HOUSTON JOURNAL OF MATHEMATICS
Volume 19, No. 2, 1993

$\mathbb{Z}$-ANALYTIC TAF ALGEBRAS AND DYNAMICAL SYSTEMS

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ABSTRACT. A classification of triangular AF algebras which are generated by integer-valued cocycles (called standard $\mathbb{Z}$-analytic TAF algebras) is given in terms of their associated ordered Bratteli diagrams. This classification is analogous to that of AF algebras by Bratteli diagrams. The ordered Bratteli diagram comes from a standard presentation which is obtained for these algebras. The dynamical system associated with an ordered Bratteli diagram is used to identify several different classes of $\mathbb{Z}$-analytic TAF algebras, and a necessary and sufficient condition is given for such an algebra to be semisimple.

In this paper we continue the study of $\mathbb{Z}$-analytic TAF algebras begun in [PPW2]. These are triangular subalgebras of AF algebras which are defined by an integer-valued cocycle on the underlying groupoid. We will give a classification of a certain class of $\mathbb{Z}$-analytic algebras which parallels that of AF algebras by Bratteli diagrams [B]. We first show in section 2 that every $\mathbb{Z}$-analytic TAF algebra in this class can be represented in a standard form. We then show in section 3 that these $\mathbb{Z}$-analytic TAF algebras can be classified by their associated *ordered Bratteli diagrams*, a concept which was introduced by Power in [Pr2] to study subalgebras of AF algebras, and used by Herman, Putnam, and Skau in [HPS] for the study of dynamical systems. In section 4, we will study the relationship between the dynamical system and the $\mathbb{Z}$-analytic TAF algebra associated with an ordered Bratteli diagram. This yields a division of $\mathbb{Z}$-analytic TAF algebras into several fundamental classes. Finally, in section 5 we use the previous

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1991 *Mathematics Subject Classification.* 46H20, 46L05.

work to obtain several results on the structure of standard $\mathbb{Z}$-analytic TAF algebras.

1. Preliminaries

An *AF algebra* is a C^*-algebra $\mathfrak{A}$ which has an increasing sequence of finite dimensional C^*-subalgebras $\{\mathfrak{A}_n : 1 \leq n < \infty\}$ such that $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{A}_n}$. In this paper, whenever we use the notation $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{A}_n}$, we will always assume that the sequence $\{\mathfrak{A}_n\}$ is increasing, $\mathfrak{A}$ is unital, and $\mathfrak{A}_1$ contains the unit 1 of $\mathfrak{A}$. In addition, $\mathfrak{A}$ is a *UHF algebra* if each $\mathfrak{A}_n$ can be chosen to be a factor.

An AF algebra $\mathfrak{A}$ can also be defined as an inductive limit $\varinjlim(\mathfrak{A}_n, j_n)$ of finite dimensional C^*-algebras $\mathfrak{A}_n$ with $j_n : \mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}$ a unital C^*-embedding [B]. Then $\mathfrak{A}_n$ is isomorphic to a C^*-subalgebra $\tilde{\mathfrak{A}}_n$ of $\mathfrak{A}$ such that $\mathfrak{A} = \overline{\bigcup_n \tilde{\mathfrak{A}}_n}$, so in this case we will identify $\mathfrak{A}_n$ and $\tilde{\mathfrak{A}}_n$.

Suppose $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{A}_n}$ is an AF algebra, where $\mathfrak{A}_n = \bigoplus_{k=1}^{r(n)} \mathbf{M}_{m(n,k)}$. By [SV], we can choose a system of matrix units $\{e_{ij}^{(nk)}\}$ for each $\mathbf{M}_{m(n,k)}$ so that (i) each $e_{ij}^{(nk)}$ is a sum of matrix units of $\mathfrak{A}_{n+1}$, and (ii) $\mathfrak{D}_n \subseteq \mathfrak{D}_{n+1}$, where $\mathfrak{D}_n$ is the linear span of $\{e_{ii}^{(nk)} : 1 \leq k \leq r(n), 1 \leq i \leq m(n,k)\}$. Then $\mathfrak{D}_n$ is a maximal abelian self-adjoint subalgebra (*masa*) of $\mathfrak{A}_n$, and, if $\mathfrak{D} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{D}_n}$, then $\mathfrak{D}$ is a masa (also called the *diagonal*) of $\mathfrak{A}$, and $\mathfrak{D}_n = \mathfrak{D} \cap \mathfrak{A}_n$. Whenever we use matrix units in $\mathfrak{A}$, we will always assume that they are chosen in this manner.

All subalgebras of AF algebras in this paper will be norm-closed. If $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ is an AF algebra with diagonal $\mathfrak{D}$, then a subalgebra $\mathcal{T}$ of $\mathfrak{A}$ is said to be *triangular AF (with diagonal* $\mathfrak{D}$), or TAF, if $\mathcal{T} \cap \mathcal{T}^* = \mathfrak{D}$. If in addition $\mathfrak{A}$ is UHF, then $\mathcal{T}$ is also said to be *triangular UHF*, or TUHF. A TAF subalgebra $\mathcal{T}$ of $\mathfrak{A}$ is said to be *maximal TAF* if $\mathcal{T}$ is the only TAF subalgebra containing $\mathcal{T}$. In addition, $\mathcal{T}$ is said to be *strongly maximal TAF* [PPW1, page 105] if the sequence $\{\mathfrak{A}_n\}$ can be chosen so that $\mathcal{T} \cap \mathfrak{A}_n$ is maximal triangular in $\mathfrak{A}_n$ with respect to $\mathfrak{D}_n$ for every n, and it is *strongly maximal triangular in factors* if in addition each $\mathfrak{A}_n$ can be chosen to be a factor. An algebra which is strongly maximal triangular in factors is clearly strongly maximal TUHF, but the converse is false [PPW2, Example 2.12].

Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ be an AF algebra with diagonal $\mathfrak{D}$, and let $X = \hat{\mathfrak{D}}$, the spectrum of $\mathfrak{D}$. Then by the spectral theorem of Muhly and Solel [MS, Theorem 3.10], elements of $\mathfrak{A}$ can be represented as continuous functions on an AF-groupoid $\mathcal{R} \subseteq X \times X$ [R]. We write $\mathfrak{A} = C^*(\mathcal{R})$. X is embedded in a natural way as the diagonal $\mathcal{R}^0$ of $\mathcal{R}$, and every closed $\mathfrak{D}$-bimodule $\mathcal{T}$ of $\mathfrak{A}$ can be represented uniquely as the set of functions in $C^*(\mathcal{R})$ which are supported on an open subset $\mathcal{P}$ of $\mathcal{R}$. In this case, we write $\mathcal{T} = \mathcal{A}(\mathcal{P})$.

We refer the reader to [MS] or [PPW2] for the details on the groupoid representation. In this paper, we only need the following facts and notation. For each projection p in $\mathfrak{D}$, $\hat{p} = \{x \in X : x(p) = 1\}$ is a closed and open (clopen) subset of X. Let v be a matrix unit in some $\mathfrak{A}_n$. Then v induces a partial homeomorphism h_v [Pr1] from $\widehat{vv^*}$ to $\widehat{v^*v}$ by $h_v(x) = x_v$, where $x_v(d) = x(vdv^*)$, and we define $\hat{v} = \{(x, x_v) \colon x(vv^*) = 1\} \subseteq X \times X$. Then the groupoid $\mathcal{R}$ associated with $\mathfrak{A}$ is defined by $\mathcal{R} = \bigcup\{\hat{v} \colon v \text{ is a matrix unit of some } \mathfrak{A}_n\}$, and $\mathcal{R}$ is given the smallest topology such that each $\hat{v}$ is clopen. Each $\hat{v}$ is compact in this topology, and $\{\hat{v}\}$ is a base for the topology.

