Reprinted from JOURNAL OF FUNCTIONAL ANALYSIS

Vol. 105, No. 1, April 1992
Printed in Belgium

M-Ideals and Quotients of Subdiagonal Algebras

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Communicated by D. Sarason

Received February 18, 1991

In this paper, we study the M-ideals and quotients of subdiagonal algebras. Particular attention is given to the subdiagonal algebras A of groupoid C^*-algebras, where all groupoids are assumed to be amenable r-discrete principal with a cover by clopen G-sets. One of the major results shows that given such A and an M-ideal J in A, both J and A/J are subdiagonal algebras of groupoid C^*-algebras.

1. INTRODUCTION

An *operator algebra* is a Banach algebra A with a matrix norm structure [3] such that A is completely isometrically isomorphic to a (not necessarily self-adjoint) norm closed subalgebra of a C^*-algebra B. In particular, every norm closed subalgebra of a C^*-algebra is an operator algebra. Throughout our discussion, unless stated otherwise, all subalgebras of C^*-algebras are assumed to be norm closed and all ideals of operator algebras are norm closed two-sided ideals. An operator algebra A is *unital* if it contains a multiplicative identity 1_A with $\|1_A\| = 1$. For non-unital operator algebras, we are interested in the case when A has a

* This research was partially supported by the National Science Foundation and the Center for Advanced Study in the University of Illinois at Urbana-Champaign.

0022-1236/92 $3.00

(contractive) *approximate identity*, i.e., a net of elements $\{a_\alpha\} \in A$ such that $\|a_\alpha\| \leqslant 1$ and

$$\|a - aa_\alpha\| \to 0 \qquad \text{and} \qquad \|a - a_\alpha a\| \to 0$$

for every $a \in A$. We note that there are many operator algebras without any approximate identity. For example, let M_n denote the operator algebra of $n \times n$ matrices and let $\{e_{ij}: 1 \leqslant i, j \leqslant n\}$ denote the canonical matrix unit in M_n. It is easy to see that $A = \operatorname{span}\{e_{12}\}$ is a subalgebra in M_2 without any approximate identity.

The existence of approximate identity plays a key role in the study of ideal structure of operator algebras. We recall that a norm closed subspace E_1 in a Banach space E is an *M-ideal* if $E_1^\perp$, the annihilator of E_1, is an *L-summand* in E^*, i.e., if there is a norm closed subspace F in E^* such that

$$E^* = F \oplus E_1^\perp$$

and

$$\|f + g\| = \|f\| + \|g\|$$

for all $f \in F$ and $g \in E_1^\perp$ (cf. [1]). It has been shown in [5] that a norm closed subspace J in a unital operator algebra A is an M-ideal if and only if J is an ideal of A with an approximate identity. This result is also true for non-unital operator algebras with an approximate identity. To see this, suppose that A is an operator algebra with an approximate identity. We may assume that A acts on a Hilbert space H; i.e., we may identify A with a subalgebra of $B(H)$, the algebra of bounded linear operators on H. Let $A^1 = A \oplus C1_H$ be the unital subalgebra of $B(H)$ obtained by joining the identity operator 1_H to A. Then A is an M-ideal in A^1. Given any norm closed subspace J in A, it follows from [1] that J is an M-ideal in A if and only if J is an M-ideal in A^1. Thus J is an M-ideal in A if and only if J has an approximate identity. This is a natural non-self-adjoint generalization of the result that every C^*-algebra has an approximate identity and its ideals coincide with the M-ideals (see [1, 16]).

Given a subalgebra A of a C^*-algebra B, we let $A^* = \{x^*: x \in A\}$. Then $A \cap A^*$ is a C^*-subalgebra of B contained in both A and A^*. It is clear that every *self-adjoint* element in A is contained in $A \cap A^*$.

If D is a C^*-subalgebra of a C^*-algebra B, a *conditional expectation* ε from B onto D is a continuous positive projection from B onto D such that

$$\varepsilon(ab) = a\varepsilon(b) \qquad \text{and} \qquad \varepsilon(ba) = \varepsilon(b)\, a$$

for all $b \in B$ and $a \in D$. It is well known that this is equivalent to ε being a projection of norm one from B onto D (cf. [18]). For our convenience, we will simply call ε a conditional expectation on B. A conditional expectation ε is *faithful* if for any $b \in B$, $\varepsilon(b^*b)=0$ implies $b=0$. Given a faithful conditional expectation ε on B, an *ε-subdiagonal algebra* (or simply a subdiagonal algebra) of B is a subalgebra A of B such that $A+A^*$ is norm dense in B, ε is multiplicative on A, and $\varepsilon(B)=A \cap A^*$ contains a positive increasing approximate identity for B.

In Section 2, we study ε-subdiagonal algebras of general C^*-algebras, and their M-ideals and quotients. We begin by showing that a subalgebra of a C^*-algebra has a self-adjoint approximate identity if and only if it has an increasing positive approximate identity. One of the major results (Theorem 2.6) in this section is to show that if A is an ε-subdiagonal algebra of a C^*-algebra B and J is an M-ideal in A such that $D_J=J \cap J^*$ contains a self-adjoint approximate identity for J, then $B_J=(J+J^*)^-$ is an ideal of B and J is an ε_J-subdiagonal algebra of B_J, where ε_J is the restriction of ε to B_J. Furthermore, A/J can be identified with the subalgebra $A_Q=(A+B_J)/B_J$ of the quotient C^*-algebra $B_Q=B/B_J$. If the conditional expectation ε_Q on B_Q induced by ε is faithful, then A_Q is an ε_Q-subdiagonal of B_Q. In this case, we get a short exact sequence

$$0 \to D_J \to A \cap A^* \to D_Q \to 0,$$

where $D_Q=A_Q \cap A_Q^*$.

In [10], Muhly and Solel have obtained a coordinate representation theorem for subalgebras and ideals of groupoid C^*-algebras of amenable r-discrete principal groupoids G that admit a cover by compact open G-sets. This provides a very useful tool for studying ε-subdiagonal algebras of groupoid C^*-algebras. Given a groupoid G, which we always assume to be amenable r-discrete principal, it is known by Renault [14] that every ideal J of the groupoid C^*-algebra $C^*(G, \sigma)$ is *-isomorphic to a groupoid C^*-algebra $C^*(G_J, \sigma_J)$, where G_J is an open subset of G and it is uniquely determined by the ideal J. However, even when G has a cover of compact open G-sets, G_J does not necessarily have a cover of compact open G_J-sets (see Example A.1 in the Appendix). This suggests us to consider a broader class of groupoids, those that admit covers by *clopen* (closed and open) G-sets.

Generalizing Muhly and Solel's result [10, Theorem 3.10], we show in the Appendix that given an amenable r-discrete principal groupoid G with a cover by clopen G-sets every norm closed $C^*(G^0)$-bimodule I of $C^*(G, \sigma)$ can be uniquely represented in the form of

$$I=A(P_I),$$

where P_I is an open subset of G and $A(P_I)$ is the (norm closed) subspace of $C^*(G, \sigma)$ consisting of all elements supported on P_I. In particular, every subalgebra A (resp., C^*-subalgebra D) of $C^*(G, \sigma)$ containing $C^*(G^0)$ can be uniquely represented as $A = A(P)$ (resp., $D = A(H)$) for some open preorder P (resp., open subgroupoid H) of G.

In Section 3, we study ε-subdiagonal algebras of groupoid C^*-algebras. We show in Theorem 3.1 that given a C^*-subalgebra $D = A(H)$ of $C^*(G, \sigma)$ containing $C^*(G^0)$, there is a conditional expectation ε from $C^*(G, \sigma)$ onto $A(H)$ if and only if the corresponding subgroupoid H is clopen in G. In this case, the conditional expectation ε is faithful and is uniquely determined by the restriction map to H, i.e., $\varepsilon(f) = f|_H$. We show in Theorem 3.2 that a subalgebra $A = A(P)$ of $C^*(G, \sigma)$ containing $C^*(G^0)$ is an ε-subdiagonal algebra if and only if P is a clopen total preorder in G. Furthermore, $A(P)$ is a maximal ε-subdiagonal algebra of $C^*(G, \sigma)$. We show in Theorem 3.3 that if $A = A(P)$ is an ε-subdiagonal algebra of $C^*(G, \sigma)$ containing $C^*(G^0)$, then a norm closed subspace J of A is an M-ideal in A if and only if it has an increasing positive approximate identity. Thus both J and A/J are subdiagonal algebras of groupoid C^*-algebras. We close this section with a study of subdiagonal algebras of AF-algebras.

In Section 4, we consider the analogous results for the σ-weakly closed M-ideals and the quotients of σ-weakly closed subdiagonal algebras of groupoid von Neumann algebras.

2. Subdiagonal Algebras of C^*-Algebras

Definition 2.1. Let A be a subalgebra of a C^*-algebra B. An approximate identity $\{a_\alpha\}$ of A is called *self-adjoint* (resp., *positive*) if each a_α is self-adjoint (resp., positive). A positive approximate identity $\{a_\alpha\}$ is *increasing* if it satisfies

$$0 \leqslant a_\alpha \leqslant a_\beta$$

whenever $\alpha \leqslant \beta$.

