq 1$.

**Proof.** To prove that $W_3(A)$ is non-empty, let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal sequence of vectors in $\mathcal{H}$. Then the sequence $\{\langle Ax_n, x_n \rangle\}_{n=1}^{\infty}$ is bounded. By choosing a subsequence if necessary, we can assume that $\langle Ax_n, x_n \rangle$ converges. Hence, $W_3(A)$ is non-empty.

Next, we show that $W_3(A)$ is closed. Suppose $a \in \text{cl}(W_3(A))$. Then for each $n \geq 1$, there exists an orthonormal sequence $\{x_k^n\}_{k=1}^{\infty}$ such that
\[ \lim_{k \to \infty} \langle Ax_k^n, x_k^n \rangle = a^n \in \mathbb{R}^m \quad \text{and} \quad \lim_{n \to \infty} a^n = a. \]

Let $\delta_n = 1/(4n^2)$. By going to subsequences if necessary, we may assume that $\|\langle Ax_k^n, x_k^n \rangle - a\| < \delta_n$ for all $n, k$. We may also assume that $\|A_1\|^2 + \cdots + \|A_m\|^2 \leq 1$. Then $\|\langle Ax, y \rangle\| \leq \|x\| \|y\|$ for all $x, y \in \mathcal{H}$. 
Choose $x_1 = x_1^1$. Then $\|\langle Ax_1, x_1 \rangle - a \| < 1$. Suppose we have chosen
\{x_1, \ldots, x_n\} orthonormal with $\|\langle Ax_k, x_k \rangle - a \| < 1/k$ for $1 \leq k \leq n$. Then
choose $N$ such that for all $1 \leq k \leq n$,
\[ |\langle x_k, x_N^{n+1} \rangle|, \|\langle Ax_k, x_N^{n+1} \rangle\| < \delta_{n+1}. \]
Let $y = x_N^{n+1} - \sum_{k=1}^n (x_N^{n+1}, x_k)x_k$. Then
\[ \|y - x_N^{n+1}\| \leq n\delta_{n+1}, \text{ so } 1 - n\delta_{n+1} \leq \|y\| \leq 1 + n\delta_{n+1}. \]
Therefore,
\[ \|\langle Ay, y \rangle - a \| \leq \|\langle A(y - x_N^{n+1}), y \rangle\| + \|\langle Ax_N^{n+1}, y - x_N^{n+1} \rangle\| + \|\langle Ax_N^{n+1}, x_N^{n+1} \rangle - a \| \leq \|y - x_N^{n+1}\|(\|y\| + \|x_N^{n+1}\|) + \delta_{n+1} \leq (2n + 2)\delta_{n+1}. \]
Let $x_{n+1} = y/\|y\|$. Then
\[ \|x_{n+1} - y\| = |1 - \|y\|| \leq n\delta_{n+1}. \]
Hence, \{x_1, \ldots, x_n, x_{n+1}\} is an orthonormal set and
\[ \|\langle Ax_{n+1}, x_{n+1} \rangle - a \| \leq \|y - x_{n+1}\|(\|y\| + \|x_{n+1}\|) + (2n + 2)\delta_{n+1} \leq (4n + 3)\delta_{n+1} < 1/(n + 1). \]
To prove the last assertion, let $a \in W_3(A)$ and $b \in \text{cl}(W(A))$. Suppose
\{x_n\} is an orthonormal sequence in $\mathcal{H}$ such that $\langle Ax_n, x_n \rangle \to a$. For $0 \leq t \leq 1$, we are going to show that $(1 - t)a + tb \in \text{cl}(W(A))$. Given $\varepsilon > 0$, let $y$ be a unit vector in $\mathcal{H}$ such that $\|\langle Ay, y \rangle - b \| < \varepsilon$. Choose $n$ such that $\|\langle Ax_n, x_n \rangle - a \| < \varepsilon$ and $\|\langle Ay, x_n \rangle \| < \varepsilon$. Choose $\theta \in \mathbb{R}$ such that $\langle e^{i\theta}y, x_n \rangle$ is imaginary. Let $z = \sqrt{t} e^{i\theta}y + \sqrt{1 - t}x_n$. Then
\[ \langle z, z \rangle = t\langle y, y \rangle + (1 - t)\langle x_n, x_n \rangle + 2\sqrt{t}\sqrt{1 - t}(\langle e^{i\theta}y, x_n \rangle + \langle x_n, e^{i\theta}y \rangle) = 1 \]
and
\[ \|\langle Az, z \rangle - ((1 - t)a + tb)\| \leq (1 - t)\|\langle Ax_n, x_n \rangle - a \| + t\|\langle Ay, y \rangle - b \| + \sqrt{t}\sqrt{1 - t}(\|e^{i\theta}Ay, x_n \rangle + \langle Ax_n, e^{i\theta}y \rangle)\| \leq 2\varepsilon. \]
Therefore, $(1 - t)a + tb \in \text{cl}(W(A))$. \]

