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AF SUBALGEBRAS OF CERTAIN CROSSED PRODUCTS

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ABSTRACT. Let (X, T) be a dynamical system with X zero dimensional. Each closed subset Y of X gives rise to a subalgebra A_Y of the crossed product C^*-algebra $C(X) \times_T \mathbf{Z}$. We give a necessary and sufficient condition on Y for A_Y to be an AF algebra. Suppose Y_1 and Y_2 are two clopen subsets satisfying the condition. We show that Y_1 and Y_2 are homeomorphic as topological spaces if and only if the AF algebras A_{Y_1} and A_{Y_2} are stably isomorphic. Finally, we show that, if the non-periodic points are dense in X and Y is a minimal subset satisfying the condition, then A_Y is a maximal AF subalgebra among the regular subalgebras of $C(X) \times_T \mathbf{Z}$.

1. Introduction. Given a compact space X, $C(X)$ will denote the C^*-algebra of complex continuous functions on X. A compact metrizable space X is said to be zero dimensional if the topology on X has a basis consisting of sets which are both closed and open (clopen). In this note we study systems (X, T) where X is a zero dimensional space and T is a homeomorphism on X. Given such a system, we have an action of the integers $\mathbf{Z}$ on $C(X)$. This gives a crossed product algebra $C(X) \times_T \mathbf{Z}$ (see Pedersen [**5**]) which is a C^*-algebra generated by $C(X)$ and a unitary U satisfying $UfU^* = f \circ T^{-1}$ for $f \in C(X)$. In [**7**], we show that the order structure on $K_0(C(X) \times_T \mathbf{Z})$ is useful in the study of classification problems of such systems and the crossed product algebras. (We will use Blackadar [**1**] and Effros [**3**] for our reference on K-theory). A system (X, T) is said to be minimal if X contains no non-empty T-invariant proper closed subsets. In recent works [**9, 10**], Putnam proved, among other results, that if X does not have isolated points and the system (X, T) is minimal, then, for every closed subset Y, the C^*-subalgebra of $C(X) \times_T \mathbf{Z}$ generated by $C(X)$ and $\{Uf : f \in C(X), f(y) = 0 \text{ for all } y \in Y\}$ is an AF algebra [**9, 10**], i.e., A_Y is the closure of an increasing sequence of finite dimensional subalgebras. This result is crucial in his study of AF-subalgebras of $C(X) \times_T \mathbf{Z}$ [**10**] and the order structure of $K_0(C(X) \times_T \mathbf{Z})$ [**9**]. In §2, given any (X, T) (not necessarily minimal) and a closed subset Y, we prove that A_Y is an AF algebra if and only if, for every clopen subset

W containing $Y, \cup_{n\in\mathbf{Z}} T^n(W) = X$. Let $D(X,T)$ be the set of closed subsets Y having the above property. In §3, we study the ordered group $K_0(A_Y)$ for Y in $D(X,T)$. Suppose $Y_1, Y_2 \in D(X,T)$ are clopen. We show that Y_1 and Y_2 are homeomorphic if and only if A_{Y_1} and A_{Y_2} are stably isomorphic (see Pedersen [**5**] or definitions in §3). Let $E(X,T)$ be the set of minimal (in the sense of inclusion) elements in $D(X,T)$. Suppose the non-periodic points are dense in X and $Y \in E(X,T)$. In §4, we prove that if A is a regular subalgebra (see definition in section 4) of $C(X) \times_T \mathbf{Z}$ which contains A_Y as a proper subalgebra, then A is not AF. In particular, if (X,T) is minimal, then, for every $y \in X$, the only regular subalgebra which properly contains $A_{\{y\}}$ is the whole crossed product algebra $C(X) \times_T \mathbf{Z}$.

We list here some facts about AF algebras and K-theory of C^*-algebras which will be used later. The details can be found in the references [**1, 2, 3** and **4**].