Obviously if A has an increasing positive approximate identity, it has a self-adjoint approximate identity. The following proposition shows that the converse is also true.

Proposition 2.2. *If A is a subalgebra of a C^*-algebra B with a self-adjoint approximate identity, then A has an increasing positive approximate identity.*

Proof. Without loss of generality, we may assume that the C^*-subalgebra $D = A \cap A^*$ contains a self-adjoint approximate identity $\{a_\alpha\}$ for A such that $\|a_\alpha\| < 1$ for all α. Taking the second duals, we may regard D''

(resp., A'') as a von Neumann subalgebra (resp., $\sigma(B'', B')$ closed subalgebra) of B''. Passing to a subnet, we may assume that $\{a_\alpha\}$ converges to a non-zero central projection $z \in D''$ in $\sigma(B'', B')$ topology. It is easy to check that z is the multiplicative identity of both D'' and A''. We note that z is not necessarily the multiplicative identity of B''.

We claim that the net of positive elements $\{a_\alpha^2\}$ in D converges to z in $\sigma(B'', B')$ topology. To see this, for any positive linear functional $\phi \in D'$ with $\|\phi\| = 1$, which is a normal state on the von Neumann algebra D'', we have $\phi(z) = 1$. By the Cauchy–Schwartz inequality, we get

$$|\phi(a_\alpha)|^2 \leqslant \phi(a_\alpha^2) \leqslant 1,$$

and thus

$$\phi(a_\alpha^2) \to \phi(z) = 1.$$

It follows that a_α^2 converges to z in $\sigma(D'', D')$ topology. Since $\psi|_D \in D'$ for every $\psi \in B'$, a_α^2 converges to z in $\sigma(B'', B')$ topology. This proves the claim.

Since z is the multiplicative identity of A'', it follows from the above claim that for every $a \in A$, the nets $\{a^* a_\alpha^2 a\}$ and $\{a a_\alpha^2 a^*\}$ in B converge to a^*a and aa^*, respectively, in the $\sigma(B'', B')$ topology, and thus in the $\sigma(B, B')$ topology. Let Ω be the set of all convex combinations of $\{a_\alpha^2\}$. It is clear that Ω is contained in $D_1^0 \cap D^+$, the intersection of the open unit ball of D and the positive part of D, and that $D_1^0 \cap D^+$ is a upward directed set with respect to the natural positive order in the C^*-algebra D (cf. [18]). We claim that $D_1^0 \cap D^+$ determines an increasing positive approximate identity for A.

To see this, for any $a \in A$, it is clear that a^*a is contained in the norm closure of the convex subset $\{a^*xa \colon x \in \Omega\}$. For any $\varepsilon > 0$, there is an element $x_0 \in \Omega \subseteq D_1^0 \cap D^+$ such that

$$\|a^* x_0 a - a^* a\| < \varepsilon^2.$$

For any $x \in D_1^0 \cap D^+$ with $x_0 \leqslant x$, we have $x \leqslant z$ and $z - x \leqslant z - x_0$ in D'', and thus

$$\begin{aligned} \|a - xa\| &= \|(z-x)\, a\| \\ &\leqslant \|(z-x)^{1/2}\| \; \|(z-x)^{1/2}\, a\| \\ &\leqslant \|a^*(z-x)\, a\|^{1/2} \\ &\leqslant \|a^*(z-x_0)\, a\|^{1/2} < \varepsilon. \end{aligned}$$

Similarly, by considering aa^* in the norm closure of $\{axa^* \colon x \in \Omega\}$, we can get $y_0 \in \Omega$ such that $\|a - ay\| < \varepsilon$ for all $y \in D_1^0 \cap D^+$ with $y \geqslant y_0$. Hence, $D_1^0 \cap D^+$ is an increasing positive approximate identity for A. ∎

Remark 2.3. Let A be a subalgebra of a C^*-algebra B. In general, $A \cap A^*$ need not contain any self-adjoint approximate identity for A. This can happen even when A has an approximate identity (see Example 2.7). On the other hand, if A is a C^*-subalgebra of B, it always has an increasing positive approximate identity.

DEFINITION 2.4. Let B be a C^*-algebra and let ε be a faithful conditional expectation on B. A subalgebra A is called an *ε-subdiagonal algebra* of B if it satisfies

(1) $A + A^*$ is norm dense in B

(2) ε is multiplicative on A

(3) $\varepsilon(B) = A \cap A^*$

(4) $A \cap A^*$ has an increasing positive approximate identity for B.

We note that we may replace condition (4) in Definition 2.4 by

(4′) $A \cap A^*$ contains a self-adjoint approximate identity for A.

If B is unital, we will assume, instead of (4), that $A \cap A^*$ contains the unit of B.

The definition of unital subdiagonal algebras of C^*-algebras was first introduced by Kawamura and Tomiyama in [9], which is motivated by an analogous definition for subalgebras of von Neumann algebras given by Arveson [2].

If B is non-unital, we may consider the unitalization B^1 of B, i.e., $B^1 = B \oplus C$ with the C^*-algebra norm defined as follows. For any $(x, \alpha) \in B^1$,

$$\|(x, \alpha)\|_B = \sup\{\|xy + \alpha y\| : y \in B, \|y\| \leqslant 1\}.$$

Given A an ε-subdiagonal algebra of a C^*-algebra B, we let $D = A \cap A^*$ and we let D^1 be the unitalization of D with the norm

$$\|(x, \alpha)\|_D = \sup\{\|xy + \alpha y\| : y \in D, \|y\| \leqslant 1\}$$

for $(x, \alpha) \in D^1$. It is a simple matter to verify that the natural embedding from $(D^1, \| \|_D)$ into $(B^1, \| \|_B)$ is a unital $*$-isomorphic injection. Thus we may identify D^1 with a unital C^*-subalgebra of B^1. If we define ε^1 on B^1 by

$$\varepsilon^1((x, \alpha)) = (\varepsilon(x), \alpha)$$

for all $(x, \alpha) \in B^1$, then ε^1 determines a conditional expectation from B^1 onto D^1. To see this, we only need to prove that ε^1 is contractive on B^1.

Since $\varepsilon^1((x, \alpha)) = (\varepsilon(x), \alpha) \in D^1$ for all $(x, \alpha) \in B_1$, we have

$$\begin{aligned}\|\varepsilon^1((x, \alpha))\|_D &= \sup\{\|\varepsilon(x)\, y + \alpha y\|: y \in D, \|y\| \leqslant 1\}\\ &= \sup\{\|\varepsilon(xy + \alpha y)\|: y \in D, \|y\| \leqslant 1\}\\ &\leqslant \|(x, \alpha)\|_B.\end{aligned}$$

If, in addition, ε is faithful on B, it is routine to check that ε^1 is faithful on B^1. Thus we have

PROPOSITION 2.5. *Let A be an ε-subdiagonal algebra of B. If we assume that A^1 is the unitalization of A with norm obtained from B^1, then A^1 is an ε^1-subdiagonal algebra of B^1.*

THEOREM 2.6. *Let A be an ε-subdiagonal algebra of a C^*-algebra B and J an M-ideal in A such that $D_J = J \cap J^*$ contains a self-adjoint approximate identity for J. Then*

(1) *$B_J = (J + J^*)^-$ is an ideal of B and ε_J, the restriction of ε to B_J, defines a faithful conditional expectation from B_J onto D_J. Furthermore, J is an ε_J-subdiagonal algebra of B_J.*

(2) *The quotient algebra A/J is completely isometrically isomorphic to the subalgebra $A_Q = (A + B_J)/B_J$ in the quotient C^*-algebra $B_Q = B/B_J$, and ε induces a conditional expectation ε_Q on B_Q given by*

$$\varepsilon_Q(b + B_J) = \varepsilon(b) + B_J$$

for all $b \in B$.

(3) *If, in addition to (2), ε_Q is faithful on B_Q, then A_Q is an ε_Q-subdiagonal algebra of B_Q. In this case, we get a short exact sequence*

$$0 \to D_J \to A \cap A^* \to D_Q \to 0,$$

where $D_Q = A_Q \cap A_Q^*$.

Proof. Without loss of generality, we may assume, by Proposition 2.5, that B is a unital C^*-algebra, and we may assume, by Proposition 2.2, that $D_J = J \cap J^*$ contains an increasing positive approximate identity $\{u_\alpha\}$ for J.

(1) First we show that $ux^* \in B_J$ for all $u \in J$ and $x \in A$. Since $ux^* \in B = (A + A^*)^-$, there are sequences $\{x_n\}$ and $\{y_n\}$ in A such that

$$ux^* = \lim_{n \to \infty} (x_n + y_n^*).$$

Notice that for every α, $u_\alpha x_n \in J$ and $u_\alpha y_n^* \in J^*$ for all $n \in N$. So, we have

$$u_\alpha u x^* = \lim_{n \to \infty} u_\alpha (x_n + y_n^*) \in B_J.$$

Since $\lim_\alpha u_\alpha u x^* = ux^*$, we have $ux^* \in B_J$. Clearly, $ux \in B_J$ for all $u \in J$ and $x \in A$. Since $A + A^*$ is dense in B, we have $ux \in B_J$ for all $u \in J$ and $x \in B$. Similarly, we have $u^* x \in B_J$ for all $u^* \in J^*$ and $x \in B$. Thus B_J is an ideal of B.

