GENERALIZED NUMERICAL RANGES, JOINT POSITIVE
DEFINITENESS AND MULTIPLE EIGENVALUES

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ABSTRACT. We prove a convexity theorem on a generalized numerical range
that combines and generalizes the following results: 1) Friedland and Loewy's
result on the existence of a nonzero matrix with multiple first eigenvalue in
subspaces of hermitian matrices, 2) Bohnenblust's result on joint positive def-
initeness of hermitian matrices, 3) the Toeplitz-Hausdorff Theorem on the
convexity of the classical numerical range and its various generalizations by

1. INTRODUCTION

The study of hermitian matrices arises naturally in many branches of math-
ematics. (See [HJ] for a nice introduction on the subject.) We are going to study
the following areas of hermitian matrices: multiplicity of the largest eigenvalue,
positive definiteness, convexity and inclusion of generalized numerical ranges.

Let $A_1, \ldots, A_p$ be $n \times n$ hermitian matrices and $\mathcal{K}$ the real linear subspace
spanned by $A_1, \ldots, A_p$. There has been considerable interest in studying the joint
behavior of $A_1, \ldots, A_p$. For example:

Friedland and Loewy [FL] proved that for every $1 < r < n$ there exists $k(r)$ such
that if $\dim \mathcal{K} \geq k(r)$, then there exists a non-zero matrix $A$ in $\mathcal{K}$ such that the
largest eigenvalue of $A$ has multiplicity $\geq r$. (See Proposition 1.3.)

Bohnenblust [Bo] gave a necessary and sufficient condition for the existence of a
positive definite matrix in $\mathcal{K}$. (See Proposition 1.2.)

Au-Yeung and Poon [AP] showed that Bohnenblust’s result is equivalent to a re-
result which generalizes the Toeplitz-Hausdorff Theorem [To], [Hau] on the convexity
of numerical range. (See Proposition 1.1.)

The Toeplitz-Hausdorff Theorem has been generalized in many directions by Au-
Yeung [A1], Halmos [Hal1] and Berger [Be], Westwick [W], Au-Yeung and Tsing
[AT1], [AT2]. Au-Yeung and Tsing [AT1], [AT2] have also shown that the con-

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definiteness property, which, in turn, is equivalent to some inclusion relations on the joint c-numerical ranges.

In this paper, we are going to prove a theorem, which will combine and generalize all of the results mentioned above.

Let \( \mathbb{R} \) be the field of real numbers, \( \mathbb{C} \) the field of complex numbers and \( \mathbb{Q} \) the skew field of real quaternions. We will use \( \mathbb{F} \) to denote any one of \( \mathbb{R} \), \( \mathbb{C} \) or \( \mathbb{Q} \). For an \( n \times m \) matrix \( A \) over \( \mathbb{F} \), \( A^* \) will denote the conjugate transpose of \( A \). For this purpose, vectors in \( \mathbb{F}^m \) will be identified with \( 1 \times n \) matrices over \( \mathbb{F} \). \( \mathcal{M}_n(\mathbb{F}) \) will denote the set of all \( n \times n \) matrices over \( \mathbb{F} \). For \( c \in \mathbb{F}^m \), \( [c] \) will denote the diagonal matrix in \( \mathcal{M}_n(\mathbb{F}) \) with diagonal \( c \). Suppose \( A_i \in \mathcal{M}_{n_i}(\mathbb{F}), 1 \leq i \leq m \). Then \( A_1 \oplus \cdots \oplus A_m \) will represent the block diagonal matrix formed by \( A_1, \ldots, A_m \).

Let \( \mathcal{H}_n(\mathbb{F}) = \{ A \in \mathcal{M}_n(\mathbb{F}) : A = A^* \} \) and \( \mathcal{U}_n(\mathbb{F}) = \{ A \in \mathcal{M}_n(\mathbb{F}) : AA^* = I_n \} \) be the hermitian and unitary matrices in \( \mathcal{M}_n(\mathbb{F}) \) respectively. Here, \( I_n \) denotes the \( n \times n \) identity matrix. For each \( n \geq 1 \), \( \mathcal{H}_n(\mathbb{F}) \) is a real vector space of dimension \( f_R(n) \), where \( f_R(n) = n(n+1)/2 \), \( f_C(n) = n^2 \), and \( f_Q(n) = n(2n-1) \). Thus, we can identify \( \mathcal{H}_n(\mathbb{F}) \) with \( \mathbb{R}^{f_R(n)} \), equipped with the usual topology.

A subset \( S \) of \( \mathbb{R}^m \) is said to be convex if for every \( x, y \in S \), we have \( t x + (1-t) y \in S \) for all \( 0 \leq t \leq 1 \). Given a subset \( S \) of \( \mathbb{R}^m \), \( \text{conv} S \) will denote the smallest convex subset of \( \mathbb{R}^m \) containing \( S \). A subset \( P \subseteq \mathbb{R}^m \) is called a hyperplane if there exist a subspace \( V \) of \( \mathbb{R}^m \) and \( a \in \mathbb{R}^m \) such that \( P = \{ a + v : v \in V \} \). A non-empty subset \( S \) of \( \mathbb{R}^m \) is said to have a convex boundary if there exists a hyperplane \( P \) that \( S \) and the boundary of \( \text{conv} S \), as a topological subspace of \( P \), is contained in \( S \).