The referee indicated that $W_3(A)$ is clearly closed, and a short proof is possible. We include a detailed proof for the sake of completeness and easy reference.

**Proof of Theorem 2.1.** For $j = 2, 3, 4, 5$, let $W_j(A)$ be the set of a satisfying condition $(j)$. Clearly, we have

\[ W_5(A) \subseteq W_3(A) \subseteq W_4(A) \subseteq W_e(A) \subseteq W_2(A). \]
Suppose $a \in W_2(A)$. We are going to show that $a \in W_5(A)$. Without loss of generality, we may assume $a = 0$. 
Since \( 0 \in W_2(A) \subseteq \text{cl}(W(A)) \), there exists a unit vector \( x_1 \in \mathcal{H} \) such
that \( \|\langle Ax_1, x_1 \rangle\| < 1/2 \). Suppose we have an orthonormal set \( \{x_1, \ldots, x_n\} \)
such that \( \|\langle Ax_n, x_n \rangle\| < 1/2^n \). Let \( Q \) be the orthogonal projection of \( \mathcal{H} \)
onto the subspace \( S \) spanned by \( x_1, \ldots, x_n \) and let
\[
B = ((I - Q)A_1(I - Q)|_{S^\perp}, \ldots, (I - Q)A_m(I - Q)|_{S^\perp}).
\]
Let \( b = (b_1, \ldots, b_m) \in W_3(B) \) and \( bI_S = (b_1I_S, \ldots, b_mI_S) \). Then for \( \overline{Q} = I - Q \), we have
\[
bI_S \oplus B = (b_1Q + \overline{Q}A_1\overline{Q}, \ldots, b_mQ + \overline{Q}A_m\overline{Q}) = A + F
\]
for some \( F \in F(\mathcal{H})^m \cap S(\mathcal{H})^m \). Therefore, \( 0 \in \text{cl}(W(bI_S \oplus B)) \).
Hence, there exists a unit vector \( x \in \mathcal{H} \) such that \( \|\langle (A + F)x, x \rangle\| < 1/2^{n+2} \). Let
\( x = y + z \), where \( y \in S \) and \( z \in S^\perp \). Then \( \|y\|^2 + \|z\|^2 = \|x\|^2 = 1 \). If \( z = 0 \),
then \( \langle (A + F)x, x \rangle = b \in W_3(B) \subseteq \text{cl}(W(B)) \). If \( z \neq 0 \), then by
Theorem 2.2, we have
\[
\langle (A + F)x, x \rangle = \|y\|^2b + \|z\|^2\langle B(z/\|z\|), z/\|z\| \rangle \in \text{cl}(W(B)).
\]
So there exists a unit vector \( x_{n+1} \in S^\perp \) such that
\[
\|\langle (A + F)x, x \rangle - \langle Bx_{n+1}, x_{n+1} \rangle\| < \frac{1}{2^{n+2}},
\]
and hence
\[
\|\langle Ax_{n+1}, x_{n+1} \rangle\| = \|\langle Bx_{n+1}, x_{n+1} \rangle\| < \frac{1}{2^{n+1}},
\]
because \( \langle Fx_{n+1}, x_{n+1} \rangle = 0 \). Inductively, we can choose an orthonormal sequence \( \{x_n\}_{n=1}^\infty \) such that
\[
(1) \quad \|\langle Ax_n, x_n \rangle\| < \frac{1}{2^n} \quad \text{for all } n \geq 1.
\]
Let \( n_1 = 1 \). For every \( 1 \leq i \leq m \), we have
\[
\sum_{n=1}^{\infty} |\langle A_ix_{n_1}, x_n \rangle|^2 \leq \|A_ix_{n_1}\|^2 \quad \text{and} \quad \sum_{n=1}^{\infty} |\langle A_ix_n, x_{n_1} \rangle|^2 \leq \|A_i^*x_{n_1}\|^2.
\]
Hence, there exists \( n_2 > n_1 \) such that
\[
\sum_{n=n_2}^{\infty} |\langle A_ix_{n_1}, x_n \rangle|^2 < \frac{1}{2} \quad \text{and} \quad \sum_{n=n_2}^{\infty} |\langle A_ix_n, x_{n_1} \rangle|^2 < \frac{1}{2},
\]
for all \( 1 \leq i \leq m \). Repeating this procedure, we get a strictly increasing sequence \( \{n_k\}_{k=1}^\infty \) of positive integers such that for all \( 1 \leq i \leq m \), we have
\[
(2) \quad \sum_{n=n_{k+1}}^{\infty} |\langle A_ix_{n_k}, x_n \rangle|^2 < \frac{1}{2^k} \quad \text{and} \quad \sum_{n=n_{k+1}}^{\infty} |\langle A_ix_n, x_{n_k} \rangle|^2 < \frac{1}{2^k}.
\]
Formulas (1) and (2) imply that