Let ε_J be the restriction of ε to B_J. For every $u \in J$, we get

$$\varepsilon_J(u) = \lim_\alpha \varepsilon_J(u_\alpha u) = \lim_\alpha u_\alpha \varepsilon_J(u) \in D_J.$$

Similar argument shows that $\varepsilon_J(u^*) \in D_J$ for all $u^* \in J^*$. It follows that $\varepsilon_J(B_J) = D_J$, since $J + J^*$ is norm dense in B_J. Thus ε_J defines a faithful conditional expectation on B_J. It is easy to see that ε_J is multiplicative on J. Thus J is an ε_J-subdiagonal algebra of B_J.

(2) We note that the subalgebras A and J (resp., A_Q) have natural matrix norms inherited from the C^*-algebra B (resp., B_Q). Thus, by identifying $M_n(A/J)$ with $M_n(A)/M_n(J)$ for each $n \geqslant 1$, the quotient algebra A/J has a natural matrix norm. It is clear that the natural homomorphism

$$\pi \colon A/J \to (A + B_J)/B_J \subseteq B_Q$$

is contractive. Since B_J is an M-ideal in the C^*-algebra B, it is proximinal (see [4]); i.e., for every element $a \in B$, there is an element $m \in B_J$ such that

$$\|a + B_J\| = \|a + m\|.$$

Since $\{u_\alpha\}$ is an increasing positive approximate identity for J and thus for B_J, we have

$$\begin{aligned}
\|a + B_J\| &= \|a + m\| \\
&\geqslant \|(a + m)(1 - u_\alpha)\| \\
&\geqslant \|a - au_\alpha\| - \|m - mu_\alpha\| \\
&\geqslant \|a + J\| - \|m - mu_\alpha\|.
\end{aligned}$$

Since $\|m - mu_\alpha\| \to 0$, this shows that $\|a + B_J\| \geqslant \|a + J\|$ and thus the homomorphism π is an isometry. By applying the above argument on $M_n(A/J)$, we can show that π is a complete isometry from A/J onto A_Q, and thus we may identify A/J with the subalgebra A_Q in B_Q. Let $A_Q^* = (A^* + B_J)/B_J$ be the involution of A_Q in B_Q. It is easy to verify that

$A_Q + A_Q^*$ is norm dense in B_Q because $A + A^*$ is norm dense in B. If we let ε_Q be the map on B_Q defined by

$$\varepsilon_Q(b + B_J) = \varepsilon(b) + B_J,$$

for every $b \in B$, it is clear that ε_Q is a conditional expectation on B_Q with range

$$\varepsilon_Q(B_Q) = \varepsilon(B) + B_J \subseteq D_Q = A_Q \cap A_Q^*,$$

and ε_Q is multiplicative on A_Q. Thus ε_Q determines a *-homomorphism from the C^*-algebra D_Q onto the C^*-subalgebra $\varepsilon_Q(B_Q)$.

(3) At this time, we do not know if ε_Q is automatically faithful or not. If we assume that ε_Q is faithful, then for any $b + B_J \in D_Q$, $\varepsilon_Q(b + B_J) = 0$ implies $\varepsilon_Q(b^*b + B_J) = \varepsilon_Q(b + B_J)^* \, \varepsilon_Q(b + B_J) = 0$. By the faithfulness of ε_Q on B_Q, we get $b^*b + B_J = 0$ and thus $b + B_J = 0$. This shows that ε_Q determines a *-isomorphism from D_Q onto $\varepsilon_Q(B_Q)$. It follows that $D_Q = \varepsilon_Q(B_Q)$, and thus A_Q is an ε_Q-subdiagonal algebra of B_Q.

Finally, since

$$D_J = J \cap J^* = A \cap A^* \cap B_J$$

is an ideal in $A \cap A^*$ and

$$D_Q = (A \cap A^* + B_J)/B_J = A \cap A^*/(A \cap A^* \cap B_J),$$

we get the short exact sequence

$$0 \to D_J \to A \cap A^* \to D_Q \to 0. \quad \blacksquare$$

We end this section with an example of a unital ε-subdiagonal algebra of a C^*-algebra which has a lot of M-ideals containing no self-adjoint approximate identities. Thus the assumption in Theorem 2.6 that the M-ideals contain self-adjoint approximate identities is not redundant.

EXAMPLE 2.7. Let $A(D)$ be the classical disc algebra and $C(T)$ the commutative C^*-algebra of all continuous functions on the unit circle T. Then $A(D)$ is an ε-subdiagonal algebra of $C(T)$ with respect to the faithful conditional expectation ε given by

$$\varepsilon(f) = \int_T f(t) \, d\mu(t),$$

where μ is the normalized Haar measure on T (cf. [9]). Letting 1_D be the constant function 1 on D, we get $A(D) \cap A(D)^* = \mathbb{C} 1_D$.

It is known by Fakhooury [6] that a subalgebra of $A(D)$ is an M-ideal if and only if it consists of all functions in $A(D)$ which are null on a closed subset of T with zero measure. Thus there are a lot of M-ideals J in $A(D)$ with $J \cap J^* = \{0\}$, since the only self-adjoint elements in $A(D)$ are the real multiples of the constant function 1_D (cf. [17]).

3. Subdiagonal Algebras of Groupoid C^*-Algebras

Let X be a second countable, locally compact Hausdorff space. An *r-discrete principal groupoid* G on X is an equivalence relation on X. The groupoid structure on G is defined as follows. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in G, set $x^{-1} = (x_2, x_1)$. If $x_2 = y_1$, then the pair (x, y) is said to be *composable* and we set $xy = (x_1, y_2)$. Let G^2 be the set of all composable pairs. We assume that G is given a topology such that

(1) G is a locally compact Hausdorff space;

(2) the maps $x \to x^{-1}$ from G onto G, and $(x, y) \to xy$ from G^2 into G, are continuous, where G^2 is given the relative topology as a subset of $G \times G$;

(3) the map $x \to (x, x)$ is a homeomorphism from X onto the unit space $G^0 = \{(x, x) : x \in X\}$;

(4) G^0 is open in G.

Following (3), we will identify X with G^0. We note that, from the above conditions, G^0 is also closed and thus clopen in G. Given $x \in G$, we call $d(x) = x^{-1}x$ the *domain* of x and $r(x) = xx^{-1}$ the *range* of x. It is clear that d and r are well-defined continuous maps from G onto its unit space G^0. A subset s of G is called a *G-set* if the restrictions of r and d to s are one-to-one. If G is an r-discrete principal groupoid with a cover by clopen G-sets, then there is a *left Haar system* $\{\lambda^x\}_{x \in X}$ such that λ^x is given by the counting measure on $G^x = r^{-1}(x)$ for each $x \in X$.

Let T be the group of complex numbers with modulus 1. A *continuous 2-cocycle* σ is a continuous map from G^2 into T such that

$$\sigma(x_0 x_1, x_2)\, \sigma(x_0, x_1) = \sigma(x_1, x_2)\, \sigma(x_0, x_1 x_2)$$

for all (x_0, x_1) and $(x_1, x_2) \in G^2$. Let $C_c(G)$ be the space of all continuous functions on G with compact supports. Given a continuous 2-cocycle σ, we can define an involutive Banach algebra structure on $C_c(G)$ as follows. For $f, g \in C_c(G)$, the multiplication is given by

$$f * g(x) = \int f(xy)\, g(y^{-1})\, \sigma(xy, y^{-1})\, d\lambda^{d(x)}(y),$$

and the involution is given by

$$f^*(x)=\overline{f(x^{-1})\,\sigma(x, x^{-1})}$$

for $x \in G$.

Let $C_c(G, \sigma)$ denote the involutive algebra with the multiplication and involution defined as above. Then the groupoid C^*-algebra $C^*(G, \sigma)$ and the reduced groupoid C^*-algebra $C^*_{red}\ (G, \sigma)$ are the completion of $C_c(G, \sigma)$ under suitable C^*-norms. When the groupoid G is *amenable*, as defined in [14, 10], these two C^*-algebras coincide.