Recall that a C^*-algebra A is said to be AF [**2**] (approximately finite dimensional) if there is an increasing sequence $\{A_n : n \geq 1\}$ of finite dimensional subalgebra of A such that $\cup_{n\geq 1} A_n$ is dense in A. Let A be an AF algebra. Then $K_0(A)$ is an ordered group with ordering given by the semisubgroup $K_0(A)^+$ of classes of projections in the matrix algebras over A (see Effros [**3**]). If X is a zero dimensional space, then $C(X)$ is a commutative AF algebra and $K_0(C(X))$ is order isomorphic to $C(X, \mathbf{Z})$, the group of integer valued continuous functions with the usual ordering [**7**]. A result of Elliot [**4**], says that the ordered group $K_0(A)$, together with a scale (see Effros [**3**]) is a complete isomorphism invariant for AF algebras. On the other hand, $K_1(A)$ is always zero for an AF algebra A. This fact can be used to show that certain C^*-algebras are not AF. For example, given any system (X,T), it follows from Pimsner and Voiculescu's exact sequence [**6**] that $K_1(C(X) \times_T \mathbf{Z}) \neq 0$. Hence, $C(X) \times_T \mathbf{Z}$ is not AF.

We wish to thank the referee for some helpful comments and the "if" part of Corollary 3.2.

2. AF subalgebras. We first establish some notation. Given a system (X,T) and a non-empty T-invariant closed subset Y of X, by restricting the functions of X and the action of T on Y, we have a C^*-homomorphism $\pi_Y : C(X) \times_T \mathbf{Z} \to C(Y) \times_T \mathbf{Z}$. Let

$\pi_Y(U) = U_Y$. Therefore, $C(Y) \times_T \mathbf{Z}$ is generated by $C(Y)$ and U_Y with $U_Y g U_Y^* = g \circ T^{-1}$ for $g \in C(Y)$. For any clopen subset W of X, let χ_W be the characteristic function on W. Then $\chi_W \in C(X)$ and $U\chi_W U^* = \chi_W \circ T^{-1} = \chi_{T(W)}$.

LEMMA 2.1. *Let A be a C^*-subalgebra of $C(X) \times_T \mathbf{Z}$ containing $C(X)$. Suppose $U\chi_{X\backslash W} \in A$ for a clopen subset W of X such that $\cup_{n\in\mathbf{Z}} T^n(W) \neq X$. Then A is not AF.*

PROOF. Let $Y = X \backslash \cup_{n\in\mathbf{Z}} T^n(W)$. Then Y is a non-empty T-invariant closed subset of X. Since $X\backslash W \supset Y, \pi_Y(U\chi_{X\backslash W}) = U_Y$ and the map $\pi_Y : A \to C(Y) \times_T \mathbf{Z}$ is surjective. Thus, the quotient $A/\ker\pi_Y \simeq C(Y) \times_T \mathbf{Z}$ is not AF. Hence, A is not AF [**2**]. □

Given a system (X, T) and a closed subset Y of X, let $C_0(X\backslash Y)$ be the set of functions in $C(X)$ which vanish on Y. Following Putnam's notation [**9, 10**], we use A_Y to denote the subalgebra of $C(X) \times_T \mathbf{Z}$ generated by $C(X)$ and $UC_0(X\backslash Y) = \{Uf : f \in C_0(X\backslash Y)\}$. Given a C^*-algebra A, let $M_n(A)$ be the $n \times n$ matrix algebra over A. The next result is essentially Putnam's construction in [**9, 10**]. We give a slight modification which allows us to compute the order structure of A_Y in §3.

LEMMA 2.2. *If Y is a clopen subset of X such that $\cup_{n\in\mathbf{Z}} T^n(Y) = X$, then A_Y is isomorphic to $\oplus_{k=1}^m M_{J_k}(C(Y_k))$ for a clopen partition $\{Y_k : 1 \leq k \leq m\}$ of Y and some positive integers $J_k, 1 \leq k \leq m$.*

PROOF. Since X is compact and Y is open, there exists an integer $n \geq 1$ such that $\cup_{k=0}^n T^k(Y) = X$. Thus, for every $y \in Y$, there exists $k \geq 1$ such that $T^k(y) \in Y$. Hence, we can define $\lambda : Y \to \mathbf{Z}$ by

$$\lambda(y) = \min\{k \geq 1 : T^k(y) \in Y\}.$$

Since Y is clopen, λ is continuous. Let $\lambda(Y) = \{J_1, \ldots, J_m\}$ with $J_1 < J_2 \cdots < J_m$. For $k = 1, \ldots, m$ and $j = 1, \ldots, J_k$, define the clopen set $Y(k, j) = T^j(\lambda^{-1}(J_k))$. Then we have:

(1) $\cup_{k=1}^{m} Y(k,1) = T(Y)$.