Let \( W^F(A_1, A_2) = \{(x A_1 x^*, x A_2 x^*) : x \in \mathbb{F}^m, x x^* = 1\} \), for \( A_1, A_2 \in \mathcal{H}_n(\mathbb{F}) \). The Toeplitz-Hausdorff Theorem [To], [Hau] states that \( W^C(A_1, A_2) \) is a convex subset of \( \mathbb{R}^2 \) for all \( A_1, A_2 \in \mathcal{H}_n(\mathbb{C}) \). Brickman [Br] proved that if \( n > 2 \), then \( W^R(A_1, A_2) \) is convex for all \( A_1, A_2 \in \mathcal{H}_n(\mathbb{R}) \). Au-Yeung [Al] gave a unified proof for the convexity of \( W^F(A_1, A_2) \) for \( F = \mathbb{R} \) (\( n > 2 \)) and \( F = \mathbb{C} \) or \( \mathbb{Q} \).

One direction of generalization of the Toeplitz-Hausdorff Theorem is to consider the set \( W^F(A_1, \cdots, A_p) = \{(x_1 A_1 x^*, \cdots, x_p A_p x^*) : x \in \mathbb{F}^m, x x^* = 1\} \) for \( A_1, \cdots, A_p \in \mathcal{H}_n(\mathbb{F}) \). Hausdorff [Hau] noted that \( W^C(A_1, A_2, A_3) \) has a convex boundary and remarked that \( W^C(A_1, A_2, A_3) \) is not convex in general. However, for \( n \geq 3 \), \( W^C(A_1, A_2, A_3) \) is convex for all \( A_1, A_2, A_3 \in \mathcal{H}_n(\mathbb{C}) \). This is a special case of the following

**Proposition 1.1.** Let \( F = \mathbb{R}, \mathbb{C}, \) or \( \mathbb{Q} \), \( 1 \leq r \leq n-1 \), \( p < f_F(r+1) - \delta_{n,r+1} \) and \( A_1, \cdots, A_p \in \mathcal{H}_n(\mathbb{F}) \). Then the set

\[
W^F_r(A_1, \cdots, A_p) = \left\{ \left( \sum_{i=1}^{r} x_i A_1 x_i^*, \cdots, \sum_{i=1}^{r} x_i A_p x_i^* \right) : x \in \mathbb{F}^m, \sum_{i=1}^{r} x_i x_i^* = 1 \right\}
\]

is convex. Here \( \delta_{i,j} \) is the Kronecker delta.

Au-Yeung and Poon showed [AP] that Proposition 1.1 is equivalent to the following result of Bohnenblust [Bo], [Ta] on joint positive definiteness of hermitian matrices.
Proposition 1.2. Let $F = R$, $C$ or $Q$, $1 \leq r \leq n - 1$, $p < f_F(r+1) - \delta_{n,r+1}$ and $A_1, \ldots, A_p \in \mathcal{H}_n(F)$. Suppose $(0, \ldots, 0) \notin W^F_r(A_1, \ldots, A_p)$. Then there exist $\alpha_1, \ldots, \alpha_p \in R$ such that the matrix $\sum_{i=1}^p \alpha_i A_i$ is positive definite.

Bohnenblust proved Proposition 1.2 for $F = R$ and $C$ but, as noted in [AP], the result remains valid for $F = Q$. Friedland and Lowey [FL] proved the next proposition and then showed that it is also equivalent to Proposition 1.2, for $F = R$ or $C$.

Proposition 1.3. Let $F = R$ or $C$, $1 < r < n$, and $k(r) = f_F(n) - f_F(n+1-r)$. Suppose $K$ is a subspace of $\mathcal{H}_n(F)$ with dimension $\geq k(r)$. Then there exists a non-zero matrix $A \in K$ such that $\lambda_1(A) = \cdots = \lambda_r(A)$.

Here, for a matrix $B \in \mathcal{H}_n(F)$, $\lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_n(B)$ are the eigenvalues of $B$ arranged in decreasing order.

Remark 1.4. a) The bound $f_F(r+1) - \delta_{n,r+1}$ in Proposition 1.1 is best possible in the sense that for every $1 \leq r \leq n - 1$ and $p \geq f_F(r+1) - \delta_{n,r+1}$, there exist $A_1, \ldots, A_p \in \mathcal{H}_n(F)$ such that $W^F_r(A_1, \ldots, A_p)$ is not convex.

b) The set $W^F_r(A_1, \ldots, A_p)$ has appeared in the study of structured singular values in control theory [D], [FT], which leads to the problem of determining the smallest $r$ such that every $w$ in $\text{conv} W^F_r(A_1, \ldots, A_p)$ can be expressed as a convex combination of no more than $r$ points in $W^F_r(A_1, \ldots, A_p)$. It has been shown in [P2] that the inequality $p < f_F(r+1) - \delta_{n,r+1}$ gives the best possible bound for $r$.

Let $c = (c_1, \ldots, c_n) \in R^n$, and $A_1, \ldots, A_p \in \mathcal{H}_n(F)$. To state another generalization of the Toeplitz-Hausdorff Theorem, define

$$W^F_c(A_1, \ldots, A_p) = \left\{ \sum_{i=1}^n c_i x_i A_1 x_i^* \cdots \sum_{i=1}^n c_i x_i A_p x_i^* : \{x_1, \ldots, x_n\} \in \mathcal{O}_n(F) \right\}.$$

Let $1 \leq k \leq n$. Halms [Hal1] conjectured that if $c$ has $k$ coordinates equal to $1$ and the rest $0$, then $W^F_c(A_1, A_2)$ is convex. The conjecture was proven by Berger [Be], [Hal2]. Berger’s result was generalized by Westwick [W], who proved the following result for $F = C$. A simple proof of Westwick’s result is given in [P1]. The following proposition was proven by Au-Yeung and Tsing [AT1].