\[ \sum_{k,l=1}^{\infty} |\langle A_i x_{n_k}, x_{n_l}\rangle|^2 < \infty. \]  

Let \( P \) be the orthogonal projection onto the subspace spanned by \( \{x_{n_k}\}_{k=1}^{\infty} \). Then it follows from (3) that \( PA_i P \) is compact for all \( 1 \leq i \leq m \).  

3. Convexity and star-shapedness

**Theorem 3.1.** Let \( A \in \mathcal{S}(\mathcal{H})^m \). Then \( W_e(A) \) is a compact convex subset of \( \text{cl}(W(A)) \). Moreover, each element in \( W_e(A) \) is a star center of the star-shaped set \( \text{cl}(W(A)) \).

**Proof.** Because \( W_e(A) \) is the intersection of compact sets, it is compact. To prove the convexity, let \( a, b \in W_e(A) \) and \( 0 \leq t \leq 1 \). Then for every \( F \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m \), we have \( a \in W_e(A) = W_e(A + F) \) and \( b \in W_e(A) \subseteq \text{cl}(W(A + F)) \). So, by Theorem 2.2, we have \( ta + (1-t)b \in \text{cl}(W(A + F)) \). Hence,

\[ ta + (1-t)b \in \bigcap \{ \text{cl}(W(A + F)) : F \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m \} = W_e(A). \]

By Theorems 2.1 and 2.2, we have the last assertion.  

Note that \( W_e(A) \cap W(\mathcal{A}) \) may be empty. For example, if

\( A = \text{diag}(1, 1/2, 1/3, \ldots) \)

acts on \( \ell^2 \), then \( W_e(A) = \{0\} \) and \( W(\mathcal{A}) = (0, 1] \). One may wonder whether a point \( a \in W_e(A) \cap W(\mathcal{A}) \) is a star center of \( W(\mathcal{A}) \). This is not true, as shown by the example below. Moreover, the example shows that for \( m \geq 4 \) there exists \( A \in \mathcal{S}(\mathcal{H})^m \) such that \( \text{cl}(W(A)) \) is convex whereas \( W(\mathcal{A}) \) is not. Of course, this is impossible for \( m \leq 3 \) as \( W(\mathcal{A}) \) is always convex.