(2) $T(Y(k,j)) = Y(k,j+1)$ for $1 \leq j < J_k$.

(3) $\cup_{k=1}^{m} Y(k,J_k) = Y$.

It follows from definitions that the sets $Y(k,j)$ $1 \leq k \leq m, 1 \leq j \leq J_k$ are pairwise disjoint. Conditions (1), (2) and (3) imply that the union of all $Y(k,j)$ is a T-invariant subset containing Y and, hence, is equal to X. Let $Y_k = Y(k,J_k)$ for $k = 1,\ldots,m$. We are going to show that A_Y is isomorphic to the AF algebra $\oplus_{k=1}^{m} M_{J_k}(C(Y_k))$.

First we identify $C(Y_k)$ with the subalgebra $\{f \in C(X) : f(y) = 0$ for all $y \notin Y_k\}$ of $C(X)$. For each $k = 1,\ldots,m, f \in C(Y_k)$ and $i,j = 1,\ldots,J_k$, define

$$e_{ij}^{(k)} \otimes f = U^{i-j} f \circ T^{J_k - j} = f \circ T^{J_k - i} U^{i-j} \in A_Y.$$

One checks directly that

$$\{e_{ij}^{(k)} \otimes f_{ij}^{(k)} : 1 \leq k \leq m, 1 \leq i,j \leq J_k \text{ and } f_{ij}^{(k)} \in C(Y_k)\}$$

generates a C^*-subalgebra isomorphic to $\oplus_{k=1}^{m} M_{J_k}(C(Y_k))$. For $f \in C(X)$, let $f_i^{(k)} = (f \circ T^{i-J_k})\chi_{Y_k}$. We have

$$(1) \quad f = \sum_{k=1}^{m}\sum_{i=1}^{J_k} f\chi_{Y(k,i)} = \sum_{k=1}^{m}\sum_{i=1}^{J_k} f_i^{(k)} \circ T^{J_k - i} = \sum_{k=1}^{m}\sum_{i=1}^{J_k} e_{ii}^{(k)} \otimes f_i^{(k)}$$

$$(2) \qquad U\chi_{X\setminus Y} = U\sum_{k=1}^{m}\sum_{i=1}^{J_k - 1} e_{ii}^{(k)} \otimes \chi_{Y_k} = \sum_{k=1}^{m}\sum_{i=1}^{J_k - 1} e_{i+1\ i}^{(k)} \otimes \chi_{Y_k}.$$

Hence, A_Y is isomorphic to $\oplus_{k=1}^{m} M(C(Y_k))$. □

THEOREM 2.3. *A_Y is an* AF *algebra if and only if* $\cup_{n\in\mathbf{Z}} T^n(W) = X$ *for every clopen subset W containing Y.*

PROOF. For necessity, suppose the contrary that there exists a clopen subset $W \supset Y$ such that $\cup_{n\in\mathbf{Z}} T^n(W) \neq X$. Since $U\chi_{X\setminus W} \in A_Y$, by Lemma 2.1, A_Y is not AF.

To prove sufficiency, suppose Y is a closed subset of X such that $\cup_{n\in\mathbf{Z}} T^n(W) = X$ for every clopen subset W containing Y. We can choose a decreasing sequence of clopen subsets $Y_1 \supseteq Y_2 \supseteq \dots$ such that $\cap_{n=1}^{\infty} Y_n = Y$. This gives an increasing sequence of AF algebras $A_{Y_1} \subseteq A_{Y_2} \subseteq \dots$ such that the closure of $\cup_{n=1}^{\infty} A_{Y_n}$ is equal to A_Y [**10**]. Since each A_{Y_n} is an AF algebra, A_Y is also AF [**2**]. □

Let $D(X,T)$ denote the set of closed subsets Y of X such that $\cup_{n\in\mathbf{Z}} T^n(W) = X$ for every clopen subset W containing Y. From the proof of the above theorem, we have

COROLLARY 2.4. *Let $Y \in D(X,T)$. Then, for every $n \geq 1$ and any clopen subset W, $U^n\chi_W \in A_Y$ if and only if $Y \cap \left(\cup_{r=0}^{n-1} T^r(W)\right) = \emptyset$.*

PROOF. Suppose $n \geq 1$ and W is a clopen subset such that $Y \cap (\cup_{r=0}^{n-1} T^r(W)) = \emptyset$. Then $U\chi_{T^r(W)} \in A_Y$ for $r = 0, \dots, n-1$. Hence

$$U^n\chi_W = U\chi_{T^{n-1}(W)} U\chi_{T^{n-2}(W)} \dots U\chi_W \in A_Y.$$

To prove the converse, we note that there is a conditional expectation [**5, 10**], $E : C(X) \times_T \mathbf{Z} \to C(X)$ such that $||E(A)|| \leq ||a||$ for $a \in C(X) \times_T \mathbf{Z}$ and $E(\sum_m U^m f_m) = f_0$, where $f_m \in C(X)$.