2. Standard $\mathbb{Z}$-analytic TAF algebras

Definition 2.1. Let $\mathfrak{A} = C^*(\mathcal{R})$ for an AF-groupoid $\mathcal{R}$ on X. A real-valued continuous function d on $\mathcal{R}$ is said to be a *cocycle* if it satisfies the cocycle condition $d(x,z) = d(x,y) + d(y,z)$ for all $(x,y),(y,z) \in \mathcal{R}$. If $d(\mathcal{R}) \subseteq \mathbb{Z}$, the integers, then d is said to be an *integer-valued cocycle*. A subalgebra $\mathcal{T} = \mathcal{A}(\mathcal{P})$ of $\mathfrak{A}$ is said to be $\mathbb{Z}$*-analytic* if there exists an integer-valued cocycle d such that $\mathcal{P} = d^{-1}[0,\infty)$. In this case, we write $\mathcal{T}_d$ for $\mathcal{T}$. $\mathcal{T}_d$ is triangular if and only if $d^{-1}(\{0\}) = \mathcal{R}^0 (\cong X)$.

The following theorem characterizes $\mathbb{Z}$-analytic TAF algebras. First, however, we note that a $\mathbb{Z}$-analytic TAF subalgebra is always strongly maximal TAF by [V]. Now if $\mathcal{T}$ is any strongly maximal TAF subalgebra of an AF algebra $\mathfrak{A}$, then [PPW2, Lemma 1.1] implies that there exists a sequence $\{\mathfrak{A}_n\}$ of finite dimensional C^*-subalgebras of $\mathfrak{A}$ and a set of matrix units $\{e_{ij}^{(nk)}\}$ for $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ such that $\mathfrak{A}_n = \bigoplus_{k=1}^{r(n)} \mathbf{M}_{m(n,k)}$ and $\mathcal{T} \cap \mathfrak{A}_n = \bigoplus_{k=1}^{r(n)} \mathbf{T}_{m(n,k)}$ for each n, where $\mathbf{T}_{m(n,k)}$ is the set of upper triangular matrices in the matrix algebra $\mathbf{M}_{m(n,k)}$.

Theorem 2.2. [c.f. PPW2, Theorem 2.2] *Let $\mathfrak{A} = C^*(\mathcal{R})$ for an AF-groupoid $\mathcal{R}$ and let $\mathcal{T}$ be a strongly maximal TAF subalgebra of $\mathfrak{A}$. Suppose $\{\mathfrak{A}_n\}$ and $\{e_{ij}^{(nk)}\}$ are chosen as above and for each $(x,y) \in \mathcal{R}$*

$$\hat{d}(x,y) = \lim_{n \to \infty} \{j - i : (x,y) \in \hat{e}_{ij}^{(nk)}\} \tag{2.1}$$

exists (as a finite number). Then $\hat{d}$ satisfies the cocycle condition in Definition 2.1. Furthermore, if $\mathcal{T}$ is $\mathbb{Z}$-analytic, then $\hat{d}$ is finite on $\mathcal{R}$. Conversely, if $\hat{d}$ is finite and continuous on $\mathcal{R}$, then $\mathcal{T}$ is $\mathbb{Z}$-analytic and $\mathcal{T} = \mathcal{T}_{\hat{d}}$.

If $\mathcal{T}$ is a $\mathbb{Z}$-analytic TAF algebra such that the function $\hat{d}$ defined above is finite and continuous, then we say that $\mathcal{T}$ is a *standard* $\mathbb{Z}$-analytic TAF algebra. There exists [PPW2, Example 2.4] a $\mathbb{Z}$-analytic TAF algebra $\mathcal{T}$ which is not standard, i.e., the function $\hat{d}$ defined above is not continuous. For the rest of this paper, unless stated otherwise, we will assume that for a standard $\mathbb{Z}$-analytic algebra $\mathcal{T}_{\hat{d}}$, the integer-valued cocycle $\hat{d}$ is given in the above form. Finally, we note that $\hat{d}$ is determined by the clopen subset $\hat{d}^{-1}(1)$.

Definition 2.3. Let $\mathfrak{A} = \bigoplus_{k=1}^{r} \mathbf{M}_{m(k)}$, and define $\mathcal{W}_{\mathfrak{A}}^0 = \{e_{ij}^{(k)} : 1 \le k \le r, 1 \le i,j \le m(k)\}$ to be the set of canonical matrix units of $\mathfrak{A}$. Also, let $\mathcal{W}_{\mathfrak{A}}$ be the set of partial isometries in $\mathfrak{A}$ which can be represented as a finite sum of matrix units in $\mathcal{W}_{\mathfrak{A}}^0$. For $w \in \mathcal{W}_{\mathfrak{A}}$, define

$$d_{\mathfrak{A}}(w) = \max\{j_p - i_p : w = \sum_k \sum_p e_{i_p j_p}^{(k)}\}.$$

Since every $w \in \mathcal{W}_{\mathfrak{A}}$ can be expressed uniquely as $w = \sum_k \sum_p e_{i_p j_p}^{(k)}$, $d_{\mathfrak{A}}(w)$ is well-defined.

Lemma 2.4. *Suppose $\mathfrak{A} = \bigoplus_{k=1}^{r} \mathbf{M}_{m(k)}$, $\mathfrak{B} = \bigoplus_{\ell=1}^{s} \mathbf{M}_{n(\ell)}$, and $\mathcal{T}_{\mathfrak{A}}$ and $\mathcal{T}_{\mathfrak{B}}$ are the upper triangular matrices in $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Denote the elements of $\mathcal{W}_{\mathfrak{B}}^0$ by $f_{ij}^{(\ell)}$, and let $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ be a unital C^*-embedding such that $\varphi(\mathcal{W}_{\mathfrak{A}} \cap \mathcal{T}_{\mathfrak{A}}) \subseteq \mathcal{W}_{\mathfrak{B}} \cap \mathcal{T}_{\mathfrak{B}}$. Now if $e_{ij}^{(k)} \in \mathcal{T}_{\mathfrak{A}}$ with $j > i$, and $\varphi(e_{ij}^{(k)}) = \sum_\ell \sum_q f_{i_q j_q}^{(\ell)}$, then*

(a) *$j_q - i_q \ge j - i$ for all q, and*

(b) *$d_{\mathfrak{B}}(\varphi(e_{ij}^{(k)})) \ge d_{\mathfrak{A}}(e_{ij}^{(k)})$.*

Furthermore, if $w \in \mathcal{W}_{\mathfrak{A}} \cap \mathcal{T}_{\mathfrak{A}}$, then $d_{\mathfrak{B}}(\varphi(w)) \ge d_{\mathfrak{A}}(w)$.

Proof. First note that φ maps non-diagonal matrix units to sums of non-diagonal matrix units. Now $e_{ij}^{(k)} = e_{i,i+1}^{(k)} e_{i+1,i+2}^{(k)} \cdots e_{j-1,j}^{(k)}$, so

$$\varphi(e_{ij}^{(k)}) = \varphi(e_{i,i+1}^{(k)})\varphi(e_{i+1,i+2}^{(k)}) \cdots \varphi(e_{j-1,j}^{(k)}). \tag{2.2}$$

Then decompose each $\varphi(e^{(k)}_{i+t,i+t+1})$ as $\sum_\ell \sum_u f^{(\ell)}_{i_u j_u}$, substitute into (2.2), and multiply out. The nonzero summands of the result are the $f^{(\ell)}_{i_q j_q}$'s in the decomposition of $\varphi(e^{(k)}_{ij})$. Since each nonzero summand must be the product of $j-i$ non-diagonal matrix units in $\mathcal{T}_{\mathfrak{B}}$, (a) follows. (b) then follows immediately from (a), and the final statement follows from (b). $\square$