Proposition 1.5. Let $c \in R^n$. If $F = R$ and $n > 2$ or $F = C$, $Q$, then $W^F_c(A_1, A_2)$ is convex for all $A_1, A_2 \in \mathcal{H}_n(F)$.

Au-Yeung [A2] has called for a unified proof of Propositions 1.1 and 1.5. The following result of Au-Yeung and Tsing [AT2] is the first step in this direction.

Proposition 1.6. Let $n \geq 3$, $c \in R^n$. Then $W^F_c(A_1, A_2, A_3)$ is convex for all $A_1, A_2, A_3 \in \mathcal{H}_n(C)$.

We note that for $c = (1, 0, \ldots, 0) \in R^n$, $W^F_c(A_1, A_2, A_3) = W^F_1(A_1, A_2, A_3)$. Thus, for $n \geq 3$ and $F = C$, Proposition 1.6 covers both Proposition 1.5 and the case when $r = 1$ in Proposition 1.1.

In this paper, we will prove a result which covers Propositions 1.1, 1.2, 1.3 1.5 and 1.6.
2. Main theorem

A square matrix with nonnegative entries is said to be doubly stochastic (d.s.) if every row sum and every column sum is equal to 1. Let $\Omega(n)$ be the set of all $n \times n$ d.s. matrices and $\Omega(n, r) = \{D = \sum_{i=1}^{m} D_i : m \geq 1, D_i \in \Omega(n_i), n_i \leq r, \sum_{i=1}^{m} n_i = n\}$. We have $\Omega(n, 1) \subseteq \Omega(n, 2) \subseteq \cdots \subseteq \Omega(n, n) = \Omega(n)$.

For $c = (c_1, \ldots, c_n)$, let $c_{[1]} \geq \cdots \geq c_{[n]}$ be the decreasing rearrangement of the components of $c$ and $c_\downarrow = (c_{[1]}, \ldots, c_{[n]})$.

**Definition 2.1.** Let $c \in \mathbb{R}^n$, $F = R$, $C$ or $Q$ and $A_1, \ldots, A_p \in \mathcal{H}_n(F)$. For $1 \leq r \leq n$, define

$$C(c, r) = \{c_{[i]} D : D \in \Omega(n, r)\},$$

and

$$W^{F}_{c, r}(A_1, \ldots, A_p) = \{(tr [b] U A_1 U^*, \ldots, tr [b] U A_p U^*) : b \in C(c, r), U \in \mathcal{U}_n(F)\},$$

where $tr A$ denotes the trace of $A$.

The main result in this paper is

**Theorem 2.2.** Let $c \in \mathbb{R}^n$, $F = R$, $C$ or $Q$, $1 \leq r \leq n - 1$, $p < f_F(r+1) - \delta_{n, r+1}$ and $A_1, \ldots, A_p \in \mathcal{H}_n(F)$. Then $W^{F}_{c, r}(A_1, \ldots, A_p)$ is convex.

**Remark 2.3.** We note that for $e = (1, 0, \ldots, 0)$ and $c \in \mathbb{R}^n$, we have

$$W^{F}_{e, r}(A_1, \ldots, A_p) = W^{F}_{r}(A_1, \ldots, A_p),$$

and

$$W^{F}_{c, 1}(A_1, \ldots, A_p) = W^{F}_{c}(A_1, \ldots, A_p).$$

For $r = 1$, $f_F(r+1) - \delta_{n, r+1} > \begin{cases} 2 & \text{if } F = R \text{ and } n \geq 3 \text{ or } F = C, Q, \\ 3 & \text{if } n \geq 3 \text{ and } F = C, Q \end{cases}$. Therefore, Theorem 2.2 is a generalization of Propositions 1.1, 1.5 and 1.6.

For the rest of this section, $F = R$, $C$ or $Q$. If $d^i \in \mathbb{R}^{n_i}$, for $1 \leq i \leq m$ with $\sum_{i=1}^{m} n_i = n$, then $(d^1, d^2, \ldots, d^m)$ will denote the vector in $\mathbb{R}^n$ formed by the juxtaposition of the components in $d^i$.

**Lemma 2.4.** a) Let $A \in \mathcal{M}_n(F)$. Then $A \in \mathcal{H}_n(F)$ if and only if there exist $U \in \mathcal{U}_n(F)$ and $c \in \mathbb{R}^n$ such that $A = U[c] U^*$.

b) Let $U \in \mathcal{U}_n(R)$. There is a path in $\mathcal{U}_n(R)$ connecting $U$ to $I_n$ if and only if det $U = 1$. For $F = C$ or $Q$, $\mathcal{U}_n(F)$ is path connected.

**Proof.** For $F = R$ or $C$, the result is well known [HJ]. For $F = Q$, a) and b) follow from Theorem 9 and 8 of Lee [Le] respectively.

For $A \in \mathcal{M}_n(F)$, let $Tr A = Re tr A$, the real part of $tr A$. Then we have

**Lemma 2.5.** For $A, B \in \mathcal{M}_n(F)$, $(A, B) = Tr AB^*$ defines a real valued inner product on $\mathcal{M}_n(F)$. Moreover, $Tr AB = Tr BA$ for all $A, B \in \mathcal{M}_n(F)$.