Suppose $U^n\chi_W \in A_Y$. Then there exists a clopen subset Y_1 containing Y and $a \in A_{Y_1}$ such that $||U^n\chi_W - a|| < 1$. Let $a = \sum_m U^m f_m$ with $f_m \in C(X)$. We have

$$||\chi_W - f_n|| = E(U^{-n}(U^n\chi_W - a)) < 1$$

Thus, $f_n^{-1}(\{0\}) \cap W = \emptyset$. Since every a in A_{Y_1} is a linear combination of $e_{ij}^{(k)} \otimes f_{ij}^{(k)} = U^{i-j} f_{ij}^{(k)} \circ T^{J_k - j}$ with $f_{ij}^{(k)} \in C(Y_1(k, J_k))$, f_n is a linear combination of those $f_{ij}^{(k)} \circ T^{J_k-j}$ with $i - j = n$ for some $i \leq J_k$. Since $f_{ij}^{(k)} \in C(Y_1(k, J_k))$, $f_{ij}^{(k)} \circ T^{J_k-j}$ vanishes off $Y_1(k,j)$. Hence,

$$W \subset X \backslash f_n^{-1}(\{0\}) \subset \bigcup_{k=1}^{m} \bigcup_{j=1}^{J_k - n} Y_1(k,j)$$

$$\Rightarrow Y \cap \left(\bigcup_{r=0}^{n-1} T^r(W)\right) \subset Y_1 \cap \left(\bigcup_{k=1}^{m} \bigcup_{j=1}^{J_k-1} Y_1(k,j)\right) = \emptyset. \ \square$$

3. The K-theory of AF subalgebras. In this section, we will use the explicit construction in Lemma 2.2 to compute the ordered group $K_0(A_Y)$.

Let $\mathbb{K}$ be the algebra of compact operators on an infinite dimensional Hilbert space. Two C^*-algebras A, B are said to be stably isomorphic if the tensor products [**5**] $A \otimes \mathbb{K}$ and $B \otimes \mathbb{K}$ are isomorphic. A result of Elliot [**4**] says that two AF algebras A, B are stably isomorphic if and only if $K_0(A)$ and $K_0(B)$ are order isomorphic. To get a complete invariant for isomorphism of AF algebras, we need to consider the order structure together with a scale [**3, 4**], $\Gamma(A)$, which is a subset of $K_0(A)^+$. If A is a unital AF algebra, then the scale for $K_0(A)^+$ is given by

$$\Gamma(A) = \{g \in K_0(A)^+ : g \leq [1_A]\},$$

where $[1_A]$ is the class containing the identity 1_A of A. $[1_A]$ is known as an order unit for $K_0(A)$ [**3**]. Two AF algebras A, B are isomorphic if and only if there exists an order isomorphism between $K_0(A)$ and $K_0(B)$ which takes $\Gamma(A)$ onto $\Gamma(B)$ (Elliot [**4**], also see Effros [**3**] for details on scales). If A, B are unital and ϕ is an order isomorphism between $K_0(A)$ and $K_0(B)$, then $\phi(\Gamma(A)) = \Gamma(B)$ if and only if $\phi([1_A]) = [1_B]$.

PROPOSITION 3.1. *If $Y \in D(X,T)$ is clopen, then $K_0(A_Y)$ is order isomorphic to $C(Y,Z)$ with order unit $u_Y = \sum_{k=1}^m J_k \chi_{Y_k}$, where J_k and $Y_k, 1 \leq k \leq m$ are as given in Lemma* 2.2.

