

## A COMPLETE ISOMORPHISM INVARIANT FOR A CLASS OF TRIANGULAR UHF ALGEBRAS

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### 1. INTRODUCTION

Let  $\mathfrak{A}$  be a unital AF algebra and  $\{\mathfrak{A}_n\}_{n=0}^\infty$  an increasing sequence of unital finite dimensional subalgebras of  $\mathfrak{A}$  such that  $\bigcup_n \mathfrak{A}_n$  is dense in  $\mathfrak{A}$ . Suppose for each  $n$  we have a maximal abelian self-adjoint subalgebras (masa)  $\mathcal{D}_n$  of  $\mathfrak{A}_n$  such that  $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$ . Then  $\mathcal{D} = \overline{\bigcup_n \mathcal{D}_n}$  is a masa in  $\mathfrak{A}$ . A norm closed subalgebra  $\mathcal{T}$  of  $\mathfrak{A}$  is said to be triangular with diagonal  $\mathcal{D}$  if  $\mathcal{T} \cap \mathcal{T}^* = \mathcal{D}$ . This class of triangular subalgebras of AF algebra (TAF algebras [6]) has been the subject of many recent studies, e.g. [1], [4–15]. A major question in this area is to classify such algebras up to isometric isomorphism. Complete isomorphism invariants for TAF algebras have been given by Power [11] and Ventura [14] and are also described implicitly in the diagonal extension theorem in Peters and Wagner ([7], 1.10) and the subgroupoid of Muhly and Solel [4]. These invariants have been used quite successfully in the study of certain types of TAF algebras, e.g. Power [10], [11], Thelwall [12], [13]. However, for most TAF algebras, it is quite difficult to determine if two TAF algebras have the same invariant. Thus, the problem of finding more explicit and computable isomorphism invariants still remains open.

Suppose  $\mathcal{T}$  is a TAF algebra with diagonal  $\mathcal{D}$ . Let  $\mathcal{P}(\mathcal{D})$  be the projections in  $\mathcal{D}$  and  $\mathcal{W}_{\mathcal{T}}$  the set of partial isometries  $v$  in  $\mathcal{T}$  such that  $v\mathcal{D}v^* \subseteq \mathcal{D}$  and  $v^*\mathcal{D}v \subseteq \mathcal{D}$ . Given  $e, f \in \mathcal{P}(\mathcal{D})$ , we write  $e \prec_{\mathcal{T}} f$  if there exists  $v \in \mathcal{W}_{\mathcal{T}}$  such that  $vv^* = e$  and  $v^*v = f$ . This diagonal ordering is defined by Peters, Poon and Wagner in [6]. Given two TAF algebras  $(\mathcal{T}, \mathcal{D})$  and  $(\mathcal{S}, \mathcal{E})$ , the orderings  $\prec_{\mathcal{T}}$  and  $\prec_{\mathcal{S}}$  are said to be isomorphic if there exists a  $C^*$ -isomorphism  $\theta : \mathcal{D} \rightarrow \mathcal{E}$  such that for each  $e, f \in \mathcal{P}(\mathcal{D})$ , we have  $e \prec_{\mathcal{T}} f$  if and only if  $\theta(e) \prec_{\mathcal{S}} \theta(f)$ . It is shown in [6] that if  $\theta : \mathcal{T} \rightarrow \mathcal{S}$  is an

isometric isomorphism, then the restriction of  $\Theta$  to  $\mathcal{D}$  gives an isomorphism of  $\prec_T$  and  $\prec_S$ . Although this ordering is not a complete invariant for isomorphism classes of TAF algebras, it is more computable and has been proven quite useful in the study of TAF algebras in [6], [7]. We are going to prove that the ordering  $\prec_T$  is a complete invariant for isometric isomorphism within a class of TAF algebras.

Let  $M_k$  denote the  $k \times k$  complex matrices and  $T_k$  (respectively,  $D_k$ ) the upper triangular (respectively, diagonal) matrices in  $M_k$ .  $I_k$  will denote the identity matrix in  $M_k$ . Suppose we are given a sequence of positive integers  $\{k_i\}_{i=0}^\infty$  with  $k_0 = 1$  and for each  $i \geq 1$  a unital embedding  $\varphi_i : M_{k_{i-1}} \rightarrow M_{k_i}$ . Then the  $C^*$ -limit [2],  $\mathfrak{A} = \varinjlim (M_{k_i}, \varphi_i)$  is called a UHF algebra [3]. In [3], Glimm showed that the generalized natural number  $\prod_{i=1}^\infty n_i$  ([3], [2]), where  $n_i = \frac{k_i}{k_{i-1}}$  for  $i \geq 1$ , is a complete isomorphism invariant for  $\mathfrak{A}$  in the class of UHF algebras. Suppose, in addition, that each  $\varphi_i$  satisfies  $\varphi_i(T_{k_{i-1}}) \subseteq T_{k_i}$  and, consequently  $\varphi_i(D_{k_{i-1}}) \subseteq D_{k_i}$ . Then the algebras  $\mathcal{T} = \varinjlim (T_{k_i}, \varphi_i)$ ,  $\mathcal{D} = \varinjlim (D_{k_i}, \varphi_i)$  can be regarded as subalgebras of  $\mathfrak{A}$  and  $\mathcal{T}$  is triangular in  $\mathfrak{A}$  with diagonal  $\mathcal{D}$ . The class of  $\mathcal{T}$  with  $\varphi_i(x) = x \otimes I_{n_i}$  for every  $i$  has been studied by Power in [9]. In [1], Baker studied the class where  $\varphi_i(x) = I_{n_i} \otimes x$  and showed that the generalized natural number  $\prod_{i=1}^\infty n_i$  is also a complete invariant for isometric isomorphism, within this class of algebras. In this paper, we study the class where each  $\varphi_i(x) = I_{n_i} \otimes x$  or  $x \otimes I_{n_i}$ . We will give a complete invariant for isometric isomorphisms within this class of algebras.

