$M$-Ideals and Quotients of Subdiagonal Algebras

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In this paper, we study the $M$-ideals and quotients of subdiagonal algebras. Particular attention is given to the subdiagonal algebras $A$ of groupoid $C^*$-algebras, where all groupoids are assumed to be amenable $r$-discrete principal with a cover by clopen $G$-sets. One of the major results shows that given such $A$ and an $M$-ideal $J$ in $A$, both $J$ and $A/J$ are subdiagonal algebras of groupoid $C^*$-algebras. © 1992

1. INTRODUCTION

An operator algebra is a Banach algebra $A$ with a matrix norm structure [3] such that $A$ is completely isometrically isomorphic to a (not necessarily self-adjoint) norm closed subalgebra of a $C^*$-algebra $B$. In particular, every norm closed subalgebra of a $C^*$-algebra is an operator algebra. Throughout our discussion, unless stated otherwise, all subalgebras of $C^*$-algebras are assumed to be norm closed and all ideals of operator algebras are norm closed two-sided ideals. An operator algebra $A$ is unital if it contains a multiplicative identity $1_A$ with $\|1_A\| = 1$. For non-unital operator algebras, we are interested in the case* when $A$ has a

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(contractive) approximate identity, i.e., a net of elements \( \{a_x\} \in A \) such that \( \|a_x\| \leq 1 \) and
\[
\|a - aa_x\| \to 0 \quad \text{and} \quad \|a - a_xa\| \to 0
\]
for every \( a \in A \). We note that there are many operator algebras without any approximate identity. For example, let \( M_n \) denote the operator algebra of \( n \times n \) matrices and let \( \{e_{ij} : 1 \leq i, j \leq n\} \) denote the canonical matrix unit in \( M_n \). It is easy to see that \( A = \text{span}\{e_{12}\} \) is a subalgebra in \( M_2 \) without any approximate identity.

The existence of approximate identity plays a key role in the study of ideal structure of operator algebras. We recall that a norm closed subspace \( E_1 \) in a Banach space \( E \) is an \( M \)-ideal if \( E_1^\perp \), the annihilator of \( E_1 \), is an \( L \)-summand in \( E^* \), i.e., if there is a norm closed subspace \( F \) in \( E^* \) such that
\[
E^* = F \oplus E_1^\perp
\]
and
\[
\|f + g\| = \|f\| + \|g\|
\]
for all \( f \in F \) and \( g \in E_1^\perp \) (cf. [1]). It has been shown in [5] that a norm closed subspace \( J \) in a unital operator algebra \( A \) is an \( M \)-ideal if and only if \( J \) is an ideal of \( A \) with an approximate identity. This result is also true for non-unital operator algebras with an approximate identity. To see this, suppose that \( A \) is an operator algebra with an approximate identity. We may assume that \( A \) acts on a Hilbert space \( H \); i.e., we may identify \( A \) with a subalgebra of \( B(H) \), the algebra of bounded linear operators on \( H \). Let \( A^1 = A \oplus C1_H \) be the unital subalgebra of \( B(H) \) obtained by joining the identity operator \( 1_H \) to \( A \). Then \( A \) is an \( M \)-ideal in \( A^1 \). Given any norm closed subspace \( J \) in \( A \), it follows from [1] that \( J \) is an \( M \)-ideal in \( A \) if and only if \( J \) is an \( M \)-ideal in \( A^1 \). Thus \( J \) is an \( M \)-ideal in \( A \) if and only if \( J \) has an approximate identity. This is a natural non-self-adjoint generalization of the result that every \( C^* \)-algebra has an approximate identity and its ideals coincide with the \( M \)-ideals (see [1, 16]).

Given a subalgebra \( A \) of a \( C^* \)-algebra \( B \), we let \( A^* = \{x^*: x \in A\} \). Then \( A \cap A^* \) is a \( C^* \)-subalgebra of \( B \) contained in both \( A \) and \( A^* \). It is clear that every self-adjoint element in \( A \) is contained in \( A \cap A^* \).

If \( D \) is a \( C^* \)-subalgebra of a \( C^* \)-algebra \( B \), a conditional expectation \( \varepsilon \) from \( B \) onto \( D \) is a continuous positive projection from \( B \) onto \( D \) such that
\[
\varepsilon(ab) = a\varepsilon(b) \quad \text{and} \quad \varepsilon(ba) = \varepsilon(b) a
\]
for all \( b \in B \) and \( a \in D \). It is well known that this is equivalent to \( \varepsilon \) being a projection of norm one from \( B \) onto \( D \) (cf. [18]). For our convenience,
we will simply call $\varepsilon$ a conditional expectation on $B$. A conditional expectation $\varepsilon$ is \textit{faithful} if for any $b \in B$, $\varepsilon(b^*b) = 0$ implies $b = 0$. Given a faithful conditional expectation $\varepsilon$ on $B$, an $\varepsilon$-\textit{subdiagonal algebra} (or simply a subdiagonal algebra) of $B$ is a subalgebra $A$ of $B$ such that $A + A^*$ is norm dense in $B$, $\varepsilon$ is multiplicative on $A$, and $\varepsilon(B) = A \cap A^*$ contains a positive increasing approximate identity for $B$.

In Section 2, we study $\varepsilon$-subdiagonal algebras of general $C^*$-algebras, and their $M$-ideals and quotients. We begin by showing that a subalgebra of a $C^*$-algebra has a self-adjoint approximate identity if and only if it has an increasing positive approximate identity. One of the major results (Theorem 2.6) in this section is to show that if $A$ is an $\varepsilon$-subdiagonal algebra of a $C^*$-algebra $B$ and $J$ is an $M$-ideal in $A$ such that $D_j = J \cap J^*$ contains a self-adjoint approximate identity for $J$, then $B_j = (J + J^*)^-$ is an ideal of $B$ and $J$ is an $\varepsilon_j$-subdiagonal algebra of $B_j$, where $\varepsilon_j$ is the restriction of $\varepsilon$ to $B_j$. Furthermore, $A/J$ can be identified with the subalgebra $A_Q = (A + B_j)/B_j$ of the quotient $C^*$-algebra $B_Q = B/B_j$. If the conditional expectation $\varepsilon_Q$ on $B_Q$ induced by $\varepsilon$ is faithful, then $A_Q$ is an $\varepsilon_Q$-subdiagonal of $B_Q$. In this case, we get a short exact sequence

$$0 \rightarrow D_j \rightarrow A \cap A^* \rightarrow D_Q \rightarrow 0,$$

where $D_Q = A_Q \cap A_Q^*$.

In [10], Muhly and Solel have obtained a coordinate representation theorem for subalgebras and ideals of groupoid $C^*$-algebras of amenable $r$-discrete principal groupoids $G$ that admit a cover by compact open $G$-sets. This provides a very useful tool for studying $\varepsilon$-subdiagonal algebras of groupoid $C^*$-algebras. Given a groupoid $G$, which we always assume to be amenable $r$-discrete principal, it is known by Renault [14] that every ideal $J$ of the groupoid $C^*$-algebra $C^*(G, \sigma)$ is *-isomorphic to a groupoid $C^*$-algebra $C^*(G_j, \sigma_j)$, where $G_j$ is an open subset of $G$ and it is uniquely determined by the ideal $J$. However, even when $G$ has a cover of compact open $G$-sets, $G_j$ does not necessarily have a cover of compact open $G_j$-sets (see Example A.1 in the Appendix). This suggests us to consider a broader class of groupoids, those that admit covers by \textit{clopen} (closed and open) $G$-sets.

Generalizing Muhly and Solel's result [10, Theorem 3.10], we show in the Appendix that given an amenable $r$-discrete principal groupoid $G$ with a cover by clopen $G$-sets every norm closed $C^*(G^0)$-bimodule $I$ of $C^*(G, \sigma)$ can be uniquely represented in the form of

$$I = A(P_I),$$

where $P_I$ is an open subset of $G$ and $A(P_I)$ is the (norm closed) subspace of $C^*(G, \sigma)$ consisting of all elements supported on $P_I$. In particular, every
subalgebra $A$ (resp., $C^*$-subalgebra $D$) of $C^*(G, \sigma)$ containing $C^*(G^0)$ can be uniquely represented as $A = A(P)$ (resp., $D = A(H)$) for some open preorder $P$ (resp., open subgroupoid $H$) of $G$.

In Section 3, we study $\varepsilon$-subdiagonal algebras of groupoid $C^*$-algebras. We show in Theorem 3.1 that given a $C^*$-subalgebra $D = A(H)$ of $C^*(G, \sigma)$ containing $C^*(G^0)$, there is a conditional expectation $\varepsilon$ from $C^*(G, \sigma)$ onto $A(H)$ if and only if the corresponding subgroupoid $H$ is clopen in $G$. In this case, the conditional expectation $\varepsilon$ is faithful and is uniquely determined by the restriction map to $H$, i.e., $\varepsilon(f) = f|_H$. We show in Theorem 3.2 that a subalgebra $A = A(P)$ of $C^*(G, \sigma)$ containing $C^*(G^0)$ is an $\varepsilon$-subdiagonal algebra if and only if $P$ is a clopen total preorder in $G$. Furthermore, $A(P)$ is a maximal $\varepsilon$-subdiagonal algebra of $C^*(G, \sigma)$. We show in Theorem 3.3 that if $A = A(P)$ is an $\varepsilon$-subdiagonal algebra of $C^*(G, \sigma)$ containing $C^*(G^0)$, then a norm closed subspace $J$ of $A$ is an $M$-ideal in $A$ if and only if it has an increasing positive approximate identity. Thus both $J$ and $A/J$ are subdiagonal algebras of groupoid $C^*$-algebras. We close this section with a study of subdiagonal algebras of $AF$-algebras.

In Section 4, we consider the analogous results for the $\sigma$-weakly closed $M$-ideals and the quotients of $\sigma$-weakly closed subdiagonal algebras of groupoid von Neumann algebras.