PROOF. From Lemma 2.2, we have a clopen partition $\{Y_k : 1 \leq k \leq m\}$ of Y and integers $J_k, 1 \leq k \leq m$ such that A_Y is isomorphic to $\oplus_{k=1}^m M_{J_k}(C(Y_k))$. Therefore

$$\begin{aligned} K_0(A_Y) &\simeq \oplus_{k=1}^m K_0[M_{J_k}(C(Y_k))] \\ &\simeq \oplus_{k=1}^m C(Y_k, \mathbf{Z}) \quad (\text{Since } K_0(M_n(A)) \simeq K_0(A)) \\ &\simeq C(Y, \mathbf{Z}). \end{aligned}$$

If $P = \oplus_{k=1}^m (p_{ij}^{(k)})$ is a projection in $\oplus_{k=1}^m M_{J_k}(C(Y_k)) \simeq A_Y$, then the class $[P]$ in $C(Y, \mathbf{Z}) \simeq K_0(A_Y)$ is given by $\sum_{k=1}^m \sum_{i=1}^{J_k} p_{ii}^{(k)}$. Thus, if f

is a projection in $C(X)$, we have

$$[f] = \sum_{k=1}^{m}\sum_{i=1}^{J_k}(f \circ T^{i-J_k})\chi_{Y_k}.$$

In particular, $[1_{A_Y}] = \sum_{k=1}^{m} J_k\chi_{Y_k}$. Hence, the ordered group $C(Y,\mathbf{Z})$ has an order unit $u_Y = \sum_{k=1}^{m} J_k\chi_{Y_k}$ and scale

$$\Gamma_Y = \{g \in C(Y,\mathbf{Z}) : 0 \leq g \leq \sum_{k=1}^{m} J_k\chi_{Y_k}\}. \square$$

COROLLARY 3.2. *Let Y_1 and Y_2 be two clopen subsets in $D(X,T)$. Y_1 and Y_2 are homeomorphic if and only if A_{Y_1} and A_{Y_2} are stably isomorphic.*

COROLLARY 3.3. *Let Y be a closed subset in $D(X,T)$ and $Y(1) \supseteq Y(2) \supseteq \ldots$ a decreasing sequence of clopen subset such that $\cap_{n=1}^{\infty} Y(n) = Y$. Then $K_0(A_Y)$ is equal to the direct limit* [3] $\lim_{n\to\infty} C(Y(n),\mathbf{Z})$ *of the scaled ordered groups $\{C(Y(n),\mathbf{Z})\}_{n\geq 1}$ where the scale of $C(Y(n),\mathbf{Z})$ is given by the order unit $u_{Y(n)} = \sum_{k=1}^{m(n)} J(n)_k\chi_{Y(n)_k}$ and the connecting homomorphism Φ_n between $C(Y(n-1),\mathbf{Z})$ and $C(Y(n),\mathbf{Z})$ is given by*

$$\Phi_n(f) = \sum_{k=1}^{m(n)} \sum_{i=0}^{J(n)_k-1} (f \circ T^{-i})\chi_{Y(n)_k}.$$

REMARK 3.4. For minimal systems (X,T), Putnam has given [**10**, Theorem 4.1] an exact sequence which relates $C(Y,\mathbf{Z}), K_0(A_Y)$ and $K_0(C(X)\times_T\mathbf{Z})$ for $Y \in D(X,T)$. This result can be easily generalized to arbitrary systems [**8**]. However, as is pointed out in [**10**], the order structure usually cannot be computed from this exact sequence.

4. Regular subalgebras.

Suppose A is a C^*-subalgebra of $C(X)\times_T\mathbf{Z}$ containing $C(X)$. Let $\mathbf{U}(A)$ be the unitary group of A. The normalizer of $C(X)$ in $\mathbf{U}(A)$ is given by

$$\mathbf{N}(C(X),A) = \{V \in \mathbf{U}(A) : VC(X)V^* = C(X)\}.$$

A is said to be regular if $\mathbf{N}(C(X), A)$ generates A. Let $Y \in D(X,T)$. Then A_Y is regular simply because every matrix algebra is generated by the permutation and diagonal matrices.