A continuous 2-cocycle σ is said to be *normalized* if for every pair $(x, y) \in G^2$ with $x=(x_1, x_2)$ and $y=(x_2, x_3)$, we have

$$\sigma(x, y)=1$$

whenever at least two of the three elements x_1, x_2, and x_3 are equal. Using an argument similar to that in [7, Proposition 7.7], one can prove that every continuous 2-cocycle is cohomologous to a normalized continuous 2-cocycle, and the corresponding groupoid C^*-algebras are *-isomorphic. If σ is normalized, then for all $h, k \in C_c(G^0)$, $f \in C_c(G, \sigma)$, and $x \in G$, we have

$$h * f(x)=\int h(xy)\, f(y^{-1})\, \sigma(xy, y^{-1})\, d\lambda^{d(x)}(y)=h(r(x))\, f(x)$$

and

$$f * k(x)=\int f(xy)\, k(y^{-1})\, \sigma(xy, y^{-1})\, d\lambda^{d(x)}(y)=f(x)\, k(d(x)).$$

Throughout this section and the Appendix, we let X be a second countable, locally compact Hausdorff space and let G be an amenable r-discrete principal groupoid on X with a cover by clopen G-sets. We also assume that every continuous 2-cocycle σ from G into T is normalized.

We note that under these hypotheses, every element in $C^*(G, \sigma)$ can be represented as a continuous function on G [14, II.4.2] and we will use $C^*(G^0)$ to denote the subalgebra of elements in $C^*(G, \sigma)$ supported on G^0. Given an open subset P of G, we let $A(P)$ denote the set of all elements in $C^*(G, \sigma)$ supported on P. An open subset P of G is called a *preoder* [10] in G if it contains G^0 and satisfies

$$P \circ P \subseteq P.$$

A preorder P is said to be *total* in G if $P \cup P^{-1}=G$. We call P a *subgroupoid* of G if P is a preorder and $P=P^{-1}$. In the Appendix

(Theorem A.7), we will generalize Theorem 3.10 of [10] which states that every norm closed $C^*(G^0)$-bimodule A in $C^*(G, \sigma)$ can be uniquely written as $A = A(P)$ for some open subset P in G. In particular, every subalgebra A (resp., C^*-subalgebra D) of $C^*(G, \sigma)$ containing $C^*(G^0)$ can be uniquely represented as $A = A(P)$ (resp., $D = A(H)$) for some open preoder P (resp., open subgroupoid H) of G.

Let E be a subset of G^0. The *reduction* of G by E is

$$G|_E = \{x \in G: \text{both } d(x) \text{ and } r(x) \in E\},$$

with the groupoid structure inherited from G. We note that $(G|_E)^0 = E$. A subset E of G^0 is said to be *invariant* if for all $x \in G$, we have $r(x) \in E$ if and only if $d(x) \in E$. Let P be an open subset of G such that $A(P)$ is an ideal of $C^*(G, \sigma)$. Then $P \cap G^0$ is an invariant subset of G^0. Conversely, given an open invariant subset E of G^0, $A(G|_E)$ is an ideal of $C^*(G, \sigma)$. By restricting the functions in $A(G|_E)$ to $G|_E$, we have $A(G|_E) \cong C^*(G|_E, \sigma|_E)$. This gives a one-to-one correspondence between ideals of $C^*(G, \sigma)$ and open invariant subsets of G^0 [14, Proposition II.4.5].

Given G as above, let E, F be subsets of G such that F is open and E is a compact subset of F. Then there exists $h \in C_c(G)$ with $\operatorname{supp}(h) \subseteq F$ such that $h(x) = 1$ for all $x \in E$ and $0 \leqslant h(x) \leqslant 1$ for all $x \in G$. We will denote such a function by

$$E \prec h \prec F.$$

THEOREM 3.1. *Let D be a C^*-subalgebra of $C^*(G, \sigma)$ containing $C^*(G^0)$ and H the corresponding open subgroupoid in G such that $D = A(H)$. Then there is a conditional expectation ε from $C^*(G, \sigma)$ onto $A(H)$ if and only if H is clopen.*

In this case, ε is faithful and uniquely determined by the restriction map to H, i.e., $\varepsilon(f) = f|_H$ for every $f \in C^*(G, \sigma)$.

Proof. If H is a clopen subgroupoid of G, then the map ε_H defined by

$$\varepsilon_H(f) = f|_H$$

for all $f \in C_c(G, \sigma)$ is a projection from $C_c(G, \sigma)$ onto $C_c(H, \sigma_H)$. Since H is open in G, the norm closure of $C_c(H, \sigma_H)$ in $C^*(G, \sigma)$ is just the C^*-subalgebra $A(H)$. Slightly modifying the proof given in [14, Proposition II 2.9 (iii)], we can show that for every $g \in C_c(G, \sigma)$, the map $f \to \varepsilon_H(g^* * f * g)$ satisfies

$$\|\varepsilon_H(g^* * f * g)\| \leqslant \|g\|^2 \, \|f\|$$

for all $f \in C_c(G, \sigma)$. We note that all norms in the above inequality are considered as the norm on $C^*(G, \sigma)$. We claim that ε_H is contractive on

$C_c(G, \sigma)$, and thus can be extended to a conditional expectation, i.e., a projection of norm one, from $C^*(G, \sigma)$ onto $A(H)$. To see this, for any $f \in C_c(G, \sigma)$, we let K denote the support of f, which is a compact subset of G. Thus $r(K) \cup d(K)$ is a compact subset of G^0, and there is a continuous function $g \in C_c(G^0)$ such that

$$r(K) \cup d(K) \prec g \prec G^0.$$

It follows that

$$f = g^* * f * g$$

and

$$\|\varepsilon_H(f)\| = \|\varepsilon_H(g^* * f * g)\| \leqslant \|f\|.$$

Conversely, let ε be any conditional expectation from $C^*(G, \sigma)$ onto $A(H)$. For any $f \in C_c(G, \sigma)$, it is clear that $\varepsilon(f)(x) = 0$ for $x \notin H$ since $\varepsilon(f)$ is supported on H. To show $\varepsilon(f)(x) = f(x)$ for every $x \in H$, we may assume that the support of f is contained in a clopen G-set s. Let $s_0 = s \cap H$ and fix any point $x \in H$. If $x \in s_0$, then we can choose continuous functions $h, k \in C_c(G^0)$ such that

$$\{r(x)\} \prec h \prec r(s_0) \qquad \text{and} \qquad \{d(x)\} \prec k \prec d(s_0).$$

Since $h * f * k$ has compact support contained in $s_0 \subseteq H$, we get $h * f * k \in A(H)$. This implies

$$\begin{aligned} f(x) &= h * f * k(x) \\ &= \varepsilon(h * f * k)(x) \\ &= h * \varepsilon(f) * k(x) \\ &= h(r(x))\, \varepsilon(f)(x)\, k(d(x)) \\ &= \varepsilon(f)(x). \end{aligned}$$

If $x \notin s_0$, then there exist open subsets U and V of $x(\cong G^0)$ such that $r(x) \in U$, $d(x) \in V$ and $(U \times V) \cap s = \varnothing$. Choose h, $k \in C_c(G^0)$ such that

$$\{r(x)\} \prec h \prec U \qquad \text{and} \qquad \{d(x)\} \prec k \prec V.$$

We have

$$h * f * k = 0 \Rightarrow f(x) = 0 = h * f * k(x) = \varepsilon(h * f * h)(x) = \varepsilon(f)(x).$$

Hence $\varepsilon(f) = f|_H$ for $f \in C_c(G, \sigma)$. For general $f \in C^*(G, \sigma)$, there is a sequence of $\{f_n\} \in C_c(G, \sigma)$ such that $f_n \to f$ in norm. It follows that $\varepsilon(f_n) \to \varepsilon(f)$ in norm. Therefore we get

$$\varepsilon(f)(x) = \lim_{n \to \infty} \varepsilon(f_n)(x) = 0$$

for all $x \notin H$, and

$$\varepsilon(f)(x) = \lim_{n \to \infty} \varepsilon(f_n)(x) = \lim_{n \to \infty} f_n(x) = f(x)$$

for all $x \in H$. This shows that $\varepsilon(f) = f|_H$ for every $f \in C^*(G, \sigma)$.

Next we show that H must be closed. Suppose not, then there exists an element $x_0 \in \bar{H} \setminus H$. Thus there is a continuous function $f \in C^*(G, \sigma)$ such that $f = 1$ on an open neighborhood V_0 of x_0 and a sequence $\{x_n\}$ in $H \cap V_0$ converging to x_0 in G. This implies

$$0 = \varepsilon(f)(x_0) = \lim_{n \to \infty} \varepsilon(f)(x_n) = \lim_{n \to \infty} f(x_n) = 1,$$

a contradiction. Hence, the subgroupoid H must be closed in G.

Finally, we show that the conditional expectation ε is faithful and uniquely determined. It follows from the above argument that any conditional expectation from $C^*(G, \sigma)$ onto $A(H)$ is given by the restriction map to H. Hence it is unique. The faithfulness of ε follows from

$$\varepsilon(f^* * f) = f^* * f|_H = 0 \Rightarrow f^* * f|_{G^0} = 0 \Rightarrow f = 0. \quad \blacksquare$$

THEOREM 3.2. *Let $A = A(P)$ be a subalgebra of a groupoid C^*-algebra $C^*(G, \sigma)$ containing $C^*(G^0)$. Then A is an ε-subdiagonal algebra of $C^*(G, \sigma)$ if and only if P is a total clopen preorder in G.*

Furthermore, A is a maximal ε-subdiagonal algebra of $C^*(G, \sigma)$.