Lemma 2.5. *Suppose* $\mathfrak{A} = \overline{\bigcup_{n=1}^\infty \mathfrak{A}_n}$ *and* $\mathcal{T} = \overline{\bigcup_{n=1}^\infty \mathcal{T}_n}$, *where* $\mathfrak{A}_n = \bigoplus_{k=1}^{r(n)} \mathbf{M}_{m(n,k)}$, $\mathcal{T}_n = \bigoplus_{k=1}^{r(n)} \mathbf{T}_{m(n,k)}$, *and* $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$ *for all* $n \geq 1$. *Let* $\mathcal{W}_n = \mathcal{W}_{\mathfrak{A}_n}$, $\mathcal{W}^0_n = \mathcal{W}^0_{\mathfrak{A}_n}$, *and* $d_n(w) = d_{\mathfrak{A}_n}(w)$ *for* $w \in \mathcal{W}_n$. *Then the function* $\hat{d}$ *defined in (2.1) is finite if for each* $N \geq 1$ *and* $w \in \mathcal{W}^0_N$ *there exists some* $M > 0$ *such that*

$$|d_n(w)| \leq M \text{ for all } n \geq N. \tag{2.3}$$

Proof. Let $\mathfrak{D}_n = \mathcal{T}_n \cap \mathcal{T}_n^*$ and $\mathfrak{D} = \overline{\bigcup_{n=1}^\infty \mathfrak{D}_n}$. Let $X = \hat{\mathfrak{D}}$ and let $\mathcal{R}$ be the AF-groupoid on X associated with $\mathfrak{A}$, as defined in section 1. Suppose $(x,y) \in \mathcal{R}$. Then there exists some $N \geq 1$ such that for every $n \geq N$ we have $(x,y) \in \hat{w}_n$ for some $w_n \in \mathcal{W}^0_n$. It follows directly from the definitions that $\hat{d}(x,y) = \lim_{n\to\infty} d_n(w_n)$. Let M satisfy (2.3) for $w = w_N$. Now for each n, w_{n+1} is one of the summands in the decomposition of w_n as a sum of matrix units in $\mathfrak{A}_{n+1}$. Thus, if $w_N \in \mathcal{T}_N$, then $w_n \in \mathcal{T}_n$ for all $n \geq N$, and it follows by Lemma 2.4(a) that $d_n(w_n)$ is increasing in n and bounded above by M, so $\lim_{n\to\infty} d_n(w_n)$ is finite. On the other hand, if $w_N \in \mathcal{T}_N^*$, then $d_n(w_n)$ is decreasing in n and bounded below by $-M$, so again $\lim_{n\to\infty} d_n(w_n)$ is finite. $\square$

Definition 2.6. Let $\mathfrak{A} = \bigoplus_{k=1}^{r} \mathbf{M}_{m(k)}$ and $\mathfrak{B} = \bigoplus_{\ell=1}^{s} \mathbf{M}_{n(\ell)}$. A unital injective embedding $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ of the form

$$\varphi\left(\bigoplus_{k=1}^{r} a_k\right) = \bigoplus_{\ell=1}^{s}\left(\bigoplus_{j=1}^{q_\ell} a_{k(\ell,j)}\right),$$

where $1 \leq k(\ell,j) \leq r$ and $\sum_{j=1}^{q_\ell} m(k(\ell,j)) = n(\ell)$, is called a *standard embedding* from $\mathfrak{A}$ to $\mathfrak{B}$.

The following result follows easily from the definitions, and it implies that a composition of standard embeddings is also standard.

Lemma 2.7. *Let* $\mathfrak{A} = \bigoplus_{k=1}^{r} \mathbf{M}_{m(k)}$ *and* $\mathfrak{B} = \bigoplus_{\ell=1}^{s} \mathbf{M}_{n(\ell)}$. *A unital* C^*-*embedding* $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ *is a standard embedding if and only if* $\varphi(\mathcal{W}_{\mathfrak{A}}^0) \subseteq \mathcal{W}_{\mathfrak{B}}$ *and* $d_{\mathfrak{B}}(\varphi(w)) = d_{\mathfrak{A}}(w)$ *for every* $w \in \mathcal{W}_{\mathfrak{A}}^0$.

Proposition 2.8. *Suppose* $\mathfrak{A}_n = \bigoplus_{k=1}^{r(n)} \mathbf{M}_{m(n,k)}$ *and* $\varphi_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ *is a standard embedding for* $n = 1, 2, \ldots$. *Let* $\mathcal{T}_n$ *be the set of upper triangular matrices in* $\mathfrak{A}_n$. *Then* $\mathcal{T} = \varinjlim(\mathcal{T}_n, \varphi_n)$ *is a standard* $\mathbb{Z}$-*analytic subalgebra of* $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \varphi_n)$.

Proof. The result follows from Lemma 2.7, Lemma 2.5, and Theorem 2.2. Note that Lemma 2.7 implies that $\hat{d}$ is continuous. □

The next theorem, the main result in this section, is the converse of Proposition 2.8.

Theorem 2.9. *Every standard* $\mathbb{Z}$-*analytic TAF algebra can be represented in the form given in Proposition 2.8, i.e., as an inductive limit generated by standard embeddings of upper triangular matrix algebras.*

Proof. We first assume that $\mathfrak{A}$, $\mathcal{T}$, $\mathfrak{D}$, and d_n are as given in Lemma 2.5. Since $O = \hat{d}^{-1}(\{1\})$ is open in $\mathcal{R}$, there exists an increasing sequence $\{O_n\}$ of compact open subsets of O such that $O = \bigcup_{n=1}^{\infty} O_n$. For each n, there exists some $N(n)$ such that $O_n = \hat{w}_n$ for some $w_n \in \mathcal{W}_{N(n)}$. Without loss of generality, we may assume that $N(n) = n$ for each n. Let $\mathfrak{B}_n$ be the C^*-algebra generated by w_n and $\mathfrak{D}_n$. Note that $\mathfrak{B}_n$ is a subalgebra of $\mathfrak{A}_n$, and $\{\mathfrak{B}_n\}$ is an increasing sequence since $\hat{w}_n \subseteq \hat{w}_{n+1}$ for all n.

$\mathfrak{A} = C^*(\mathcal{R})$ and $\mathcal{T} = \mathcal{A}(\mathcal{P})$, where $\mathcal{R} = \bigcup_{m=-\infty}^{\infty} \hat{d}^{-1}(\{m\})$ and $\mathcal{P} = \bigcup_{m=0}^{\infty} \hat{d}^{-1}(\{m\})$. Now suppose $\hat{e}_{ij}^{(pk)} \subseteq \hat{d}^{-1}(\{2\})$. By compactness, $e_{ij}^{(pk)}$ can be written as $\sum_{k'} \sum_q e_{i_q j_q}^{(p'k')}$ with $j_q - i_q = 2$ for all q. Then for each q, $e_{i_q j_q}^{(p'k')} = e_{i_q, i_q+1}^{(p'k')} e_{i_q+1, j_q}^{(p'k')}$ and $\hat{e}_{i_q, i_q+1}^{(p'k')}, \hat{e}_{i_q+1, j_q}^{(p'k')} \subseteq \hat{d}^{-1}(\{1\})$. Thus, $e_{ij}^{(pk)} \in \overline{\bigcup_{n=1}^{\infty} \mathcal{T} \cap \mathfrak{B}_n}$. Similar arguments work for any matrix unit in $\hat{d}^{-1}(\{m\})$, so it follows that $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{B}_n}$ and $\mathcal{T} = \overline{\bigcup_{n=1}^{\infty} \mathcal{T} \cap \mathfrak{B}_n}$.

Since $\mathfrak{B}_n \subseteq \mathfrak{A}_n$, $\mathcal{T} \cap \mathfrak{B}_n$ is the set of upper triangular matrices in $\mathfrak{B}_n$. Now every nondiagonal matrix unit w in $\mathcal{T} \cap \mathfrak{B}_n$ is a product of matrix units v_q with $\hat{v}_q \subseteq \hat{w}_n$. It follows that $d_m(w) = d_n(w)$ for every $m \geq n$. A similar argument gives the same result for $w \in \mathcal{T}^* \cap \mathfrak{B}_n$. Hence, by Lemma 2.7, the embedding of $\mathfrak{B}_n$ into $\mathfrak{B}_{n+1}$ is a standard embedding. $\square$

3. Ordered Bratteli diagrams

Definition 3.1. Let V and W be two non-empty finite sets. An *ordered diagram* from V to W consists of a partially ordered set E and surjective maps $r\colon E \to W$ and $s\colon E \to V$ such that e and e' are comparable iff $r(e) = r(e')$. Sometimes we just write E for (E, r, s). The elements of V and W are the *vertices* and the elements of E are the *edges* of the diagram.