Given a decreasing chain $\{Y_i : i \in I\}, Y_i \in D(X,T)$, let $Y = \cap_{i\in I} Y_i$. If W is any clopen subset containing Y, then W contains some Y_i. Hence, $Y \in D(X,T)$. Thus we can choose a minimal (in terms of inclusion) element in $D(X,T)$. Let $E(X,T)$ be the set of minimal elements of $D(X,T)$. We are going to study regular subalgebras A of $C(X) \times_T \mathbf{Z}$ such that $A \supset A_Y$ for some $Y \in E(X,T)$. First, we need the following description of $\mathbf{N}(C(X), C(X) \times_T \mathbf{Z})$ by Putnam [**10**, Lemma 5.1]:

LEMMA 4.1. *Let (X,T) be a system where the set of non-periodic points $X_0 = \{x \in X : T^n(x) \neq x \text{ for } n \neq 0\}$ is dense in X. Then every $V \in \mathbf{N}(C(x), C(X) \times_T \mathbf{Z})$ can be decomposed into the form*

$$V = f \sum_{n\in\mathbf{Z}} p_n U^n,$$

where $f \in \mathbf{U}(C(X))$, each p_n is a projection in $C(X)$ with finitely many p_n different from $0, p_n p_m = 0$ for $n \neq m$, and

$$\sum_n p_n = \sum_n p_n \circ T^n = 1.$$

Moreover, this decomposition is unique.

REMARKS 4.2. Putnam proved the above result for minimal systems. But with slight modification, the proof also works when X_0 is dense in X.

The main result in this section is

THEOREM 4.3. *Let (X,T) be a system with X_0 dense in X. If A is a regular subalgebra of $C(X) \times_T \mathbf{Z}$ such that $A \supset A_Y$ for some $Y \in E(X,T)$, then A is not* AF.

PROOF. Since A is regular, there exists $V \in \mathrm{N}(C(X), A)$ such that $V \notin A_Y$. Let $V = f\sum_{n\in\mathbf{Z}} p_n U^n$ be the decomposition as given in Lemma 4.1. Hence, $p_n U^n \notin A_Y$ for some n. Without loss of generality, we may assume $n \geq 1$. Writing $p_n U^n = U^n \chi_W$ for a clopen set W, we have $U^n\chi_W = p_n \bar{f} V \in A$ and, from Corollary 2.4, $Y \cap (\cup_{k=0}^{n-1} T^k(W)) \neq \emptyset$. We are going to prove by induction on n that if for a clopen subset W of X such that for some $n \geq 1, U^n_{\chi_W} \in A$ and $Y \cap (\cup_{k=0}^{n-1} T^k(W)) \neq \emptyset$, then A is not AF.

(1) If $n = 1$, then $Y \cap W \neq \emptyset$. Thus, $Y \backslash W$ is a proper closed subset of Y. By the minimality of Y, there exists a clopen subset O of X containing $Y\backslash W$ such that $\cup_{n\in\mathbf{Z}} T^n(O) \neq X$. Therefore, $O \cup W \supset Y$ and $U\chi_{X\backslash O} = U\chi_W\chi_{X\backslash O} + U\chi_{X\backslash(O\cup W)} \in A$. Hence by Lemma 2.1, A is not AF.

(2) If $n > 1$, let $k = \min\{i : 0 \leq i \leq n-1, Y \cap T^i(W) \neq \emptyset\}$. We divide the proof into three cases:

(a) $k > 0$. So, $Y \cap W = \emptyset$ and $U\chi_W \in A_Y \subset A$. We have,

$$U^{n-1}\chi_{T(W)} = U^n\chi_W(U\chi_W)^* \in A$$

and

$$Y \cap \Big(\bigcup_{i=0}^{n-2} T^i(T(W))\Big) \supset Y \cap T^k(W) \neq \emptyset.$$

Hence, by the induction hypothesis, A is not AF.

(b) $k = 0$ and $T^{n-1}(Y \cap W)\backslash Y \neq \emptyset$. Choose $y \in Y \cap W$ and a clopen subset O with $y \in O$ such that $T^{n-1}(O) \cap Y = \emptyset$. Thus, $U\chi_{T^{n-1}(O)} \in A_Y \subset A$. We have, $U^{n-1}\chi_{O\cap W} = (U\chi_{T^{n-1}(O)})^* U^n \chi_W \in A$ and $Y \cap (\cup_{k=0}^{n-2} T^k(O \cap W)) \supseteq Y \cap (O \cap W) \neq \emptyset$. Hence, by the induction hypotheses, A is not AF.