2. Subdiagonal Algebras of $C^*$-Algebras

Definition 2.1. Let $A$ be a subalgebra of a $C^*$-algebra $B$. An approximate identity $\{a_\alpha\}$ of $A$ is called self-adjoint (resp., positive) if each $a_\alpha$ is self-adjoint (resp., positive). A positive approximate identity $\{a_\alpha\}$ is increasing if it satisfies

$$0 \leq a_\alpha \leq a_\beta$$

whenever $\alpha \leq \beta$.

Obviously if $A$ has an increasing positive approximate identity, it has a self-adjoint approximate identity. The following proposition shows that the converse is also true.

Proposition 2.2. If $A$ is a subalgebra of a $C^*$-algebra $B$ with a self-adjoint approximate identity, then $A$ has an increasing positive approximate identity.

Proof. Without loss of generality, we may assume that the $C^*$-subalgebra $D = A \cap A^*$ contains a self-adjoint approximate identity $\{a_\alpha\}$ for $A$ such that $\|a_\alpha\| < 1$ for all $\alpha$. Taking the second duals, we may regard $D^*$
(resp., $A''$) as a von Neumann subalgebra (resp., $\sigma(B'', B')$ closed sub-
algebra) of $B''$. Passing to a subnet, we may assume that $\{a_\alpha\}$ converges to a non-zero central projection $z \in D''$ in $\sigma(B'', B')$ topology. It is easy to check that $z$ is the multiplicative identity of both $D''$ and $A''$. We note that $z$ is not necessarily the multiplicative identity of $B''$.

We claim that the net of positive elements $\{a_\alpha^2\}$ in $D$ converges to $z$ in $\sigma(B'', B')$ topology. To see this, for any positive linear functional $\phi \in D'$ with $\|\phi\| = 1$, which is a normal state on the von Neumann algebra $D''$, we have $\phi(z) = 1$. By the Cauchy–Schwartz inequality, we get

$$|\phi(a_\alpha)|^2 \leq \phi(a_\alpha^2) \leq 1,$$

and thus

$$\phi(a_\alpha^2) \to \phi(z) = 1.$$

It follows that $a_\alpha^2$ converges to $z$ in $\sigma(D'', D')$ topology. Since $\psi\mid_D \in D'$ for every $\psi \in B'$, $a_\alpha^2$ converges to $z$ in $\sigma(B'', B')$ topology. This proves the claim.

Since $z$ is the multiplicative identity of $A''$, it follows from the above claim that for every $a \in A$, the nets $\{a^*a_\alpha^2a\}$ and $\{aa_\alpha^2a^*\}$ in $B$ converge to $a^*a$ and $aa^*$, respectively, in the $\sigma(B'', B')$ topology, and thus in the $\sigma(B, B')$ topology. Let $\Omega$ be the set of all convex combinations of $\{a_\alpha^2\}$. It is clear that $\Omega$ is contained in $D_1^0 \cap D^+$, the intersection of the open unit ball of $D$ and the positive part of $D$, and that $D_1^0 \cap D^+$ is an upward directed set with respect to the natural positive order in the C*-algebra $D$ (cf. [18]). We claim that $D_1^0 \cap D^+$ determines an increasing positive approximate identity for $A$.

To see this, for any $a \in A$, it is clear that $a^*a$ is contained in the norm closure of the convex subset $\{a^*xa: x \in \Omega\}$. For any $\varepsilon > 0$, there is an element $x_0 \in \Omega \subseteq D_1^0 \cap D^+$ such that

$$\|a^*x_0a - a^*a\| < \varepsilon^2.$$

For any $x \in D_1^0 \cap D^+$ with $x_0 \leq x$, we have $x \leq z$ and $z - x \leq z - x_0$ in $D''$, and thus

$$\|a - xa\| = \|(z - x) a\|$$

$$\leq \|(z - x)^{1/2}\| \|(z - x)^{1/2} a\|$$

$$\leq \|a^*(z - x) a\|^{1/2}$$

$$\leq \|a^*(z - x_0) a\|^{1/2} < \varepsilon.$$

Similarly, by considering $aa^*$ in the norm closure of $\{axa^*: x \in \Omega\}$, we can get $y_0 \in \Omega$ such that $\|a - ay\| < \varepsilon$ for all $y \in D_1^0 \cap D^+$ with $y \geq y_0$. Hence, $D_1^0 \cap D^+$ is an increasing positive approximate identity for $A$. \qed
Remark 2.3. Let $A$ be a subalgebra of a $C^*$-algebra $B$. In general, $A \cap A^*$ need not contain any self-adjoint approximate identity for $A$. This can happen even when $A$ has an approximate identity (see Example 2.7). On the other hand, if $A$ is a $C^*$-subalgebra of $B$, it always has an increasing positive approximate identity.

Definition 2.4. Let $B$ be a $C^*$-algebra and let $\varepsilon$ be a faithful conditional expectation on $B$. A subalgebra $A$ is called an $\varepsilon$-subdiagonal algebra of $B$ if it satisfies

1. $A + A^*$ is norm dense in $B$
2. $\varepsilon$ is multiplicative on $A$
3. $\varepsilon(B) = A \cap A^*$
4. $A \cap A^*$ has an increasing positive approximate identity for $B$.

We note that we may replace condition (4) in Definition 2.4 by

(4') $A \cap A^*$ contains a self-adjoint approximate identity for $A$.

If $B$ is unital, we will assume, instead of (4), that $A \cap A^*$ contains the unit of $B$.

The definition of unital subdiagonal algebras of $C^*$-algebras was first introduced by Kawamura and Tomiyama in [9], which is motivated by an analogous definition for subalgebras of von Neumann algebras given by Arveson [2].

If $B$ is non-unital, we may consider the unitalization $B^1$ of $B$, i.e., $B^1 = B \oplus C$ with the $C^*$-algebra norm defined as follows. For any $(x, z) \in B^1$,

$$
\|(x, z)\|_B = \sup\{\|xy + zy\| : y \in B, \|y\| \leq 1\}.
$$

Given $A$ an $\varepsilon$-subdiagonal algebra of a $C^*$-algebra $B$, we let $D = A \cap A^*$ and we let $D^1$ be the unitalization of $D$ with the norm

$$
\|(x, z)\|_D = \sup\{\|xy + zy\| : y \in D, \|y\| \leq 1\}
$$

for $(x, z) \in D^1$. It is a simple matter to verify that the natural embedding from $(D^1, \|\cdot\|_D)$ into $(B^1, \|\cdot\|_B)$ is a unital $^*$-isomorphic injection. Thus we may identify $D^1$ with a unital $C^*$-subalgebra of $B^1$. If we define $\varepsilon^1$ on $B^1$ by

$$
\varepsilon^1((x, z)) = (\varepsilon(x), z)
$$

for all $(x, z) \in B^1$, then $\varepsilon^1$ determines a conditional expectation from $B^1$ onto $D^1$. To see this, we only need to prove that $\varepsilon^1$ is contractive on $B^1$. 

Since \( \varepsilon^1(x, \alpha) = (\varepsilon(x), \alpha) \in D^1 \) for all \( (x, \alpha) \in B_1 \), we have

\[
\|\varepsilon^1((x, \alpha))\|_D = \sup\{\|\varepsilon(y) + \alpha y\| : y \in D, \|y\| \leq 1\}
\leq \sup\{\|\varepsilon(xy + \alpha y)\| : y \in D, \|y\| \leq 1\}
\leq \|\varepsilon((x, \alpha))\|_B.
\]

If, in addition, \( \varepsilon \) is faithful on \( B \), it is routine to check that \( \varepsilon^1 \) is faithful on \( B^1 \). Thus we have

**Proposition 2.5.** Let \( A \) be an \( \varepsilon \)-subdiagonal algebra of \( B \). If we assume that \( A^1 \) is the unitalization of \( A \) with norm obtained from \( B^1 \), then \( A^1 \) is an \( \varepsilon^1 \)-subdiagonal algebra of \( B^1 \).

**Theorem 2.6.** Let \( A \) be an \( \varepsilon \)-subdiagonal algebra of a \( C^* \)-algebra \( B \) and \( J \) an \( M \)-ideal in \( A \) such that \( D_J = J \cap J^* \) contains a self-adjoint approximate identity for \( J \). Then

1. \( B_J = (J + J^*)^- \) is an ideal of \( B \) and \( \varepsilon_J \), the restriction of \( \varepsilon \) to \( B_J \), defines a faithful conditional expectation from \( B_J \) onto \( D_J \). Furthermore, \( J \) is an \( \varepsilon_J \)-subdiagonal algebra of \( B_J \).

2. The quotient algebra \( A/J \) is completely isometrically isomorphic to the subalgebra \( A_Q = (A + B_J)/B_J \) in the quotient \( C^* \)-algebra \( B_Q = B/B_J \), and \( \varepsilon \) induces a conditional expectation \( \varepsilon_Q \) on \( B_Q \) given by

\[
\varepsilon_Q(b + B_J) = \varepsilon(b) + B_J
\]

for all \( b \in B \).

3. If, in addition to (2), \( \varepsilon_Q \) is faithful on \( B_Q \), then \( A_Q \) is an \( \varepsilon_Q \)-subdiagonal algebra of \( B_Q \). In this case, we get a short exact sequence

\[
0 \to D_J \to A \cap A^* \to D_Q \to 0,
\]

where \( D_Q = A_Q \cap A_Q^* \).

**Proof.** Without loss of generality, we may assume, by Proposition 2.5, that \( B \) is a unital \( C^* \)-algebra, and we may assume, by Proposition 2.2, that \( D_J = J \cap J^* \) contains an increasing positive approximate identity \( \{u_n\} \) for \( J \).