An ordered diagram (E, r, s) from $V = \{1, \ldots, k\}$ to $W = \{1, \ldots, \ell\}$ determines a standard embedding $\varphi\colon \bigoplus_{i=1}^{k} \mathbf{M}_{m(i)} \to \bigoplus_{j=1}^{\ell} \mathbf{M}_{n(j)}$ by the formula

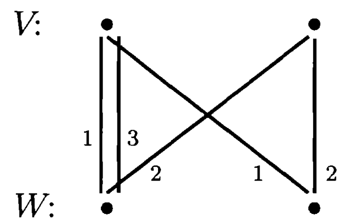
$$\varphi\left(\bigoplus_{i=1}^{k} a_i\right) = \bigoplus_{j=1}^{\ell}\left(\bigoplus_{p=1}^{q_j} a_{s(e(j,p))}\right),$$

where $\{e(j,p)\colon p = 1, \ldots, q_j\}$ is the ordered set of edges e which satisfy $r(e) = j$, and $n(j) = \sum_{p=1}^{q_j} m\left(s(e(j,p))\right)$. Conversely, every standard embedding can be represented by an ordered diagram.

Example 3.2. Let $\varphi\colon \mathbf{M}_3 \oplus \mathbf{M}_2 \to \mathbf{M}_8 \oplus \mathbf{M}_5$ be given by

$$\varphi(a_1 \oplus a_2) = \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & a_1 \end{bmatrix} \oplus \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}.$$

Then the corresponding ordered diagram is



Definition 3.3. [Pr2, HPS] An *ordered Bratteli diagram* $(\mathcal{V}, \mathcal{E})$ consists of a vertex set

$$\mathcal{V} = V_0 \cup V_1 \dots \text{ (disjoint union of finite sets),}$$

where V_0 is a singleton, and

$$\mathcal{E} = \{(E_n, r_n, s_n) : n \geq 1\},$$

where (E_n, r_n, s_n) is an ordered diagram from V_{n-1} to V_n. If $e_i \in E_i$ with $r_{i-1}(e_{i-1}) = s_i(e_i)$ for all i, $m < i \leq n$, then $(e_{m+1}, e_{m+2}, \dots, e_n)$ is a *path* from V_m to V_n. We also use the term *path* to denote an infinite sequence $(e_1, e_2, \dots)$ of edges with $e_i \in E_i$ and $r_{i-1}(e_{i-1}) = s_i(e_i)$ for all i.

Given an ordered Bratteli diagram $(\mathcal{V}, \mathcal{E})$, let $\mathfrak{A}_0 = \mathbb{C}$, and then use (E_n, r_n, s_n) to define $\mathfrak{A}_n$ and standard embeddings $\varphi_n \colon \mathfrak{A}_{n-1} \to \mathfrak{A}_n$ for $n \geq 1$. We will denote $\varinjlim(\mathfrak{A}_n, \varphi_n)$ by $\mathrm{AF}(\mathcal{V}, \mathcal{E})$. Let $\mathcal{T}_n$ and $\mathfrak{D}_n$ be the sets of upper triangular and diagonal matrices, respectively, in $\mathfrak{A}_n$. Then $\mathcal{T} = \overline{\bigcup_n \mathcal{T}_n}$ is a standard $\mathbb{Z}$-analytic TAF algebra of $\mathfrak{A}$ with diagonal $\mathfrak{D} = \overline{\bigcup_n \mathfrak{D}_n}$ by Proposition 2.8. We will write $\mathcal{T} = \mathrm{TAF}(\mathcal{V}, \mathcal{E})$. It follows from Theorem 2.9 that every standard $\mathbb{Z}$-analytic algebra $\mathcal{T}$ can be represented as $\mathrm{TAF}(\mathcal{V}, \mathcal{E})$ for some $(\mathcal{V}, \mathcal{E})$.

Definition 3.4. Two ordered diagrams (E, r, s) and (E', r', s') from V to W are said to be *(order) equivalent* if there exists a (order-preserving) bijection $\Phi \colon E \to E'$ such that

$$r(e) = r'(\Phi(e)) \quad \text{and} \quad s(e) = s'(\Phi(e)) .$$

We will denote equivalence of diagrams by $(E, r, s) \cong (E', r', s')$ and order equivalence by $(E, r, s) \cong^{\mathrm{ord}} (E', r', s')$. Note that, given $\mathfrak{A} = \bigoplus_{i=1}^k \mathbf{M}_{m(i)}$ and $\mathfrak{B} = \bigoplus_{j=1}^\ell \mathbf{M}_{n(j)}$, order equivalent diagrams induce identical corresponding standard embeddings from $\mathfrak{A}$ to $\mathfrak{B}$.

Definition 3.5. Given $V_0, \dots, V_n$ and ordered diagrams (E_i, r_i, s_i) from V_{i-1} to V_i $(i = 1, \dots, n)$, define an ordered diagram (E, r, s) from V_0 to V_n by

$$E = \{(e_1, \dots, e_n) \colon e_i \in E_i \text{ and } r_{i-1}(e_{i-1}) = s_i(e_i), 1 < i \leq n\},$$
$$r(e_1, \dots, e_n) = r_n(e_n), \qquad \text{and} \qquad s(e_1, \dots, e_n) = s_1(e_1).$$

The order on E is defined by $(e_1, \dots, e_n) > (f_1, \dots, f_n)$ iff there is some i, $1 \leq i \leq n$, such that $e_i > f_i$ ¿and $e_j = f_j$ for $i < j \leq n$ (if $i < n$). E is called a *contraction* [HPS] of $E_1, \dots, E_n$, and we write $E = E_n \circ \cdots \circ E_1$. Note that a contraction satisfies the following property: If $\varphi_1, \dots, \varphi_n$ are standard embeddings with corresponding ordered diagrams $E_1, \dots, E_n$, and F is the ordered diagram corresponding to $\varphi = \varphi_n \circ \cdots \circ \varphi_1$, then $F \cong^{\mathrm{ord}} E_n \circ \cdots \circ E_1$.

Definition 3.6. Now we define two (ordered) Bratteli diagrams $(\mathcal{V}, \mathcal{E})$ and $(\mathcal{W}, \mathcal{F})$ to be *(order) equivalent* if there exist strictly increasing functions $f, g \colon \mathbb{N} \to \mathbb{N}$ with $f(0) = 0 = g(0)$ and ordered diagrams E'_n from V_n to $W_{f(n)}$ and F'_n from W_n to $V_{g(n)}$ such that

$$F'_{f(n)} \circ E'_n \cong E_{g(f(n))} \circ \cdots \circ E_{n+1} \quad (\cong^{\mathrm{ord}})$$

and

$$E'_{g(n)} \circ F'_n \cong F_{f(g(n))} \circ \cdots \circ F_{n+1} \quad (\cong^{\mathrm{ord}})$$

for all n. In this case, we write $(\mathcal{V}, \mathcal{E}) \cong (\mathcal{W}, \mathcal{F})$ $((\mathcal{V}, \mathcal{E}) \cong^{\mathrm{ord}} (\mathcal{W}, \mathcal{F}))$.

Bratteli [B] classified the AF-algebras by showing that $\mathrm{AF}(\mathcal{V}, \mathcal{E})$ is C^*-isomorphic to $\mathrm{AF}(\mathcal{W}, \mathcal{F})$ iff $(\mathcal{V}, \mathcal{E}) \cong (\mathcal{W}, \mathcal{F})$. Similarly, we have

Theorem 3.7. *Suppose $(\mathcal{V}, \mathcal{E})$ and $(\mathcal{W}, \mathcal{F})$ are two ordered Bratteli diagrams. Then $\mathrm{TAF}(\mathcal{V}, \mathcal{E})$ is isometrically isomorphic to $\mathrm{TAF}(\mathcal{W}, \mathcal{F})$ if and only if $(\mathcal{V}, \mathcal{E}) \cong^{\mathrm{ord}} (\mathcal{W}, \mathcal{F})$.*

Remark 3.8. [HPS] has a more restrictive definition of equivalence and order equivalence of diagrams which requires a bijection between the vertices (after contraction). Equivalent diagrams under their definition yield isomorphic algebras, but not the converse.