**Proof.** Since the result is well known for $F = R$ or $C$, it remains to prove the case when $F = Q$. Recall that $Q$ is generated by $C$ and $j$ such that $aj = j\bar{a}$ for all $a \in C$. Thus, every $A \in \mathcal{M}_n(Q)$ can be expressed uniquely as $A = A_1 + jA_2$, with $A_1, A_2 \in \mathcal{M}_n(C)$. This induces (see Lee [Le]) a unital real algebra isomorphism $f$ of $\mathcal{M}_n(Q)$ into a subalgebra of $\mathcal{M}_{2n}(C)$ by

$$f(A_1 + jA_2) = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}.$$
Here, \( \overline{A} \) denotes the \textit{conjugate} of \( A \). Since \( Tr \ A = (tr \ f(A)) / 2 \), the result follows from the case when \( F = C \).

Let \( b, c \in \mathbb{R}^n \). We will write \( b \prec c \) if \( b \in C(c, n) \). By the Hardy-Littlewood-Pólya Theorem [HLP], [MO], we have \( b \prec c \) if and only if

\[
\sum_{i=1}^{k} b_{[i]} \leq \sum_{i=1}^{k} c_{[i]} \quad (k = 1, \ldots, n - 1) \text{ and } \sum_{i=1}^{n} b_{[i]} = \sum_{i=1}^{n} c_{[i]}.
\]

\textbf{Lemma 2.6.} For \( c \in \mathbb{R}^n \), let \( \mathcal{U}_F(c, r) = \{U^*[b]U : b \in C(c, r), U \in \mathcal{U}_n(F)\} \). We have

a) For \( 1 \leq r \leq n \), \( \text{conv} \mathcal{U}_F(c, r) = \{U^*[b]U : b \prec c, U \in \mathcal{U}_n(F)\} = \mathcal{U}_F(c, n) \).

b) \( \{b : b \prec c\} = \{\text{diag}U[c]U^* : U \in \mathcal{U}_n(F)\} \)

\[= \{\text{diag}U[b]U^* : U \in \mathcal{U}_n(F), b \prec c\}.\]

c) \( \mathcal{U}_F(c, n - 1) \) has a convex boundary.

\textbf{Proof.} We may assume that \( c = c_1 \).

a) Since \( \mathcal{U}_F(c, 1) \subseteq \cdots \subseteq \mathcal{U}_F(c, n) \), it suffices to prove that \( \text{conv} \mathcal{U}_F(c, 1) = \mathcal{U}_F(c, n) \). Thompson [Th]\(^1\) proved the result for \( F = C \) and it was pointed out by Au-Yeung and Tsing [AT1] that the result also holds for \( F = R \) and \( Q \).

b) The first equality is due to Horn [Ho] and Au-Yeung and Tsing [AT1]. The second equality follows from the first equality.

c) Let \( P_c = \{A \in \mathcal{H}_n(F) : Tr \ A = \sum_{i=1}^{n} c_i\} = \{[c] + B : B \in \mathcal{H}_n(F), tr \ B = 0\} \). Then \( P_c \) is a hyperplane containing \( \mathcal{U}_F(c, n) \). From a), \( \mathcal{U}_F(c, n - 1) = \mathcal{U}_F(c, n) \subseteq P_c \). Suppose \( B \in \mathcal{U}_F(c, n) \). Then there exists \( U \in \mathcal{U}_n(F) \) and \( b \in C(c, n) \) such that \( B = U[b]U^* \). If equality holds in (1) for some \( 1 \leq k < n \), then \( B \in \mathcal{U}_F(c, n - 1) \). Suppose strict inequality holds in (1) for all \( 1 \leq k < n \). Let \( A \in P_c \) have eigenvalues \( \lambda_1 \geq \cdots \geq \lambda_n \). If \( A \) is sufficiently close to \( B \) in \( P_c \), then we have \( \sum_{i=1}^{k} \lambda_i < \sum_{i=1}^{k} c_i \) for \( k = 1, \ldots, n - 1 \). Hence, \( A \in \mathcal{U}_F(c, n) \). Therefore the boundary of the convex set \( \mathcal{U}_F(c, n) \) in \( P_c \) is contained in \( \mathcal{U}_F(c, n - 1) \).

\textbf{Remark 2.7.} Let \( b \in \mathbb{R}^n, A \in \mathcal{H}_n(F) \) and \( U \in \mathcal{U}_n(F) \). It follows from Lemma 2.5 and 2.6 b) that \( Tr \ [b]U[AU]^* = Tr \ [b]U[AU]^* = Tr \ U^*[b]U A \). Thus we have

\[
W_{c,r}^F(A_1, \ldots, A_p) = \{(Tr \ BA_1, \ldots, Tr \ BA_p) : B \in \mathcal{U}_F(c, r)\},
\]

\[
\text{conv} W_{c,r}^F(A_1, \ldots, A_p) = W_{c,n}^F(A_1, \ldots, A_p) \text{ for all } 1 \leq r \leq n.
\]

First, we prove a special case of Theorem 2.2.

\textbf{Lemma 2.8.} Let \( p \leq f_F(n) - 2 \). Then \( W_{c,n-1}^F(A_1, \ldots, A_p) \) is convex for all \( A_1, \ldots, A_p \in \mathcal{H}_n(F) \).

\textbf{Proof.} Let \( w = (w_1, \ldots, w_p) \in \text{conv} W_{c,n-1}^F(A_1, \ldots, A_p) = W_{c,n}^F(A_1, \ldots, A_p) \).