1. First we show that \( ux^* \in B_J \) for all \( u \in J \) and \( x \in A \). Since \( ux^* \in B = (A + A^*)^- \), there are sequences \( \{x_n\} \) and \( \{y_n\} \) in \( A \) such that

\[
ux^* = \lim_{n \to \infty} (x_n + y_n^*).
\]
Notice that for every \( a, u_n x_n \in J \) and \( u_n y_n^* \in J^* \) for all \( n \in N \). So, we have
\[
u x^* = \lim_{n \to x} u_n (x_n + y_n^*) \in B_J.
\]
Since \( \lim_n u_n x^* = u x^* \), we have \( u x^* \in B_J \). Clearly, \( u x \in B_J \) for all \( u \in J \) and \( x \in A \). Since \( A + A^* \) is dense in \( B \), we have \( u x \in B_J \) for all \( u \in J \) and \( x \in B \). Similarly, we have \( u^* x \in B_J \) for all \( u^* \in J^* \) and \( x \in B \). Thus \( B_J \) is an ideal of \( B \).

Let \( \varepsilon_J \) be the restriction of \( \varepsilon \) to \( B_J \). For every \( u \in J \), we get
\[
\varepsilon_J(u) = \lim_n \varepsilon_J(u_n u) = \lim_n u_n \varepsilon_J(u) \in D_J.
\]
Similar argument shows that \( \varepsilon_J(u^*) \in D_J \) for all \( u^* \in J^* \). It follows that \( \varepsilon_J(B_J) = D_J \), since \( J + J^* \) is norm dense in \( B_J \). Thus \( \varepsilon_J \) defines a faithful conditional expectation on \( B_J \). It is easy to see that \( \varepsilon_J \) is multiplicative on \( J \). Thus \( J \) is an \( \varepsilon_J \)-subdiagonal algebra of \( B_J \).

(2) We note that the subalgebras \( A \) and \( J \) (resp., \( A_Q \)) have natural matrix norms inherited from the \( C^* \)-algebra \( B \) (resp., \( B_Q \)). Thus, by identifying \( M_n(A/J) \) with \( M_n(A)/M_n(J) \) for each \( n \geq 1 \), the quotient algebra \( A/J \) has a natural matrix norm. It is clear that the natural homomorphism
\[
\pi: A/J \to (A + B_J)/B_J \subseteq B_Q
\]
is contractive. Since \( B_J \) is an \( M \)-ideal in the \( C^* \)-algebra \( B \), it is proximinal (see \([4]\)); i.e., for every element \( a \in B \), there is an element \( m \in B_J \) such that
\[
\|a + B_J\| = \|a + m\|.
\]
Since \( \{u_n\} \) is an increasing positive approximate identity for \( J \) and thus for \( B_J \), we have
\[
\|a + B_J\| = \|a + m\|
\geq \|(a + m)(1 - u_n)\|
\geq \|a - au_n\| - \|m - mu_n\|
\geq \|a + J\| - \|m - mu_n\|.
\]
Since \( \|m - mu_n\| \to 0 \), this shows that \( \|a + B_J\| \geq \|a + J\| \) and thus the homomorphism \( \pi \) is an isometry. By applying the above argument on \( M_n(A/J) \), we can show that \( \pi \) is a complete isometry from \( A/J \) onto \( A_Q \), and thus we may identify \( A/J \) with the subalgebra \( A_Q \) in \( B_Q \). Let \( A_Q^* = (A^* + B_J)/B_J \) be the involution of \( A_Q \) in \( B_Q \). It is easy to verify that
$A_Q + A_Q^*$ is norm dense in $B_Q$ because $A + A^*$ is norm dense in $B$. If we let $\varepsilon_Q$ be the map on $B_Q$ defined by

$$\varepsilon_Q(b + B_J) = \varepsilon(b) + B_J,$$

for every $b \in B$, it is clear that $\varepsilon_Q$ is a conditional expectation on $B_Q$ with range

$$\varepsilon_Q(B_Q) = \varepsilon(B) + B_J \subseteq D_Q = A_Q \cap A_Q^*,$$

and $\varepsilon_Q$ is multiplicative on $A_Q$. Thus $\varepsilon_Q$ determines a *-homomorphism from the $C^*$-algebra $D_Q$ onto the $C^*$-subalgebra $\varepsilon_Q(B_Q)$.

(3) At this time, we do not know if $\varepsilon_Q$ is automatically faithful or not. If we assume that $\varepsilon_Q$ is faithful, then for any $b + B_J \in D_Q$, $\varepsilon_Q(b + B_J) = 0$ implies $\varepsilon_Q(b^*b + B_J^*) = \varepsilon(b + B_J) = 0$ by the faithfulness of $\varepsilon_Q$ on $B_Q$. We get $b^*b + B_J = 0$ and thus $b + B_J = 0$. This shows that $\varepsilon_Q$ determines a *-isomorphism from $D_Q$ onto $\varepsilon_Q(B_Q)$. It follows that $D_Q = \varepsilon_Q(B_Q)$, and thus $A_Q$ is an $\varepsilon_Q$-subdiagonal algebra of $B_Q$.

Finally, since

$$D_J = J \cap J^* = A \cap A^* \cap B_J$$

is an ideal in $A \cap A^*$ and

$$D_Q = (A \cap A^* + B_J)/B_J = A \cap A^*/(A \cap A^* \cap B_J),$$

we get the short exact sequence

$$0 \to D_J \to A \cap A^* \to D_Q \to 0.$$ 

We end this section with an example of a unital $\varepsilon$-subdiagonal algebra of a $C^*$-algebra which has a lot of $M$-ideals containing no self-adjoint approximate identities. Thus the assumption in Theorem 2.6 that the $M$-ideals contain self-adjoint approximate identities is not redundant.

**Example 2.7.** Let $A(D)$ be the classical disc algebra and $C(T)$ the commutative $C^*$-algebra of all continuous functions on the unit circle $T$. Then $A(D)$ is an $\varepsilon$-subdiagonal algebra of $C(T)$ with respect to the faithful conditional expectation $\varepsilon$ given by

$$\varepsilon(f) = \int_T f(t) \, d\mu(t),$$

where $\mu$ is the normalized Haar measure on $T$ (cf. [9]). Letting $1_\varnothing$ be the constant function 1 on $D$, we get $A(D) \cap A(D)^* = C1_\varnothing$. 
It is known by Fakhoury [6] that a subalgebra of \( A(D) \) is an \( M \)-ideal if and only if it consists of all functions in \( A(D) \) which are null on a closed subset of \( T \) with zero measure. Thus there are a lot of \( M \)-ideals \( J \) in \( A(D) \) with \( J \cap J^* = \{0\} \), since the only self-adjoint elements in \( A(D) \) are the real multiples of the constant function \( 1_D \) (cf. [17]).

3. Subdiagonal Algebras of Groupoid \( C^* \)-Algebras

Let \( X \) be a second countable, locally compact Hausdorff space. An \( r \)-discrete principal groupoid \( G \) on \( X \) is an equivalence relation on \( X \). The groupoid structure on \( G \) is defined as follows. For \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) in \( G \), set \( x^{-1} = (x_2, x_1) \). If \( x_2 = y_1 \), then the pair \( (x, y) \) is said to be \( \text{composable} \) and we set \( xy = (x_1, y_2) \). Let \( G^2 \) be the set of all composable pairs. We assume that \( G \) is given a topology such that

1. \( G \) is a locally compact Hausdorff space;
2. the maps \( x \mapsto x^{-1} \) from \( G \) onto \( G \), and \( (x, y) \mapsto xy \) from \( G^2 \) into \( G \), are continuous, where \( G^2 \) is given the relative topology as a subset of \( G \times G \);
3. the map \( x \mapsto (x, x) \) is a homeomorphism from \( X \) onto the unit space \( G^0 = \{(x, x) : x \in X\} \);
4. \( G^0 \) is open in \( G \).

Following (3), we will identify \( X \) with \( G^0 \). We note that, from the above conditions, \( G^0 \) is also closed and thus clopen in \( G \). Given \( x \in G \), we call \( d(x) = x^{-1} \) the \( \text{domain} \) of \( x \) and \( r(x) = xx^{-1} \) the \( \text{range} \) of \( x \). It is clear that \( d \) and \( r \) are well-defined continuous maps from \( G \) onto its unit space \( G^0 \). A subset \( s \) of \( G \) is called a \( G \)-set if the restrictions of \( r \) and \( d \) to \( s \) are one-to-one. If \( G \) is an \( r \)-discrete principal groupoid with a cover by clopen \( G \)-sets, then there is a \( \text{left Haar system} \) \( \{\hat{x}^y\}_{x \in X} \) such that \( \hat{x}^y \) is given by the counting measure on \( G^x = r^{-1}(x) \) for each \( x \in X \).

Let \( T \) be the group of complex numbers with modulus 1. A \( \text{continuous} \) \( 2 \)-cocycle \( \sigma \) is a continuous map from \( G^2 \) into \( T \) such that

\[
\sigma(x_0, x_1, x_2) \sigma(x_0, x_1) = \sigma(x_1, x_2) \sigma(x_0, x_1 x_2)
\]

for all \( (x_0, x_1) \) and \( (x_1, x_2) \in G^2 \). Let \( C_c(G) \) be the space of all continuous functions on \( G \) with compact supports. Given a continuous \( 2 \)-cocycle \( \sigma \), we can define an involutive Banach algebra structure on \( C_c(G) \) as follows. For \( f, g \in C_c(G) \), the multiplication is given by

\[
f \ast g(x) = \int f(xy) g(y^{-1}) \sigma(xy, y^{-1}) d\hat{x}^y(y),
\]
and the involution is given by
\[ f^*(x) = f(x^{-1}) \sigma(x, x^{-1}) \]
for \( x \in G \).

Let \( C_c(G, \sigma) \) denote the involutive algebra with the multiplication and involution defined as above. Then the groupoid \( C^* \)-algebra \( C^*(G, \sigma) \) and the reduced groupoid \( C^* \)-algebra \( C^*_{red}(G, \sigma) \) are the completion of \( C_c(G, \sigma) \) under suitable \( C^* \)-norms. When the groupoid \( G \) is amenable, as defined in [14, 10], these two \( C^* \)-algebras coincide.