**Example 3.2.** Consider \( \mathcal{H} = \ell^2 \) with canonical basis \( \{e_n : n \geq 1\} \). Let \( A = (A_1, \ldots, A_4) \) with

\[ A_1 = \text{diag}(1, 0, 1/3, 1/4, \ldots), \quad A_2 = \text{diag}(1, 0) \oplus 0, \]

\[ A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 0, \quad A_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus 0. \]

Then \((1, 1, 0, 0) \in W(\mathcal{A}) \) and \((0, 0, 0, 0) \in W(\mathcal{A}) \cap W_e(\mathcal{A}) \), but \((1/2, 1/2, 0, 0) \notin W(\mathcal{A}) \). Hence, \( W(\mathcal{A}) \) is not convex. However, \( \text{cl}(W(\mathcal{A})) \) is convex.

**Proof.** Note that \((1, 1, 0, 0) = \langle Ae_1, e_1 \rangle \in W(\mathcal{A}) \) and

\[ (0, 0, 0, 0) = \langle Ae_2, e_2 \rangle = \lim_{n \to \infty} \langle Ae_n, e_n \rangle \in W(\mathcal{A}) \cap W_e(\mathcal{A}). \]
To show that \((1/2, 1/2, 0, 0) \notin W(A)\), consider a unit vector \(x = \sum x_j e_j\) such that \(\sum_{n=1}^{\infty} |x_n|^2 = 1\). If \(\langle A_1 x, x \rangle = \langle A_2 x, x \rangle = 1/2\), then

\[
|x_1|^2 + \sum_{n=3}^{\infty} |x_n|^2/n = |x_1|^2 = 1/2.
\]

Thus, \(x_n = 0\) for all \(n \geq 3\) and \(|x_1|^2 = |x_2|^2 = 1/2\). It then follows that \((\langle A_3 x, x \rangle, \langle A_4 x, x \rangle) \neq (0, 0)\). This proves that \((1/2, 1/2, 0, 0) \notin W(A)\). Hence, \((0, 0, 0, 0) \in W_e(A) \cap W(A)\) is not a star center of \(W(A)\), and \(W(A)\) is not convex.

To see that \(\text{cl}(W(A))\) is convex, note that \(0 \in W_e(A)\). Thus, by Theorem 3.1, for every \(b \in \text{cl}(W(A))\) we have \(t0 + (1 - t)b \in \text{cl}(W(A))\) for any \(t \in [0, 1]\).

Let \(B = (B_1, B_2, B_3, B_4)\), where

\[
B_1 = \text{diag}(0, 1, 0), \quad B_2 = \text{diag}(0, 1, 0),
\]

\[
B_3 = [0] \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_4 = [0] \oplus \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},
\]

and \(C = (C_1, C_2, C_3, C_4)\), where \(C_1 = \text{diag}(1/3, 1/4, \ldots) \oplus [0], \quad C_2 = C_3 = C_4 = \text{diag}(0, 0, \ldots) \oplus [0]\). Then it is easy to verify that

\[
W(B) = \{(r, r, s, t) \in \mathbb{R}^4 : 4(r - 1/2)^2 + s^2 + t^2 \leq 1\}
\]

and

\[
W(C) = \{(c, 0, 0, 0) : c \in [0, 1/3]\}
\]

are both compact and convex. Hence, \(W(B \oplus C) = \text{conv}(W(B) \cup W(C))\) is compact and convex and

\[
W(A) \subseteq W(B \oplus C) \Rightarrow \text{cl}(W(A)) \subseteq W(B \oplus C).
\]

On the other hand, \(B \oplus C = [0] \oplus A \oplus [0]\). Therefore,

\[
W(B \oplus C) = \{t0 + (1 - t)b : b \in W(A)\} \subseteq \text{cl}(W(A)).
\]

So, \(\text{cl}(W(A)) = W(B \oplus C)\) is convex.

4. Other descriptions of \(W_e(A)\). For \(c = (c_1, \ldots, c_m) \in \mathbb{R}^m\) and \(A = (A_1, \ldots, A_m) \in S(\mathcal{H})^m\), let \(c \cdot A = \sum_{i=1}^{m} c_i A_i\). Using the convexity of \(W_e(A)\), we obtain additional conditions equivalent to membership in \(W_e(A)\) in terms of \(c \cdot A \in S(\mathcal{H})\) so that the joint behavior of \(A_1, \ldots, A_m\) can be understood from their linear combinations. For \(A \in S(\mathcal{H})\) and a positive integer \(k\), let

\[
\lambda_k(A) = \inf \{\max \sigma(A + F) : F \in S(\mathcal{H}) \text{ with rank}(F) < k\}.
\]