Proof. Let $A = A(P)$ be an ε-subdiagonal algebra of $C^*(G, \sigma)$ containing $C^*(G^0)$. To verify that P is a total clopen preorder in G, we only need to verify that $P \cup P^{-1} = G$ and P is closed in G.

By definition, $f(x) = 0$ for all $f \in A$ and $x \notin P$. Since $A + A^*$ is norm dense in $C^*(G, \sigma)$, it follows that for every $x \notin P \cup P^{-1}$ we have $h(x) = 0$ for all $h \in C^*(G, \sigma)$. Hence, we must have $P \cup P^{-1} = G$.

Let $D = A \cap A^* = A(H)$, where $H = P \cap P^{-1}$. Since there is a faithful conditional expectation ε from $C^*(G, \sigma)$ onto the C^*-subalgebra $A(H)$, it follows from Theorem 3.1 that H is clopen in G. This implies that $P^{-1} \setminus H$ is open in G, and thus $P = G \setminus (P^{-1} \setminus H)$ is closed in G.

Conversely, suppose P is a clopen total preorder in G. Then $A = A(P)$ is a norm closed subalgebra of $C^*(G, \sigma)$ containing $C^*(G^0)$. Since $H = P \cap P^{-1}$ is a clopen subgroupoid in G, it follows from Theorem 3.1 that the map given by the restriction to H is a faithful conditional expectation from $C^*(G, \sigma)$ onto $D = A(H)$.

Given any f, $g \in A(P)$, and any $x \in H$, we have

$$\begin{aligned} f * g(x) &= \int_{y \in G} f(xy)\, g(y^{-1})\, \sigma(xy, y^{-1})\, d\lambda^{d(x)}(y) \\ &= \int_{xy \in P,\, y \in P^{-1}} f(xy)\, g(y^{-1})\, \sigma(xy, y^{-1})\, d\lambda^{d(x)}(y) \\ &= \int_{y \in H} f(xy)\, g(y^{-1})\, \sigma(xy, y^{-1})\, d\lambda^{d(x)}(y) \\ &= \int_{y \in G} f|_H(xy)\, g|_H(y^{-1})\, \sigma(xy, y^{-1})\, d\lambda^{d(x)}(y) \\ &= f|_H * g|_H(x). \end{aligned}$$

If $x \notin H$, it is easy to see that

$$\int_{y \in G} f|_H(xy)\, g|_H(y^{-1})\, \sigma(xy, y^{-1})\, d\lambda^{d(x)}(y) = 0.$$

This shows that ε is multiplicative on A, i.e., for any f, $g \in A$ we have

$$\varepsilon(f * g) = \varepsilon(f) * \varepsilon(g).$$

The norm density of $A + A^*$ in $C^*(G, \sigma)$ follows from $G = P \cup P^{-1}$. Thus A is an ε-subdiagonal algebra of $C^*(G, \sigma)$.

Finally, we need to show that $A(P)$ is a maximal ε-subdiagonal algebra. Let

$$A_m = \{f \in C^*(G, \sigma) : \varepsilon(g * f * h) = \varepsilon(h * f * g) = 0 \quad \text{for all } g, h \in A(P) \text{ such that } \varepsilon(g) = 0\}.$$

In [9, Theorem 3.1], Kawamura and Tomiyama prove that for any unital ε-subdiagonal algebra A of a C^*-algebra B, A_m is a maximal ε-subdiagonal algebra of B containing A. By using Proposition 2.5, we can generalize this result to ε-subdiagonal algebras which are not necessarily unital. Thus it suffices to prove that $A_m \subseteq A(P)$.

Following the idea of [10], for $f \in A_m$, we only need to show that

$$f(x) = 0$$

for all $x \notin P$. Given $x \notin P$, we have $y = x^{-1} \in P \backslash H$. So, there exists a clopen G-set $s \subseteq P$ such that $y \in s$ and $s \cap H = \varnothing$. Choose continuous functions a and b in $C_c(G)$ such that

$$\{y\} \prec a \prec s, \qquad \text{and} \qquad \{r(y)\} \prec b \prec G^0.$$

It is clear that a, $b \in A(P)$ and $\varepsilon(a)=0$. This implies that

$$\varepsilon(a * f * b)=0,$$

and thus

$$f(x)=a(y)\,f(x)\,b(r(y))=(a * f * b)(r(y))=\varepsilon(a * f * b)(r(y))=0. \quad \blacksquare$$

Next we turn our consideration to M-ideals and quotients of ε-subdiagonal algebras. Given a groupoid G, we will assume that all faithful conditional expectations ε on $C^*(G, \sigma)$ contain $C^*(G^0)$ in their ranges. In particular, we let ε_0 denote the faithful conditional expectation from $C^*(G, \sigma)$ onto $C^*(G^0)$. If $A=A(P)$ is an ε-subdiagonal algebra of $C^*(G, \sigma)$, then every ideal J in A can be uniquely represented in the form of $J=A(P_J)$ for some open subset P_J contained in P such that $P \circ P_J \circ P \subseteq P_J$ (see Theorem A.8). Letting $G_J^0=G^0 \cap P_J$, we write

$$P'_J=\{x \in P\colon \text{either } r(x) \text{ or } d(x) \in G_J^0\}$$

and

$$P''_J=\{x \in P\colon \text{both } r(x) \text{ and } d(x) \in G_J^0\}.$$

THEOREM 3.3. *The following are equivalent.*

(1) *J is an M-ideal in A, i.e., J admits an approximate identity.*

(2) *$P_J=P'_J=P''_J$.*

(3) *J has an increasing positive approximate identity contained in $C^*(G_J^0)$.*

Proof. (1) $\Rightarrow$ (2). *It is clear that* $P''_J \subseteq P'_J \subseteq P_J$. For each $x_0 \in P_J$, there is an open G-set s such that $x_0 \in s \subseteq P_J$. We can find a continuous function $a \in C_c(G)$ such that

$$\{x_0\} \prec a \prec s.$$

Since $s \subseteq P_J$, a is an element in $J=A(P_J)$. Letting $\{a_\alpha\}$ be an approximate identity for J, we have

$$\|a_\alpha * a-a\|_\infty \leqslant \|a_\alpha * a-a\| \to 0.$$

It follows that

$$a_\alpha * a(x_0)=a_\alpha(r(x_0))\,a(x_0) \to a(x_0)=1.$$

Therefore, we get $a_\alpha(r(x_0)) \neq 0$ for some α. Thus $r(x_0) \in G_J^0$. Similarly, we can show that $d(x_0) \in G_J^0$. Hence we get $P_J \subseteq P''_J$.

(2) $\Rightarrow$ (3). Since G is a second countable, locally compact Hausdorff space and G_J^0 is an open subset of G^0, G_J^0 is also a second

countable, locally compact Hausdorff space. Thus G_J^0 is σ-compact and there exists a sequence $\{U_n\}$ of open sets such that $G_J^0 = \bigcup_n U_n$ and that $\overline{U_n}$ is a compact subset of U_{n+1} for all n. Let $h_n \in C_c(G^0)$ such that $\overline{U_n} \prec h_n \prec U_{n+1}$. It follows from the hypothesis (2) that $\{h_n\} \in C^*(G_J^0)$ is an increasing positive approximate identity for J.

(3) $\Rightarrow$ (1). This is trivial. ∎

The above theorem shows that an M-ideal $J = A(P_J)$ in an ε-subdiagonal algebra $A = A(P)$ of $C^*(G, \sigma)$ always has increasing positive approximate identities. Writing

$$G_J = P_J \cup P_J^{-1},$$

we have

$$\begin{aligned} G_J &= \{x \in G\colon \text{either } r(x) \text{ or } d(x) \in G_J^0\} \\ &= \{x \in G\colon \text{both } r(x) \text{ and } d(x) \in G_J^0\}. \end{aligned}$$

This shows that G_J^0 is an open invariant subset of G^0 and G_J is the reduction of G by G_J^0. It is clear that G_J is an amenable r-discrete principal groupoid with a cover by clopen G_J-sets and G_J^0 is the clopen unit space of G_J.

Let $G_Q = G \backslash G_J$, $P_Q = P \backslash P_J$, and $G_Q^0 = G^0 \backslash G_J^0$. Then G_Q^0 is a closed invariant subset of G^0 and G_Q is the reduction of G by G_Q^0, which is closed in G. It is also clear that G_Q is an amenable r-discrete principal groupoid with a cover by clopen G_Q-sets and G_Q^0 is the clopen unit space of G_Q. The corresponding groupoid C^*-algebra $C^*(G_Q, \sigma_Q)$ is $*$-isomorphic to the quotient C^*-algebra $C^*(G, \sigma)/C^*(G_J, \sigma_J)$ [14, II.4.5]. Here, we use σ_J and σ_Q to denote the restrictions of σ to G_J and G_Q, respectively.