To prove the theorem, we first need the following lemma.

Lemma 3.9. *Suppose $\mathfrak{A} = \bigoplus_{k=1}^{r} \mathrm{M}_{m(k)}$, $\mathfrak{B} = \bigoplus_{\ell=1}^{s} \mathrm{M}_{n(\ell)}$, and $\mathfrak{C} = \bigoplus_{q=1}^{t} \mathrm{M}_{p(q)}$. Let $\mathcal{T}_{\mathfrak{A}}$, $\mathcal{T}_{\mathfrak{B}}$, and $\mathcal{T}_{\mathfrak{C}}$ be the upper triangular matrices of $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, respectively. Suppose $\varphi \colon \mathfrak{A} \to \mathfrak{B}$ and $\psi \colon \mathfrak{B} \to \mathfrak{C}$ are unital C^*-embeddings of finite dimensional C^*-algebras such that*

(a) $\varphi(\mathcal{W}_{\mathfrak{A}} \cap \mathcal{T}_{\mathfrak{A}}) \subseteq \mathcal{W}_{\mathfrak{B}} \cap \mathcal{T}_{\mathfrak{B}}$,
(b) $\psi(\mathcal{W}_{\mathfrak{B}} \cap \mathcal{T}_{\mathfrak{B}}) \subseteq \mathcal{W}_{\mathfrak{C}} \cap \mathcal{T}_{\mathfrak{C}}$, *and*
(c) $\psi \circ \varphi$ *is a standard embedding from $\mathfrak{A}$ to $\mathfrak{B}$.*

Then φ is also a standard embedding.

Proof. Let $w \in \mathcal{W}^0_{\mathfrak{A}} \cap \mathcal{T}_{\mathfrak{A}}$. Then Lemma 2.7 and Lemma 2.4 imply that

$$\begin{aligned} d_{\mathfrak{A}}(w) &= d_{\mathfrak{C}}((\psi \circ \varphi)(w)) && \text{since } \psi \circ \varphi \text{ is a standard embedding} \\ &\geq d_{\mathfrak{B}}(\varphi(w)) && \text{by (b)} \\ &\geq d_{\mathfrak{A}}(w) && \text{by (a).} \end{aligned}$$

Therefore, $d_{\mathfrak{A}}(w) = d_{\mathfrak{B}}(\varphi(w))$. Hence, φ is a standard embedding by Lemma 2.7. □

Proof of Theorem 3.7. First, suppose that $\mathcal{S} = \mathrm{TAF}(\mathcal{V}, \mathcal{E})$ is isometrically isomorphic to $\mathcal{T} = \mathrm{TAF}(\mathcal{W}, \mathcal{F})$. Let $\mathfrak{A} = \mathrm{AF}(\mathcal{V}, \mathcal{E})$ and $\mathfrak{B} = \mathrm{AF}(\mathcal{W}, \mathcal{F})$. Then by [PW, Corollary 1.14], there is an isomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ such that for each n, $\varphi(\mathfrak{A}_n) \subseteq \mathfrak{B}_{f(n)}$ and $\varphi(\mathcal{S} \cap \mathfrak{A}_n) \subseteq \mathcal{T} \cap \mathfrak{B}_{f(n)}$ for some $f(n)$. Since φ is unital, we can define $f(0) = 0$, and since the sequence $\{\mathfrak{B}_k\}$ is increasing, we can define $f(n)$ inductively such that f is increasing. Similarly, if $\psi = \varphi^{-1}$, then there is an increasing function $g\colon \mathbb{N} \to \mathbb{N}$ with $g(0) = 0$ such that $\psi(\mathfrak{B}_n) \subseteq \mathfrak{A}_{g(n)}$ and $\psi(\mathcal{T} \cap \mathfrak{B}_n) \subseteq \mathcal{S} \cap \mathfrak{A}_{g(n)}$ for all n. Define $\varphi_n = \varphi|_{\mathfrak{A}_n}$ and $\psi_n = \psi|_{\mathfrak{B}_n}$. Denote the (standard) embeddings of $\mathfrak{A}_{n-1}$ into $\mathfrak{A}_n$ and $\mathfrak{B}_{n-1}$ into $\mathfrak{B}_n$ by i_n and j_n, respectively, and let E_n and F_n be the corresponding ordered diagrams.

Now $(\psi \circ \varphi)(\mathfrak{A}_n) = (\psi_{f(n)} \circ \varphi_n)(\mathfrak{A}_n) = (i_{g(f(n))} \circ \cdots \circ i_{n+1})(\mathfrak{A}_n)$. Since the embedding on the right is standard, Lemma 3.9 implies that φ_n is standard. A similar argument shows that each ψ_n is standard. Let E'_n be the ordered diagram from V_n to $W_{f(n)}$ corresponding to φ_n, and let F'_n be the ordered diagram from W_n to $V_{g(n)}$ corresponding to ψ_n. Then $F'_{f(n)} \circ E'_n$ is order equivalent to the ordered diagram corresponding to $\psi_{f(n)} \circ \varphi_n$, and $E_{g(f(n))} \circ \cdots \circ E_{n+1}$ is order equivalent to the ordered diagram corresponding to $i_{g(f(n))} \circ \cdots \circ i_{n+1}$, so $F'_{f(n)} \circ E'_n \cong^{\mathrm{ord}} E_{g(f(n))} \circ \cdots \circ E_{n+1}$. Similarly, $E'_{g(n)} \circ F'_n \cong^{\mathrm{ord}} F_{f(g(n))} \circ \cdots \circ F_{n+1}$. Thus, $(\mathcal{V}, \mathcal{E})$ and $(\mathcal{W}, \mathcal{F})$ are order equivalent.

Conversely, suppose $(\mathcal{V}, \mathcal{E}) \cong^{\mathrm{ord}} (\mathcal{W}, \mathcal{F})$. Let $f, g\colon \mathbb{N} \to \mathbb{N}$, E'_n, and F'_n be the functions and diagrams defined above which implement this order equivalence. Define $\varphi_n\colon \mathfrak{A}_n \to \mathfrak{B}_{f(n)}$ and $\psi_n\colon \mathfrak{B}_n \to \mathfrak{A}_{g(n)}$ to be the standard embeddings corresponding to E'_n and F'_n, respectively. Then the ordered diagram corresponding to $\psi_{f(n)} \circ \varphi_n$ is order equivalent to the one corresponding to $i_{g(f(n))} \circ \cdots \circ i_{n+1}$, so $(\psi_{f(n)} \circ \varphi_n)(\mathfrak{A}_n) = (i_{g(f(n))} \circ \cdots \circ i_{n+1})(\mathfrak{A}_n)$ for all n. Similarly, $(\varphi_{g(n)} \circ \psi_n)(\mathfrak{B}_n) = (j_{f(g(n))} \circ \cdots \circ j_{n+1})(\mathfrak{B}_n)$ for all n. Since φ_n and ψ_n map upper triangular matrices to upper triangular matrices, it follows by [PW, Corollary 1.14] that TAF$(\mathcal{V}, \mathcal{E})$ is isometrically isomorphic to TAF$(\mathcal{W}, \mathcal{F})$. $\square$

Several examples using the ordered Bratteli diagram will be included in the next section. We will also use the diagram to identify a number of different classes of standard $\mathbb{Z}$-analytic TAF algebras.