A continuous 2-cocycle \( \sigma \) is said to be normalized if for every pair \((x, y) \in G^2 \) with \( x = (x_1, x_2) \) and \( y = (x_2, x_3) \), we have
\[ \sigma(x, y) = 1 \]
whenever at least two of the three elements \( x_1, x_2, \) and \( x_3 \) are equal. Using an argument similar to that in [7, Proposition 7.7], one can prove that every continuous 2-cocycle is cohomologous to a normalized continuous 2-cocycle, and the corresponding groupoid \( C^* \)-algebras are *-isomorphic. If \( \sigma \) is normalized, then for all \( h, k \in C_c(G^0) \), \( f \in C_c(G, \sigma) \), and \( x \in G \), we have
\[ h \ast f(x) = \int h(xy) \, f(y^{-1}) \, \sigma(xy, y^{-1}) \, d\lambda^{d(x)}(y) = h(r(x)) \, f(x) \]
and
\[ f \ast k(x) = \int f(xy) \, k(y^{-1}) \, \sigma(xy, y^{-1}) \, d\lambda^{d(x)}(y) = f(x) \, k(d(x)) . \]

Throughout this section and the Appendix, we let \( X \) be a second countable, locally compact Hausdorff space and let \( G \) be an amenable r-discrete principal groupoid on \( X \) with a cover by clopen \( G \)-sets. We also assume that every continuous 2-cocycle \( \sigma \) from \( G \) into \( T \) is normalized.

We note that under these hypotheses, every element in \( C^*(G, \sigma) \) can be represented as a continuous function on \( G \) [14, II.4.2] and we will use \( C^*(G^0) \) to denote the subalgebra of elements in \( C^*(G, \sigma) \) supported on \( G^0 \). Given an open subset \( P \) of \( G \), we let \( A(P) \) denote the set of all elements in \( C^*(G, \sigma) \) supported on \( P \). An open subset \( P \) of \( G \) is called a preorder [10] in \( G \) if it contains \( G^0 \) and satisfies
\[ P \ast P \subseteq P . \]

A preorder \( P \) is said to be total in \( G \) if \( P \cup P^{-1} = G \). We call \( P \) a subgroupoid of \( G \) if \( P \) is a preorder and \( P = P^{-1} \). In the Appendix
(Theorem A.7), we will generalize Theorem 3.10 of [10] which states that every norm closed $C^*(G^0)$-bimodule $A$ in $C^*(G, \sigma)$ can be uniquely written as $A = A(P)$ for some open subset $P$ in $G$. In particular, every subalgebra $A$ (resp., $C^*$-subalgebra $D$) of $C^*(G, \sigma)$ containing $C^*(G^0)$ can be uniquely represented as $A = A(P)$ (resp., $D = A(H)$) for some open preorder $P$ (resp., open subgroupoid $H$) of $G$.

Let $E$ be a subset of $G^0$. The reduction of $G$ by $E$ is

$$G|_E = \{ x \in G: \text{both } d(x) \text{ and } r(x) \in E \},$$

with the groupoid structure inherited from $G$. We note that $(G|_E)^0 = E$. A subset $E$ of $G^0$ is said to be invariant if for all $x \in G$, we have $r(x) \in E$ if and only if $d(x) \in E$. Let $P$ be an open subset of $G$ such that $A(P)$ is an ideal of $C^*(G, \sigma)$. Then $P \cap G^0$ is an invariant subset of $G^0$. Conversely, given an open invariant subset $E$ of $G^0$, $A(G|_E)$ is an ideal of $C^*(G, \sigma)$. By restricting the functions in $A(G|_E)$ to $G|_E$, we have $A(G|_E) \cong C^*(G|_E, \sigma|_E)$. This gives a one-to-one correspondence between ideals of $C^*(G, \sigma)$ and open invariant subsets of $G^0$ [14, Proposition II.4.5].

Given $G$ as above, let $E, F$ be subsets of $G$ such that $F$ is open and $E$ is a compact subset of $F$. Then there exists $h \in C_c(G)$ with supp$(h) \subseteq F$ such that $h(x) = 1$ for all $x \in E$ and $0 \leq h(x) \leq 1$ for all $x \in G$. We will denote such a function by

$$E < h < F.$$

**Theorem 3.1.** Let $D$ be a $C^*$-subalgebra of $C^*(G, \sigma)$ containing $C^*(G^0)$ and $H$ the corresponding open subgroupoid in $G$ such that $D = A(H)$. Then there is a conditional expectation $\varepsilon$ from $C^*(G, \sigma)$ onto $A(H)$ if and only if $H$ is clopen.

In this case, $\varepsilon$ is faithful and uniquely determined by the restriction map to $H$, i.e., $\varepsilon(f) = f|_H$ for every $f \in C^*(G, \sigma)$.

**Proof.** If $H$ is a clopen subgroupoid of $G$, then the map $\varepsilon_H$ defined by

$$\varepsilon_H(f) = f|_H$$

for all $f \in C_c(G, \sigma)$ is a projection from $C_c(G, \sigma)$ onto $C_c(H, \sigma_H)$. Since $H$ is open in $G$, the norm closure of $C_c(H, \sigma_H)$ in $C^*(G, \sigma)$ is just the $C^*$-subalgebra $A(H)$. Slightly modifying the proof given in [14, Proposition II 2.9 (iii)], we can show that for every $g \in C_c(G, \sigma)$, the map $f \mapsto \varepsilon_H(g \ast f \ast g)$ satisfies

$$\|\varepsilon_H(g \ast f \ast g)\| \leq \|g\|^2 \|f\|$$

for all $f \in C_c(G, \sigma)$. We note that all norms in the above inequality are considered as the norm on $C^*(G, \sigma)$. We claim that $\varepsilon_H$ is contractive on


\( C_c(G, \sigma) \), and thus can be extended to a conditional expectation, i.e., a projection of norm one, from \( C^*(G, \sigma) \) onto \( A(H) \). To see this, for any \( f \in C_c(G, \sigma) \), we let \( K \) denote the support of \( f \), which is a compact subset of \( G \). Thus \( r(K) \cup d(K) \) is a compact subset of \( G^0 \), and there is a continuous function \( g \in C_c(G^0) \) such that

\[
    r(K) \cup d(K) \prec g \prec G^0.
\]

It follows that

\[
    f = g \ast f \ast g
\]

and

\[
    \| \varepsilon_H(f) \| = \| \varepsilon_H(g \ast f \ast g) \| \leq \| f \|.
\]

Conversely, let \( \varepsilon \) be any conditional expectation from \( C^*(G, \sigma) \) onto \( A(H) \). For any \( f \in C_c(G, \sigma) \), it is clear that \( \varepsilon(f)(x) = 0 \) for \( x \notin H \) since \( \varepsilon(f) \) is supported on \( H \). To show \( \varepsilon(f)(x) = f(x) \) for every \( x \in H \), we may assume that the support of \( f \) is contained in a clopen \( G \)-set \( s \). Let \( s_0 = s \cap H \) and fix any point \( x \in H \). If \( x \in s_0 \), then we can choose continuous functions \( h, k \in C_c(G^0) \) such that

\[
    \{ r(x) \} \prec h \prec r(s_0) \quad \text{and} \quad \{ d(x) \} \prec k \prec d(s_0).
\]

Since \( h \ast f \ast k \) has compact support contained in \( s_0 \subseteq H \), we get \( h \ast f \ast k \in A(H) \). This implies

\[
    f(x) = h \ast f \ast k(x)
    = \varepsilon(h \ast f \ast k)(x)
    = h \ast \varepsilon(f) \ast k(x)
    = h(r(x)) \varepsilon(f)(x) \, k(d(x))
    = \varepsilon(f)(x).
\]

If \( x \notin s_0 \), then there exist open subsets \( U \) and \( V \) of \( x(\equiv G^0) \) such that \( r(x) \in U \), \( d(x) \in V \) and \( (U \times V) \cap s = \emptyset \). Choose \( h, k \in C_c(G^0) \) such that

\[
    \{ r(x) \} \prec h \prec U \quad \text{and} \quad \{ d(x) \} \prec k \prec V.
\]

We have

\[
    h \ast f \ast k = 0 \Rightarrow f(x) = 0 = h \ast f \ast h(x) = \varepsilon(h \ast f \ast h)(x) = \varepsilon(f)(x).
\]

Hence \( \varepsilon(f) = f \mid_H \) for \( f \in C_c(G, \sigma) \). For general \( f \in C^*(G, \sigma) \), there is a sequence of \( \{ f_n \} \in C_c(G, \sigma) \) such that \( f_n \to f \) in norm. It follows that \( \varepsilon(f_n) \to \varepsilon(f) \) in norm. Therefore we get

\[
    \varepsilon(f)(x) = \lim_{n \to \infty} \varepsilon(f_n)(x) = 0
\]
for all \( x \not\in H \), and

\[
e(f)(x) = \lim_{n \to \infty} e(f_n)(x) = \lim_{n \to \infty} f_n(x) = f(x)
\]

for all \( x \in H \). This shows that \( e(f) = f|_H \) for every \( f \in C^*(G, \sigma) \).

Next we show that \( H \) must be closed. Suppose not, then there exists an element \( x_0 \in H \setminus H \). Thus there is a continuous function \( f \in C^*(G, \sigma) \) such that \( f = 1 \) on an open neighborhood \( V_0 \) of \( x_0 \) and a sequence \( \{x_n\} \) in \( H \cap V_0 \) converging to \( x_0 \) in \( G \). This implies

\[
0 = e(f)(x_0) = \lim_{n \to \infty} e(f)(x_n) = \lim_{n \to \infty} f(x_n) = 1,
\]

a contradiction. Hence, the subgroupoid \( H \) must be closed in \( G \).