THEOREM 3.4. *Let $A = A(P)$ be an ε-subdiagonal algebra of $C^*(G, \sigma)$ and let $J = A(P_J)$ be an M-ideal in A. Then*

(1) *J is an ε_J-subdiagonal algebra of $C^*(G_J, \sigma_J)$.*

(2) *The quotient algebra A/J is completely isometrically isomorphic to the ε_Q-subdiagonal algebra $A(P_Q)$ of $C^*(G_Q, \sigma_Q)$.*

(3) *We have the short exact sequence*

$$0 \to D_J \to D \to D_Q \to 0,$$

where $D_J = A(P_J \cap P_J^{-1})$, $D = A(P \cap P^{-1})$, and $D_Q = A(P_Q \cap P_Q^{-1})$.

Proof. (1) Let $B_J = (J + J^*)^- = A(P_J \cup P_J^{-1})$. We note that $B_J \cong C^*(G_J, \sigma_J)$ and the $*$-isomorphism is given [14, II.4.5] by the restriction of the functions in B_J to G_J. So the result follows from Theorem 2.6 (1).

(2) Let $B_Q = C^*(G, \sigma)/B_J$. The restriction of functions in $C^*(G, \sigma)$ to G_Q induces [14, II.4.5] a *-isomorphism between B_Q and $C^*(G_Q, \sigma_Q)$. Let ε_Q be the conditional expectation on $C^*(G_Q, \sigma_Q)$ induced by ε. It follows from Theorem 3.1 that ε_Q is faithful. Thus the result follows from Theorem 2.6 (2) and (3).

(3) This follows immediately from parts (1), (2), and Theorem 2.6 (3). ∎

We conclude this section by looking at a special class of groupoid C^*-algebras. A C^*-algebra B is called an AF algebra if there exists an increasing sequence of finite dimensional C^*-subalgebras $\{B_n\}$ such that $B = (\cup B_n)^-$. If B is unital, we require that B_1 contains the unit 1 of B. A maximal abelian self-adjoint subalgebra (masa) D of an AF algebra $B = (\cup B_n)^-$ is called *standard* if there exists an increasing sequence $\{D_n\}$, such that each D_n is a masa in B_n and $D = (\cup D_n)^-$. It has been shown by Stratila and Voiculescu in [15] that every AF algebra B has a standard masa D and there exists a unique faithful conditional expectation ε_0 from B onto D. Let B be an AF algebra with a standard masa D and $X = \hat{D}$, the maximal ideal space of D. Then there is an AF-groupoid G on X such that $B \cong C^*(G)$ [14]. Let P be an open subset of G such that $P \circ P \subseteq P$, then the subalgebra $A(P)$ is an ε_0-subdiagonal algebra of B if and only if $P \cup P^{-1} = G$ and $P \cap P^{-1} = G^0$ [10, Theorem 4.2]. Following the results of [10, 19, 20], we have that ε_0-subdiagonal subalgebras are the same as the *strongly maximal triangular* subalgebras of B as defined in [12]. Let B be an AF algebra with a standard masa D and A an ε-subdiagonal algebra of B containing D. Suppose J is an M-ideal in A and $Q = A/J$. It follows that both B_J and B_Q are AF. Thus we have

COROLLARY 3.5. *Let B be an AF algebra with a standard masa D and A an ε-subdiagonal algebra of B containing D. Suppose J is an M-ideal in A and $Q = A/J$. Then J* (resp., Q) *is an εJ (resp., ε_Q)-subdiagonal algebra of the AF algebra B_J (resp., B_Q). In particular, if A is a strongly maximal triangular subalgebra of B, then both J and Q are strongly maximal triangular.*

Remark 3.6. In a forthcoming paper [13], we are going to study the class $\mathscr{A}$ of subdiagonal algebras of AF algebras in more details. In particular, we obtain a converse of Corollary 3.5, i.e., if A is a subdiagonal algebra of a C^*-algebra such that the sequence

$$0 \to J \to A \to Q \to 0$$

is exact for some $J, Q \in \mathscr{A}$, then $A \in \mathscr{A}$.

The following is an example of an ideal (but not an M-ideal) of a strongly maximal triangular subalgebra of AF-algebra such that neither the ideal nor its quotient is an ε-subdiagonal algebra of any C^*-algebra.

EXAMPLE 3.7. Let M_3 be the 3×3 matrix algebra and D_3 the diagonal matrices of M_3. Then the algebra of upper triangular matrices T_3 in M_3 is a finite dimensional strongly maximal triangular subalgebra of M_3. Let $J = \text{span}\ \{e_{13}\}$. It is easy to see that J is an ideal of T_3. Since J has no approximate identity, it is clear that J can not be isometrically isomorphic to an ε-subdiagonal algebra of any C^*-algebra. We are going to show that the quotient algebra $Q = T_3/J$ is also not isometrically isomorphic to an ε-subdiagonal algebra of any C^*-algebra.

Let $\pi\colon T_3 \to Q$ be the natural surjection and $f_{ij} = \pi(e_{ij})$ for $1 \leqslant i \leqslant j \leqslant 3$. Then Q is a unital non-commutative operator algebra of dimension 5. Suppose $\phi\colon Q \to A$ is an isometric isomorphism of Q onto an ε-subdiagonal algebra A of some C^*-algebra B. Given $x \in Q$, then x is an idempotent of norm 1 if and only if $\phi(x)$ is a (self-adjoint) projection in $A \cap A^*$. An easy calculation shows that for every idempotent $x \in Q$ of norm 1, there exists a self-adjoint projection $a \in D_3$ such that $\pi(a) = x$. This shows that $A \cap A^* = \phi \circ \pi(D_3)$ and $B = A + A^*$ is a non-commutative C^*-algebra of dimension 7. It follows that B is $*$-isomorphic to the C^*-algebra $M_2 \oplus C \oplus C \oplus C$. Then direct computation shows that $\phi(f_{12})$ and $\phi(f_{23})$ are linearly dependent, a contradiction.

4. $\mathscr{M}$-IDEALS AND QUOTIENTS FOR SUBDIAGONAL ALGEBRAS OF VON NEUMANN ALGEBRAS

In this section, we study the von Neumann algebra version of the previous results. First recall the definition of subdiagonal algebras of von Neumann algebras (cf. [11]). Let B_0 be a von Neumann subalgebra of a von Neumann algebra B and ε a faithful normal conditional expectation from B onto B_0.

DEFINITION 4.1. A σ-weakly closed unital subalgebra A of B is called an *ε-subdiagonal algebra* of B if it satisfies

(1) $A + A^*$ is σ-weakly dense in B

(2) ε is multiplicative on A

(3) $\varepsilon(B) = A \cap A^*$.

THEOREM 4.2. *Let A be an ε-subdiagonal algebra of a von Neumann algebra B and J a σ-weakly closed subspace of A. Then J is an M-ideal in A if and only if $J = pA$ for a central projection p in B.*

In this case, $J = pA$ is a subdiagonal algebra of pB and $A/J \cong (1-p)A$ is a subdiagonal algebra of $(1-p)B$.

Proof. If J is an M-ideal in A, then $J = pA$ for a central projection in A by [5, Theorem 2.2]. Since $A + A^*$ is σ-weakly dense in the von Neumann algebra B, then p must be a central projection in B. The proof for the converse is trivial.

In this case, we have

$$J \cap J^* = p(A \cap A^*)$$

and

$$(J + J^*)^{-\sigma} = p(A + A^*)^{-\sigma} = pB.$$

Hence, J is a subdiagonal algebra of the von Neumann algebra pB, since

$$\varepsilon(pB) = p\varepsilon(B) = p(A \cap A^*) = J \cap J^*.$$

A similar argument shows that $A/J \cong (1-p)A$ is a subdiagonal algebra of the von Neumann algebra $(1-p)B$. ∎

Now we study the properties of M-ideals and quotients of subdiagonal algebras of groupoid von Neumann algebras. Let $(X, \mathscr{B}, \mu)$ be a standard Borel measure space. An equivalence relation $R \subseteq X \times X$ is called *standard* if R is a Borel subset in the product σ-field. The standard equivalence relation R is called countable if for every $x \in X$

$$R(x) = \{y \in X: (x, y) \in R\}$$

is a countable set. Throughout this section, R will denote a standard and countable equivalence relation on X.

Given R associated with X, define two maps π_l and π_r from R onto X by

$$\pi_l(x, y) = x \qquad \text{and} \qquad \pi_r(x, y) = y,$$

for all $(x, y) \in R$.

Given a Borel subset C of X, we let

$$R(C) = \{y \in X: (x, y) \in R \text{ for some } x \in C\}.$$

The set C is called *saturated* if $\mu(R(C) \setminus C) = 0$. Given a standard groupoid R and a Borel 2-cocycle s, one can define a groupoid von Neumann algebra $M(R, s)$ and a Cartan subalgebra $A(R, s)$ of $M(R, s)$ as in [8]. Again we may assume that the 2-cocycle s is normalized. Given a subset C of X, let $\mathscr{X}_C$ denote the characteristic function on C.