4. Dynamical systems associated with ordered Bratteli diagrams

Let $\mathfrak{A} = C^*(\mathcal{R})$ be an AF algebra ($\mathcal{R} \subseteq X \times X$) and $\mathcal{T}$ a standard $\mathbb{Z}$-analytic TAF subalgebra defined by an integer-valued cocycle d. Define a partial homeomorphism ϕ on X by $\phi(x) = y$, where y is the unique element which satisfies $d(x, y) = \min\{d(x, z) \colon d(x, z) > 0\}$. Then ϕ is defined on the open subset $\{x \in X \colon d(x, z) > 0 \text{ for some } z\}$. ϕ can be described in terms of the ordered Bratteli diagram as follows. First, if $\mathfrak{A} = \mathrm{AF}(\mathcal{V}, \mathcal{E})$, then we can represent X as $X = \{(e_1, e_2, \dots) \colon e_i \in E_i,\ r_{i-1}(e_{i-1}) = s_i(e_i)\}$ [HPS]. Thus, each point of X corresponds to a path in the diagram E of $\mathfrak{A}$. Now define $X_{\max}$ to be the set of maximal paths, i.e., $X_{\max} = \{(e_i) \in X \colon e_i \text{ is maximal in } E_i \text{ for all } i\}$. Similarly, define $X_{\min} = \{(e_i) \in X \colon e_i \text{ is minimal in } E_i \text{ for all } i\}$. Since X is compact, it follows that these sets are always nonempty. Now for every $(e_i) \in X \setminus X_{\max}$, let $k = \min\{i \colon e_i \text{ is not maximal in } E_i\}$ and let f_k be the successor of e_k in E_k. For $1 \leq i < k$, define f_i so that $(f_1, \dots, f_{k-1})$ is the unique minimal path from V_0 to V_{k-1} such that $r_{k-1}(f_{k-1}) = s_k(f_k)$ (i.e., each f_i is minimal in E_i). Finally, let $f_n = e_n$ for $n > k$. Then $\phi((e_i)) = (f_i)$.

Herman, Putnam, and Skau [HPS] studied the ordered Bratteli diagrams $(\mathcal{V}, \mathcal{E})$ for which $X_{\max}$ and $X_{\min}$ are single points. They call these diagrams *essentially simple*, and in this case ϕ can be extended to a homeomorphism (also denoted by ϕ) of X which is *essentially minimal.* This means that X has a unique minimal closed ϕ-invariant set. If this set is X itself, then ϕ is said to be *minimal* (equivalently, the ϕ-orbit of each $x \in X$ is dense in X). For essentially minimal dynamical systems (X, ϕ), we can use the results of Putnam [Pu] and its generalization in [Po] to study AF$(\mathcal{V}, \mathcal{E})$ and TAF$(\mathcal{V}, \mathcal{E})$. In particular, we will examine the question of when two such algebras are isomorphic.

Let X be a compact zero-dimensional space and ϕ an essentially minimal homeomorphism of X. The crossed product $\mathbb{Z} \times_\phi C(X)$ is the C^*-algebra generated by $C(X)$ and a unitary U such that $UfU^* = f \circ \phi$ for $f \in C(X)$. Let $\mathbf{x}$ be a point in the unique minimal closed ϕ-invariant set S. Then the C^*-subalgebra $\mathfrak{A}(\phi, \mathbf{x})$ of $\mathbb{Z} \times_\phi C(X)$ generated by $C(X)$ and $UC_0(X) = \{Uf : f \in C(X), f(\mathbf{x}) = 0\}$ is an AF algebra [Pu] with diagonal $\mathfrak{D} = C(X)$, and the subalgebra $\mathcal{T}(\phi, \mathbf{x})$ generated by $C(X)$ and $UC_0(X)$ is a standard $\mathbb{Z}$-analytic subalgebra of $\mathfrak{A}(\phi, \mathbf{x})$ [PPW2, Proposition 2.6]. The cocycle d which generates $\mathcal{T}(\phi, \mathbf{x})$ is defined on the set $\mathcal{P} = \{(\mathbf{z}, \phi^n(\mathbf{z})) : \phi^i(\mathbf{z}) \neq \mathbf{x} \text{ for } 1 \leq i \leq n, n \geq 0\} \subseteq X \times X$, and is given by the formula $d(\mathbf{z}, \phi^n(\mathbf{z})) = n$. It was shown in [HPS] that $\mathfrak{A}(\phi, \mathbf{x}) \cong \mathfrak{A}(\phi, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in S$ (the case when ϕ is minimal was proved earlier in [Pu]). However, the next theorem shows that the same result is rarely true for the corresponding standard $\mathbb{Z}$-analytic subalgebras.

Theorem 4.1. *Let X be a compact zero-dimensional space and ϕ an essentially minimal homeomorphism of X. Then $\mathcal{T}(\phi, \mathbf{x}) \cong \mathcal{T}(\phi, \mathbf{y})$ if and only if there is a homeomorphism ψ of X such that $\psi \circ \phi = \phi \circ \psi$ and $\psi(\mathbf{x}) = \mathbf{y}$.*

Proof. Let d_x and d_y be the cocycles which generate $\mathcal{T}(\phi, \mathbf{x})$ and $\mathcal{T}(\phi, \mathbf{y})$, as given above. Suppose Ψ is an isometric isomorphism of $\mathcal{T}(\phi, \mathbf{x})$ onto $\mathcal{T}(\phi, \mathbf{y})$. Then Ψ restricts to an automorphism of the diagonal $\mathfrak{D}$, which in turn induces a homeomorphism ψ of X. In fact, thinking of Ψ as a map on the groupoid, we have $\Psi(\mathbf{u}, \mathbf{v}) = (\psi(\mathbf{u}), \psi(\mathbf{v}))$. Thus, ψ preserves the orderings on X defined by d_x and d_y. In particular, since $\mathbf{x}$ and $\mathbf{y}$ are the unique minimal points in X in the orderings defined by d_x and d_y, respectively, it follows that $\psi(\mathbf{x}) = \mathbf{y}$. In addition, Ψ must map $d_x^{-1}(\{1\})$ onto $d_y^{-1}(\{1\})$. Therefore, $\Psi(\mathbf{u}, \phi(\mathbf{u})) = (\psi(\mathbf{u}), \psi(\phi(\mathbf{u}))) = (\psi(\mathbf{u}), \phi(\psi(\mathbf{u})))$, so $\psi(\phi(\mathbf{u})) = \phi(\psi(\mathbf{u}))$ for all $\mathbf{u}$ satisfying $\phi(\mathbf{u}) \neq \mathbf{x}$. Since this set is dense, it follows that $\psi \circ \phi = \phi \circ \psi$.

For the converse, simply define a map $\Psi(\mathbf{u}, \phi(\mathbf{u})) = (\psi(\mathbf{u}), \psi(\phi(\mathbf{u})))$. Since ψ and ϕ commute, Ψ maps $d_x^{-1}(\{1\})$ onto $d_y^{-1}(\{1\})$, and therefore extends to an isomorphism of $\mathcal{T}(\phi, \mathbf{x})$ onto $\mathcal{T}(\phi, \mathbf{y})$. □

As an application, we will show that for certain dynamical systems, those defined by an "odometer", $\mathcal{T}(\phi, \mathbf{x})$ is always isomorphic to $\mathcal{T}(\phi, \mathbf{y})$. An "odometer" dynamical system is defined as follows: Given a sequence

$\{p_k\}$ of positive integers greater than one, let $X = \prod_{k=1}^{\infty} \{0, \ldots, p_k - 1\}$. Let $[i_1, \ldots, i_j]$ be the cylinder set $\{\mathbf{x} = \{x_k\} \in X : x_1 = i_1, x_2 = i_2, \ldots, x_j = i_j\}$, and endow X with the smallest topology for which the cylinder sets are open. For $\mathbf{x} = \{x_k\}$, $\mathbf{y} = \{y_k\} \in X$, define $\mathbf{x} + \mathbf{y} = \{z_k\}$, where

$$z_1 = x_1 + y_1 \pmod{p_1} \qquad \text{and} \qquad c_2 = \begin{cases} 0 & \text{if } x_1 + y_1 < p_1 \\ 1 & \text{if } x_1 + y_1 \geq p_1 \,, \end{cases}$$

and for $n > 1$,

$$z_n = x_n + y_n + c_n \pmod{p_n} \qquad \text{and} \qquad c_{n+1} = \begin{cases} 0 & \text{if } x_n + y_n + c_n < p_n \\ 1 & \text{if } x_n + y_n + c_n \geq p_n \,. \end{cases}$$

Let $\mathbf{1} = \{1, 0, 0, \ldots\}$. Define $\phi : X \to X$ by $\phi(\mathbf{x}) = \mathbf{x} + \mathbf{1}$. Then ϕ is a minimal homeomorphism, and given any $\mathbf{x} \in X$ we can form $\mathcal{T}(\phi, \mathbf{x})$ as usual.