Then, by Remark 2.7, there exists \( B \in \mathcal{U}_F(c, n) \) such that \( w_j = Tr \ BA_j \) for all \( 1 \leq j \leq p \). Since \( p \leq f_F(n) - 2 \), there exists a non-zero \( X \in \mathcal{H}_n(F) \) such that \( Tr \ X = Tr \ XA_j = 0 \) for all \( 1 \leq j \leq p \). For all \( t \in \mathbb{R} \), we have \( B + tX \in P_c \) (as defined in the proof of Lemma 2.6 c)) and \( Tr \ (B + tX)A_j = w_j \) for all \( 1 \leq j \leq p \). Since \( \mathcal{U}_F(c, n) \) is bounded, we have, by Lemma 2.6 c), \( B + tX \in \mathcal{U}_F(c, n - 1) \) for some \( t \).

\(^1\)The author is thankful to Professor Tin Yau Tam for pointing out this reference.
Lemma 2.9. Let \( p = f_F(n) - 1 \) and \( K \) be the subspace spanned by \( A_1, \ldots, A_p \) in \( \mathcal{H}_n(F) \). Then we have
a) If \( I_n \in K \) or \( \dim K < p \), then \( W^F_{c,n-1}(A_1, \ldots, A_p) \) is convex.
b) \( W^F_{c,n-1}(A_1, \ldots, A_p) \) has a convex boundary.

Proof. a) Suppose \( B_1, \ldots, B_q \) is a basis of \( K \). Then \( W^F_{c,n-1}(A_1, \ldots, A_p) \) is convex if and only if \( W^F_{c,n-1}(B_1, \ldots, B_q) \) is convex. If \( q < p \), the result follows from Lemma 2.8.

Suppose \( I_n \in K \) and \( q = p \). Then \( A_1, \ldots, A_p \) is a basis of \( K \). We may assume that \( A_p = I_n \). Since \( \text{Tr} X I_n = \sum_{i=1}^{n} c_i = s \) is independent of \( X \in \mathcal{U}_F(c, n-1) \), we have \( W^F_{c,n-1}(A_1, \ldots, A_p) = \{(w, s) : w \in W^F_{c,n-1}(A_1, \ldots, A_p-1)\} \). Hence, the convexity of \( W^F_{c,n-1}(A_1, \ldots, A_p) \) follows from that of \( W^F_{c,n-1}(A_1, \ldots, A_p-1) \).

b) From a), we only need to consider the case when \( \dim K = p \) and \( I_n \notin K \); then \( \{I_n, A_1, \ldots, A_p\} \) is a basis for \( \mathcal{H}_n(F) \). We have
\[
W^F_{c,n-1}(A_1, \ldots, A_p) \cong W^F_{c,n-1}(A_1, \ldots, A_p, I_n) \cong \mathcal{U}_F(c, n-1).
\]
Hence, by Lemma 2.6 c), \( W^F_{c,n-1}(A_1, \ldots, A_p) \) has a convex boundary.

Let \( \mathbf{b} \in \mathbb{R}^n \) and \( 1 \leq k \leq \ell \leq n \); we will denote \( (b_k, \ldots, b_\ell) \) by \( \mathbf{b}[k, \ell] \). Similarly, for \( A = (a_{i,j}) \in \mathcal{M}_n(F) \), \( A[k, \ell] \) will denote the principal submatrix \( (a_{i,j})_{k \leq i, j \leq \ell} \) of \( A \).

Lemma 2.10. Let \( K \) be a subspace of \( \mathcal{H}_n(F) \) of dimension \( f_F(n-1) - 1 \). Then there exist a nonzero \( A \in K \) and \( U \in \mathcal{U}_n(F) \) such that \( (UAU^*)_{[1, n-1]} = \alpha I_{n-1} \) for some \( \alpha \in \mathbb{R} \). Moreover, \( U \) can be chosen in the path connected component of \( I_n \) in \( \mathcal{U}_n(F) \).

Proof. If \( I_n \in K \), then we are done. Suppose \( I_n \notin K \). Let \( K_1 \) be the subspace spanned by \( I_n \) and \( K \). Let \( \mathcal{L} = K_1^\perp \) be the orthogonal complement of \( K_1 \) in \( \mathcal{H}_n(F) \), with respect to the inner product defined in Lemma 2.5. Let \( m = \dim \mathcal{L} = f_F(n) - f_F(n-1) \). By a result of Adams, Lax and Phillips ([ALP, Theorem 1]), there exists a nonzero singular matrix \( B \) in \( \mathcal{L} \). Choose \( U \in \mathcal{U}_n(F) \) such that the last row and last column of \( UBU^* \) are zero. Let \( \mathcal{D} = \{D \in \mathcal{H}_n(F) : D_{[1, n-1]} = 0\} \).

Since \( UBU^* \in \mathcal{D} \cap (ULU^*) \) and \( \dim ULU^* = \dim \mathcal{L} = \dim \mathcal{D} \), we have
\[
\dim (\mathcal{D} \cap (ULU^*)^\perp) = \dim \mathcal{D} + \dim (ULU^*)^\perp - \dim (\mathcal{D} + (ULU^*)^\perp) = f_F(n) - \dim (\mathcal{D} + (ULU^*)^\perp) = \dim (\mathcal{D} + (ULU^*)^\perp)^\perp = \dim (\mathcal{D} \cap (ULU^*)) > 0.
\]

Let \( D \) be a nonzero matrix in \( \mathcal{D} \cap (ULU^*)^\perp = D \cap UK_1 U^* \). Then there exist \( A \in K \) and \( \alpha \in \mathbb{R} \) such that \( U(A - \alpha I_n)U^* = D \in \mathcal{D} \). Thus, \( (UAU^*)_{[1, n-1]} = \alpha I_{n-1} \). If \( A = 0 \), then \( \alpha = 0 \) because \( D_{[1, n-1]} = 0 \). This would imply \( D = 0 \), a contradiction. Hence, we have \( \alpha \neq 0 \).