Recall that (Glimm [3] or Effros [2], p. 28) a generalized natural number  $\mathbf{n} = \prod_{i=1}^\infty n_i$  is a mapping  $f : \mathbb{P} \rightarrow \{0, 1, \dots, \infty\}$ , where  $\mathbb{P} = \{p_i : p_i \text{ is the } i^{\text{th}} \text{ prime number}\}$ , such that

$$f(p) = \sup\{j \geq 0 : p^j \text{ divides } \prod_{i=1}^m n_i \text{ for some } m\}.$$

If  $f(p_i) = r_i$ , we can write  $\mathbf{n} = 2^{r_1} 3^{r_2} 5^{r_3} \dots$ . An (ordinary) natural number  $n = p_1^{r_1} \dots p_m^{r_m}$  can be regarded as a generalized natural number with  $r_i = 0$  for  $i > m$ . Two generalized natural numbers can be multiplied as follows

$$(2^{r_1} 3^{r_2} \dots)(2^{s_1} 3^{s_2} \dots) = 2^{r_1+s_1} 3^{r_2+s_2} \dots, \text{ where we let } \infty + t = t + \infty = \infty.$$

To simplify notations for later discussion, let  $\sigma_n(x) = I_n \otimes x$  and  $\nu_n(x) = x \otimes I_n$ .  $\sigma_n$  and  $\nu_n$  are called [6] the standard embedding and nest embedding respectively. Let  $\Phi = \{\varphi : \varphi = \sigma_n \text{ or } \nu_n \text{ for some } n \geq 1\}$ . Suppose  $\varphi = \{\varphi_i\}_{i=1}^\infty$  is a sequence in  $\Phi$ . Define

$$\sigma(\varphi) = \prod \{n_i : \varphi_i = \sigma_{n_i}\},$$

$$\nu(\varphi) = \prod \{n_i : \varphi_i = \nu_{n_i}\} \text{ and}$$

$$\tau(\varphi) = \sigma(\varphi)\nu(\varphi).$$

For  $\varphi_1, \dots, \varphi_n \in \Phi$  and  $\pi$  a permutation on  $\{1, \dots, n\}$ , we note that  $\varphi_1 \circ \dots \circ \varphi_n = \varphi_{\pi(1)} \circ \dots \circ \varphi_{\pi(n)} = \nu_r \circ \sigma_s$  where  $r = \prod \{n_i : \varphi_i = \nu_{n_i}\}$  and  $s = \prod \{n_i : \varphi_i = \sigma_{n_i}\}$ .

The main result in this paper is the following

**THEOREM 1.1.** Let  $\{k_i\}_{i=0}^\infty, \{l_i\}_{i=0}^\infty$  be two sequences of positive integers with  $k_0 = l_0 = 1$  and  $n_i = \frac{k_i}{k_{i-1}}, m_i = \frac{l_i}{l_{i-1}}$  for  $i \geq 1$ . Suppose  $\varphi = \{\varphi_i\}_{i=1}^\infty, \psi = \{\psi_i\}_{i=1}^\infty$  are sequences in  $\Phi$  such that  $\varphi_i = \sigma_{n_i}$  or  $\nu_{n_i}$  and  $\psi_i = \sigma_{m_i}$  or  $\nu_{m_i}$  for each  $i$ . For each  $i \geq 0$ , let  $\mathcal{T}_i = T_{k_i}, \mathcal{D}_i = D_{k_i}, \mathcal{S}_i = T_{l_i}$  and  $\mathcal{E}_i = \mathcal{D}_{l_i}$ . Since  $\varphi_i(\mathcal{T}_{i-1}) \subseteq \mathcal{T}_i, \varphi_i(\mathcal{D}_{i-1}) \subseteq \mathcal{D}_i, \psi_i(\mathcal{S}_{i-1}) \subseteq \mathcal{S}_i$  and  $\psi_i(\mathcal{E}_{i-1}) \subseteq \mathcal{E}_i$ , we can form the direct limits  $\mathcal{T} = \varinjlim(\mathcal{T}_i, \varphi_i), \mathcal{D} = \varinjlim(\mathcal{D}_i, \varphi_i), \mathcal{S} = \varinjlim(\mathcal{S}_i, \psi_i)$  and  $\mathcal{E} = \varinjlim(\mathcal{E}_i, \psi_i)$ . Then  $\mathcal{D}, \mathcal{E}$  are the diagonals of  $\mathcal{T}, \mathcal{S}$  respectively. Let  $\prec_{\mathcal{T}}$  (and  $\prec_{\mathcal{S}}$ ) be the diagonal orderings on  $\mathcal{D}$  (and  $\mathcal{E}$ ) defined by  $\mathcal{T}$  (and  $\mathcal{S}$ ). Then the following conditions are equivalent.