Finally, we show that the conditional expectation \( e \) is faithful and uniquely determined. It follows from the above argument that any conditional expectation from \( C^*(G, \sigma) \) onto \( A(H) \) is given by the restriction map to \( H \). Hence it is unique. The faithfulness of \( e \) follows from

\[
e(f^* \ast f) = f^* \ast f|_H = 0 \Rightarrow f^* \ast f|_{G^0} = 0 \Rightarrow f = 0. \]

**Theorem 3.2.** Let \( A = A(P) \) be a subalgebra of a groupoid \( C^*- \)algebra \( C^*(G, \sigma) \) containing \( C^*(G^0) \). Then \( A \) is an \( e \)-subdiagonal algebra of \( C^*(G, \sigma) \) if and only if \( P \) is a total clopen preorder in \( G \).

Furthermore, \( A \) is a maximal \( e \)-subdiagonal algebra of \( C^*(G, \sigma) \).

**Proof.** Let \( A = A(P) \) be an \( e \)-subdiagonal algebra of \( C^*(G, \sigma) \) containing \( C^*(G^0) \). To verify that \( P \) is a total clopen preorder in \( G \), we only need to verify that \( P \cup P^{-1} = G \) and \( P \) is closed in \( G \).

By definition, \( f(x) = 0 \) for all \( f \in A \) and \( x \not\in P \). Since \( A + A^* \) is norm dense in \( C^*(G, \sigma) \), it follows that for every \( x \not\in P \cup P^{-1} \) we have \( h(x) = 0 \) for all \( h \in C^*(G, \sigma) \). Hence, we must have \( P \cup P^{-1} = G \).

Let \( D = A \cap A^* = A(H) \), where \( H = P \cap P^{-1} \). Since there is a faithful conditional expectation \( e \) from \( C^*(G, \sigma) \) onto the \( C^* \)-subalgebra \( A(H) \), it follows from Theorem 3.1 that \( H \) is clopen in \( G \). This implies that \( P^{-1} \setminus H \) is open in \( G \), and thus \( P = G \setminus (P^{-1} \setminus H) \) is closed in \( G \).

Conversely, suppose \( P \) is a clopen total preorder in \( G \). Then \( A = A(P) \) is a norm closed subalgebra of \( C^*(G, \sigma) \) containing \( C^*(G^0) \). Since \( H = P \cap P^{-1} \) is a clopen subgroupoid in \( G \), it follows from Theorem 3.1 that the map given by the restriction to \( H \) is a faithful conditional expectation from \( C^*(G, \sigma) \) onto \( D = A(H) \).

Given any \( f, g \in A(P) \), and any \( x \in H \), we have
\[
f \ast g(x) = \int_{y \in G} f(xy) g(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y)
\]
\[
= \int_{xy \in P, y \in P^{-1}} f(xy) g(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y)
\]
\[
= \int_{y \in H} f(xy) g(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y)
\]
\[
= \int_{y \in G} f|_H(xy) g|_H(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y)
\]
\[
= f|_H \ast g|_H(x).
\]

If \( x \notin H \), it is easy to see that

\[
\int_{y \in G} f|_H(xy) g|_H(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y) = 0.
\]

This shows that \( \varepsilon \) is multiplicative on \( A \), i.e., for any \( f, g \in A \) we have

\[ \varepsilon(f \ast g) = \varepsilon(f) \ast \varepsilon(g). \]

The norm density of \( A + A^* \) in \( C^*(G, \sigma) \) follows from \( G = P \cup P^{-1} \). Thus \( A \) is an \( \varepsilon \)-subdiagonal algebra of \( C^*(G, \sigma) \).

Finally, we need to show that \( A(P) \) is a maximal \( \varepsilon \)-subdiagonal algebra. Let

\[ A_m = \{f \in C^*(G, \sigma): \varepsilon(g \ast f \ast h) = \varepsilon(h \ast f \ast g) = 0 \}
\]

for all \( g, h \in A(P) \) such that \( \varepsilon(g) = 0 \).

In [9, Theorem 3.1], Kawamura and Tomiyama prove that for any unital \( \varepsilon \)-subdiagonal algebra \( A \) of a \( C^* \)-algebra \( B \), \( A_m \) is a maximal \( \varepsilon \)-subdiagonal algebra of \( B \) containing \( A \). By using Proposition 2.5, we can generalize this result to \( \varepsilon \)-subdiagonal algebras which are not necessarily unital. Thus it suffices to prove that \( A_m \subseteq A(P) \).

Following the idea of [10], for \( f \in A_m \), we only need to show that

\[ f(x) = 0 \]

for all \( x \notin P \). Given \( x \notin P \), we have \( y = x^{-1} \in P \setminus H \). So, there exists a clopen \( G \)-set \( s \subseteq P \) such that \( y \in s \) and \( s \cap H = \emptyset \). Choose continuous functions \( a \) and \( b \) in \( C_c(G) \) such that

\[ \{y\} < a < s, \quad \text{and} \quad \{r(y)\} < b < G^0. \]
It is clear that \( a, b \in A(P) \) and \( \varepsilon(a) = 0 \). This implies that
\[
\varepsilon(a \ast f \ast b) = 0,
\]
and thus
\[
f(x) = a(y) f(x) b(r(y)) = (a \ast f \ast b)(r(y)) = \varepsilon(a \ast f \ast b)(r(y)) = 0.
\]

Next we turn our consideration to \( M \)-ideals and quotients of \( \varepsilon \)-subdiagonal algebras. Given a groupoid \( G \), we will assume that all faithful conditional expectations \( \varepsilon \) on \( C^*(G, \sigma) \) contain \( C^*(G^0) \) in their ranges. In particular, we let \( \varepsilon_0 \) denote the faithful conditional expectation from \( C^*(G, \sigma) \) onto \( C^*(G^0) \). If \( A = A(P) \) is an \( \varepsilon \)-subdiagonal algebra of \( C^*(G, \sigma) \), then every ideal \( J \) in \( A \) can be uniquely represented in the form of \( J = A(P_J) \) for some open subset \( P_J \) contained in \( P \) such that \( P \ast P_J \ast P \subseteq P_J \) (see Theorem A.8). Letting \( G^0_J = G^0 \cap P_J \), we write
\[
P'_J = \{ x \in P : \text{either } r(x) \text{ or } d(x) \in G^0_J \}
\]
and
\[
P''_J = \{ x \in P : \text{both } r(x) \text{ and } d(x) \in G^0_J \}.
\]

**Theorem 3.3.** The following are equivalent.

1. \( J \) is an \( M \)-ideal in \( A \), i.e., \( J \) admits an approximate identity.
2. \( P_J = P'_J = P''_J \).
3. \( J \) has an increasing positive approximate identity contained in \( C^*(G^0_J) \).

**Proof.** (1) \( \Rightarrow \) (2). It is clear that \( P'_J \subseteq P_J \subseteq P_J \). For each \( x_0 \in P_J \), there is an open \( G \)-set \( s \) such that \( x_0 \in s \subseteq P_J \). We can find a continuous function \( a \in C_c(G) \) such that
\[
\{ x_0 \} < a < s.
\]
Since \( s \subseteq P_J \), \( a \) is an element in \( J = A(P_J) \). Letting \( \{ a_z \} \) be an approximate identity for \( J \), we have
\[
\| a_z \ast a - a \|_\infty \leqslant \| a_z \ast a - a \| \to 0.
\]
It follows that
\[
a_z \ast a(x_0) = a_z(r(x_0)) \ast a(x_0) \to a(x_0) = 1.
\]
Therefore, we get \( a_z(r(x_0)) \neq 0 \) for some \( z \). Thus \( r(x_0) \in G^0_J \). Similarly, we can show that \( d(x_0) \in G^0_J \). Hence we get \( P_J \subseteq P''_J \).

(2) \( \Rightarrow \) (3). Since \( G \) is a second countable, locally compact Hausdorff space and \( G^0_J \) is an open subset of \( G^0 \), \( G^0_J \) is also a second
countable, locally compact Hausdorff space. Thus $G^0_J$ is $\sigma$-compact and there exists a sequence $\{U_n\}$ of open sets such that $G^0_J = \bigcup_n U_n$ and that $\overline{U}_n$ is a compact subset of $U_{n+1}$ for all $n$. Let $h_n \in C_c(G^0_J)$ such that $\overline{U}_n < h_n < U_{n+1}$. It follows from the hypothesis (2) that $\{h_n\} \in C^*(G^0_J)$ is an increasing positive approximate identity for $J$.

(3) $\Rightarrow$ (1). This is trivial. 

The above theorem shows that an $M$-ideal $J = A(P_J)$ in an $\varepsilon$-subdiagonal algebra $A = A(P)$ of $C^*(G, \sigma)$ always has increasing positive approximate identities. Writing $G_J = P_J \cup P_J^{-1}$, we have $G_J = \{x \in G: \text{either } r(x) \text{ or } d(x) \in G^0_J\}$

$= \{x \in G: \text{both } r(x) \text{ and } d(x) \in G^0_J\}$.

This shows that $G^0_J$ is an open invariant subset of $G^0$ and $G_J$ is the reduction of $G$ by $G^0_J$. It is clear that $G_J$ is an amenable $r$-discrete principal groupoid with a cover by clopen $G_J$-sets and $G^0_J$ is the clopen unit space of $G_J$.

Let $G_Q = G \setminus G_J$, $P_Q = P \setminus P_J$, and $G^0_Q = G^0 \setminus G^0_J$. Then $G^0_Q$ is a closed invariant subset of $G^0$ and $G_Q$ is the reduction of $G$ by $G^0_Q$, which is closed in $G$. It is also clear that $G_Q$ is an amenable $r$-discrete principal groupoid with a cover by clopen $G_Q$-sets and $G^0_Q$ is the clopen unit space of $G_Q$. The corresponding groupoid $C^*$-algebra $C^* (G_Q, \sigma_Q)$ is $*$-isomorphic to the quotient $C^*$-algebra $C^* (G, \sigma) / C^* (G_J, \sigma_J)$ [14, II.4.5]. Here, we use $\sigma_J$ and $\sigma_Q$ to denote the restrictions of $\sigma$ to $G_J$ and $G_Q$, respectively.