By Lemma 2.4 b), \( U \) is in the connected component of \( I_n \) in \( \mathcal{U}_n(F) \) for \( F = \mathbb{C} \) and \( \mathbb{Q} \). For \( F = \mathbb{R} \), by multiplying \( U \) by \( I_{n-1} \oplus [-1] \), if necessary, we get \( \det U = 1 \) and the result follows.

Remark 2.11. Suppose \( A \in \mathcal{H}_n(F) \) has eigenvalues \( \lambda_1 \geq \cdots \geq \lambda_n \) and \( \alpha \in \mathbb{R} \). Then direct computation shows that \( \lambda_2 = \cdots = \lambda_{n-1} = \alpha \) if and only if there
exists $U \in \mathcal{U}_n(F)$ such that $UAU^*[1, n-1] = \alpha I_{n-1}$. Therefore Lemma 2.10 can be restated as follows.

**Proposition 2.12.** Let $K$ be a subspace of $\mathcal{H}_n(F)$ of dimension $f_{F}(n-1) - 1$. Then there exists a nonzero $A \in K$ such that $\lambda_2(A) = \cdots = \lambda_{n-1}(A)$.

It would be interesting to see if there is any direct relationship between Propositions 1.3 and 2.12.

The following lemma is an extension of an idea contained in a paper by Au-Yeung and Tsing [AT2].

**Lemma 2.13.** Let $r = n - 2 \geq 1$, $p = f_{F}(r+1) - 1$ and $A_1, \ldots, A_p \in \mathcal{H}_n(F)$. Suppose $c = c_i \in \mathbb{R}^{r-1}$, $b \in C(c, n-1)$ and $e \in \mathbb{R}$ such that either $e \leq c_{n-1}$ or $e \geq c_1$. Then there exist $d \in C(c, r)$ and $U \in \mathcal{U}_n(F)$ such that $Tr \left( [b] \oplus [e] \right) A_j = Tr \left( [d] \oplus [e] \right) UA_j U^*$ for all $1 \leq j \leq p$.

**Proof.** Let $K$ be the subspace spanned by $A_1, \ldots, A_p$. If $\dim K < p$, let $B_i = A_i[1, n-1]$ for $1 \leq i \leq p$. Then, by Lemma 2.9 a), $W_{p,F} (B_1, \ldots, B_p)$ is convex and the result follows.

Suppose $\dim K = p$. Then by Lemma 2.10, we can get $U_1$ in the path connected component of $I_n$ in $\mathcal{U}_n(F)$ such that $(U_1AU_1^*[1, n-1] = \alpha I_{n-1}$ for some nonzero $A \in K$ and $\alpha \in \mathbb{R}$.

Let $U(t), 0 \leq t \leq 1$, be a path in $\mathcal{U}_n(F)$ such that $U(0) = I_n$ and $U(1) = U_1$. For every $0 \leq t \leq 1$ and $1 \leq j \leq p$, let $(U(t)A_jU(t)^*)[1, n-1] = B_j(t)$ and $(U(t)A_jU(t)^*)[n, n] = a_j(t)$. Define

$$W(t) = W_{p,F}(B_1(t), \ldots, B_p(t)) + \{ \epsilon (a_1(t), \ldots, a_p(t)) \}.$$ 

Then $W(t)$ varies continuously with $t$. Let $K_1$ be the subspace of $\mathcal{H}_n(F)$ spanned by $B_1(1), \ldots, B_p(1)$. Then either $I_{n-1} \in K_1$ or $\dim K_1 < p$. We have

a) $q = (Tr \left( [b] \oplus [e] \right) A_j)_{1 \leq j \leq p} \in \text{conv } W(0)$.

b) For each $0 \leq t \leq 1$, $W(t)$ has a convex boundary and is contained in

$$\left\{ (Tr \left( [d] \oplus [e] \right) UA_j U^*)_{1 \leq j \leq p} : d \in C(c, r), U \in \mathcal{U}_n(F) \right\}.$$ 

c) $W(1)$ is convex.

If $q \in W(0)$, then the result follows from b). Suppose $W(0)$ is not convex and $q$ lies in the interior of conv $W(0)$.

Since $W(t)$ varies continuously with $t$ and each $W(t)$ has a convex boundary, we have either 1) $W(t)$ is convex for some $t > 0$ and $q \in W(t)$ or 2) there exists $t < 1$ such that $q$ lies on the boundary of conv $W(t)$. In both cases, the result follows from b).

**Proof of Theorem 2.2.** For $r = n - 1$, the result follows from Lemma 2.8. So, we may assume that $1 \leq r \leq n - 2$. Therefore, $2 \leq r + 1 \leq n - 1$ and $p \leq f_{F}(r+1) - 1$.

Suppose $w \in W_{p,F} (A_1, \ldots, A_p)$. Let $s = \min \{ i : w \in W_{p-i} (A_1, \ldots, A_p) \}$. It suffices to show that $s \leq r$. Let $w = (w_1, \ldots, w_p)$. There exist $b \in C(c, s)$ and $U \in \mathcal{U}_n(F)$ such that $w_j = Tr \left( [b] \oplus [e] \right) A_j$ for all $1 \leq j \leq p$. Moreover, there exist $D_i \in \Omega(n_i)$ with $n_i \leq s$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} n_i = n$ such that $b = c \oplus [e] D_i$. Let $k_0 = 0$, $k_i = \sum_{j=1}^{i} n_j$, $c_i = c[k_{i-1} + 1, k_i]$ and $b_i = b[k_{i-1} + 1, k_i]$ for $1 \leq i \leq m$. Let $A_j^i = (UA_j U^*)[k_{i-1} + 1, k_i]$ for $1 \leq j \leq p$. Then $w_j = \sum_{i=1}^{m} Tr \left( [b_i] A_j^i \right)$ for all $1 \leq j \leq p$. 