**Theorem 4.1.** Let \(A \in S(\mathcal{H})^m\) and \(a = (a_1, \ldots, a_m) \in \mathbb{R}^m\). Then \(a \in W_e(A)\) if and only if any one (and hence all) of the following conditions holds:
(1) For every $c \in \mathbb{R}^m$, $c \cdot a \in W_e(c \cdot A)$.
(2) For every $c \in \mathbb{R}^m$, $c \cdot a \in \{ \text{cl}(W(c \cdot A + F)) : F \in \mathcal{F}(\mathcal{H}) \cap S(\mathcal{H}) \}$.
(3) For every $c \in \mathbb{R}^m$, there is an orthonormal sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ such that
\[
\lim_{n \to \infty} \langle c \cdot Ax_n, x_n \rangle = c \cdot a.
\]
(4) For every $c \in \mathbb{R}^m$, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ of unit vectors such that $\{x_n\}_{n=1}^{\infty}$ converges weakly to $0$ in $\mathcal{H}$ and
\[
\lim_{n \to \infty} \langle c \cdot Ax_n, x_n \rangle = c \cdot a.
\]
(5) For every $c \in \mathbb{R}^m$, there is an infinite-dimensional projection $P \in S(\mathcal{H})$ such that $P(c \cdot A - c \cdot a I)P \in \mathcal{K}(\mathcal{H})$.
(6) For every $c \in \mathbb{R}^m$ and $k \geq 1$, $\lambda_k(c \cdot A - c \cdot a I) \geq 0$.

Proof. By the convexity of $W_e(A)$, we can apply the separation theorem to Theorem 2.1 to show that $a \in W_e(A)$ if and only if any one of the conditions (1) to (5) holds.

To prove the equivalence of condition (6), suppose $a \in \mathbb{R}^m$. Without loss of generality, we may assume that $a = 0$. Suppose 0 satisfies condition (6). Then for every $c \in \mathbb{R}^m$ and $F \in \mathcal{F}(\mathcal{H}) \cap S(\mathcal{H})$ with rank($F$) = $k$, we have
\[
\lambda_1(c \cdot A + F) \geq \lambda_{k+1}(c \cdot A) \geq 0 \quad \text{and} \quad \lambda_1(-(c \cdot A + F)) \geq \lambda_{k+1}(-c \cdot A) \geq 0.
\]

Hence, $c \cdot 0 = 0 \in \text{cl}(W(c \cdot A + F))$. Therefore, condition (2) is satisfied.

Conversely, if 0 does not satisfy condition (6), then there exist $c \in \mathbb{R}^m$ and $k \geq 1$ such that $\lambda_k(c \cdot A) < 0$. Thus there exists $F \in \mathcal{F}(\mathcal{H}) \cap S(\mathcal{H})$ such that $c \cdot A + F < 0$ and 0 does not satisfy condition (2).  

Let $A \in S(\mathcal{H})^m$. Although the set $\text{cl}(W(A))$ may not be convex if $m \geq 4$, we have the following analog of the separation theorem for a convex set.

THEOREM 4.2. Let $A = (A_1, \ldots, A_m) \in S(\mathcal{H})^m$ and $d = (d_1, \ldots, d_m) \in \mathbb{R}^m$. Then $d \notin W_e(A)$ if and only if any one (and hence all) of the following conditions holds:

(a) There exists $K \in \mathcal{K}(\mathcal{H})^m \cap S(\mathcal{H})^m$ such that $d \notin \text{cl}(W(A + K))$.
(b) There exists $F \in \mathcal{F}(\mathcal{H})^m \cap S(\mathcal{H})^m$ with $d \notin \text{conv}(\text{cl}(W(A + F)))$.
(c) There exist $F \in \mathcal{F}(\mathcal{H}) \cap S(\mathcal{H})$, $r > 0$ and $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$ such that
\[
\left( \sum_{i=1}^{m} c_i(A_i - d_i I) \right) + F > r I_{\mathcal{H}}.
\]

Proof. For simplicity, replace $(A_1, \ldots, A_m)$ by $(A_1 - d_1 I, \ldots, A_m - d_m I)$ and assume that $d = (0, \ldots, 0)$. 