**Theorem 3.4.** Let $A = A(P)$ be an $\varepsilon$-subdiagonal algebra of $C^*(G, \sigma)$ and let $J = A(P_J)$ be an $M$-ideal in $A$. Then

1. $J$ is an $\varepsilon_J$-subdiagonal algebra of $C^*(G_J, \sigma_J)$.
2. The quotient algebra $A/J$ is completely isometrically isomorphic to the $\varepsilon_Q$-subdiagonal algebra $A(P_Q)$ of $C^*(G_Q, \sigma_Q)$.
3. We have the short exact sequence

$$0 \to D_J \to D \to D_Q \to 0,$$

where $D_J = A(P_J \cap P_J^{-1})$, $D = A(P \cap P^{-1})$, and $D_Q = A(P_Q \cap P_Q^{-1})$.

**Proof.** (1) Let $B_J = (J + J^*)^- = A(P_J \cup P_J^{-1})$. We note that $B_J \cong C^*(G_J, \sigma_J)$ and the $*$-isomorphism is given [14, II.4.5] by the restriction of the functions in $B_J$ to $G_J$. So the result follows from Theorem 2.6 (1).
(2) Let $B_Q = C^*(G, \sigma)/B_J$. The restriction of functions in $C^*(G, \sigma)$ to $G_Q$ induces [14, II.4.5] an $*$-isomorphism between $B_Q$ and $C^*(G_Q, \sigma_Q)$. Let $\varepsilon_Q$ be the conditional expectation on $C^*(G_Q, \sigma_Q)$ induced by $\varepsilon$. It follows from Theorem 3.1 that $\varepsilon_Q$ is faithful. Thus the result follows from Theorem 2.6 (2) and (3).

(3) This follows immediately from parts (1), (2), and Theorem 2.6 (3).

We conclude this section by looking at a special class of groupoid $C^*$-algebras. A $C^*$-algebra $B$ is called an AF algebra if there exists an increasing sequence of finite dimensional $C^*$-subalgebras $\{B_n\}$ such that $B = (\cup B_n)^{-}$. If $B$ is unital, we require that $B_1$ contains the unit 1 of $B$. A maximal abelian self-adjoint subalgebra (masa) $D$ of an AF algebra $B = (\cup B_n)^{-}$ is called standard if there exists an increasing sequence $\{D_n\}$, such that each $D_n$ is a masa in $B_n$ and $D = (\cup D_n)^{-}$. It has been shown by Stratila and Voiculescu in [15] that every AF algebra $B$ has a standard masa $D$ and there exists a unique faithful conditional expectation $\varepsilon_0$ from $B$ onto $D$. Let $B$ be an AF algebra with a standard masa $D$ and $X = \hat{D}$, the maximal ideal space of $D$. Then there is an AF-groupoid $G$ on $X$ such that $B \cong C^*(G)$ [14]. Let $P$ be an open subset of $G$ such that $P \circ P \subseteq P$, then the subalgebra $A(P)$ is an $\varepsilon_0$-subdiagonal algebra of $B$ if and only if $P \cup P^{-1} = G$ and $P \cap P^{-1} = G^0$ [10, Theorem 4.2]. Following the results of [10, 19, 20], we have that $\varepsilon_0$-subdiagonal subalgebras are the same as the strongly maximal triangular subalgebras of $B$ as defined in [12]. Let $B$ be an AF algebra with a standard masa $D$ and $A$ an $\varepsilon$-subdiagonal algebra of $B$ containing $D$. Suppose $J$ is an $M$-ideal in $A$ and $Q = A/J$. It follows that both $B_J$ and $B_Q$ are AF. Thus we have

**Corollary 3.5.** Let $B$ be an AF algebra with a standard masa $D$ and $A$ an $\varepsilon$-subdiagonal algebra of $B$ containing $D$. Suppose $J$ is an $M$-ideal in $A$ and $Q = A/J$. Then $J$ (resp., $Q$) is an $\varepsilon J$ (resp., $\varepsilon Q$)-subdiagonal algebra of the AF algebra $B_J$ (resp., $B_Q$). In particular, if $A$ is a strongly maximal triangular subalgebra of $B$, then both $J$ and $Q$ are strongly maximal triangular.

**Remark 3.6.** In a forthcoming paper [13], we are going to study the class $\mathcal{A}$ of subdiagonal algebras of AF algebras in more details. In particular, we obtain a converse of Corollary 3.5, i.e., if $A$ is a subdiagonal algebra of a $C^*$-algebra such that the sequence

$$0 \to J \to A \to Q \to 0$$

is exact for some $J, Q \in \mathcal{A}$, then $A \in \mathcal{A}$. 
The following is an example of an ideal (but not an \( M \)-ideal) of a strongly maximal triangular subalgebra of \( AF \)-algebra such that neither the ideal nor its quotient is an \( \varepsilon \)-subdiagonal algebra of any \( C^* \)-algebra.

**Example 3.7.** Let \( M_3 \) be the \( 3 \times 3 \) matrix algebra and \( D_3 \) the diagonal matrices of \( M_3 \). Then the algebra of upper triangular matrices \( T_3 \) in \( M_3 \) is a finite dimensional strongly maximal triangular subalgebra of \( M_3 \). Let \( J = \text{span} \{ e_{13} \} \). It is easy to see that \( J \) is an ideal of \( T_3 \). Since \( J \) has no approximate identity, it is clear that \( J \) can not be isometrically isomorphic to an \( \varepsilon \)-subdiagonal algebra of any \( C^* \)-algebra. We are going to show that the quotient algebra \( Q = T_3/J \) is also not isometrically isomorphic to an \( \varepsilon \)-subdiagonal algebra of any \( C^* \)-algebra.

Let \( \pi: T_3 \to Q \) be the natural surjection and \( f_{ij} = \pi(e_{ij}) \) for \( 1 \leq i \leq j \leq 3 \). Then \( Q \) is a unital non-commutative operator algebra of dimension 5. Suppose \( \phi: Q \to A \) is an isometric isomorphism of \( Q \) onto an \( \varepsilon \)-subdiagonal algebra \( A \) of some \( C^* \)-algebra \( B \). Given \( x \in Q \), then \( x \) is an idempotent of norm 1 if and only if \( \phi(x) \) is a (self-adjoint) projection in \( A \cap A^* \). An easy calculation shows that for every idempotent \( x \in Q \) of norm 1, there exists a self-adjoint projection \( a \in D_3 \) such that \( \pi(a) = x \). This shows that \( A \cap A^* = \phi \circ \pi(D_3) \) and \( B = A + A^* \) is a non-commutative \( C^* \)-algebra of dimension 7. It follows that \( B \) is \( * \)-isomorphic to the \( C^* \)-algebra \( M_2 \oplus C \oplus C \oplus C \). Then direct computation shows that \( \phi(f_{12}) \) and \( \phi(f_{23}) \) are linearly dependent, a contradiction.

4. **\( M \)-Ideals and Quotients for Subdiagonal Algebras of von Neumann Algebras**

In this section, we study the von Neumann algebra version of the previous results. First recall the definition of subdiagonal algebras of von Neumann algebras (cf. [11]). Let \( B_0 \) be a von Neumann subalgebra of a von Neumann algebra \( B \) and \( \varepsilon \) a faithful normal conditional expectation from \( B \) onto \( B_0 \).

**Definition 4.1.** A \( \sigma \)-weakly closed unital subalgebra \( A \) of \( B \) is called an \( \varepsilon \)-subdiagonal algebra of \( B \) if it satisfies

1. \( A + A^* \) is \( \sigma \)-weakly dense in \( B \)
2. \( \varepsilon \) is multiplicative on \( A \)
3. \( \varepsilon(B) = A \cap A^* \).

**Theorem 4.2.** Let \( A \) be an \( \varepsilon \)-subdiagonal algebra of a von Neumann algebra \( B \) and \( J \) a \( \sigma \)-weakly closed subspace of \( A \). Then \( J \) is an \( M \)-ideal in \( A \) if and only if \( J = pA \) for a central projection \( p \) in \( B \).
In this case, \( J = pA \) is a subdiagonal algebra of \( pB \) and \( A/J \cong (1 - p) A \) is a subdiagonal algebra of \( (1 - p) B \).

Proof. If \( J \) is an \( M \)-ideal in \( A \), then \( J = pA \) for a central projection in \( A \) by [5, Theorem 2.2]. Since \( A + A^* \) is \( \sigma \)-weakly dense in the von Neumann algebra \( B \), then \( p \) must be a central projection in \( B \). The proof for the converse is trivial.

In this case, we have

\[
J \cap J^* = p(A \cap A^*)
\]

and

\[
(J + J^*)^{-\sigma} = p(A + A^*)^{-\sigma} = pB.
\]

Hence, \( J \) is a subdiagonal algebra of the von Neumann algebra \( pB \), since

\[
e(pB) = pe(B) = p(A \cap A^*) = J \cap J^*.
\]

A similar argument shows that \( A/J \cong (1 - p) A \) is a subdiagonal algebra of the von Neumann algebra \( (1 - p) B \).

Now we study the properties of \( M \)-ideals and quotients of subdiagonal algebras of groupoid von Neumann algebras. Let \((X, \mathcal{B}, \mu)\) be a standard Borel measure space. An equivalence relation \( R \subseteq X \times X \) is called **standard** if \( R \) is a Borel subset in the product \( \sigma \)-field. The standard equivalence relation \( R \) is called countable if for every \( x \in X \)

\[
R(x) = \{ y \in X : (x, y) \in R \}
\]

is a countable set. Throughout this section, \( R \) will denote a standard and countable equivalence relation on \( X \).

Given \( R \) associated with \( X \), define two maps \( \pi_l \) and \( \pi_r \) from \( R \) onto \( X \) by

\[
\pi_l(x, y) = x \quad \text{and} \quad \pi_r(x, y) = y,
\]

for all \( (x, y) \in R \).