- 1)  $\mathcal{T}$  is isometrically isomorphic to  $\mathcal{S}$ .
- 2) The orderings  $\prec_{\mathcal{T}}$  and  $\prec_{\mathcal{S}}$  are isomorphic.
- 3) There exist positive integers  $n$  and  $m$  such that  $n\sigma(\varphi) = m\sigma(\psi)$  and  $m\nu(\varphi) = n\nu(\psi)$ .

The proof of this theorem will be given in Section 2. We end this section with some simple corollaries.

Given a generalized natural number  $\mathbf{m}$ , let  $P(\mathbf{m}) = \{p \in \mathbb{P} : p|\mathbf{m}\}$  and  $P_0(\mathbf{m}) = \{p \in P(\mathbf{m}) : p^\infty \nmid \mathbf{m}\}$ . Suppose  $\mathcal{S}$  and  $\mathcal{T}$  are constructed from sequences  $\varphi = \{\varphi_i\}$  and  $\psi = \{\psi_i\}, \varphi_i, \psi_i \in \Phi$  such that  $\tau(\varphi) = \tau(\psi) = \mathbf{m}$ . Then condition 3 in Theorem 1.1 holds if and only if there exists a finite subset  $\mathbf{Q} \subseteq P_0(\mathbf{m})$  such that for every prime  $p \notin \mathbf{Q}$  and every positive integer  $d$ , we have  $p^d \mid \sigma(\varphi)$  if and only if  $p^d \mid \sigma(\psi)$ , and  $p^d \mid \nu(\varphi)$  if and only if  $p^d \mid \nu(\psi)$ . We note that since  $\tau(\varphi) = \mathbf{m}$ , therefore, for each prime number  $p$  such that  $p^\infty \mid \mathbf{m}$ , exactly one of the following three conditions holds:

- i)  $p^\infty \mid \sigma(\varphi)$  and  $p^\infty \mid \nu(\varphi)$
- ii)  $p^\infty \mid \sigma(\varphi)$  and  $p^\infty \nmid \nu(\varphi)$
- iii)  $p^\infty \nmid \sigma(\varphi)$  and  $p^\infty \mid \nu(\varphi)$

Thus, for a fixed generalized natural number  $\mathbf{m}$ , if we let  $\mathcal{T}(\mathbf{m})$  be the collection of isomorphism classes of  $\mathcal{T}$  constructed from a sequence  $\varphi$  in  $\Phi$  with  $\tau(\varphi) = \mathbf{m}$ , we have

**COROLLARY 1.2.** For each generalized natural number  $\mathbf{m}$ , we have

- 1) if  $P(\mathbf{m})$  is infinite, then  $\mathcal{T}(\mathbf{m})$  is uncountable, and

2) if  $P(\mathfrak{m})$  is finite, then  $T(\mathfrak{m})$  has exactly  $3^n$  elements, where  $n$  is the number of  $p \in P(\mathfrak{m})$  such that  $p^\infty \mid \mathfrak{m}$ .

In particular, if  $\mathfrak{m} = p^\infty$  for some prime number  $p$ , we have the following.

COROLLARY 1.3. Let  $p$  be a prime. Then  $T(p^\infty)$  is equal to the set of isomorphism classes of the algebras constructed from the following sequences

- 1)  $\{\varphi_i\}_{i=1}^\infty, \varphi_i = \nu_p$  for all  $i$ ,
- 2)  $\{\varphi_i\}_{i=1}^\infty, \varphi_i = \sigma_p$  for all  $i$ ,
- 3)  $\{\varphi_i\}_{i=1}^\infty, \varphi_i = \sigma_p$  when  $i$  is even,  $\varphi_i = \nu_p$  when  $i$  is odd.

For  $p = 2$ , the fact that all three algebras defined in Corollary 1.3 are non-isomorphic has been proven by Peters, Poon and Wagner in [6] (Examples 1.1 and 3.27).