(c)⇒(b). If (c) holds, we may perturb \( (c_1, \ldots, c_m) \) so that \( c_j \neq 0 \) for all \( j \in \{1, \ldots, m\} \) and condition (4) still holds true. In particular, \( c_1 \neq 0 \). Then let \( F = (F/c_1, 0, \ldots, 0) \). We have \( c \cdot a > r > 0 \) for all \( a \in W(A + F) \). Therefore, \( 0 \notin \text{conv}(\text{cl}(W(A + F))) \).

Clearly, we have (b)⇒(a), which implies that \( 0 \notin W_e(A) \).

Finally, suppose \( 0 \notin W_e(A) \). Then by Theorem 4.1(2), there exist a real vector \( c = (c_1, \ldots, c_m) \) and \( F \in \mathcal{F}(\mathcal{H}) \cap S(\mathcal{H}) \) such that \( 0 = c \cdot 0 \notin \text{cl}(W(c \cdot A + F)) \). Since \( \text{cl}(W(c \cdot A + F)) \) is a closed subinterval \([s, t]\) of \( \mathbb{R} \), we may assume that \( 0 < s \leq t \). Let \( r = s/2 \). Then \( \sum_{i=1}^m c_i A_i + F > r I_H \). Hence, (c) holds. ■

Let \( \Omega = \{ c \in \mathbb{R}^m : \langle c, c \rangle = 1 \} \). By Theorem 4.2, we have the following result showing that \( W_e(A) \) can be expressed as the intersection of half-spaces.

**Corollary 4.3.** Let \( A = (A_1, \ldots, A_m) \in S(\mathcal{H})^m \). Then

\[
W_e(A) = \bigcap_{c \in \Omega} \{ d \in \mathbb{R}^m : \langle c, d \rangle \leq \max_{c \in \Omega} W_e(c \cdot A) \}
= \{ d \in \mathbb{R}^m : \langle c, d \rangle \in W_e(c \cdot A) \text{ for all } c \in \Omega \}.
\]

For \( A \in \mathcal{B}(\mathcal{H}) \), let \( \sigma_e(A) = \bigcap \{ \sigma(A + K) : K \in \mathcal{K}(\mathcal{H}) \} \) be the essential spectrum of \( A \). Then for \( A \in S(\mathcal{H}) \), we have

\[
W_e(A) = \text{conv} \sigma_e(A).
\]

Thus, one may replace \( \max W_e(c \cdot A) \) by \( \max \sigma_e(c \cdot A) \) in Corollary 4.3.

**Corollary 4.4.** Let \( A = (A_1, \ldots, A_m) \in S(\mathcal{H})^m \). If \( d \notin \text{cl}(W(A)) \), then for any \( i \in \{1, \ldots, m\} \) there exists \( F \in \mathcal{F}(\mathcal{H}) \cap S(\mathcal{H}) \) such that \( d \notin \text{conv}(\text{cl}(W(A))) \), where \( \hat{A} = (A_1, \ldots, A_{i-1}, A_i + F, A_{i+1}, \ldots, A_m) \).

**Proof.** If \( d \notin \text{cl}(W(A)) \), then \( d \notin W_e(A) \). The result readily follows from the arguments in the last paragraph in the proof of Theorem 4.2. ■

It follows from Theorem 2.1 that the intersection of the non-convex sets \( \text{cl}(W(A + K)) \), which equals \( W_e(A) \), is a convex set. By Theorem 4.2 and Corollary 4.4, we see that one can replace \( \text{cl}(W(A + K)) \) by its convex hull in the intersection to obtain the same convex set \( W_e(A) \). It is known that for any \( B = (B_1, \ldots, B_m) \in \mathcal{B}(\mathcal{H})^m \),

\[
\text{conv}(\text{cl}(W(B))) = \{(f(B_1), \ldots, f(B_m)) : f \in \Xi\},
\]