Given a Borel subset \( C \) of \( X \), we let

\[
R(C) = \{ y \in X : (x, y) \in R \text{ for some } x \in C \}.
\]

The set \( C \) is called **saturated** if \( \mu(R(C) \setminus C) = 0 \). Given a standard groupoid \( R \) and a Borel 2-cocycle \( s \), one can define a groupoid von Neumann algebra \( M(R, s) \) and a Cartan subalgebra \( A(R, s) \) of \( M(R, s) \) as in [8]. Again we may assume that the 2-cocycle \( s \) is normalized. Given a subset \( C \) of \( X \), let \( \mathcal{C} \) denote the characteristic function on \( C \).
Theorem 4.3. A Borel subset $C$ of $X$ is saturated if and only if $\mathcal{X}_C$ is a central projection of $M(R, s)$ contained in $A(R, s)$.

Proof. Recall from [8] that elements in $M(R, s)$ can be represented as functions on $R$ and $A(R, s)$ consists of functions in $M(R, s)$ which are supported on the diagonal $\Delta(X) = \{(x, x) : x \in X\}$ of $X \times X$. Identifying $\Delta(X)$ with $X$, we have $\mathcal{X}_C$ is an element in $A(R, s)$ for every Borel subset $C$ of $X$. Let $\phi$ be a partial Borel isomorphism of $X$ such that its graph $\Gamma(\phi) \subseteq R$. Then it follows from [8, 2.4] that the linear span of all $f \cdot \mathcal{X}_{\Gamma(\phi)}$, where $f \in A(R, s)$ and $\Gamma(\phi) \subseteq R$, is dense in $M(R, s)$. Thus, for a Borel subset $C$ of $X$, $\mathcal{X}_C$ is a central projection of $M(R, s)$ if and only if $\mathcal{X}_C$ commutes with all $\mathcal{X}_{\Gamma(\phi)}$. Let $\xi \in L^2(R, v)$, then we have

$$(\mathcal{X}_C \cdot \mathcal{X}_{\Gamma(\phi)})(\xi)(x, z) = \begin{cases} \mathcal{X}_C(x) \xi(\phi(x), z) s(x, \phi(x), z) & \text{if } x \in D(\phi) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(\mathcal{X}_{\Gamma(\phi)} \cdot \mathcal{X}_C)(\xi)(x, z) = \begin{cases} \mathcal{X}_C(\phi(x)) \xi(\phi(x), z) s(x, \phi(x), z) & \text{if } x \in D(\phi) \\ 0 & \text{otherwise,} \end{cases}$$

where $D(\phi)$ indicates the domain of $\phi$. Thus if $C$ is saturated, $\mathcal{X}_C$ commutes with every $\mathcal{X}_{\Gamma(\phi)}$ and hence, is a central projection in $M(R, s)$.

Conversely, suppose that $\mathcal{X}_C$ is a central projection of $M(R, s)$, we may choose [7] a sequence of partial Borel isomorphisms $\{\phi_i\}$ such that $\pi^{-1}_i(C) = \bigcup_{i=1}^\infty \Gamma(\phi_i)$. Let $\xi = \mathcal{X}_x$. For each $i$, we have

$$(\mathcal{X}_C \cdot \mathcal{X}_{\Gamma(\phi_i)})(\xi)(x, z) = \begin{cases} \mathcal{X}_C(x) & \text{if } x \in D(\phi_i) \text{ and } z = \phi_i(x) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(\mathcal{X}_{\Gamma(\phi_i)} \cdot \mathcal{X}_C)(\xi)(x, z) = \begin{cases} 1 & \text{if } x \in D(\phi_i) \text{ and } z = \phi_i(x) \\ 0 & \text{otherwise,} \end{cases}$$

Hence, $\mathcal{X}_C \mathcal{X}_{\Gamma(\phi_i)} = \mathcal{X}_{\Gamma(\phi_i)} \mathcal{X}_C$ implies that $\mu(D(\phi_i) \setminus C) = 0$ for all $i$. Since $R(C) = \pi, \pi^{-1}(C) = \cup D(\phi_i)$, we have $\mu(R(C) \setminus C) = 0$.

Given a Borel subset $Q$ of $R$, we write

$$\mathcal{F}(Q) = \{a \in M(R, s) : a \text{ is supported on } Q\}.$$  

It follows from [11] that every $\sigma$-weakly closed $A(R, s)$-bimodule $A$ of $M(R, s)$ containing $A(R, s)$ can be written as $A = \mathcal{F}(Q)$. $A$ is a subalgebra of $M(R, s)$ if and only if $Q \circ Q \subseteq Q$ and $A$ contains $A(R, s)$ if and only if
\[ A(X) \subseteq Q \cap Q^{-1}. \] If \( B \) is a von Neumann subalgebra of \( M(R, s) \) containing \( A(R, s) \) associated with the Borel subset \( Q \), then the restriction map \( e(a) = a|_Q \) defines a conditional expectation from \( M(R, s) \) onto \( B \) (cf. [11, Theorem 3.4]). It follows that a \( \sigma \)-weakly closed subalgebra \( A \) containing \( A(R, s) \) is a subdiagonal algebra of \( M(R, s) \) if and only if \( A + A^* \) is \( \sigma \)-weakly dense in \( M(R, s) \). The following result is an immediate consequence of Theorems 4.2 and 4.3.

**Corollary 4.4.** Let \( B = M(R, s) \) and \( A \) a subdiagonal algebra of \( B \) containing \( A(R, s) \). Then a \( \sigma \)-weakly closed ideal \( J = \mathcal{F}(Q) \) of \( A \) is an \( M \)-ideal of \( A \) if and only if the Borel subset \( C = \pi_1(Q) \) is saturated. In this case, \( J \) is a subdiagonal algebra of the groupoid von Neumann algebra \( M(R|_C, s) \) and the quotient \( A/J \) is a subdiagonal algebra of the groupoid von Neumann algebra \( M(R|_{X \setminus C}, s) \).

**Appendix**

We begin with an example which has motivated us to consider the class of amenable \( r \)-discrete principal groupoid \( G \) with a cover by clopen \( G \)-sets.

**Example A.1.** Let \( B = C([0, 1], M_2) \) be the \( C^* \)-algebra of all continuous maps from the unit interval \([0, 1]\) into \( M_2 \). Then \( B \) can be represented as a groupoid \( C^* \)-algebra \( C^*G, \sigma \) where

\[ G = \{ (e_{ij}, x): x \in [0, 1] \text{ for } i, j = 1, 2 \}. \]

The groupoid structure on \( G \) is given by:

1. \((e_{ij}, x) \text{ and } (e_{kl}, y) \in G \) are composable if and only if \( j = k \) and \( x = y \), where \((e_{ij}, x)(e_{ij}, x) = (e_{ij}, x)\).
2. \((e_{ij}, x)^{-1} = (e_{ji}, x)\).
3. The topology on \( G \) is that induced by the usual topology on \([0, 1]\).
4. \( \sigma \) is the trivial 2-cocycle on \( G \).

It is easy to see that \( G \) is an amenable \( r \)-discrete principal groupoid with a cover by compact open \( G \)-sets.

Let \( I = \{ f \in B: f(0) = 0 \} \). Then \( I \) is an ideal of \( B \). Let \( H \) be the reduction of the groupoid \( G \) by the invariant open subset \( = (0, 1] \). Thus, \( I \) is \(*\)-isomorphic to the groupoid \( C^* \)-algebra of \( H \). Since the only compact open subset in \( H \) is the empty set \( \emptyset \), \( H \) has no cover by compact open \( H \)-sets.
Let \( \mathcal{G} \) be the class of amenable \( r \)-discrete principal groupoid \( G \) which has a cover by clopen \( G \)-sets. Suppose \( G \in \mathcal{G} \) and \( S \) is an open invariant subset of \( G^0 \). Let \( H \) (resp., \( K \)) be the reduction groupoid of \( G \) by \( S \) (resp., \( G^0 \setminus S \)), then it is easy to show that both \( H \) and \( K \) are in \( \mathcal{G} \). Thus, if \( G \in \mathcal{G} \) and \( I \) is an ideal of \( C^*(G, \sigma) \), then both \( I \) and \( C^*(G, \sigma)/I \) are isomorphic to \( C^* \)-algebras of groupoids in \( \mathcal{G} \).

In the rest of this section, we assume that every groupoid \( G \) is amenable \( r \)-discrete principal with a cover by clopen \( G \)-sets, the unit space \( G^0 \) is a second countable, locally compact Hausdorff space, and every continuous 2-cocycle from \( G \) into \( T \) is normalized. We show that the major results in [10] can be generalized to this context. The main difference between our argument and that in [10] is that for a clopen \( G \)-set \( K \), the characteristic function \( \chi_K \) may not lie in \( C^*(G, \sigma) \). Except for some changes to accommodate this difference, our proofs are borrowed directly from those in [10].

First, we recall that every element \( f \in C^*(G, \sigma) \) can be represented as a function in \( C_0(G) \) with

\[
\|f\|_{\infty} \leq \|f\|.
\]

The following proposition is an easy consequence of [14, Proposition II.4.2 (ii)], which will be very useful in our argument.

**Proposition A.2.** If \( f \in C^*(G, \sigma) \) with the support contained in a \( G \)-set, then

\[
\|f\|_{\infty} = \|f\|.
\]

Let \( \mathcal{A} \subseteq C^*(G, \sigma) \) be a norm closed \( C^*(G^0) \)-bimodule. We write

\[
Q(\mathcal{A}) = \{ x \in G : a(x) = 0 \text{ for all } a \in \mathcal{A} \}.
\]

It is clear that \( Q(\mathcal{A}) \) is a closed subset of \( G \). On the other hand, if \( Q \) is a closed subset of \( G \), we write

\[
I(Q) = \{ a \in C^*(G, \sigma) : a = 0 \text{ on } Q \}.
\]

Since for every \( h, k \in C_c(G^0) \) and \( f \in C^*(G, \sigma) \) we have

\[
h * f(x) = \int h(xy) f(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y) = h(r(x)) f(x)
\]

and

\[
f * k(x) = \int f(xy) k(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y) = f(x) k(d(x))
\]
for all $x, y \in G$, it is clear that $I(Q)$ is a $C^*(G^0)$-bimodule in $C^*(G, \sigma)$. The space $I(Q)$ is norm closed in $C^*(G, \sigma)$. To see this, suppose that $\{f_n\}$ is a sequence in $I(Q)$ converging to some $f \in C^*(G, \sigma)$ in norm. Then we have

$$\|f_n - f\|_\infty \leq \|f_n - f\|_2 \to 0.$$ 

This implies that $\{f_n\}$ is pointwise convergent to $f$ on $G$. Hence, we must have $f \in I(Q)$.