(c) $k = 0$ and $T^{n-1}(Y \cap W) \subset Y$. We are going to find a T-invariant closed subset Y_1 such that the image of A under the map $\pi_{Y_1} : A \to C(Y_1) \times_T \mathbf{Z}$ is not AF. For every $y \in Y \cap W$, let $\lambda(y) = \min\{i \geq 1 : T^i(y) \in Y\}$. Thus $1 \leq \lambda(y) \leq n-1$ for all $y \in Y \cap W$. Choose $y_0 \in Y \cap W$ such that $r = \lambda(y_0)$ is a maximum. We will show that $T^r(y_0) = y_0$.

Suppose the contrary that $T^r(y_0) \neq y_0$. Choose a clopen subset O of X containing y_0 such that $T^r(O) \cap O = \emptyset$ and $T^i(O) \cap Y = \emptyset$ for

$1 \le i < r$. From the definition of r, we have $T^r(O \cap W \cap Y) \subset Y$. Therefore $Y_2 = Y \backslash (O \cap W)$ is a proper closed subset of Y and $Y_2 \supset T^r(O \cap W \cap Y)$. Hence, for every clopen subset $W_2 \supset Y_2$, we have $T^{-r}(W_2) \supset (O \cap W \cap Y)$. This gives $W_2 \cup T^{-r}(W_2) \supset Y$ which implies

$$\bigcup_{m \in \mathbf{Z}} T^m(W_2) = \bigcup_{m \in \mathbf{Z}} T^m(W_2 \cup T^{-r}(W_2)) = X.$$

Thus Y_2 is also in $D(X,T)$, contradicting the minimality of Y.

Let $Y_1 = \{T^i(y_0) : 0 \le i < r-1\}$. Then Y_1 is a T-invariant closed subset of X. We choose a clopen subset Q containing y_0 such that $T^i(Q) \cap Y = \emptyset$ for $1 \le i \le r-1$. Therefore $U\chi_{T(Q) \cup \ldots T^{r-1}(Q)} \in A_Y \subset A$. Hence,

$$V = U^n \chi_W \chi_Q + U\chi_{T(Q) \ldots T^{r-1}(Q)} \in A.$$

Since $T^{n-1}(y_0) \in Y$, we have that r divides $n-1$. Let $n = rs+1$ for some integer $s \ge 0$. Since Y_1 contains r points permuted cyclically by T, we can describe $C(Y_1) \times_T \mathbf{Z}$ very explicitly:

Let S be the set of complex numbers of modulus 1. Then $C(Y_1) \times_T \mathbf{Z}$ is isomorphic to $M_r(C(S))$. Under this isomorphism, $f \in C(Y_1)$ is given by a diagonal matrix with diagonal equal to $[f(y_0), f(T(y_0)), \ldots, f(T^{r-1}(y_0))]$ and U_{Y_1} is equal to (u_{ij}) with $u_{ii-1} = 1$ for $2 \le i \le r, u_{1r} = z$, the identity function on S and $u_{ij} = 0$ elsewhere. Therefore

$$\pi_{Y_1}(V) = U_{Y_1}^n \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & & \end{bmatrix} + U_{Y_1} \begin{bmatrix} 0 & & \ldots & 0 \\ & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \ldots & 0 & z \\ z^s & 0 & & & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & 0 \end{bmatrix}.$$

Given a unitary matrix $B \in M_r(C(S)) \simeq C(Y_1) \times_T \mathbf{Z}$, the corresponding class [B] in $K_1(M_r(C(S))) \simeq \mathbf{Z}$ is given by the winding number of $\det B$. Thus $[\pi_{Y_1}(V)] = (-1)^{r-1}(s+1) \neq 0$ in $\mathbf{Z}$. Hence, $\pi_Y(A)$ and consequently, A is not AF. □

If (X, T) is minimal, then $\{y\} \in E(X, T)$ for every $y \in X$. Since X does not have periodic points, Case 2(c) in the proof of the above theorem does not occur. Thus, by induction, we can assume $n = 1$ and (1) shows that $U \in A$. Hence we have

COROLLARY 4.4. *Let (X, T) be a minimal system and $y \in X$. If A is a regular subalgebra such that $A \supset A_{\{y\}}$, then $A = C(X) \times_T M$.*

REMARK 4.5. T-invariant sets in $D(X, T)$ and $E(X, T)$ have shown [**8**, **11**], to be useful in determining when the invertible elements in $C(X) \times_T \mathbf{Z}$ are dense.

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