## 2. THE PROOF

Let  $\{e_{ij}^n : i, j = 1, \dots, n\}$  be the usual matrix units for  $M_n$ . Thus,  $T_n$  and  $D_n$  are spanned by  $\{e_{ij}^n : 1 \leq i \leq j \leq n\}$  and  $\{e_{ii}^n : 1 \leq i \leq n\}$  respectively. We will write  $e_i^n$  for  $e_{ii}^n$ . Then, for non-zero projections  $e, f \in \mathcal{P}(D_n)$ , we have  $e \prec_{T_n} f$  if and only if  $e = \sum_{t=1}^k e_{i_t}^n, f = \sum_{t=1}^k e_{j_t}^n$  for some  $1 \leq i_1 < \dots < i_k \leq n, 1 \leq j_1 < \dots < j_k \leq n$  and  $i_t \leq j_t$  for every  $1 \leq t \leq k$ .

Let  $\{k_i\}, \{l_i\}, \{\varphi_i\}, \{\psi_i\}, \mathcal{D}, \mathcal{E}, \mathcal{S}$  and  $T$  be as given in Theorem 1.1. If  $e, f \in \mathcal{P}(\mathcal{D})$  such that  $e \prec_T f$ , then there exists some  $i$  such that  $e, f \in \mathcal{D}_i$ . By Corollary 3.7 in [6], every  $v$  in  $W_T$  is of the form  $v = dw$  for some unitary  $d$  in  $\mathcal{D}$  and  $w \in W_{T_j}$  for some  $j$ . Thus, we may assume  $e \prec_{T_i} f$ . Hence, the discussion in the previous paragraph gives an explicit description of  $\prec_T$ . Clearly, similar description holds for  $\prec_S$ .

Suppose  $\Theta : \mathcal{D} \rightarrow \mathcal{E}$  is an isomorphism between the orderings  $\prec_T$  and  $\prec_S$ . Then for each  $n$ , there exist  $i(n)$  and  $j(n)$  such that  $\Theta(\mathcal{D}_n) \subseteq \mathcal{E}_{j(n)}$  and  $\Theta^{-1}(\mathcal{E}_n) \subseteq \mathcal{D}_{i(n)}$ . Furthermore, we may assume that for each  $n$  and  $e, f$  in  $\mathcal{D}_n$  (respectively,  $\mathcal{E}_n$ )  $e \prec_{T_n} f \Rightarrow \Theta(e) \prec_{S_{j(n)}} \Theta(f)$  (respectively,  $e \prec_{S_n} f \Rightarrow \Theta^{-1}(e) \prec_{T_{i(n)}} \Theta^{-1}(f)$ ).

In the following three lemmas,  $\Theta$  ( and  $\Theta_i$  ) :  $D_n \rightarrow D_m$  will always denote a 1-1 unital  $C^*$ -homomorphism such that  $e \prec_{T_n} f \Rightarrow \Theta(e) \prec_{T_m} \Theta(f)$ . To simplify notations, we will write  $e^n[k, l] = \sum_{i=k}^l e_i^n$  for  $1 \leq k \leq l \leq n$ . If  $r, s$  are positive integers and  $\nu_r \circ \sigma_s : D_n \rightarrow D_m$ , then direct computation shows that for  $1 \leq i \leq n$ ,

$$\nu_r \circ \sigma_s(e_i^n) = \sum_{t=0}^{s-1} e^{m[1 + (i-1)r + nrt, ir + nrt]}.$$

LEMMA 2.1. Let  $\Theta : D_n \rightarrow D_m$ . For each  $1 \leq i \leq n, 1 \leq j \leq m$ , let  $\mu(i, j)$  be the number of  $k$  such that  $1 \leq k \leq j$  and  $\Theta(e_i^n) \geq e_k^m$  (the usual ordering of projections in  $D_m$ ). Then, we have

- 1)  $\mu(i, j) \geq \mu(i', j)$  and  $\mu(i, j) \leq \mu(i, j')$  for  $1 \leq i \leq i' \leq n$  and  $1 \leq j \leq j' \leq m$ .
- 2) For each  $1 \leq j \leq m, \sum_{i=1}^n \mu(i, j) = j$ .
- 3) Suppose  $\Theta(e^n[1, i]) \geq e^m[1, j]$  and  $\Theta(e^n[1, i])e_{j+1}^m = 0$ . Then  $\Theta(e_{i+1}^n) \geq e_{j+1}^m$ .

*Proof.*

1) For  $1 \leq i \leq n$  and  $1 \leq j \leq j' \leq m$ , it follows from definitions that  $\mu(i, j) \leq \mu(i, j')$ . Suppose  $1 \leq i \leq i' \leq n$ . Then  $e_i^n \prec_{T_n} e_{i'}^n \Rightarrow \Theta(e_i^n) \prec_{T_m} \Theta(e_{i'}^n)$ . So

$$\Theta(e_i^n) = \sum_{t=1}^k e_{j'_t}^m \text{ and } \Theta(e_{i'}^n) = \sum_{t=1}^k e_{j'_t}^m$$

for some  $1 \leq j_1 < \dots < j_k \leq m, 1 \leq j'_1 < \dots < j'_k \leq m$  and  $j_t \leq j'_t$  for every  $1 \leq t \leq k$ . Hence, for each  $1 \leq j \leq m$  we have  $\mu(i, j) \geq \mu(i', j)$

2) follows from  $\sum_{i=1}^n \Theta(e_i^n) \geq e^m[1, j]$ .