Given $\mathcal{U}$ a norm closed $C^*(G^0)$-bimodule in $C^*(G, \sigma)$, it is clear that $\mathcal{U} \subseteq I(Q(\mathcal{U}))$. Our first goal is to show that $\mathcal{U} = I(Q(\mathcal{U}))$, which is a generalization of [10, Theorem 3.10]. First we need to generalize some lemmas in [10, Sect. 3].

**Lemma A.3.** Let $s$ be a clopen $G$-set in $G$. Then the map $f \mapsto f|_s$, the restriction of $f$ to $s$, is a contractive linear map on $C^*(G, \sigma)$.

**Proof.** Let $s$ be a clopen $G$-set. If $f \in C_c(G, \sigma)$, it is clear that $f|_s$ has a compact support $K$ contained in the $G$-set $s$. From Proposition A.2,

$$\|f|_s\| = \|f|_s\|_\infty \leq \|f\|_\infty \leq \|f\|.$$ 

Hence, the map $f \mapsto f|_s$ is a linear contraction on $C_c(G, \sigma)$ and has a natural extension to a linear contraction on $C^*(G, \sigma)$. $\blacksquare$

Next, we note that Lemma 3.2 through Proposition 3.6 in [10] are also valid when we replace the condition "$t$ is a compact open $G$-set" by "$t$ is a clopen $G$-set." Thus, we simply state the generalized Proposition 3.6 in [10] as follows.

**Proposition A.4.** Let $\mathcal{U}$ be a norm closed $C^*(G^0)$-bimodule in $C^*(G, \sigma)$. For every $a \in \mathcal{U}$ and any clopen $G$-set $s$, we have $a|_s \in \mathcal{U}$.

Let $Q$ be a closed subset of the groupoid $G$ and $Q^c = G \setminus Q$. We denote $C_c(Q^c)$ the space of all $f \in I(Q)$ with compact support $\text{supp}(f) \subseteq Q^c$ and we denote $I_c(Q)$ the space of all $f \in I(Q)$ with compact support. It is clear that $C_c(Q^c) \subseteq I_c(Q) \subseteq I(Q)$. In general $C_c(Q^c)$ is a proper subspace of $I_c(Q)$ since we might have $f \in I_c(Q)$ such that $\text{supp}(f) \cap Q \neq \emptyset$. The following lemma shows that these two spaces have the same norm closure when the groupoid $G$ admits a cover of clopen $G$-sets. Our discussion differs from that given in [10, Proposition 3.8].

**Lemma A.5.** $C_c(Q^c)^- = I_c(Q)^-$. 

**Proof.** We only need to show that every $f \in I_c(Q)$ is contained in $C_c(Q^c)^-$. Given $f \in I_c(Q)$, the support of $f$ can be covered by finitely many clopen $G$-sets, say $s_1, \ldots, s_n$. We may assume that $s_1, \ldots, s_n$ are pairwise
disjoint and thus we can write $f = \sum f|_{s_i}$. Obviously each $f|_{s_i} \in C^*(G, \sigma)$ by Lemma A.3 and $f|_{s_i} = 0$ on $Q$. Hence, we may assume $f \in I_c(Q)$ with $\text{supp}(f)$ contained in a clopen $G$-set $s$. Let $O_f = \{x \in G : f(x) \neq 0\}$. Thus $O_f$ is an open subset of $s \cap Q^c$ with the closure $O_f^- = \text{supp}(f)$. Since $r(O_f)$ is an open subset of $G^0$, it is $\sigma$-compact. Hence, we can choose an increasing sequence $\{U_n\}$ of open subsets of $r(O_f)$ such that $r(O_f) = \bigcup n U_n$ and for every $n$, $U_n^-$ is compact and contained in $U_{n+1}$. Therefore there is a sequence of functions $h_n \in C_c(r(O_f))$ such that

$$U_n^- < h_n < U_{n+1}^- .$$

We get $h_n * f \in C_c(Q^c)$ with compact support contained in the open $G$-set $O_f$. Since $f \in C_0(O_f)$, we have, by Proposition A.2, that

$$\|h_n * f - f\| = \|h_n * f - f\|_{\infty} \to 0 .$$

This shows $f \in C_c(Q^c)^-$. \]

**Lemma A.6.** For every norm closed $C^*(G^0)$-bimodule $\mathcal{A}$ in $C^*(G, \sigma)$, we have

$$C_c(Q(\mathcal{A}))^c \subseteq \mathcal{A} .$$

**Proof.** Given $f \in C_c(Q(\mathcal{A}))^c$, we may assume, without loss of generality, that the support $K$ of $f$ is contained in a clopen $G$-set $s$. Let $s_0 = s \cap Q(\mathcal{A})^c \supseteq K$. For any $x \in K$, there is an element $a_x \in \mathcal{A}$ such that $a_x(x) > 0$. It follows from Proposition A.4 that we can get $a_x$ with the support contained in the clopen $G$-set $s$. It is clear that there is an open subset $V_x$ of $s_0$ containing $\{x\}$ such that $a_x(x) > \varepsilon > 0$ on $V_x$. Since $K$ is compact, there are finitely many such $V_x$'s covering $K$, say $V_1, \ldots, V_n$. Let $V = \bigcup_{i=1}^n V_i$ and $a = \sum_{i=1}^n a_i \in \mathcal{A}$. Then we have $K \subseteq V \subseteq s_0$ and $a > \varepsilon$ on $V$ for some $\varepsilon > 0$.

We can find a function $u \in C^*(G^0)$ with $r(K) < u < r(V)$ and define a function $h \in C_c(G^0)$ by

$$h(t, t) = u(t, t) \cdot \frac{1}{a(t, s(t))}$$

for all $(t, t) \in r(V)$ and $h(t, t) = 0$ otherwise. Since $\mathcal{A}$ is a $C^*(G^0)$-bimodule, we get $h * a \in \mathcal{A}$ with $h * a(x) = 1$ for all $x \in K$ and $h * a(x) = 0$ for all $x \in G \setminus V$. Define $g(t, t) = f(t, s(t))$. Then $g \in C^*(G^0)$. Thus, we have $f = g * (h * a) \in \mathcal{A}$. \]

**Theorem A.7.** For every norm closed $C^*(G^0)$-bimodule $\mathcal{A}$ in $C^*(G, \sigma)$ we have

$$\mathcal{A} = I(Q(\mathcal{A})).$$
For every closed subset $Q_0$ of $G$, we have

$$Q(I(Q_0)) = Q_0.$$ 

**Proof.** Given a norm closed $C^*(G^0)$-bimodule $\mathfrak{U}$ in $C^*(G, \sigma)$, it follows from Lemma A.6 that $C_c(\mathfrak{U})^* \subseteq \mathfrak{U} \subseteq I(Q(\mathfrak{U}))$. By Lemma A.5, $C_c(Q(\mathfrak{U}))^* = I_c(Q(\mathfrak{U}))$. It remains to show that $I_c(Q(\mathfrak{U}))^* = I(Q(\mathfrak{U}))$. But this has been proved by Muhly and Solel [10, 3.9 and 3.10].

The second statement is Lemma 3.11 in [10].

Theorem A.7 established a one-to-one correspondence between the norm closed $C^*(G^0)$-bimodules $\mathfrak{U}$ in $C^*(G, \sigma)$ and the closed subsets $Q(\mathfrak{U})$ in $G$. Given an open subset $P$ in $G$, we write $A(P) = I(G \setminus P)$. Let $P(\mathfrak{U}) = G \setminus Q(\mathfrak{U})$. This gives a one-to-one correspondence between the norm closed $C^*(G^0)$-bimodules $\mathfrak{U}$ in $C^*(G, \sigma)$ and open subsets $P(\mathfrak{U})$ in $G$, and it is easy to show that the correspondence preserves inclusion, i.e., if $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are norm closed $C^*(G^0)$-bimodules in $C^*(G, \sigma)$, then $\mathfrak{U}_1 \subseteq \mathfrak{U}_2$ if and only if $P(\mathfrak{U}_1) \subseteq P(\mathfrak{U}_2)$. In particular, $C^*(G^0) \subseteq \mathfrak{U}$ if and only if $G^0 \subseteq P(\mathfrak{U})$.

**THEOREM A.8.** Let $A = A(P)$ a norm closed $C^*(G^0)$-bimodules in $C^*(G, \sigma)$. We have

1. $A$ is a subalgebra of $C^*(G, \sigma)$ containing $C^*(G^0)$ if and only if $P$ is an open preorder in $G$. In this case, $A^* = A(P^{-1})$ and $A \cap A^* = A(P \cap P^{-1})$.

2. $A$ is a $C^*$-subalgebra of $C^*(G, \sigma)$ containing $C^*(G^0)$ if and only if $P$ is an open subgroupoid in $G$.

3. There is a one-to-one correspondence between all ideals $J = A(P_J)$ of $A = A(P)$ and open subsets $P_J$ of $P$ satisfying $P : P_J \subseteq P \subseteq P_J$.

**Proof.** Owing to Theorem A.7, (1) can be proved by using a similar argument as that in [10, Theorem 4.1]. Part (2) follows from (1). Part (3) is an easy generalization of [10, Lemma 4.3].

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