Proposition 4.2. *If (X, ϕ) is given by an odometer, then $\mathcal{T}(\phi, \mathbf{x}) \cong \mathcal{T}(\phi, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$.*

Proof. Without loss of generality, we may assume $\mathbf{x} = (0, 0, 0, \ldots)$. Define $\psi \colon X \to X$ by $\psi(\mathbf{z}) = \mathbf{z} + \mathbf{y}$. Then ψ is a homeomorphism which commutes with ϕ and takes $\mathbf{x}$ to $\mathbf{y}$. Therefore, $\mathcal{T}(\phi, \mathbf{x}) \cong \mathcal{T}(\phi, \mathbf{y})$ by Theorem 4.1. $\square$

Remark 4.3. An odometer can be characterized by its ordered Bratteli diagram. It has a *single vertex* diagram, i.e., a diagram in which each vertex V_i consists of a singleton. Conversely, any standard $\mathbb{Z}$-analytic TAF algebra with a single vertex diagram is an odometer.

Consider the following classes of standard $\mathbb{Z}$-analytic TAF algebras:

$\mathcal{O} = \{\mathcal{T}(\phi, \mathbf{x}) \colon (X, \phi) \text{ is an odometer}\}$

$\mathcal{F} = \{$standard $\mathbb{Z}$-analytic TAF algebras which are strongly maximal in factors$\}$

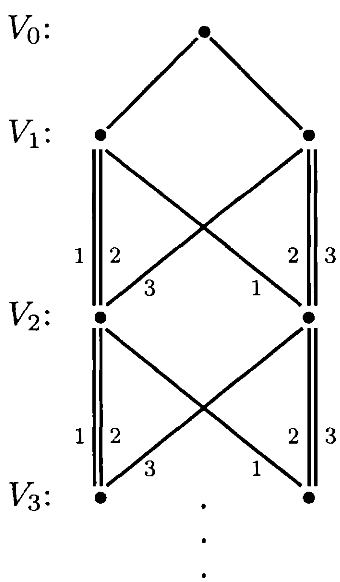
$\mathcal{U} = \{$standard $\mathbb{Z}$-analytic TUHF algebras$\}$

$\mathcal{S} = \{$standard $\mathbb{Z}$-analytic TAF algebras which have an essentially simple ordered Bratteli diagram$\}$

$\mathcal{E} = \{$standard $\mathbb{Z}$-analytic TAF algebras for which ϕ extends to a homeomorphism of X$\}$

$\mathcal{O} \subseteq \mathcal{F}$ by Remark 4.3, $\mathcal{F} \subseteq \mathcal{S}$ by [PPW2, Lemma 1.6], and, as noted earlier in this section, $\mathcal{S} \subseteq \mathcal{E}$. Also, clearly $\mathcal{F} \subseteq \mathcal{U}$. One could furthur divide $\mathcal{E}$ into those algebras for which the extension is minimal, essentially minimal, or neither. For example, it was shown in [PPW2, Corollary 2.11] that for algebras in $\mathcal{F}$, ϕ extends to a minimal homeomorphism. The following examples show that each of these inclusions is strict and that neither $\mathcal{U}$ nor $\mathcal{S}$ contains the other. Finally, Example 4.6 below also shows that not every standard $\mathbb{Z}$-analytic TAF algebra belongs to $\mathcal{E}$.

Example 4.4. The standard $\mathbb{Z}$-analytic TAF algebra corresponding to the ordered Bratteli diagram $(\mathcal{V}, \mathcal{E})$ below is essentially simple. However, it is easy to see from the (unordered) Bratteli diagram that the dimension group [E] of $\mathrm{AF}(\mathcal{V}, \mathcal{E})$ has rank 2. Therefore, $\mathrm{AF}(\mathcal{V}, \mathcal{E})$ is not UHF, so this TAF algebra is not in $\mathcal{U}$ (and therefore not in $\mathcal{F}$).



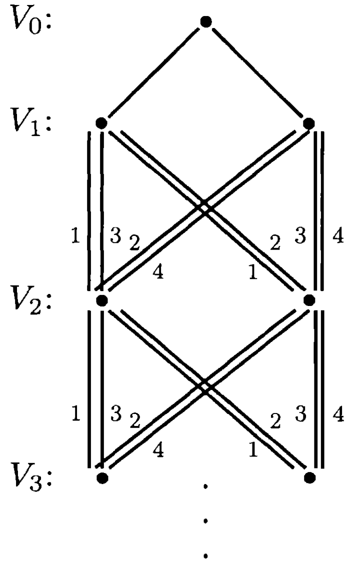
Example 4.5. This example is similar to the one following Corollary 6.3 in [HPS]. Let $\mathfrak{A}_n = \mathbf{M}_{2\cdot 4^n}$ and let $\mathcal{T}_n$ be the set of upper triangular matrices

in $\mathfrak{A}_n$. Define embeddings $j_n\colon \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ on block matrices by

$$j_n\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cccc|cccc} A & B & & & & & & \\ C & D & & & & & & \\ & & A & B & & & & \\ & & C & D & & & & \\ \hline & & & & A & & B & \\ & & & & & A & & B \\ & & & & C & & D & \\ & & & & & C & & D \end{array}\right]$$

Note that $j_n(\mathcal{T}_n) \subseteq \mathcal{T}_{n+1}$, so $\mathcal{T} = \varinjlim(\mathcal{T}_n, j_n)$ is a TAF algebra which is strongly maximal in factors. By Theorem 2.2, $\mathcal{T}$ is standard $\mathbb{Z}$-analytic, so $\mathcal{T} \in \mathcal{F}$. We will show, however, that $\mathcal{T} \notin \mathcal{O}$.

First, it is easy to show that $\mathcal{T}$ is isomorphic to the standard $\mathbb{Z}$-analytic TAF algebra given by the following ordered Bratteli diagram:



Thus, it suffices to show that this diagram is not order equivalent to a single vertex diagram.

Label the vertices of the vertex sets V_i, $0 \leq i$, in the above diagram by L_i and R_i (left and right, respectively). For simplicity of notation, let

$L = L_1$ and $R = R_1$. Also, let $N(A, B)$ denote the number of edges from a vertex A to a vertex B. Now suppose the ordered Bratteli diagram above is order equivalent to a single vertex ordered Bratteli diagram. Then there is a contraction $E = E_n \circ \ldots E_2$ from V_1 to some V_n, an ordered diagram F from V_1 to a single vertex W, and an ordered diagram G from W to V_n such that $G \circ F \cong^{\mathrm{ord}} E$. Now $N(L, L_n) = N(L, R_n) = N(R, L_n) = N(R, R_n) = 4^{n-1}/2$. Since $N(L, W) \cdot N(W, L_n) = N(L, L_n)$ and $N(R, W) \cdot N(W, L_n) = N(R, L_n)$, it follows that $N(L, W) = N(R, W)$. Similarly, $N(W, L_n) = N(W, R_n)$.

Now consider the sequence $S_L = \{s(e) \colon e \in G \circ F \text{ and } r(e) = L_n\}$, listed according to the order of the edges in the contraction $G \circ F$. S_L consists of 4^{n-1} elements labeled either L or R, in a certain pattern. But now note that $S_R = \{s(e) \colon e \in G \circ F \text{ and } r(e) = R_n\}$ is actually equal to S_L, since the pattern of L's and R's is determined by the order in F and the numbers $N(W, L_n) = N(W, R_n)$, which are equal. It follows that $T_L(n) = \{s(e) \colon e \in E \text{ and } r(e) = L_n\}$ and $T_R(n) = \{s(e) \colon e \in E \text{ and } r(e) = R_n\}$ must also be equal. We will obtain a contradiction by showing that this cannot be true for any n.