where \( \Xi \) is the set of linear functionals \( f \) on \( \mathcal{B}(\mathcal{H}) \) satisfying \( 1 = f(I) = \max \{ f(X) : X \in \mathcal{B}(\mathcal{H}), \|X\| \leq 1 \} \) (for example, see [10, 11]). So, it is easier to determine \( \text{conv}(\text{cl}(W(A + K))) \) than \( \text{cl}(W(A + K)) \). In fact, we have the following.
\textbf{Corollary 4.5.} Let $A \in S(\mathcal{H})^m$ and $i \in \{1, \ldots, m\}$. Then
\[
W_e(A) = \bigcap \{\text{cl}(W(A + F)) : F \in \{0\}^{i-1} \times (F(\mathcal{H}) \cap S(\mathcal{H})) \times \{0\}^{m-i}\}
\]
\[
= \bigcap \{\text{conv}(\text{cl}(W(A + F))) : F \in \{0\}^{i-1} \times (F(\mathcal{H}) \cap S(\mathcal{H})) \times \{0\}^{m-i}\}.
\]

\textbf{Proof.} Let $F \in \{0\}^{i-1} \times (F(\mathcal{H}) \cap S(\mathcal{H})) \times \{0\}^{m-i}$. Clearly,
\[
W_e(A) \subseteq \text{cl}(W(A + F)) \subseteq \text{conv}(\text{cl}(W(A + F))).
\]

So, we may take the intersection of the second and third sets over all $F \in \{0\}^{i-1} \times (F(\mathcal{H}) \cap S(\mathcal{H})) \times \{0\}^{m-i}$, and get an inclusion involving the three sets in the corollary. Finally, if $d \notin W_e(A)$, then $d$ will not belong to the third set by Corollary 4.4. So, the third set is a subset of $W_e(A)$. Hence, the three sets in the corollary are equal. \hfill \blacksquare

\section{5. Additional results.} The following result shows that $W_e(A)$ is unchanged under certain operations on $A$.

\textbf{Theorem 5.1.} Let $A = (A_1, \ldots, A_m) \in S(\mathcal{H})^m$.

(a) Suppose $\mathcal{H}_1$ is a closed subspace of $\mathcal{H}$ such that $\mathcal{H}_1^\perp$ is finite-dimensional. If $X : \mathcal{H}_1 \rightarrow \mathcal{H}$ is such that $X^*X = I_{\mathcal{H}_1}$, then
\[
W_e(A) = W_e(X^*A_1X, \ldots, X^*A_mX).
\]

(b) For each $j \in \{1, \ldots, m\}$, suppose $P_j : \mathcal{H} \rightarrow \mathcal{H}$ is an orthogonal projection such that $I - P_j$ has finite rank. Then
\[
W_e(A) = W_e(P_1A_1P_1, \ldots, P_mA_mP_m).
\]

\textbf{Proof.} Use Theorem 2.1. \hfill \blacksquare

We will establish some additional relationships between the sets $W_e(A)$ and $W(A)$. The next theorem generalizes the results of [29] and [14].

\textbf{Theorem 5.2.} Let $A \in S(\mathcal{H})^m$. Then $W_e(A) = \text{cl}(W(A))$ if and only if $\text{Ext}(W(A)) \subseteq W_e(A)$.

\textbf{Proof.} If $W_e(A) = \text{cl}(W(A))$, then
\[
\text{Ext}(W(A)) \subseteq W(A) \subseteq W_e(A).
\]

Conversely, if $\text{Ext}(W(A)) \subseteq W_e(A)$, then by (P5),
\[
\text{Ext}(\text{cl}(W(A))) \subseteq W_e(A).
\]

Hence,
\[
\text{cl}(W(A)) \subseteq \text{conv}(\text{Ext}(\text{cl}(W(A)))) \subseteq \text{conv}(W_e(A)) = W_e(A).
\]