If \( n_i > r + 1 \) for some \( i \), then it follows from \( p \leq f_{\mathcal{F}}(r + 2) - 2 \) and Lemma 2.8 that there exist \( d^i \in \mathcal{C} (c^i, r + 1) \) and \( U_i \in \mathcal{U}_{n_i}(\mathcal{F}) \) such that \( Tr [b^j] A^j_j = Tr [d^i] U_i \mathcal{A}^j_j U_i^* \) for all \( 1 \leq j \leq p \). Hence we may assume that \( s \leq r + 1 \). Suppose \( n_i = r + 1 \). If \( p \leq f_{\mathcal{F}}(r + 1) - 2 \), then by Lemma 2.8, \( W_{c^i,n_i}^{\mathcal{F}} (\mathcal{A}^i_1, \cdots, \mathcal{A}^i_p) \) is convex. If \( p = f_{\mathcal{F}}(r + 1) - 1 \), then we can apply Lemma 2.13 to \( b^i, c^i \) and \( (U A U^*) [k_{i-1} + 1, k_i] \) if \( i < m \) (or \( (U A U^*) [k_{i-1}, k_i] \) if \( i = m \)). In all cases, we can get \( d^i \in \mathcal{C} (c^i, r) \), \( b = (b^1, \cdots, b^{i-1}, d^i, b^{i+1}, \cdots, b^m) \) and \( V \in \mathcal{U}_n(\mathcal{F}) \) such that \( w_j = Tr [d] V A_j V^* \) for all \( 1 \leq j \leq p \). Repeating this procedure, we get \( s \leq r \). 

\[ \square \]

3. Some related results

**Proposition 3.1.** Let \( \mathcal{F} = \mathcal{R}, \mathcal{C} \) or \( \mathcal{Q} \) and \( A_1, \cdots, A_p \in \mathcal{H}_n(\mathcal{F}) \). Then for every \( c = (c_1, \cdots, c_n) \in \mathbb{R}^n \) and \( 1 \leq r \leq n \), the following conditions are equivalent:

- a) \( W_{c,r}^{\mathcal{F}} (A_1, \cdots, A_p) \) is convex.
- b) If \( (r_1, \cdots, r_p) \in \mathbb{R}^p \setminus W_{c,r}^{\mathcal{F}} (A_1, \cdots, A_p) \), then there exist \( \alpha_1, \cdots, \alpha_p \in \mathbb{R} \) such that \( Tr X \left( \sum_{j=1}^{p} \alpha_j A_j \right) > \sum_{j=1}^{p} \alpha_j r_j \), for all \( X \in \mathcal{U}_n(c,n) \).
- c) If \( (r_1, \cdots, r_p) \in \mathbb{R}^p \setminus W_{c,r}^{\mathcal{F}} (A_1, \cdots, A_p) \), then there exist \( \alpha_1, \cdots, \alpha_p \in \mathbb{R} \) such that the eigenvalues of \( A = \sum_{j=1}^{p} \alpha_j A_j \) satisfies \( \sum_{i=1}^{n} c_{\sigma(i)} \lambda_i(A) > \sum_{j=1}^{p} \alpha_j r_j \), for all permutation \( \sigma \) of \( \{1, \cdots, n\} \).
- d) \( W_{b,r}^{\mathcal{F}} (A_1, \cdots, A_p) \subset W_{c,r}^{\mathcal{F}} (A_1, \cdots, A_p) \) for all \( b < c \).

Some special cases of Proposition 3.1 are given by Au-Yeung and Tsing in [AT1] (for \( p = 2 \) and \( r = 1 \)) and [AT2] (for \( \mathcal{F} = \mathcal{C} \) and \( r = 1 \)). The proof of the general case is similar to the one given in [AT1]. Also, see [A2] for some related results.

Suppose \( 1 \leq r \leq n - 1 \) and \( p < f_{\mathcal{F}}(r + 1) - \delta_{n,r+1} \). It follows from Theorem 2.2 that all the conditions in Proposition 3.1 hold. In particular, by putting \( (r_1, \cdots, r_p) = (0, \cdots, 0) \) in Proposition 3.1 c), we have the following generalization of Proposition 1.2.

**Theorem 3.2.** Let \( \mathcal{F} = \mathcal{R}, \mathcal{C} \) or \( \mathcal{Q} \), \( 1 \leq r \leq n - 1 \), \( p < f_{\mathcal{F}}(r + 1) - \delta_{n,r+1} \) and \( A_1, \cdots, A_p \in \mathcal{H}_n(\mathcal{F}) \). Suppose \( (0, \cdots, 0) \notin W_{c,r}^{\mathcal{F}} (A_1, \cdots, A_p) \). Then there exist \( \alpha_1, \cdots, \alpha_p \in \mathbb{R} \) such that the eigenvalues of \( A = \sum_{j=1}^{p} \alpha_j A_j \) satisfy \( \sum_{i=1}^{n} c_{\sigma(i)} \lambda_i(A) > 0 \) for every permutation \( \sigma \) of \( \{1, \cdots, n\} \).