3) From the given conditions, we have  $\sum_{k=1}^i \mu(k, j) = j = \sum_{k=1}^i \mu(k, j+1)$  and  $\mu(i', j) = 0$  for  $1 \leq i < i' \leq n$ . From 1) and 2) we have  $\mu(i+1, j+1) \geq \mu(i', j+1)$  for all  $i' > i+1$  and  $\sum_{k=i+1}^n \mu(k, j+1) = 1$ . Therefore,  $\mu(i+1, j+1) = 1$ . So, it follows from  $\mu(i+1, j) = 0$  that  $\Theta(e_{i+1}^n) \geq e_{j+1}^m$ .

LEMMA 2.2. Let  $\Theta_1 : D_n \rightarrow D_m, \Theta_2 : D_m \rightarrow D_p$  and  $\Theta_2 \circ \Theta_1 = \nu_r \circ \sigma_s$  for some positive integers  $r, s$ . Then there exists a positive integer  $k$  such that  $k|r$  and for  $1 \leq i \leq n$

$$(*) \quad \Theta_1(e_i^n) \geq e^m[1 + (i-1)k, ik],$$

$$\Theta_2(e^m[1 + (i-1)k, ik]) \geq e^p[1 + (i-1)r, ir].$$

Furthermore, if  $nk < m$ , then  $\Theta_1(e_1^n) \geq e_{nk+1}^m$ .

*Proof.* We first note that for  $1 \leq i \leq n$ ,

$$\Theta_2(\Theta_1(e_i^n))e^p[1, nr] = \nu_r \circ \sigma_s(e_i^n)e^p[1, nr] = e^p[1 + (i-1)r, ir].$$

Hence, for every  $1 \leq j \leq m$ , if  $\Theta_2(e_j^m)e^p[1 + (i-1)r, ir] \neq 0$ , then  $\Theta_1(e_i^n) \geq e_j^m$ .

From 1) and 2) in Lemma 2.1 with  $j = 1$ , we have  $\Theta_1(e_1^n) \geq e_1^m$  and  $\Theta_2(e_1^m) \geq e_1^p$ . Let  $k_1$  be the greatest integer  $k$  such that  $\Theta_2(e_k^m)e^p[1, r] \neq 0$ . By putting  $j = 1, \dots, r$

successively in 3) of Lemma 2.1, we have  $\Theta_2(e^m[1, k_1]) \geq e^p[1, r]$  and  $\Theta_2(e_l^m)e^p[1, r] \neq 0$  for all  $1 \leq l \leq k_1$ . Hence, we have  $\Theta_1(e_1^n) \geq e^m[1, k_1]$ . Furthermore,  $\Theta_2 \circ \Theta_1(e_2^n) \geq e_{r+1}^m$ . Therefore,  $\Theta_2(e^m[1, k_1])e_{r+1}^p = 0$ . Hence, again by 3) of Lemma 2.1,  $\Theta_2(e_{k_1+1}^m) \geq e_{r+1}^p$ . This gives  $\Theta_1(e_2^n) \geq e_{k_1+1}^m$ . Applying Lemma 2.1 inductively on  $i$  and  $j$ , we can get integers  $0 = k_0 < k_1 < \dots < k_n$  such that

$$\Theta_2(e^m[1 + k_{i-1}, k_i]) \geq e^p[1 + (i-1)r, ir]$$

and

$$\Theta_1(e_i^n) \geq e^m[1 + k_{i-1}, k_i]$$

for  $1 \leq i \leq n$ .

Applying 1) in Lemma 2.1 for  $\Theta_1$  we have

$$k_i - k_{i-1} \geq k_{i'} - k_{i'-1} \text{ for } 1 \leq i \leq i' \leq n.$$

For each  $1 \leq j \leq k_n$ , let  $\mu_j = \mu(j, nr)$  as defined in Lemma 2.1 for  $\Theta_2$ . We have  $\mu_j \geq \mu_{j'}$  for  $1 \leq j \leq j' \leq k_n$ . But, for every  $1 < i \leq n$  we have

$$r = \sum_{j=1}^{k_1} \mu_j \geq k_1 \mu_{k_1} \geq (k_i - k_{i-1}) \mu_{k_{i-1}} \geq \sum_{j=k_{i-1}+1}^{k_i} \mu_j = r.$$

This gives  $k_i - k_{i-1} = k_1 = k$  for some constant  $k$  and  $\mu_j = \frac{r}{k}$  for all  $1 \leq j \leq k_n = nk$ . Hence,  $k \mid r$  and  $k_i = ik$  for  $1 \leq i < n$ . Furthermore, if  $nk < m$ , applying 1) and 2) of Lemma 2.1 to  $\Theta_1$  with  $j = nk + 1$ , we have  $\Theta_1(e_1^n) \geq e_{nk+1}^m$ .