Since $W_e(A) \subseteq \text{cl}(W(A))$, we have $W_e(A) = \text{cl}(W(A))$. \hfill \blacksquare
For \( k \geq 1 \), let \( I_k \) denote the \( k \times k \) identity matrix. Then for \( A = (A_1, \ldots, A_m) \in S(\mathcal{H})^m \), we have
\[
A \otimes I_k = (A_1 \otimes I_k, \ldots, A_m \otimes I_k) \in S(\mathcal{H} \oplus \cdots \oplus \mathcal{H})^m.
\]
Similarly, let \( I_\infty \) denote the identity operator acting on \( \ell_2 \). Then for \( A = (A_1, \ldots, A_m) \in S(\mathcal{H})^m \), we have
\[
A \otimes I_\infty = (A_1 \otimes I_\infty, \ldots, A_m \otimes I_\infty) \in S(\mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H})^m.
\]

**Theorem 5.3.** Let \( A = (A_1, \ldots, A_m) \in S(\mathcal{H})^m \). Then for any positive integer \( k > \sqrt{m} - 1 \),
\[
W(A \otimes I_k) = \text{conv}(W(A)).
\]
Moreover,
\[
W_e(A \otimes I_\infty) = \text{cl}(\text{conv}(W(A))).
\]

**Proof.** Suppose \( k > \sqrt{m} - 1 \). By the result in [34], every \( a \in \text{conv}(W(A)) \) can be written as \( a = \sum_{j=1}^k t_j \langle Ax_j, x_j \rangle \) for some unit vectors \( x_1, \ldots, x_k \in \mathcal{H} \). Thus, for \( x = (\sqrt{t_1} x_1, \ldots, \sqrt{t_k} x_k) \in \mathcal{H} \oplus \cdots \oplus \mathcal{H} \), we have \( \langle A \otimes I_k x, x \rangle = a \). Conversely, if \( a = \langle A \otimes I_k x, x \rangle \in W(A \otimes I_k) \), one can decompose the unit vector \( x \) into \( k \) parts \( y_1, \ldots, y_k \) according to the structure of \( \mathcal{H} \otimes I_k \). Then
\[
a = \sum_{j=1}^k \|y_j\|^2 \langle Ay_j/\|y_j\|, y_j/\|y_j\| \rangle \in \text{conv}(W(A)).
\]
If \( a \in \text{cl}(\text{conv}(W(A))) \), then there is a sequence \( \{x_n\} \) of unit vectors in \( \mathcal{H} \) such that \( \langle Ax_n, x_n \rangle \rightarrow a \). Let
\[
\tilde{x}_n = (0, \ldots, 0, x_n, 0, \ldots) \in \mathcal{H} \oplus \mathcal{H} \oplus \cdots
\]
Then \( \{\tilde{x}_n\} \) is an orthonormal sequence in \( \mathcal{H} \oplus \mathcal{H} \oplus \cdots \) and \( \langle A \otimes I_\infty \tilde{x}_n, \tilde{x}_n \rangle \rightarrow a \). Therefore, \( a \in W_e(A \otimes I_\infty) \). Since
\[
W_e(A \otimes I_\infty) \subseteq \text{cl}(W(A \otimes I_\infty)) = \text{cl} \left( \bigcup_{k=1}^{\infty} W(A \otimes I_k) \right) \subseteq \text{cl}(\text{conv}(W(A))),
\]
we get the reverse inclusion. \( \square \)

**Corollary 5.4.** Let \( S \) be a compact convex subset of \( \mathbb{R}^m \). Then there are \( A, \tilde{A} \in S(\mathcal{H})^m \) with \( \mathcal{H} = \ell^2 \) such that \( W(A) \) is convex and
\[
W(A) \subseteq S = \text{cl}(W(A)) = W_e(\tilde{A}).
\]

**Proof.** For \( j = 1, \ldots, m \), let \( A_j = \text{diag}(a_{1j}, a_{2j}, \ldots) \) act on \( \ell^2 \) with the standard canonical basis \( \{e_n : n \geq 1\} \) and be such that \( \{(a_{i1}, \ldots, a_{im}) : \).
\( i \geq 1 \) is a dense subset of \( S \). Then for \( A = (A_1, \ldots, A_m) \) the set
\[
W(A) = \text{conv}\{(a_{i1}, \ldots, a_{im}) : i \geq 1\}
\]
is convex, and \( \bar{A} = A \otimes I_\infty \) satisfies the assertion by Theorem 5.3.

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