We will use induction on n. First, $T_L(2) = \{L, R, L, R\}$, but $T_R(2) = \{L, L, R, R\}$. Thus, $T_L(2) \neq T_R(2)$. Now suppose $T_L(n-1) \neq T_R(n-1)$. To obtain $T_L(n)$, simply replace each L in $T_L(2)$ with the entire sequence $T_L(n-1)$, and replace each R in $T_L(2)$ with the entire sequence $T_R(n-1)$. $T_R(n)$ is obtained similarly by replacing L and R in $T_R(2)$ with $T_L(n-1)$ and $T_R(n-1)$, respectively. Thus, for example,

$$T_L(3) = \{L, R, L, R, L, L, R, R, L, R, L, R, L, L, R, R\}$$

and

$$T_R(3) = \{L, R, L, R, L, R, L, R, L, L, R, R, L, L, R, R\}.$$

It follows that $T_L(n) \neq T_R(n)$.

Example 4.6. [PPW2, Example 2.12] is an example of a standard $\mathbb{Z}$-analytic TAF algebra $\mathcal{T}$ in $\mathcal{U}$ for which ϕ does not extend to a homeomorphism. Thus, $\mathcal{T} \notin \mathcal{E}$, and in particular, $\mathcal{T} \notin \mathcal{S}$. In fact, it was shown that X_{max} and X_{min} each consists of two points.

It is easy to obtain a standard $\mathbb{Z}$-analytic TAF algebra which is not essentially simple, but for which ϕ does extend to a homeomorphism. For example, just take the direct sum of two standard $\mathbb{Z}$-analytic TAF algebras which are given by odometers. Thus, $\mathcal{S}$ is strictly contained in $\mathcal{E}$.

Question. Another subclass of $\mathcal{S}$ is the set $\mathcal{C} = \{\mathcal{T}(\phi, \mathbf{x})\colon (X, \phi)$ satisfies the property given in Theorem 4.1$\}$. As indicated earlier, $\mathcal{O} \subseteq \mathcal{C}$, but in general the property in Theorem 4.1 is quite strong (indeed, it is not hard to show that $\mathcal{C} \subsetneq \mathcal{S}$). It would be interesting to know whether or not every algebra in $\mathcal{C}$ comes from an odometer.

5. Semisimplicity and structured presentations

Definition 5.1. Recall that if $\mathfrak{A} = \mathrm{AF}(\mathcal{V}, \mathcal{E})$, then each point of X, the spectrum of $\mathfrak{D}$, corresponds to a path in the diagram of $\mathfrak{A}$. We say that $(e_1, e_2, \dots)$ is a *single path* if there is some N such that for each $n \geq m \geq N$, there is no path $(f_1, f_2, \dots)$ with $s_m(e_m) = s_m(f_m)$, $r_n(e_n) = r_n(f_n)$, and $e_i \neq f_i$ for some i, $m \leq i \leq n$.

Theorem 5.2. *Let $\mathcal{T}$ be a standard $\mathbb{Z}$-analytic TAF subalgebra of an AF algebra $\mathfrak{A}$. Then $\mathcal{T}$ semisimple if and only if the ordered Brattelli diagram of $\mathcal{T}$ contains no single paths. In particular, $\mathcal{T}$ is semisimple if $\mathfrak{A}$ is simple.*

Proof. Note that the property of having single paths is preserved under order equivalence, so we can use any ordered Bratteli diagram which generates $\mathcal{T}$. We will need the following terminology from [D]: if e is a matrix unit in $\mathcal{T}$, then another matrix unit f is a *link* for e if ff^* is a subprojection of e^*e and f^*f is a subprojection of ee^*. Now if the ordered Bratteli diagram of $\mathcal{T}$ contains no single paths, then for each matrix unit $e = e_{ij}^{(nk)} \in \mathcal{T}$, there is some n', k', such that the decomposition of e into a sum of matrix units in $\mathcal{T}_{n'}$ is of the form $e = e_{I_1 J_1}^{(n'k')} + e_{I_2 J_2}^{(n'k')} +$ other terms (i.e., the multiplicity of the embedding of $\mathbf{M}^{(nk)}$ into $\mathbf{M}^{(n'k')}$ is at least two). It follows from [D, Lemma 5], and the fact that the embeddings are standard, that e has a link. [D, Theorem 4] then implies that $\mathcal{T}$ is semisimple.

On the other hand, if there is a single path, then there is a matrix unit $e \in \mathcal{T}_m$ which does not decompose as a sum of more than one matrix unit in each $\mathcal{T}_n$, $n > m$. In other words, ee^* and e^*e are minimal projections in $\mathfrak{D}$. Thus, there is no link for e, so [D, Theorem 4] implies that $\mathcal{T}$ is not semisimple.

Finally, if $\mathfrak{A}$ is simple, then [B, Corollary 3.5] implies that there are no single paths in the ordered Bratteli diagram of $\mathcal{T}$. $\square$

Larson recently introduced the notion of a structured presentation for a Banach algebra [L]. Algebras of this type have many useful properties.

Definition 5.3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be finite dimensional Banach algebras, and suppose $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ is an injective homomorphism of norm one. Then φ is said to be *structured* if there exists a homomorphism $\alpha\colon \mathfrak{B} \to \mathfrak{A}$ of norm one such that $\alpha \circ \varphi = 1_{\mathfrak{A}}$.

Definition 5.4. Let $\mathfrak{A}$ be a Banach algebra. $\mathfrak{A}$ is said to have a *structured presentation* if there exists a sequence of finite dimensional Banach algebras $\{\mathfrak{A}_n\}$ and a sequence of structured embeddings $\varphi_n\colon \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ such that $\mathfrak{A}$ is isometrically isomorphic to the direct limit $\varinjlim(\mathfrak{A}_n, \varphi_n)$.

Suppose $\mathfrak{A}$ is a Banach algebra with a structured presentation. Then Larson [L] showed that for each $n \geq 1$, there is a conditional expectation $\psi_n\colon \mathfrak{A} \to \mathfrak{A}_n$ which is also an algebra homomorphism. Thus, $\mathfrak{A}$ has many finite dimensional representations. In addition, $\lim\limits_{n\to\infty} \psi_n(a) = a$ for each $a \in \mathfrak{A}$, and $\psi_n \circ \psi_m = \psi_n$ for $n < m$.

Using the results of §2, we will show that every standard $\mathbb{Z}$-analytic TAF algebra has a structured presentation.

Theorem 5.5. *Let $\mathcal{T}$ be a standard $\mathbb{Z}$-analytic TAF algebra. Then $\mathcal{T}$ has a structured presentation consisting of standard embeddings of direct sums of upper triangular matrix algebras.*

Proof. By Theorem 2.9, $\mathcal{T}$ can be represented as $\varinjlim(\mathcal{T}_n, \varphi_n)$, where $\mathcal{T}_n$ and φ_n are chosen as in Proposition 2.8. Since φ_n is injective, for each $1 \leq k \leq r(n)$ there exists some k', $1 \leq k' \leq r(n+1)$, such that $\varphi(\mathcal{T}_{m(n,k)}) \cap \mathcal{T}_{m(n+1,k')} \neq \{0\}$. It follows that we can define a unital homomorphism $\tilde{\psi}_k\colon \mathcal{T}_{m(n+1,k')} \to \mathcal{T}_{m(n,k)}$ such that $(\tilde{\psi}_k \circ \varphi_n)(t) = t$ for all $t \in \mathcal{T}_{m(n,k)}$ ($\tilde{\psi}_k$ is essentially just a compression of $\mathcal{T}_{m(n+1,k')}$ to an appropriate direct summand of $\mathcal{T}_n$). Now combine the $\tilde{\psi}_k$'s to form $\psi_n\colon \mathcal{T}_{n+1} \to \mathcal{T}_n$, which then satisfies $\psi_n \circ \varphi_n = 1_{\mathcal{T}_n}$. $\square$

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