LEMMA 2.3. Suppose  $n_1 > 1$ . Let  $\Theta_i : D_{n_i} \rightarrow D_{n_{i+1}}$  for  $i = 1, 2, 3$  such that  $\Theta_2 \circ \Theta_1 = \nu_{r_1} \circ \sigma_{s_1}$  and  $\Theta_3 \circ \Theta_2 = \nu_{r_2} \circ \sigma_{s_2}$  for some positive integers  $r_1, r_2, s_1, s_2$ . Then there exist  $k_1$  and  $k_2$  dividing  $r_1$  such that, with  $q_1 = n_2/(n_1 k_1)$ ,  $\Theta_1 = \nu_{k_1} \circ \sigma_{q_1}$  and

$$\Theta_2(e_j^{n_2}) \geq e^{n_3}[1 + (j-1)k_2, jk_2] \text{ for } 1 \leq j \leq n_2.$$

If, in addition, there exists  $\Theta_4 : D_{n_4} \rightarrow D_{n_5}$  such that  $\Theta_4 \circ \Theta_3 = \nu_{r_3} \circ \sigma_{s_3}$  for some positive integers  $r_3, s_3$ , then  $\Theta_2 = \nu_{k_2} \circ \sigma_{q_2}$ , where  $q_2 = s_1/q_1$ .

Proof. We apply Lemma 2.2 to  $\Theta_2 \circ \Theta_1$  and  $\Theta_3 \circ \Theta_2$  respectively, and from (\*), we have  $k_1$ , and  $k_2$  such that

$$\Theta_1(e_i^{n_1}) \geq e^{n_2}[1 + (i-1)k_1, ik_1] \quad 1 \leq i \leq n_1$$

and

$$\Theta_2(e_j^{n_2}) \geq e^{n_3}[1 + (j-1)k_2, jk_2] \quad 1 \leq j \leq n_2.$$

Since

$$\Theta_2 \circ \Theta_1(e_i^{n_1}) = \nu_{r_1} \circ \sigma_{s_1}(e_i^{n_1}) = \sum_{t=0}^{s_1-1} e^{n_3}[1 + (i-1)r_1 + n_1 r_1 t, ir_1 + n_1 r_1 t]$$

and  $n_1 > 1$ , we have  $k_1 k_2 = r_1$ . For each  $1 \leq i \leq n_1$  and  $1 \leq j \leq n_2$  such that  $\Theta_1(e_i^{n_1}) \geq e_j^{n_2}$ , we have

$$\begin{aligned} \Theta_2 \circ \Theta_1(e_i^{n_1}) &\geq \Theta_2(e_j^{n_2}) \Rightarrow \\ &\Rightarrow \Theta_2 \circ \Theta_1(e_i^{n_1}) e^{n_3}[1, k_2 n_2] \geq e^{n_3}[1 + (j-1)k_2, jk_2] \Rightarrow \\ &\Rightarrow \sum_{t=0}^{q_1-1} e^{n_3}[1 + (i-1)r_1 + n_1 r_1 t, ir_1 + n_1 r_1 t] \geq e^{n_3}[1 + (j-1)k_2, jk_2], \end{aligned}$$

$$\text{where } q_1 = \frac{n_2 k_2}{n_1 r_1} = \frac{n_2}{n_1 k_1}$$

$$\Rightarrow 1 + (i-1)r_1 + n_1 r_1 t \leq 1 + (j-1)k_2 \leq jk_2 \leq ir_1 + n_1 r_1 t$$

for some  $0 \leq t \leq q_1 - 1$

$$\Rightarrow 1 + (i-1)k_1 + n_1 k_1 t \leq j \leq ik_1 + n_1 k_1 t \text{ for some } 0 \leq t \leq q_1 - 1.$$

Hence,  $\Theta_1(e_i^{n_1}) = \sum_{t=0}^{q_1-1} e^{n_2}[1 + (i-1)k_1 + n_1 k_1 t, ik_1 + n_1 k_1 t] = \nu_{k_1} \circ \sigma_{q_1}(e_i^{n_1})$  for  $1 \leq i \leq n_1$ .

If, in addition, there exists  $\Theta_4 : T_{n_4} \rightarrow T_{n_5}$  such that  $\Theta_4 \circ \Theta_3 = \nu_{r_3} \circ \sigma_{s_3}$  for some positive integers  $r_3, s_3$ , then we can apply the results in the first part of this lemma to  $\Theta_2, \Theta_3$  and  $\Theta_4$  and use

$$q_2 = \frac{n_3}{n_2 k_2} = \frac{n_1 r_1 s_1}{n_1 k_1 q_1 k_2} = \frac{s_1}{q_1}.$$

*Proof of Theorem 1.1.* Since the result is trivial if either  $\mathcal{T}$  or  $\mathcal{S}$  is finite dimensional, we may assume that both  $\tau(\varphi)$  and  $\tau(\psi)$  are infinite.

1)  $\Rightarrow$  2) is proved in Proposition 3.20 of [6].

For 2)  $\Rightarrow$  3), suppose that  $\Theta : \mathcal{D} \rightarrow \mathcal{E}$  is an isomorphism of  $\prec_{\mathcal{T}}$  and  $\prec_{\mathcal{S}}$ . Then for every natural number  $n | \tau(\varphi)$ , there exist mutually orthogonal projections  $e_1, \dots, e_n$  in  $\mathcal{D}$  such that

$$e_i \prec_{\mathcal{T}} e_{i+1} \text{ for } 1 \leq i < n \text{ and } \sum_{i=1}^n e_i = 1_{\mathcal{D}}, \text{ the identity in } \mathcal{D}$$

$$\Rightarrow \Theta(e_i) \prec_{\mathcal{S}} \Theta(e_{i+1}) \text{ for } 1 \leq i < n \text{ and } \sum_{i=1}^n \Theta(e_i) = 1_{\mathcal{E}}, \text{ the identity in } \mathcal{E}.$$

Also,  $\Theta(e_1), \dots, \Theta(e_n)$  are mutually orthogonal projections in  $\mathcal{E}$ . Therefore,  $n \mid \tau(\psi)$ . Applying the same argument for  $\Theta^{-1}$ , we have  $\tau(\varphi) = \tau(\psi)$ .

In order to prove 3), we may assume, without loss of generality, that  $n_1, m_1 > 1$ . Choose  $j(1)$  and  $i(1)$  such that

- i)  $\Theta(\mathcal{D}_1) \subseteq \mathcal{E}_{j(1)}$  and for  $e, f \in \mathcal{D}_1, e \prec_{\mathcal{T}_1} f \Rightarrow \Theta(e) \prec_{\mathcal{S}_{j(1)}} \Theta(f)$ , and
- ii)  $\Theta^{-1}(\mathcal{E}_1) \subseteq \mathcal{D}_{i(1)}$  and for  $e, f \in \mathcal{E}_1, e \prec_{\mathcal{S}_1} f \Rightarrow \Theta^{-1}(e) \prec_{\mathcal{T}_{i(1)}} \Theta^{-1}(f)$ .

Let  $\mathcal{Q}$  be the set of all primes  $p$  such that  $p \mid k_{i(1)}l_{j(1)}$  and  $p^\infty$  does not divide anyone of  $\sigma(\varphi), \sigma(\psi), \nu(\varphi)$  and  $\nu(\psi)$ . Given a prime  $p \notin \mathcal{Q}$  and positive integer  $d$  such that  $p^d \mid \nu(\psi)$ , we can choose  $l > j(1)$  such that  $p^d \mid \prod \{m_i : j(1) < i \leq l, \psi_i = \nu_{m_i}\}$ . Then  $\psi_l \circ \psi_{l-1} \circ \dots \circ \psi_{j(1)+1} = \nu_t \circ \sigma_u$  for some positive integers  $t, u$  with  $p^d \mid t$ . Choose  $k > 1$  such that  $\Theta^{-1}(\mathcal{E}_1) \subseteq \mathcal{D}_k$  and for  $e, f \in \mathcal{E}_1, e \prec_{\mathcal{S}_1} f \Rightarrow \Theta^{-1}(e) \prec_{\mathcal{T}_k} \Theta^{-1}(f)$ . Then the embedding  $\mathcal{D}_1 \rightarrow \mathcal{D}_k$  is given by  $\nu_r \circ \sigma_s$  where  $r = \prod \{n_i : \varphi_i = \nu_{n_i}, 1 < i \leq k\}$ . Applying Lemma 2.2 for  $\Theta_1 = \nu_t \circ \sigma_u \circ \Theta$  on  $\mathcal{D}_1$  and  $\Theta_2 = \Theta^{-1}$  on  $\mathcal{E}_1$ , we have  $p^d \mid r$  and consequently,  $p^d \mid \nu(\varphi)$ .

By repeating the above argument with  $\Theta^{-1}$ , we can show that for every prime  $p \notin \mathcal{Q}$  and positive integer  $d, p^d \mid \nu(\varphi)$  if and only if  $p^d \mid \nu(\psi)$ . Similarly, by applying Lemma 2.3, we can prove the same condition for  $\sigma(\varphi)$  and  $\sigma(\psi)$ .

For 3)  $\Rightarrow$  1), suppose  $\varphi$  and  $\psi$  satisfies

$$n\nu(\varphi) = m\nu(\psi) \quad \text{and} \quad m\sigma(\varphi) = n\sigma(\psi)$$

for some positive integers  $n$  and  $m$ . Without loss of generality, we may assume  $n$  and  $m$  are relatively prime and all  $n_i$  and  $m_i$  are prime numbers. For each prime  $p$  dividing  $n$  (or  $m$ ), we have  $p \mid \sigma(\varphi)$  (or  $p \mid \nu(\varphi)$  respectively). Since switching a finite number of  $\sigma_{n_i}$  to  $\nu_{n_i}$  (or  $\nu_{n_i}$  to  $\sigma_{n_i}$ ) will not change the isomorphism class of  $\mathcal{T}$ , we may assume that  $\sigma(\varphi) = \sigma(\psi)$  and  $\nu(\varphi) = \nu(\psi)$ . Hence, there exists a permutation  $\pi$  of the positive integers such that  $\psi_i = \varphi_{\pi(i)}$ .

So the result follows from the next lemma.

**LEMMA 2.4.** Let  $\{\varphi_i\}_{i=1}^\infty, \{\psi_i\}_{i=1}^\infty, \mathcal{T}$  and  $\mathcal{S}$  be as given in Theorem 1.1. Suppose there is a permutation  $\pi$  of the positive integers such that  $\psi_i = \varphi_{\pi(i)}$ . Then  $\mathcal{T}$  and  $\mathcal{S}$  are isomorphic.

*Proof.* Let  $E = \{i_1, \dots, i_r\}$  be a non-empty finite subset of positive integers. Then from the discussion in the paragraph preceding Theorem 1.1, the map  $\varphi_E = \varphi_{i_1} \circ \dots \circ \varphi_{i_r}$  is well defined on  $M_n$  for every fixed  $n$ . If  $E_1^i$  and  $E_2$  are disjoint subsets, then we have  $\varphi_{E_1} \circ \varphi_{E_2} = \varphi_{E_1 \cup E_2}$ .

We note that the sequence  $\{l_i\}$  of  $\mathcal{S}$  is determined by  $\{k_i\}$  and the permutation  $\pi$ . Specifically, we have  $m_i = n_{\pi(i)}$  and  $l_i = \prod_{j=1}^i m_j$ . We are going to construct two



sequences

$$i(0) < i(1) < i(2) < \dots, j(1) < j(2) < \dots$$

and mappings  $\theta_i$  such that the following diagram commutes

$$\begin{array}{ccccccc} T_0 & \rightarrow & T_{i(0)} & \rightarrow & T_{i(1)} & \rightarrow & T_{i(2)} \rightarrow \dots \\ \parallel & & \theta_1 & \searrow & \uparrow \theta_2 & \searrow & \theta_3 \uparrow \theta_4 \\ S_0 & \rightarrow & & & S_{j(1)} & \rightarrow & S_{j(2)} \rightarrow \dots \end{array}$$

For each  $i \leq j$ , let  $[i, j] = \{k : i \leq k \leq j\}$ . First choose  $i(0) = 1$  and  $j(1) > \pi^{-1}(1)$ . Let  $E_1 = \pi([1, j(1)]) \setminus \{1\}$  and  $\theta_1 = \varphi_{E_1}$ . Next, we choose  $i(1) > \max\{\pi(i) : 1 \leq i \leq j(1)\}$  and set  $E_2 = [1, i(1)] \setminus \pi([1, j(1)])$ ,  $\theta_2 = \varphi_{E_2}$ . We have

$$1) E_1 \cup \{1\} = \pi([1, j(1)]) \Rightarrow \theta_1 \circ \varphi_1 = \varphi_{\pi(j(1))} \circ \dots \circ \varphi_{\pi(1)} \text{ and}$$

$$2) E_2 \cup E_1 = [2, i(1)] \Rightarrow \theta_2 \circ \theta_1 = \varphi_{i(1)} \circ \dots \circ \varphi_2.$$

Suppose we have chosen sequences

$$1 = i(0) < i(1) < \dots < i(r),$$

$$0 = j(0) < j(1) < \dots < j(r)$$

and subsets  $E_0 = \{1\}, E_1, E_2, \dots, E_{2r}$  such that

$$\begin{aligned} E_{2t} \cap E_{2t-1} &= \emptyset \\ E_{2t-1} \cap E_{2t-2} &= \emptyset \\ E_{2t} \cup E_{2t-1} &= [i(t-1) + 1, i(t)] \\ E_{2t-1} \cup E_{2t-2} &= \pi([j(t-1) + 1, j(t)]) \end{aligned} \quad (**)$$

for  $1 \leq t \leq r$ . Then, we can choose  $j(r+1) > \max\{\pi^{-1}(t) : 1 \leq t \leq i(r)\}$ ,  $i(r+1) > \max\{\pi(t) : 1 \leq t \leq j(r+1)\}$  and put  $E_{2r+1} = \pi([j(r) + 1, j(r+1)]) \setminus E_{2r}$ ,  $E_{2r+2} = [i(r) + 1, i(r+1)] \setminus E_{2r+1}$ . Then (\*\*) holds for  $1 \leq t \leq (r+1)$ . Thus  $\{i(t)\}, \{j(t)\}$  and  $E_t$  can be defined inductively and the diagram commutes with  $\theta_i = \varphi_{E_i}$ .

REMARK 2.5. After this paper had been submitted, we learned that the equivalence of 1) and 3) in Theorem 1.1 has also been obtained by Hopenwesser and Power in [16]. Their proof uses the invariant defined by Power in [11] and is quite different from the one given here.

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