From Theorem 3.2, we have the following generalization of Proposition 1.3.

**Theorem 3.3.** Let \( \mathcal{F} = \mathcal{R}, \mathcal{C} \) or \( \mathcal{Q} \), \( 1 < r < n \), and \( k(r) = f_{\mathcal{F}}(n) - f_{\mathcal{F}}(n+1-r) \). Suppose \( \mathcal{K} \) be a subspace of \( \mathcal{H}_n(\mathcal{F}) \) with dimension \( k(r) \). Then for every \( c \in \mathbb{R}^n \), with \( c_i \neq c_j \) for some \( i, j \), there exists a nonzero matrix \( A \in \mathcal{K} \) such that \( A = \lambda I + tB \) for some \( \lambda, t \in \mathbb{R} \) and \( B \in \mathcal{U}_n(c, n-r) \). Then 1) \( I_n \notin \mathcal{K} \) and 2) \( \mathcal{K} \cap \mathcal{U}_n(c, n-r) = \emptyset \), where \( \mathcal{K} \) is the subspace spanned by \( \mathcal{K} \) and \( I_n \). Let \( \mathcal{L} = \mathcal{K}_1^+ \). It follows from 1) that
dim $L = p < f_F(n + 1 - r)$. Since $1 < r < n$, we have $0 < n - r < n - 1$. Let \( \{A_1, \cdots, A_p\} \) be a basis of \( L \). 2) implies that \((0, \cdots, 0) \notin W^F_{c_i, n-r} (A_1, \cdots, A_p) \).

Hence, by Theorem 3.2, there exists \( A \in L \) such that \( \sum_{i=1}^{n} \sigma(i) \lambda_i (A) > 0 \) for every permutation \( \sigma \) of \( \{1, \cdots, n\} \). This implies that \( Tr A = \sum_{i=1}^{n} \lambda_i (A) > 0 \) because \( \sum_{i=1}^{n} c_i > 0 \). But \( I_n \in K_1 = L^\perp \). So we have \( Tr A = Tr AI_n = 0 \), a contradiction. \( \square \)

As we have pointed out in Remark 1.4 a), the bound \[ f_F(r + 1) - \delta_{n, r+1} \] is best possible for Proposition 1.1. The following theorem is an extension of this result.

**Theorem 3.4.** Suppose \( c_1 \geq \cdots \geq c_n \). If \( c_1 > c_{r+1} \) or \( c_{n-r} > c_n \) and \( p \geq f_F(r+1) - \delta_{n, r+1} \), then there exist \( A_1, \cdots, A_p \in \mathcal{H}_n (F) \) such that \( W^F_{c_i, r} (A_1, \cdots, A_p) \) is not convex.

**Proof.** It suffices to prove the case when \( p = f_F(r + 1) - \delta_{n, r+1} \).

We may assume that \( c_1 > c_{r+1} \geq c_n > 0 \). Let \( d = (\sum_{i=1}^{r+1} c_i) (r + 1)^{-1} \) and \( s = \sum_{i=1}^{n} c_i \). Let \( K = \{ A \in \mathcal{H}_n (F) : A[1, r+1] = \lambda I_{r+1} \) and \( dtr A = \lambda s \} \). Then \( K \) has dimension \( f_F(n) - p \). Let \( A_1, \cdots, A_p \in \mathcal{H}_n (F) \) be a basis of \( K^\perp \). Since \( dI_{r+1} [c[r+2, n]] \in K \cap \mathcal{U}_F (c, r+1) \), we have \((0, \cdots, 0) \notin W^F_{c_i, r} (A_1, \cdots, A_p) \). We are going to show that \((0, \cdots, 0) \notin W^F_{c_i, r} (A_1, \cdots, A_p) \). So, \( W^F_{c_i, r}(A_1, \cdots, A_p) \) is not convex.

Suppose \((0, \cdots, 0) \in W^F_{c_i, r} (A_1, \cdots, A_p) \). Then there exists \( A = (a_{ij}) \in K \cap \mathcal{U}_F (c, r) \). We have \( a_{ij} = d \) for \( 1 \leq i \leq r + 1 \). By Lemma 2.6 b), we have

\[
\sum_{i=1}^{r+1} c_i = \sum_{i=1}^{r+1} a_{ii} \leq \sum_{i=1}^{r+1} \lambda_i (A) \leq \sum_{i=1}^{r+1} c_i.
\]

Hence, \( \sum_{i=1}^{r+1} a_{ii} = \sum_{i=1}^{r+1} \lambda_i (A) \). By a result of Li [Li, Lemma 4.1], \( a_{ij} = a_{ji} = 0 \) for all \( 1 \leq i \leq r+1 \) and \( r+2 \leq j \leq n \). This implies that \( \lambda_i (A) = d \) for \( 1 \leq i \leq r+1 \). Since \( A \in \mathcal{U}_F (c, r) \), we have \( \sum_{i=1}^{m} d = \sum_{i=1}^{m} c_i \) for some \( 1 \leq m \leq r \). This gives

\[
\sum_{i=1}^{m} \sum_{j=1}^{r+1} c_j = \sum_{j=1}^{r+1} \sum_{i=1}^{m} c_i \Rightarrow \sum_{i=1}^{m} \sum_{j=m+1}^{r+1} c_j = \sum_{j=m+1}^{r+1} \sum_{i=1}^{m} c_i \Rightarrow \sum_{i=1}^{m} \sum_{j=m+1}^{r+1} (c_i - c_j) = 0 \Rightarrow c_1 = c_{r+1},
\]
a contradiction. \( \square \)

**References**


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