THEOREM 4.3. *A Borel subset C of X is saturated if and only if $\mathscr{X}_C$ is a central projection of $M(R, s)$ contained in $A(R, s)$.*

Proof. Recall from [8] that elements in $M(R, s)$ can be represented as functions on R and $A(R, s)$ consists of functions in $M(R, s)$ which are supported on the diagonal $\Delta(X) = \{(x, x): x \in X\}$ of $X \times X$. Identifying $\Delta(X)$ with X, we have $\mathscr{X}_C$ is an element in $A(R, s)$ for every Borel subset C of X. Let ϕ be a partial Borel isomorphism of X such that its graph $\Gamma(\phi) \subseteq R$. Then it follows from [8, 2.4] that the linear span of all $f \cdot \mathscr{X}_{\Gamma(\phi)}$, where $f \in A(R, s)$ and $\Gamma(\phi) \subseteq R$, is dense in $M(R, s)$. Thus, for a Borel subset C of X, $\mathscr{X}_C$ is a central projection of $M(R, s)$ if and only if $\mathscr{X}_C$ commutes with all $\mathscr{X}_{\Gamma(\phi)}$. Let $\xi \in L^2(R, \nu)$, then we have

$$(\mathscr{X}_C \cdot \mathscr{X}_{\Gamma(\phi)})(\xi)(x, z) = \begin{cases} \mathscr{X}_C(x)\, \xi(\phi(x), z)\, s(x, \phi(x), z) & \text{if } x \in D(\phi) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(\mathscr{X}_{\Gamma(\phi)} \cdot \mathscr{X}_C)(\xi)(x, z) = \begin{cases} \mathscr{X}_C(\phi(x))\, \xi(\phi(x), z)\, s(x, \phi(x), z) & \text{if } x \in D(\phi) \\ 0 & \text{otherwise,} \end{cases}$$

where $D(\phi)$ indicates the domain of ϕ. Thus if C is saturated, $\mathscr{X}_C$ commutes with every $\mathscr{X}_{\Gamma(\phi)}$ and hence, is a central projection in $M(R, s)$.

Conversely, suppose that $\mathscr{X}_C$ is a central projection of $M(R, s)$, we may choose [7] a sequence of partial Borel isomorphisms $\{\phi_i\}$ such that $\pi_l^{-1}(C) = \bigcup_{i=1}^{\infty} \Gamma(\phi_i)$. Let $\xi = \mathscr{X}_X$. For each i, we have

$$(\mathscr{X}_C \cdot \mathscr{X}_{\Gamma(\phi_i)})(\xi)(x, z) = \begin{cases} \mathscr{X}_C(x) & \text{if } x \in D(\phi_i) \text{ and } z = \phi_i(x) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(\mathscr{X}_{\Gamma(\phi_i)} \cdot \mathscr{X}_C)(\xi)(x, z) = \begin{cases} 1 & \text{if } x \in D(\phi_i) \text{ and } z = \phi_i(x) \\ 0 & \text{otherwise,} \end{cases}$$

Hence, $\mathscr{X}_C \mathscr{X}_{\Gamma(\phi_i)} = \mathscr{X}_{\Gamma(\phi_i)} \mathscr{X}_C$ implies that $\mu(D(\phi_i) \backslash C) = 0$ for all i. Since $R(C) = \pi_r \pi_l^{-1}(C) = \cup D(\phi_i)$, we have $\mu(R(C) \backslash C) = 0$. ∎

Given a Borel subset Q of R, we write

$$\mathscr{T}(Q) = \{a \in M(R, s): a \text{ is supported on } Q\}.$$

It follows from [11] that every σ-weakly closed $A(R, s)$-bimodule A of $M(R, s)$ containing $A(R, s)$ can be written as $A = \mathscr{T}(Q)$. A is a subalgebra of $M(R, s)$ if and only if $Q \circ Q \subseteq Q$ and A contains $A(R, s)$ if and only if

$\Delta(X) \subseteq Q \cap Q^{-1}$. If B is a von Neumann subalgebra of $M(R, s)$ containing $A(R, s)$ associated with the Borel subset Q, then the restriction map $\varepsilon(a) = a|_Q$ defines a conditional expectation from $M(R, s)$ onto B (cf. [11, Theorem 3.4]). It follows that a σ-weakly closed subalgebra A containing $A(R, s)$ is a subdiagonal algebra of $M(R, s)$ if and only if $A + A^*$ is σ-weakly dense in $M(R, s)$. The following result is an immediate consequence of Theorems 4.2 and 4.3.

COROLLARY 4.4. *Let $B = M(R, s)$ and A a subdiagonal algebra of B containing $A(R, s)$. Then a σ-weakly closed ideal $J = \mathscr{T}(Q)$ of A is an M-ideal of A if and only if the Borel subset $C = \pi_l(Q)$ is saturated. In this case, J is a subdiagonal algebra of the groupoid von Neumann algebra $M(R|_C, s)$ and the quotient A/J is a subdiagonal algebra of the groupoid von Neumann algebra $M(R|_{(X \cdot C)}, s)$.*

APPENDIX

We begin with an example which has motivated us to consider the class of amenable r-discrete principal groupoid G with a cover by clopen G-sets.

EXAMPLE A.1. Let $B = C([0, 1], M_2)$ be the C^*-algebra of all continuous maps from the unit interval $[0, 1]$ into M_2. Then B can be represented as a groupoid C^*-algebra $C^*(G, \sigma)$ where

$$G = \{(e_{ij}, x) : x \in [0, 1] \text{ for } i, j = 1, 2\}.$$

The groupoid structure on G is given by:

(1) (e_{ij}, x) and $(e_{kl}, y) \in G$ are composable if and only if $j = k$ and $x = y$, where $(e_{ij}, x) \circ (e_{jl}, x) = (e_{il}, x)$.

(2) $(e_{ij}, x)^{-1} = (e_{ji}, x)$.

(3) The topology on G is that induced by the usual topology on $[0, 1]$.

(4) σ is the trivial 2-cocycle on G.

It is easy to see that G is an amenable r-discrete principal groupoid with a cover by compact open G-sets.

Let $I = \{f \in B : f(0) = 0\}$. Then I is an ideal of B. Let H be the reduction of the groupoid G by the invariant open subset $= (0, 1]$. Thus, I is $*$-isomorphic to the groupoid C^*-algebra of H. Since the only compact open subset in H is the empty set $\varnothing$, H has no cover by compact open H-sets.

Let $\mathscr{G}$ be the class of amenable r-discrete principal groupoid G which has a cover by clopen G-sets. Suppose $G \in \mathscr{G}$ and S is an open invariant subset of G^0. Let H (resp., K) be the reduction groupoid of G by S (resp., $G^0 \backslash S$), then it is easy to show that both H and K are in $\mathscr{G}$. Thus, if $G \in \mathscr{G}$ and I is an ideal of $C^*(G, \sigma)$, then both I and $C^*(G, \sigma)/I$ are isomorphic to C^*-algebras of groupoids in $\mathscr{G}$.

In the rest of this section, we assume that every groupoid G is amenable r-discrete principal with a cover by clopen G-sets, the unit space G^0 is a second countable, locally compact Hausdorff space, and every continuous 2-cocycle from G into T is normalized. We show that the major results in [10] can be generalized to this context. The main difference between our argument and that in [10] is that for a clopen G-set K, the characteristic function $\mathscr{X}_K$ may not lie in $C^*(G, \sigma)$. Except for some changes to accommodate this difference, our proofs are borrowed directly from those in [10].

First, we recall that every element $f \in C^*(G, \sigma)$ can be represented as a function in $C_0(G)$ with

$$\|f\|_\infty \leqslant \|f\|.$$

The following proposition is an easy consequence of [14, Proposition II.4.2 (ii)], which will be very useful in our argument.

PROPOSITION A.2. *If $f \in C^*(G, \sigma)$ with the support contained in a G-set, then*

$$\|f\|_\infty = \|f\|.$$

Let $\mathfrak{A} \subseteq C^*(G, \sigma)$ be a norm closed $C^*(G^0)$-bimodule. We write

$$Q(\mathfrak{A}) = \{x \in G\colon a(x) = 0 \text{ for all } a \in \mathfrak{A}\}.$$

It is clear that $Q(\mathfrak{A})$ is a closed subset of G. On the other hand, if Q is a closed subset of G, we write

$$I(Q) = \{a \in C^*(G, \sigma)\colon a = 0 \text{ on } Q\}.$$

Since for every $h, k \in C_c(G^0)$ and $f \in C^*(G, \sigma)$ we have

$$h * f(x) = \int h(xy)\, f(y^{-1})\, \sigma(xy, y^{-1})\, d\lambda^{d(x)}(y) = h(r(x))\, f(x)$$

and

$$f * k(x) = \int f(xy)\, k(y^{-1})\, \sigma(xy, y^{-1})\, d\lambda^{d(x)}(y) = f(x)\, k(d(x))$$

for all $x, y \in G$, it is clear that $I(Q)$ is a $C^*(G^0)$-bimodule in $C^*(G, \sigma)$. The space $I(Q)$ is norm closed in $C^*(G, \sigma)$. To see this, suppose that $\{f_n\}$ is a sequence in $I(Q)$ converging to some $f \in C^*(G, \sigma)$ in norm. Then we have

$$\|f_n - f\|_\infty \leqslant \|f_n - f\| \to 0.$$

This implies that $\{f_n\}$ is pointwise convergent to f on G. Hence, we must have $f \in I(Q)$.

Given $\mathfrak{A}$ a norm closed $C^*(G^0)$-bimodule in $C^*(G, \sigma)$, it is clear that $\mathfrak{A} \subseteq I(Q(\mathfrak{A}))$. Our first goal is to show that $\mathfrak{A} = I(Q(\mathfrak{A}))$, which is a generalization of [10, Theorem 3.10]. First we need to generalize some lemmas in [10, Sect. 3].

LEMMA A.3. *Let s be a clopen G-set in G. Then the map $f \to f|_s$, the restriction of f to s, is a contractive linear map on $C^*(G, \sigma)$.*

Proof. Let s be a clopen G-set. If $f \in C_c(G, \sigma)$, it is clear that $f|_s$ has a compact support K contained in the G-set s. From Proposition A.2,

$$\|f|_s\| = \|f|_s\|_\infty \leqslant \|f\|_\infty \leqslant \|f\|.$$

Hence, the map $f \to f|_s$ is a linear contraction on $C_c(G, \sigma)$ and has a natural extension to a linear contraction on $C^*(G, \sigma)$. ∎

Next, we note that Lemma 3.2 through Proposition 3.6 in [10] are also valid when we replace the condition "t is a compact open G-set" by "t is a clopen G-set." Thus, we simply state the generalized Proposition 3.6 in [10] as follows.

PROPOSITION A.4. *Let $\mathfrak{A}$ be a norm closed $C^*(G^0)$-bimodule in $C^*(G, \sigma)$. For every $a \in \mathfrak{A}$ and any clopen G-set s, we have $a|_s \in \mathfrak{A}$.*

Let Q be a closed subset of the groupoid G and $Q^c = G \backslash Q$. We denote $C_c(Q^c)$ the space of all $f \in I(Q)$ with compact support $\operatorname{supp}(f) \subseteq Q^c$ and we denote $I_c(Q)$ the space of all $f \in I(Q)$ with compact support. It is clear that $C_c(Q^c) \subseteq I_c(Q) \subseteq I(Q)$. In general $C_c(Q^c)$ is a proper subspace of $I_c(Q)$ since we might have $f \in I_c(Q)$ such that $\operatorname{supp}(f) \cap Q \neq \varnothing$. The following lemma shows that these two spaces have the same norm closure when the groupoid G admits a cover of clopen G-sets. Our discussion differs from that given in [10, Proposition 3.8].

LEMMA A.5. $C_c(Q^c)^- = I_c(Q)^-$.

Proof. We only need to show that every $f \in I_c(Q)$ is contained in $C_c(Q^c)^-$. Given $f \in I_c(Q)$, the support of f can be covered by finitely many clopen G-sets, say $s_1, \ldots, s_n$. We may assume that $s_1, \ldots, s_n$ are pairwise disjoint and thus we can write $f = \sum f|_{s_i}$. Obviously each $f|_{s_i} \in C^*(G, \sigma)$ by Lemma A.3 and $f|_{s_i} = 0$ on Q. Hence, we may assume $f \in I_c(Q)$ with $\operatorname{supp}(f)$ contained in a clopen G-set s. Let $O_f = \{x \in G: f(x) \neq 0\}$. Thus O_f is an open subset of $s \cap Q^c$ with the closure $O_f^- = \operatorname{supp}(f)$. Since $r(O_f)$ is an open subset of G^0, it is σ-compact. Hence, we can choose an increasing sequence $\{U_n\}$ of open subsets of $r(O_f)$ such that $r(O_f) = \bigcup_n U_n$ and for every n, U_n^- is compact and contained in U_{n+1}. Therefore there is a sequence of functions $h_n \in C_c(r(O_f))$ such that

$$U_n^- \prec h_n \prec U_{n+1}.$$

We get $h_n * f \in C_c(Q^c)$ with compact support contained in the open G-set O_f. Since $f \in C_0(O_f)$, we have, by Proposition A.2, that

$$\|h_n * f - f\| = \|h_n * f - f\|_\infty \to 0.$$

This shows $f \in C_c(Q^c)^-$. ∎

LEMMA A.6. *For every norm closed $C^*(G^0)$-bimodule $\mathfrak{A}$ in $C^*(G, \sigma)$, we have*

$$C_c(Q(\mathfrak{A})^c) \subseteq \mathfrak{A}.$$

Proof. Given $f \in C_c(Q(\mathfrak{A})^c)$, we may assume, without loss of generality, that the support K of f is contained in a clopen G-set s. Let $s_Q = s \cap Q(\mathfrak{A})^c \supseteq K$. For any $x \in K$, there is an element $a_x \in \mathfrak{A}$ such that $a_x(x) > 0$. It follows from Proposition A.4 that we can get a_x with the support contained in the clopen G-set s. It is clear that there is an open subset V_x of s_Q containing $\{x\}$ such that $a_x(x) > \varepsilon_x > 0$ on V_x. Since K is compact, there are finitely many such V_x's covering K, say $V_1, ..., V_n$. Let $V = \bigcup_{i=1}^n V_i$ and $a = \sum_{i=1}^n a_i \in \mathfrak{A}$. Then we have $K \subseteq V \subseteq s_Q$ and $a > \varepsilon$ on V for some $\varepsilon > 0$.

We can find a function $u \in C^*(G^0)$ with $r(K) \prec u \prec r(V)$ and define a function $h \in C_c(G^0)$ by

$$h(t, t) = u(t, t) \cdot \frac{1}{a(t, s(t))}$$

for all $(t, t) \in r(v)$ and $h(t, t) = 0$ otherwise. Since $\mathfrak{A}$ is a $C^*(G^0)$-bimodule, we get $h * a \in \mathfrak{A}$ with $h * a(x) = 1$ for all $x \in K$ and $h * a(x) = 0$ for all $x \in G \backslash V$. Define $g(t, t) = f(t, s(t))$. Then $g \in C^*(G^0)$. Thus, we have $f = g * (h * a) \in \mathfrak{A}$. ∎

THEOREM A.7. *For every norm closed $C^*(G^0)$-bimodule $\mathfrak{A}$ in $C^*(G, \sigma)$ we have*

$$\mathfrak{A} = I(Q(\mathfrak{A})).$$

For every closed subset Q_0 of G, we have

$$Q(I(Q_0)) = Q_0.$$

Proof. Given a norm closed $C^*(G^0)$-bimodule $\mathfrak{A}$ in $C^*(G, \sigma)$, it follows from Lemma A.6 that $C_c(Q(\mathfrak{A})^c)^- \subseteq \mathfrak{A} \subseteq I(Q(\mathfrak{A}))$. By Lemma A.5, $C_c(Q(\mathfrak{A})^c)^- = I_c(Q(\mathfrak{A}))^-$. It remains to show that $I_c(Q(\mathfrak{A}))^- = I(Q(\mathfrak{A}))$. But this has been proved by Muhly and Solel [10, 3.9 and 3.10].

The second statement is Lemma 3.11 in [10].

Theorem A.7 established a one-to-one correspondence between the norm closed $C^*(G^0)$-bimodules $\mathfrak{A}$ in $C^*(G, \sigma)$ and the closed subsets $Q(\mathfrak{A})$ in G. Given an open subset P in G, we write $A(P) = I(G \backslash P)$. Let $P(\mathfrak{A}) = G \backslash Q(\mathfrak{A})$. This gives a one-to-one correspondence between the norm closed $C^*(G^0)$-bimodules $\mathfrak{A}$ in $C^*(G, \sigma)$ and open subsets $P(\mathfrak{A})$ in G, and it is easy to show that the correspondence preserves inclusion, i.e., if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are norm closed $C^*(G^0)$-bimodules in $C^*(G, \sigma)$, then $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ if and only if $P(\mathfrak{A}_1) \subseteq P(\mathfrak{A}_2)$. In particular, $C^*(G^0) \subseteq \mathfrak{A}$ if and only if $G^0 \subseteq P(\mathfrak{A})$.

THEOREM A.8. *Let $A = A(P)$ a norm closed $C^*(G^0)$-bimodules in $C^*(G, \sigma)$. We have*

(1) *A is a subalgebra of $C^*(G, \sigma)$ containing $C^*(G^0)$ if and only if P is an open preoder in G. In this case, $A^* = A(P^{-1})$ and $A \cap A^* = A(P \cap P^{-1})$.*

(2) *A is a C^*-subalgebra of $C^*(G, \sigma)$ containing $C^*(G^0)$ if and only if P is an open subgroupoid in G.*

(3) *There is a one-to-one correspondence between all ideals $J = A(P_J)$ of $A = A(P)$ and open subsets P_J of P satisfying $P \circ P_J \circ P \subseteq P_J$.*

Proof. Owing to Theorem A.7, (1) can be proved by using a similar argument as that in [10, Theorem 4.1]. Part (2) follows from (1). Part (3) is an easy generalization of [10, Lemma 4.3]. ∎

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium