THE VANISHING OF THE FUNDAMENTAL GAP OF CONVEX DOMAINS IN $\mathbb{H}^n$

THEODORA BOURNI, JULIE CLUTTERBUCK, XUAN HIEN NGUYEN, ALINA STANCU, GUOFANG WEI, AND VALENTINA-MIRA WHEELER

Abstract. For the Laplace operator with Dirichlet boundary conditions on convex domains in $\mathbb{H}^n$, $n \geq 2$, we prove that the product of the fundamental gap with the square of the diameter can be arbitrarily small for domains of any diameter.

1. Introduction

We consider the low eigenvalues of the Laplace operator $-\Delta$ with Dirichlet boundary conditions on a convex, compact domain Ω of $\mathbb{H}^n$. This operator has a discrete spectrum with ∞ as its accumulation point. If the sequence of eigenvalues is arranged in increasing order $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, the *fundamental gap* is the difference between the first two eigenvalues

$$\lambda_2 - \lambda_1 > 0.$$

In 1983, while investigating the thermodynamic functions of a free boson gas, van den Berg observed that for many convex domains in Euclidean space, the fundamental gap has a lower bound $\lambda_2 - \lambda_1 \geq 3\pi^2/D^2$, where D is the diameter of the domain [17]; the lower bound was conjectured to hold for all convex domains by Yau, and Ashbaugh and Benguria [3, 18]. It is known that for non-convex domains the fundamental gap has no such lower bound; for non-connected domains, the gap may vanish.

In 2011, Andrews and Clutterbuck showed that the conjecture holds [1]. The result is sharp, with the limiting case being rectangles that collapse to a line. We refer to this paper for history and earlier work on this important subject, see also the survey article [8].

The question of a lower bound on the fundamental gap in other spaces of constant curvature is well defined, but more difficult to investigate. Recently, Dai, He, Seto, Wang, and Wei (in various subsets) [7, 11, 15] generalized the estimate to convex domains in $\mathbb{S}^n$, showing that the same bound holds: $\lambda_2 - \lambda_1 \geq 3\pi^2/D^2$.

The research of Julie Clutterbuck was supported by grant FT1301013 of the Australian Research Council. The research of Xuan Hien Nguyen was supported by grant 579756 of the Simons Foundation. The research of Alina Stancu was supported by NSERC Discovery Grant RGPIN 327635. The research of Guofang Wei was supported by NSF Grant DMS 1811558. The research of Valentina-Mira Wheeler was supported by grants DP180100431 and DE190100379 of the Australian Research Council.

Not much was known in the case of hyperbolic spaces. In a previous paper [5], the authors found a convex domain in $\mathbb{H}^2$ in which the above lower bound is breached, thereby raising the question of estimating the fundamental gap for convex domains with small diameter. It is reasonable to believe that, as the diameters get smaller, the distortion from the metric becomes negligible and one would get a lower bound for the fundamental gap in terms of the diameter approaching $3\pi^2/D^2$, the bound for Euclidean space, from below. The main result of this paper is the construction of explicit examples showing that, on the contrary, for any diameter, *there is no lower bound on the gap.*

Theorem 1.1. In hyperbolic spaces $\mathbb{H}^n$, $n \geq 2$, for any constants $\varepsilon > 0$, $D > 0$, there is a convex domain Ω with diameter D whose fundamental gap satisfies

$$\lambda_2(\Omega) - \lambda_1(\Omega) < \frac{\varepsilon\pi^2}{D^2}.$$

From the discussion above, Theorem 1.1 shows that the behavior of the fundamental gap in hyperbolic spaces is drastically different from $\mathbb{R}^n$ and sphere cases. We explain the intuition behind the phenomenon in Section 2. We further remark that the quantity $D^2(\lambda_2 - \lambda_1)$ is invariant under the scaling of the metric. Hence the same result also holds for any simply connected negative constant curvature space forms.

For hyperbolic spaces, many explicit estimates on the upper and lower bounds of the first eigenvalue exist [2,10,13,14]. For the fundamental gap, Benguria and Linde [4] obtained a beautiful upper bound for any open bounded domain $\Omega \subset \mathbb{H}^n$. Namely, the gap $\lambda_2(\Omega) - \lambda_1(\Omega) \leq \lambda_2(B_\Omega) - \lambda_1(B_\Omega)$, where B_Ω is a ball in $\mathbb{H}^n$ such that $\lambda_1(B_\Omega) = \lambda_1(\Omega)$.

Our work here draws strongly on work of Shih [16], who constructed a domain in $\mathbb{H}^2$ with a first eigenfunction that is not log-concave. Shih's result highlights another difference from the situation in Euclidean cases, where the first Dirichlet eigenfunction is always log-concave [6]. Log-concavity implies that the superlevel sets of the eigenfunction are convex. We use domains similar to the ones in [16] and find that the first eigenfunction has two distinct maxima. This means that the superlevel sets are very far from being convex: they are not even connected. Thus, this article also gives a simpler proof of the existence of domains where the first eigenfunction is not log-concave.

The organization of this paper is as follows. We begin in dimension 2 for simplicity and because most of the insight can be garnered here. In Section 2, we give a heuristic explanation for the phenomenon with reference to a simple case in $\mathbb{R}^2$. In Section 3, we explicitly construct the domain for the example, and describe its shape and diameter. In Section 4, we sketch the main strategy. In Section 5, we make estimates on the first eigenvalue of the domain. In Section 6, we describe precisely the way in which the first eigenfunction is very small in the middle of the domain. In Section 7, we show that the gap goes to zero. The generalization to higher dimensions is left until Section 8.

Acknowledgements. This research originated at the workshop "Women in Geometry 2" at the Casa Matemática Oaxaca (CMO) from June 23 to June 28, 2019. We would like to thank CMO-BIRS for creating the opportunity to start work on this problem through their support of the workshop. Julie Clutterbuck thanks Ben Andrews for a useful discussion about the existence of a double-peaked eigenfunction.

2. Heuristic argument

For a bounded, connected domain, the first eigenvalue is simple, and so the fundamental gap is always positive. In order to understand our example, we begin by describing a simple situation in $\mathbb{R}^2$ where the first eigenvalue is not simple.

Let U be the disjoint union of two unit balls; this domain is not convex and not connected. The Dirichlet eigenfunctions of U are given by combinations of the eigenfunctions on each ball, which are given by Bessel functions. Let u_i be the i^{th} eigenfunction on the ball. Let $\mu_i(B)$ be the i^{th} eigenvalue on the ball. Then the first eigenfunction for U is given by two copies of u_1, translated to each ball. The first eigenvalue of U is $\mu_1(B)$. The *second* eigenfunction is given by a copy of u_1 on one ball, and a copy of $-u_1$ on the other ball. We can see this is orthogonal to the first eigenfunction, but has the same eigenvalue: the fundamental gap is zero.

The eigenvalues are continuous under perturbations of the domain. Specifically, if we join the two components of U by a small tube of width ϵ to create a new domain U_ϵ, then $\lambda_k(U_\epsilon) \to \lambda_k(U)$ as $\epsilon \to 0$ [12, Th 2.3.20]. On such a domain, the second eigenfunction is very close to u_1 on the first ball and $-u_1$ on the second ball. In the neck joining the balls, the first and second eigenfunctions are very small, and thus contribute very little to the Rayleigh quotient for either the first or second eigenvalue. Therefore the eigenvalues are ϵ-close to those on U, and the fundamental gap is close to zero.

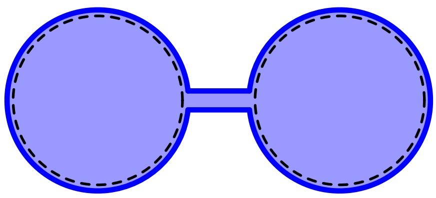
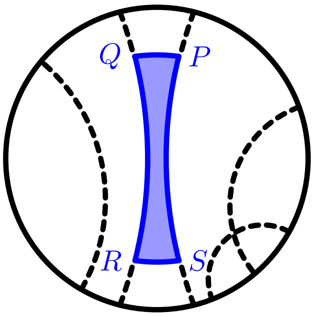


Figure 1. The fundamental gap can be small if the domain has a neck.

In the case of convex domains in $\mathbb{R}^n$ and in $\mathbb{S}^n$, such dumbbell-shaped domains are excluded: they are not convex. However, in hyperbolic space, geodesics diverge, and thus we can find a convex domain with a narrow region separating regions of relatively large area. These domains support eigenfunctions similar to that on the dumbbell domain described above, and therefore have very small gap.



Our domain in the Poincaré disc model of $\mathbb{H}^2$.

The picture of the two balls is not entirely accurate in our case because the size of the neck of our domains is not arbitrarily small compared to the distance from R to S as seen in (2). Nevertheless, the presence of a neck of shrinking width allows for the vanishing fundamental gap.

3. The domain

Let $\mathbb{H}^2$ be the hyperbolic space modeled by the Poincaré half-plane $\{(x, y) \mid y > 0\} = \{(r, \varphi) \mid r > 0, \varphi \in (-\pi/2, \pi/2)\}$ with the metric $g = ds^2 = \frac{dx^2+dy^2}{y^2}$. Note that the coordinates (r, φ) are not standard polar coordinates and are related to (x, y) by $x = r \sin\varphi$, $y = r\cos\varphi$.

Let our domain be given by

$$\Omega_{\sqrt{\mu},L} := \{(r, \varphi) : 1 \le r \le e^{\pi/\sqrt{\mu}}, -L \le \varphi \le L\},$$

where we start with an arbitrary fixed $L < \pi/2$, but will choose a suitable (large) positive μ later. The boundaries $r = 1$ and $r = e^{\pi/\sqrt{\mu}}$ are geodesics, but the other two boundaries are not.

For easier reference, we label some points of our domains (see Figure 2)

$$P = (\sin L, \cos L), \qquad Q = e^{\pi/\sqrt{\mu}}(\sin L, \cos L), \qquad R = e^{\pi/\sqrt{\mu}}(-\sin L, \cos L),$$
$$S = (-\sin L, \cos L), \qquad T = (0, 1), \qquad U = (0, e^{\pi/\sqrt{\mu}}).$$

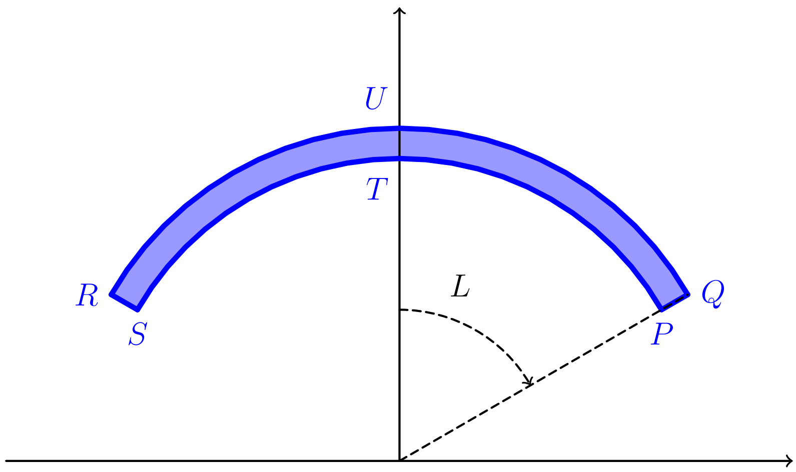


Figure 2. Domain $\Omega_{\sqrt{\mu},L} = \{(r, \varphi) \mid 1 < r < e^{\pi/\sqrt{\mu}},\ -L < \varphi < L\}$.

In an earlier paper studying the fundamental gap [5], the authors considered a similar domain $\Omega_{c,\theta_0,\theta_1}$ that may not be symmetric with respect to the geodesic $\varphi = 0$. The domains of the two papers differ by a slight change of coordinates in which the new variables are:

$$\varphi = \frac{\pi}{2} - \theta, \quad L = \frac{\pi}{2} - \theta_*, \quad \mu = c^2,$$

where $\theta_* = \min\{\theta_0, \pi - \theta_1\}$.

Lemmas 3.3 and 4.3 in our earlier paper [5] showed that the gap of $\Omega_{\sqrt{\mu},L}$ goes to $\frac{3\pi^2}{D^2}$ when μ is fixed and L goes to zero. Those convex domains are thin strips along a segment of y-axis. In this paper we will focus on the convex domains with L fixed and $\mu \to \infty$, namely thin strips along part of the upper unit circle as in Figure 2.

3.1. **The diameter.** From Proposition 4.1 of [5], the diameter $D_{\sqrt{\mu},L}$ of $\Omega_{\sqrt{\mu},L}$ is given by

$$D_{\sqrt{\mu},L} = \max\{\mathrm{dist}(P,Q), \mathrm{dist}(P,R), \mathrm{dist}(R,S)\}.$$

Since this domain is symmetric with respect to $\varphi = 0$, we have $\mathrm{dist}(R,S) = \mathrm{dist}(P,Q)$. Hence we conclude that:

Lemma 3.1. On a domain $\Omega_{\sqrt{\mu},L}$, the diameter is realized on the diagonal geodesic joining P and R.

Proof. We recall that distances in hyperbolic half-plane Poincaré model are given by

$$\mathrm{dist}\left((x_1,y_1),(x_2,y_2)\right) = \mathrm{arcosh}\left(\frac{(x_1^2+y_1^2)+(x_2^2+y_2^2)-2x_1x_2}{2y_1y_2}\right). \tag{1}$$

Thus,

$$\mathrm{dist}(R,S) = \mathrm{arcosh}\left(\frac{1+e^{2\pi/\sqrt{\mu}} - 2e^{\pi/\sqrt{\mu}}(\sin L)^2}{2e^{\pi/\sqrt{\mu}}(\cos L)^2}\right)$$
$$\mathrm{dist}(P,R) = \mathrm{arcosh}\left(\frac{1+e^{2\pi/\sqrt{\mu}} + 2e^{\pi/\sqrt{\mu}}(\sin L)^2}{2e^{\pi/\sqrt{\mu}}(\cos L)^2}\right)$$

and, since the argument of the latter is strictly greater than the argument of the former, and arcosh is increasing, the diameter must be realized on the geodesic joining P and R. ☐

We emphasize that Figure 2 may be deceiving as in $\mathbb{H}^2$, the distance from T to U is smaller than the distance from P to Q (or that from R to S). Indeed, using the formula for distance (1), the inequalities $|\sinh x| \geq |x|$ and $\mathrm{arcosh}(x^2+1) \geq \sqrt{2}x$, we get

$$\begin{aligned}\mathrm{dist}(R,S) &= \mathrm{arcosh}\left(\frac{(e^{\pi/\sqrt{\mu}}-1)^2}{2e^{\pi/\sqrt{\mu}}}\frac{1}{(\cos L)^2}+1\right)\\ &= \mathrm{arcosh}\left(2\left(\sinh\left(\tfrac{\pi}{2\sqrt{\mu}}\right)\right)^2\frac{1}{(\cos L)^2}+1\right)\\ &\geq \frac{1}{\cos L}\frac{\pi}{\sqrt{\mu}} = \frac{1}{\cos L}\,\mathrm{dist}(T,U).\end{aligned} \tag{2}$$

Finally, we remark that, for all μ larger than a fixed constant μ_2, we can bound the diameter in terms of L.

Proposition 3.2. Given any positive constant μ_2, for all $\mu > \mu_2$, the diameter $D_{\sqrt{\mu},L}$ of $\Omega_{\sqrt{\mu},L}$ is bounded by

$$\operatorname{arcosh}(1 + 2(\tan L)^2) \leq D_{\sqrt{\mu},L} \leq \operatorname{arcosh}(1 + 2(\tan L)^2) + \pi/\sqrt{\mu_2}. \tag{3}$$

Proof. The diameter is bounded below by $\operatorname{dist}(S, P)$:

$$D_{\sqrt{\mu},L} \geq \operatorname{dist}(S, P) = \operatorname{arcosh}\left(\frac{1 + (\sin L)^2}{(\cos L)^2}\right) = \operatorname{arcosh}\left(1 + 2\,(\tan L)^2\right).$$

For the bound from above, the distance formula (1) gives that $\operatorname{dist}(U, R) = \operatorname{dist}(T, S)$, so we have

$$\begin{aligned} D_{\sqrt{\mu},L} &\leq \operatorname{dist}(P, T) + \operatorname{dist}(T, U) + \operatorname{dist}(U, R) \\ &= \operatorname{dist}(P, S) + \operatorname{dist}(T, U) \\ &= \operatorname{arcosh}(1 + 2(\tan L)^2) + \pi/\sqrt{\mu}. \end{aligned}$$

□

4. Sketch of the proof of Theorem 1.1 for $n = 2$

With the upper and lower bounds on the diameter from Proposition 3.2, to prove Theorem 1.1, it suffices to show that given $L \in (0, \frac{\pi}{2})$, the fundamental gap $\lambda_2(\Omega_{\sqrt{\mu},L}) - \lambda_1(\Omega_{\sqrt{\mu},L}) \to 0$ as $\mu \to \infty$.

The domains $\Omega_{\sqrt{\mu},L}$ were chosen because they allow for separation of variables [5,16]. The eigenfunctions for the Laplace operator can be obtained by $u(r, \varphi) = h(\varphi)f(r)$, with

$$\begin{aligned} r^2 f_{rr} + r f_r &= -\mu f \text{ on } r \in (1, e^{\pi/\sqrt{\mu}}) \\ h_{\varphi\varphi} + \lambda(\sec\varphi)^2 h &= \mu h \text{ on } \varphi \in (-L, L), \end{aligned} \tag{4}$$

where λ, f, and h all depend on μ and where f and h satisfy Dirichlet boundary conditions. As pointed out in our earlier paper [5, Section 2.3], the first eigenvalue λ_1 of (4) is equal to $\lambda_1(\Omega_{\sqrt{\mu},L})$, and the second eigenvalue λ_2 of (4) is not necessarily equal to $\lambda_2(\Omega_{\sqrt{\mu},L})$[1], the second eigenvalue of the Laplace operator on our domain $\Omega_{\sqrt{\mu},L}$, but it is certainly true that

$$\lambda_2(\Omega_{\sqrt{\mu},L}) \leq \lambda_2.$$

As a consequence, $\lambda_2(\Omega_{\sqrt{\mu},L}) - \lambda_1(\Omega_{\sqrt{\mu},L}) \leq \lambda_2 - \lambda_1$. Therefore it suffices to show that $\lambda_2 - \lambda_1 \to 0$ as $\mu \to \infty$. A large part of this paper is concerned with studying eigenvalues and eigenfunctions of (4).

[1]It is the case that $\lambda_2 = \lambda_2(\Omega_{\sqrt{\mu},L})$ for μ large. Lemma 5.1 and Proposition 5.2 show that for $\eta \in (0, L)$ and $\mu > \mu_2$, we have $\lambda_1^{4\mu} \geq (\cos L)^2 4\mu > (\cos\eta)^2\mu \geq \lambda_2^{\mu}$, where $\lambda_1^{4\mu}$ is the first eigenvalue of $h_{\varphi\varphi} + \lambda(\sec\varphi)^2 h = 4\mu h$.

The first eigenvalue. Note that λ_1 is not bounded as $\mu \to \infty$. If it were, we would have that $h_{\varphi\varphi} = h\,(\mu - \lambda_1(\sec\varphi)^2) \geq 0$ for μ large when h is the first (nonnegative) eigenfunction, which contradicts the fact that h_1 vanishes at the boundary. The first step is to capture how fast λ_1 grows as $\mu \to \infty$. This is done in Section 5.

Rayleigh quotients. The first eigenvalue is a minimum of the Rayleigh quotient

$$R[h] := \frac{\displaystyle\int_{-L}^{L} (h_\varphi)^2 + \mu h^2 \, d\varphi}{\displaystyle\int_{-L}^{L} h^2 (\sec\varphi)^2 \, d\varphi}, \tag{5}$$

over the Sobolev space $\mathcal{H} = \{h \in W^{1,2}\left((-L, L)\right) \mid h(-L) = h(L) = 0\}$.

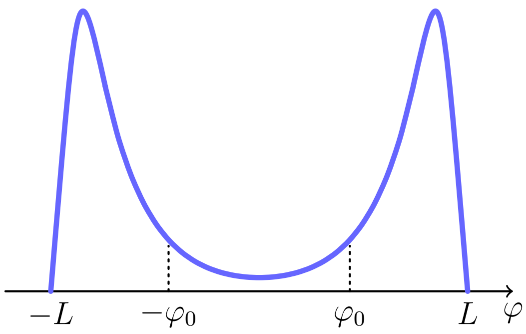


FIGURE 3. Expected graph of h_1. The maxima move towards $-L$ and L and $h_1(0) \to 0$ as $\mu \to \infty$. The constant $\varphi_0 \in (0, L/2)$ is fixed and used in Section 6.

The first eigenfunction h_1**.** A good grasp of h_1 is needed for estimating the Rayleigh quotient so in Section 6, we make precise the characteristics of the first eigenfunction: it has two maxima points as expected; as $\mu \to \infty$, the points where the maxima occur move towards $-L$ and L respectively; the value $h_1(0)$ decays to zero and the first eigenfunction becomes more flat near $\varphi = 0$.

Upper bound on the fundamental gap with Rayleigh quotients. For the first eigenvalue, we just take $\lambda_1 = R[h_1]$. The second eigenvalue λ_2 is not computed directly, but bounded above by the Rayleigh quotient of an appropriate test function. The simplest way to obtain such a function is to multiply h_1 by the following odd function: let $\psi(s)$ be the continuous piecewise linear function

$$\psi = \begin{cases} 1 \text{ for } \varphi < -\varphi_1, \\ -\varphi/\varphi_1 \text{ for } |\varphi| \leq \varphi_1, \\ -1 \text{ for } \varphi > \varphi_1, \end{cases}$$

where $\varphi_1 = \frac{\varphi_0}{\mu}$ (see Figure 4). The function ψh_1 is odd and matches h_1 on $(-L, -\varphi_1)$ and $-h_1$ on (φ_1, L).

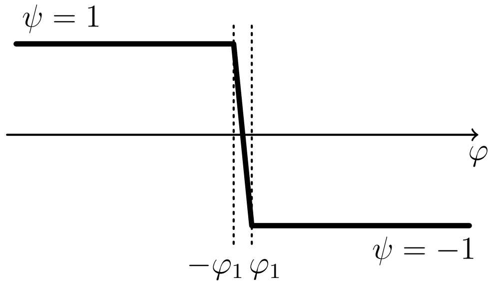


FIGURE 4. Graph of ψ.

As mentioned above, we infer that

$$\lambda_2 - \lambda_1 \leq R[\psi h_1] - R[h_1]. \tag{6}$$

Hence for estimating the fundamental gap from above, it suffices to find an upper bound on the right-hand side of the inequality consisting of the quotients' difference. The difference is concentrated on the interval $(-\varphi_1, \varphi_1)$, where we have that h_1 and its derivatives are small. The computation is done in Section 7 using estimates on h_1 from Section 6.

5. The first eigenvalue

Before we can prove that the first eigenfunction has the shape given in Figure 3, we need estimates on the first eigenvalue.

Recall the equation for the eigenfunctions

$$h_{\varphi\varphi} + (\lambda(\sec\varphi)^2 - \mu)h = 0. \tag{4}$$

We emphasize that the first eigenvalue $\lambda_1 = \lambda_1^\mu$ and the corresponding nonnegative eigenfunction $h_1 = h_1^\mu$ both depend on μ, even though it will not always be showcased in the notation.

The bound from below is a straightforward application of Wirtinger inequality. We will use the weaker $\lambda_1 \geq (\cos L)^2\mu$ in the rest of the article but include the stronger estimate for completeness. Note that Lemma 5.1 gives us that $\lambda_1 \to \infty$ as $\mu \to \infty$.

Lemma 5.1 (Bound from below). $\lambda_1 \geq (\cos L)^2 \left(\frac{\pi^2}{4L^2} + \mu\right)$.

Proof. From the characterization of the first eigenvalue through the Rayleigh quotient (5) on the Sobolev space $\mathcal{H} = \{h \in W^{1,2}((-L, L)) \mid h(-L) = h(L) = 0\}$, we have

$$\lambda_1^\mu = \inf_{h\in\mathcal{H}} \frac{\displaystyle\int_{-L}^{L} (h_\varphi)^2 + \mu h^2 \, d\varphi}{\displaystyle\int_{-L}^{L} h^2 (\sec\varphi)^2 \, d\varphi} \geq \inf_{h\in\mathcal{H}} \frac{\displaystyle\int_{-L}^{L} \left(\tfrac{\pi}{2L}\right)^2 h^2 + \mu h^2 \, d\varphi}{\displaystyle\int_{-L}^{L} h^2 (\sec\varphi)^2 \, d\varphi}$$

$$\geq \inf_{h\in\mathcal{H}} \frac{\left(\left(\tfrac{\pi}{2L}\right)^2 + \mu\right)\displaystyle\int_{-L}^{L} h^2 \, d\varphi}{(\sec L)^2 \displaystyle\int_{-L}^{L} h^2 \, d\varphi} = (\cos L)^2 \left(\left(\tfrac{\pi}{2L}\right)^2 + \mu\right). \qquad \square$$

We now control the rate of growth $\lambda_1 = \lambda_1^\mu$ from above.

Proposition 5.2. For every $\eta \in (0, L)$, there exists a μ_2 such that $\mu > \mu_2$ implies

$$\mu(\cos\eta)^2 \geq \lambda_1. \tag{7}$$

Proof. We argue by contradiction and assume that there is an $\eta \in (0, L)$ and a sequence $\mu_k \to \infty$ so that

$$\frac{\lambda_1^{\mu_k}}{\mu_k} \geq (\cos\eta)^2.$$

For those μ's and corresponding λ's,

$$\lambda_1^{\mu_k}(\sec\varphi)^2 - \mu_k \geq \lambda_1^{\mu_k}\left((\sec\varphi)^2 - (\sec\eta)^2\right).$$

For $\varphi \in \left(-L, -\frac{L+\eta}{2}\right)$, the coefficient of $\lambda_1^{\mu_k}$ is positive, bounded away from zero. Taking μ_k and therefore $\lambda_1^{\mu_k}$ large enough, we can make the right-hand side larger than $\frac{\pi^2}{4(L-\eta)^2}$. For these large μ_k's, Sturm's Comparison Theorem applied to (4) and $h_{\varphi\varphi} + \frac{\pi^2}{4(L-\eta)^2} h = 0$ would imply that $h_1^{\mu_k}$ has a zero in $\left(-L, -\frac{L+\eta}{2}\right)$. This contradicts the fact that the first eigenfunction is positive in $(-L, L)$. $\square$

Combining Lemma 5.1 and Proposition 5.2, we get the following asymptotic behavior for the ratio of λ_1 and μ:

Corollary 5.3. $\frac{\lambda_1}{\mu} \searrow (\cos L)^2$ as $\mu \to \infty$.

6. The shape of h_1

In this section, we show that the first eigenfunction behaves as claimed in Figure 3 and obtain estimates for the rate at which $h_1(0)$ tends to zero. This is done first by an integral estimate, then a pointwise estimate, then an improved integral estimate, then finally an integral estimate on the derivative.

The first eigenfunction of (4) is even because all the coefficients of (4) are even. We also assume that $h_1 > 0$ on $(-L, L)$ and is normalized so that $\int_{-L}^{L} h_1^2 (\sec\varphi)^2 d\varphi = 1$.

The first eigenfunction has two inflection points $\pm\varphi_{\mathrm{IP}}$ situated where $(\cos\varphi_{\mathrm{IP}})^2 = \lambda_1/\mu$. From Corollary 5.3, we know that those inflection points exist (i.e. the equation is satisfied in $(-L, L)$) and that $\varphi_{\mathrm{IP}} \to L$ as $\mu \to \infty$. Going to Proposition 5.2, we can describe the behavior a little better:

Corollary 6.1. For any $\eta \in (0, L)$, there is a positive constant μ_2 so that the inflection points for h_1^μ are outside of the interval $(-\eta, \eta)$ whenever $\mu > \mu_2$. Therefore the maxima of h_1 are at points outside of the interval $(-\eta, \eta)$ and the function h_1 is increasing on $(0, \eta)$.

The last property is a consequence of the concavity of h_1 and the fact that $h_1'(0) = 0$. If we seek integral bounds on some fixed interval $(-\varphi_0, \varphi_0)$, we can take $\eta \in (\varphi_0, L)$. With the corresponding μ_2 from Proposition 5.2, we have that h_1^μ is increasing on $(0, \varphi_0)$ for all $\mu > \mu_2$.

6.1. **A uniform integral bound on a subinterval.**

Lemma 6.2. Given $\varphi_0 \in (0, L)$, there exists a function $b(\mu, \varphi_0)$ with $b(\mu, \varphi_0) \to 0$ as $\mu \to \infty$ so that

$$\int_{-\varphi_0}^{\varphi_0} h_1^2 (\sec\varphi)^2 d\varphi < b(\mu, \varphi_0), \tag{8}$$

where the first eigenfunction is normalized so that $\int_{-L}^{L} h_1^2 (\sec\varphi)^2 d\varphi = 1$.

Proof. We first give an estimate of λ_1 from below. By (5) and the normalization of h_1, we have

$$\begin{aligned}
\lambda_1 = R[h_1] &> \mu \int_{-L}^{L} h_1^2 d\varphi \\
&\geq 2\mu(\cos L)^2 \int_{\varphi_0}^{L} h_1^2 (\sec\varphi)^2 d\varphi + 2\mu(\cos\varphi_0)^2 \int_0^{\varphi_0} h_1^2 (\sec\varphi)^2 d\varphi \\
&= \mu(\cos L)^2 + \mu\left((\cos\varphi_0)^2 - (\cos L)^2\right) \int_{-\varphi_0}^{\varphi_0} h_1^2 (\sec\varphi)^2 d\varphi.
\end{aligned}$$

Hence

$$\frac{\lambda_1}{\mu} - (\cos L)^2 > \left((\cos\varphi_0)^2 - (\cos L)^2\right) \int_{-\varphi_0}^{\varphi_0} h_1^2 (\sec\varphi)^2 d\varphi.$$

Simply set $b(\mu, \varphi_0) = \frac{\lambda_1/\mu - (\cos L)^2}{(\cos\varphi_0)^2 - (\cos L)^2}$. The fact that $b(\mu, \varphi_0) \to 0$ as $\mu \to \infty$ follows from Corollary 5.3. □

6.2. **A pointwise lower and upper bound on h_1 near 0.**

From now on, φ_0 is a constant in $(0, L/2)$.

We use the following Sturm comparison for Jacobi equations to obtain a lower bound for the first eigenfunction h_1 near 0.

Theorem 6.3 (Sturm Comparison Theorem)*.* For $i = 1, 2$, let f_i satisfy

$$f_i'' + b_i f_i = 0 \text{ on } (0, l),$$

and $f_1(0) = f_2(0) > 0$, $f_i'(0) = 0$. Suppose that $b_1 \geq b_2$ and $f_1 > 0$ on $(0, l)$. Then $f_1 \leq f_2$ on $(0, l)$. If $f_1 = f_2$ at $t_1 \in (0, l)$, then $b_1 \equiv b_2$ on $(0, t_1)$.

Proof. The theorem is well known. For example, it is stated in [9, Page 238-239] for the initial conditions $f_1(0) = f_2(0) = 0$, $f_1'(0) = f_2'(0) > 0$. Clearly, the same proof works for the above dual initial conditions. □

Recall once again that h_1 satisfies the Jacobi equation:

$$h_{\varphi\varphi} + (\lambda(\sec\varphi)^2 - \mu)h = 0. \tag{4}$$

with $h_1'(0) = 0$. Since, by Proposition 5.2, $\lambda_1 \leq \mu(\cos(2\varphi_0))^2$ for all μ sufficiently large, and on $(-\varphi_0, \varphi_0)$, $(\sec\varphi)^2 \leq (\sec\varphi_0)^2$, we have

$$\lambda_1(\sec\varphi)^2 - \mu \leq \mu(\sec\varphi_0)^2(\cos(2\varphi_0))^2 - \mu = -c_1\mu,$$

where we set $c_1 = 1 - (\sec\varphi_0 \cos(2\varphi_0))^2$. Remark that $c_1 > 0$.

Let $\bar{h}(\varphi) = h_1(0)\cosh(\sqrt{\mu c_1}\varphi)$. Then $\bar{h}$ satisfies the Jacobi equation

$$h_{\varphi\varphi} - c_1\mu h = 0$$

with $\bar{h}'(0) = 0$ and $\bar{h}(0) = h_1(0)$.

By the Sturm Comparison Theorem above, we have $h_1(\varphi) \geq \bar{h}(\varphi)$ on $[0, \varphi_0)$, thereby on $(-\varphi_0, \varphi_0)$ because both functions are even.

We formulate this as a lemma below.

Lemma 6.4. Fix $\varphi_0 \in (0, L/2)$. Then for all μ sufficiently large,

$$h_1(\varphi) \geq h_1(0)\cosh(\sqrt{\mu c_1}\,\varphi), \tag{9}$$

for all $|\varphi| < \varphi_0$. Here $c_1 = 1 - (\sec\varphi_0\cos(2\varphi_0))^2$.

Similarly, using the lower bound for λ_1 (Lemma 5.1) and $(\sec\varphi)^2 \geq 1$, we have

$$\lambda_1(\sec\varphi)^2 - \mu \geq (\cos L)^2\left(\frac{\pi^2}{4L^2} + \mu\right) - \mu \geq \mu\left[(\cos L)^2 - 1\right] = -(\sin L)^2\mu,$$

therefore

$$h_1(\varphi) \leq h_1(0)\cosh(\sqrt{\mu}\sin L\,\varphi). \tag{10}$$

This last estimate is used to improve the integral bound. In the meantime, (9) allows us to estimate $h_1(0)$.

Lemma 6.5. Fix $\varphi_0 \in (0, L/2)$. Then for all μ sufficiently large depending on φ_0 and L, we have that

$$h_1(0)^2 \leq 4b(\mu, \varphi_0)\exp(-\sqrt{\mu c_1}\varphi_0/2) \leq C\exp(-\sqrt{\mu c_1}\varphi_0/2), \tag{11}$$

where $c_1 = 1 - \left(\frac{\cos(2\varphi_0)}{\cos\varphi_0}\right)^2$, $b(\mu, \varphi_0)$ is the function from Lemma 6.2, and C is a positive constant independent of φ_0 and μ.

From the explicit form of $b(\mu, \varphi_0)$ in the proof of Lemma 6.2, we see that $b(\mu, \varphi_0)$ is bounded above uniformly for $\varphi_0 \in (0, L/2)$.

Proof. Inserting (9) into (8), we obtain a rough estimate on $h(0)$ for μ large as follows

$$\begin{aligned} b(\mu, s_0) &\geq \int_{-\varphi_0}^{\varphi_0} h_1^2 (\sec\varphi)^2 d\varphi \geq 2\int_0^{\varphi_0} h_1^2 d\varphi \geq 2\int_0^{\varphi_0} h_1(0)^2 (\cosh(\sqrt{\mu c_1}\varphi))^2 d\varphi \\ &\geq \frac{1}{2}\int_0^{\varphi_0} h_1(0)^2 \exp(2\sqrt{\mu c_1}\varphi) d\varphi = \frac{1}{4} h_1(0)^2 \frac{\exp(2\sqrt{\mu c_1}\varphi_0) - 1}{\sqrt{\mu c_1}} \\ &\geq \frac{1}{4} h_1(0)^2 \frac{\exp(\sqrt{\mu c_1}\varphi_0)}{\sqrt{\mu c_1}}, \end{aligned}$$

where we have used, in the first line, the even property of h_1.

The smaller coefficient for φ_0 in the exponential in (11) is to compensate for the powers of μ outside of the exponential. □

6.3. **An improved integral bound.**

Lemma 6.6. For $\varphi_1 := \varphi_0/\mu$, we have

$$\lim_{\mu\to\infty} \mu^2 \int_{-\varphi_1}^{\varphi_1} h_1^2 d\varphi = 0. \tag{12}$$

Proof. Now, using the upper bound (11) on $h_1(0)$ in (10), we obtain for $|\varphi| < \varphi_1$

$$h_1^2(\varphi) \leq C\exp(-\sqrt{\mu c_1}\varphi_0/2)\cosh^2(\sqrt{\mu}\sin L\varphi).$$

Then

$$\begin{aligned} \mu^2 \int_{-\varphi_1}^{\varphi_1} h_1^2 d\varphi &\leq 2\mu^2 C\exp(-\sqrt{\mu c_1}\varphi_0/2)\int_0^{\varphi_1} \cosh^2(\sqrt{\mu}\sin L\varphi) d\varphi \\ &= C\mu^2 \exp(-\sqrt{\mu c_1}\varphi_0/2)\left[\frac{\varphi_0}{\mu} + \frac{e^{2\sin L\frac{\varphi_0}{\sqrt{\mu}}} - e^{-2\sin L\frac{\varphi_0}{\sqrt{\mu}}}}{4\sqrt{\mu}\sin L}\right]. \end{aligned}$$

Since C is independent of μ, the last term goes to zero as $\mu \to \infty$. □

6.4. **An integral estimate on the derivative of h_1.**

Using the bound for the first eigenvalue in Lemma 5.1 we obtain a bound on h_1' for φ small.

Lemma 6.7. For $\varphi_1 := \varphi_0/\mu$, we have

$$\lim_{\mu\to\infty} \int_{-\varphi_1}^{\varphi_1} (h_1')^2 d\varphi = 0. \tag{13}$$

Proof. Choose μ_2 large such that h_1 is increasing and convex on $(0,\varphi_0)$ for $\mu > \mu_2$ (see Corollary 6.1). We first estimate $h_1'(\varphi)$ for $\varphi \in (0,\varphi_1)$. From equation (4) and the fact that $h_1'(0) = 0$, we have

$$\begin{aligned} h_1'(\varphi) &= \int_0^\varphi h_1''(t)dt = \int_0^\varphi \left(\mu - \lambda_1(\sec t)^2\right) h_1(t)dt \\ &\leq \int_0^\varphi \left(\mu - \mu(\cos L)^2(\sec t)^2\right) h_1(t)dt \leq \mu(\sin L)^2 \int_0^\varphi h_1(t)dt, \end{aligned}$$

where we used the lower bound on λ_1 from Lemma 5.1. Since h_1' is increasing on $(0,\varphi_1)$, the Cauchy-Schwarz inequality then implies

$$\left(h_1'(\varphi_1)\right)^2 \leq \mu^2(\sin L)^4\varphi_1 \int_0^{\varphi_1} h_1^2 d\varphi.$$

And the right-hand side goes to zero as $\mu \to \infty$ by (12).

□

7. Estimating the Rayleigh quotient difference

In the beginning of Section 4, we argued that Theorem 1.1 is a corollary of the bounds on the diameter (3) and the following proposition.

Theorem 7.1. Given the equation

$$h_{\varphi\varphi} + (\lambda(\sec\varphi)^2 - \mu)h = 0, \quad \varphi \in (-L,L) \tag{4}$$

with zero Dirichlet boundary conditions, the difference between the first and second eigenvalues satisfies

$$\lambda_2 - \lambda_1 \to 0, \text{ as } \mu \to \infty.$$

Proof. From inequality (6) in Section 4, it suffices to show that $R[\psi h_1] - R[h_1] \to 0$ as $\mu \to \infty$ where ψ is defined in Section 4. The difference between $|\psi h_1|$ and $|h_1|$ is supported on the interval $(-\varphi_1,\varphi_1)$ (see Figure 5).

Before we start, we set the notation for the denominator of $R[\psi h_1]$

$$\int_{-L}^{L} (\psi h_1)^2(\sec\varphi)^2 d\varphi = 1 - \int_{-L}^{L} \left((h_1)^2 - (\psi h_1)^2\right)(\sec\varphi)^2 d\varphi = 1 - \mathrm{A},$$

where $\mathrm{A} := \int_{-\varphi_1}^{\varphi_1} \left((h_1)^2 - (\psi h_1)^2\right)(\sec\varphi)^2 d\varphi$.

The difference of the Rayleigh quotients is then

$$\begin{aligned} R[\psi h_1] - R[h_1] &= \frac{1}{1-\mathrm{A}}\left\{(1-\mathrm{A})R[\psi h_1] - R[h_1] + R[h_1]\mathrm{A}\right\} \\ &= \frac{1}{1-\mathrm{A}}\left(\mathrm{B} + \mathrm{C} + \mathrm{D}\right) \end{aligned}$$

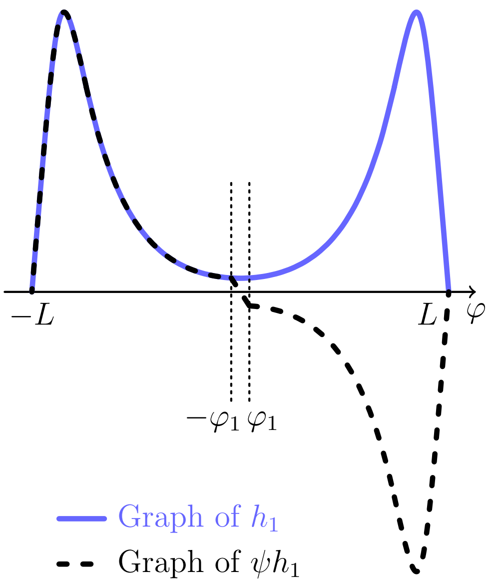


FIGURE 5. Graphs of h_1 and ψh_1.

where

$$\begin{aligned}
\mathrm{B} &= \int_{-L}^{L} \left[(\psi h_1)'^2 - h_1'^2\right] d\varphi = \int_{-\varphi_1}^{\varphi_1} \left[(\psi h_1)'^2 - h_1'^2\right] d\varphi, \\
\mathrm{C} &= \int_{-L}^{L} \mu \left[(\psi h_1)^2 - h_1^2\right] d\varphi = \int_{-\varphi_1}^{\varphi_1} \mu \left[(\psi h_1)^2 - h_1^2\right] d\varphi, \\
\mathrm{D} &= \lambda_1 \mathrm{A},
\end{aligned}$$

because $R[h_1] = \lambda_1$. Recall that $\varphi_0 \in (0, L/2)$ and $\varphi_1 = \varphi_0/\mu$. We finish the proof by showing that A, B, C, and D all go to zero as μ goes to infinity. Note that $0 < \mathrm{A} \leq \mathrm{D}$ so we can skip A.

We have

$$\begin{aligned}
|\mathrm{B}| &\leq \int_{-\varphi_1}^{\varphi_1} (h_1\psi' + h_1'\psi)^2 \, d\varphi + \int_{-\varphi_1}^{\varphi_1} h_1'^2 d\varphi \\
&\leq 2 \int_{-\varphi_1}^{\varphi_1} \left(h_1^2(\psi')^2 + (h_1')^2\psi^2\right) d\varphi + \int_{-\varphi_1}^{\varphi_1} h_1'^2 d\varphi \\
&\leq 2\frac{\mu^2}{\varphi_0^2} \int_{-\varphi_1}^{\varphi_1} h_1^2 d\varphi + 3 \int_{-\varphi_1}^{\varphi_1} (h_1')^2 d\varphi.
\end{aligned}$$

Both terms go to zero as $\mu \to \infty$ by (12) and (13) respectively.

For C, note that

$$|\mathrm{C}| \leq \mu \int_{-\varphi_1}^{\varphi_1} h_1^2 d\varphi$$

which tends to zero by (12).

Finally, $\lambda_1 \leq \mu(\cos(L/2))^2$ for μ sufficiently large by Proposition 5.2. For such μ's, we have

$$0 < \mathrm{D} \leq \mu(\cos(L/2))^2 \int_{-\varphi_1}^{\varphi_1} h_1^2(\sec\varphi)^2 d\varphi \leq \mu \int_{-\varphi_1}^{\varphi_1} h_1^2 d\varphi,$$

which tends to zero for $\mu \to \infty$ by (12). This completes the proof. □

8. Higher dimensions

In this section we generalise the above result to higher dimensions. It is a computation in coordinates. We have included minute details for ease of understanding. The coordinates are standard spherical coordinates (unlike the nonstandard coordinates (r, φ) in the rest of the article).

8.1. **Coordinates and the Laplace operator in coordinates.** Let us recall the n-dimensional spherical coordinates $(r, \omega_2, \omega_3, \dots, \omega_n)$:

$$\begin{aligned}
x_1 &= r\cos\omega_2 \\
x_2 &= r\sin\omega_2\cos\omega_3 \\
x_3 &= r\sin\omega_2\sin\omega_3\cos\omega_4 \\
&\vdots \\
x_{n-1} &= r\sin\omega_2\sin\omega_3\cdots\sin\omega_{n-1}\cos\omega_n \\
x_n &= r\sin\omega_2\sin\omega_3\cdots\sin\omega_{n-1}\sin\omega_n.
\end{aligned}$$

The metric g_{ij} in these coordinates is given by $g_{ij} = 0$ for $i \neq j$ and

$$\begin{aligned}
g_{11} &= 1 \\
g_{22} &= r^2 \\
g_{33} &= r^2(\sin\omega_2)^2 \\
g_{ii} &= r^2(\sin\omega_2)^2\cdots(\sin\omega_{i-1})^2, \qquad \text{for } i = 4, \dots, n
\end{aligned}$$

The determinant of the matrix g_{ij} is $g = \det(g_{ij}) = r^{2n-2}(\sin\omega_2)^{2n-4}\cdots(\sin\omega_{n-1})^2$. The Laplacian (in $\mathbb{R}^n$) in these coordinates is (where $\omega_1 = r$)

$$\Delta_{\mathbb{R}^n} u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial\omega_j}\left(\sqrt{g}g^{ij}\frac{\partial u}{\partial\omega_i}\right) = \sum_{i=1}^{n} \frac{g^{ii}}{\sqrt{g}}\frac{\partial}{\partial\omega_i}\left(\sqrt{g}\frac{\partial u}{\partial\omega_i}\right),$$

because g_{ij} is diagonal and the entry g_{ii} does not depend on ω_i. Replacing the values of the metric and its inverse in the equation above, we get

$$\Delta_{\mathbb{R}^n} u = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) + \sum_{i=2}^{n} \frac{g^{ii}}{(\sin\omega_i)^{n-i}}\frac{\partial}{\partial\omega_i}\left((\sin\omega_i)^{n-i}\frac{\partial u}{\partial\omega_i}\right) \tag{14}$$

8.2. **The domains.** In order to have a well-defined and computation-suited metric, we center our domain around $\omega_i = \pi/2$. The only coordinate that is not close to zero is x_n and should be the one used for the weight in the hyperbolic half-space model.

The natural generalization of our domains $\Omega_{\sqrt{\mu},L}$ is

$$\Omega_{\sqrt{\mu},\delta_2,\dots,\delta_{n-1},L} = \{(r,\omega_2,\dots,\omega_n) \mid 1 < r < e^{\pi/\sqrt{\mu}},\ |\omega_i - \pi/2| < \delta_i, |\omega_n - \pi/2| < L \text{ for } i = 2,\dots,n-1\}$$

for L and $\delta_i \in (0,\pi/2)$, $i = 2,\dots,n-1$. The δ_i's don't necessarily have to be small but it is easier to picture the domains and convince oneself that the diameter is bounded independently of μ.

As in the two-dimension case, we will study the fundamental gap for $\mu \to \infty$.

The metric on $\mathbb{H}^n$ we take is $ds^2_{\mathbb{H}^n} = \frac{1}{x_n^2} ds^2_{\mathbb{R}^n}$.

Under a conformal change of metric given by $\tilde{g} = e^{2f}g$ for a smooth function f, the Laplacian changes as $\Delta_{\tilde{g}} = e^{-2f}\Delta_g - (n-2)e^{-2f}g^{ij}\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}$. In our case $f = -\ln x_n$ and $\frac{\partial f}{\partial r} = -\frac{1}{r}$, $\frac{\partial f}{\partial \omega_i} = -\frac{\cos\omega_i}{\sin\omega_i}$. The Laplacian is given by

$$\Delta_{\mathbb{H}^n} = x_n^2 \Delta_{\mathbb{R}^n} + (n-2)x_n^2\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\cos\omega_2}{\sin\omega_2}\frac{\partial}{\partial\omega_2} + \sum_{i=3}^n g^{ii}\frac{\cos\omega_i}{\sin\omega_i}\frac{\partial}{\partial\omega_i}\right).$$

An eigenvalue-eigenfunction pair λ, u for Ω in hyperbolic space satisfies $-\lambda u = \Delta_{\mathbb{H}^n} u$, which is written in coordinates as

$$\begin{aligned}(15)\quad \frac{-\lambda}{r^2(\sin\omega_2)^2\cdots(\sin\omega_n)^2}u &= \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) + (n-2)\frac{1}{r}\frac{\partial u}{\partial r}\\ &+ \frac{1}{r^2}\frac{1}{(\sin\omega_2)^{n-2}}\frac{\partial}{\partial\omega_2}\left((\sin\omega_2)^{n-2}\frac{\partial u}{\partial\omega_2}\right) + (n-2)\frac{1}{r^2}\frac{\cos\omega_2}{\sin\omega_2}\frac{\partial u}{\partial\omega_2}\\ +\sum_{i=3}^n \frac{1}{r^2(\sin\omega_2)^2\cdots(\sin\omega_{i-1})^2}&\left(\frac{1}{(\sin\omega_i)^{n-i}}\frac{\partial}{\partial\omega_i}\left((\sin\omega_i)^{n-i}\frac{\partial u}{\partial\omega_i}\right) + (n-2)\frac{\cos\omega_i}{\sin\omega_i}\frac{\partial u}{\partial\omega_i}\right).\end{aligned}$$

8.3. **Separation of variables.** As in the two-dimensional case, let us separate variables. We write

$$u = f(r)\mathsf{f}_2(\omega_2)\cdots\mathsf{f}_n(\omega_n)$$

then divide both sides of equation (15) by u/r^2 to obtain

$$\begin{aligned}(16)\quad \frac{-\lambda}{(\sin\omega_2)^2\cdots(\sin\omega_n)^2} &= \frac{1}{fr^{n-3}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial f}{\partial r}\right)+(n-2)\frac{r}{f}\frac{\partial f}{\partial r}\\ &+\frac{1}{\mathsf{f}_2(\sin\omega_2)^{n-2}}\frac{\partial}{\partial\omega_2}\left((\sin\omega_2)^{n-2}\frac{\partial\mathsf{f}_2}{\partial\omega_2}\right)+(n-2)\frac{\cos\omega_2}{\mathsf{f}_2(\sin\omega_2)}\frac{\partial\mathsf{f}_2}{\partial\omega_2}\\ +\sum_{i=2}^{n}\frac{1}{\mathsf{f}_i(\sin\omega_2)^2\cdots(\sin\omega_{i-1})^2}&\left(\frac{1}{(\sin\omega_i)^{n-i}}\frac{\partial}{\partial\omega_i}\left((\sin\omega_i)^{n-i}\frac{\partial\mathsf{f}_i}{\partial\omega_i}\right)+(n-2)\frac{\cos\omega_i}{\sin\omega_i}\frac{\partial\mathsf{f}_i}{\partial\omega_i}\right)\end{aligned}$$

First, we get that the only piece depending on r has to be a constant, say $-\kappa_1$. We rewrite this fact as

$$(17)\qquad r^{n-1}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial f}{\partial r}\right)+(n-2)r^{n-2}\left(r^{n-1}\frac{\partial f}{\partial r}\right)=-\kappa_1 fr^{2n-4}.$$

For $n\geq 3$, to solve the ode we change to the variable t so that $\frac{dr}{dt}=-r^{n-1}$, i.e. $t=r^{2-n}/(n-2)$ and (17) becomes $t^2\partial^2_{tt}f-t\partial_t f=-\frac{1}{(n-2)^2}\kappa_1 f$. Now let $s=\log t$, we have

$$(18)\qquad \partial^2_{ss}f-2\partial_s f=-\frac{1}{(n-2)^2}\kappa_1 f,$$

where $s\in(-(n-2)\frac{\pi}{\sqrt{\mu}}-\log(n-2),\ -\log(n-2))$.

This is a linear second ordinary differential equation whose characteristic polynomial has the roots $r_{1,2}=1\pm\sqrt{1-\frac{\kappa_1}{(n-2)^2}}$. For f to satisfy the Dirichlet boundary conditions, we must have two complex conjugate roots, thus $\frac{\kappa_1}{(n-2)^2}-1=\frac{k^2\mu}{(n-2)^2}>0$, for k non-zero integers. Furthermore, as the first eigenfunction f has to be positive and satisfy the Dirichlet boundary conditions,

$$(19)\qquad \frac{\kappa_1}{(n-2)^2}-1:=\frac{\mu}{(n-2)^2}>0.$$

so $f(s)=-e^s\sin\left(\frac{\sqrt{\mu}}{n-2}(s+\log(n-2)\right)$. From here, one can get an explicit solution $f(r)$, but it is not important for the rest of the argument.

Let us continue with our separation of variables.

Using (17), we replace the first two terms on the right-hand side of (16) by $-\kappa_1$, then multiply by $(\sin\omega_2)^2$ to get

$$\frac{-\lambda}{(\sin\omega_3)^2\cdots(\sin\omega_n)^2} = -\kappa_1(\sin\omega_2)^2 + \frac{1}{\mathsf{f}_2(\sin\omega_2)^{n-4}}\frac{\partial}{\partial\omega_2}\left((\sin\omega_2)^{n-2}\frac{\partial\mathsf{f}_2}{\partial\omega_2}\right) + (n-2)\frac{(\cos\omega_2)(\sin\omega_2)}{\mathsf{f}_2}\frac{\partial\mathsf{f}_2}{\partial\omega_2}$$
$$+\sum_{i=2}^{n}\frac{1}{\mathsf{f}_i(\sin\omega_3)^2\cdots(\sin\omega_{i-1})^2}\left(\frac{1}{(\sin\omega_i)^{n-i}}\frac{\partial}{\partial\omega_i}\left((\sin\omega_i)^{n-i}\frac{\partial\mathsf{f}_i}{\partial\omega_i}\right) + (n-2)\frac{\cos\omega_i}{\sin\omega_i}\frac{\partial\mathsf{f}_i}{\partial\omega_i}\right).$$

The only piece depending on ω_2 has to be constant, so

(20)
$$-\kappa_1(\sin\omega_2)^2+\frac{1}{\mathsf{f}_2(\sin\omega_2)^{n-4}}\frac{\partial}{\partial\omega_2}\left((\sin\omega_2)^{n-2}\frac{\partial\mathsf{f}_2}{\partial\omega_2}\right)+(n-2)\frac{(\cos\omega_2)(\sin\omega_2)}{\mathsf{f}_2}\frac{\partial\mathsf{f}_2}{\partial\omega_2} = -\kappa_2$$

For $i = 3, \dots, n-1$, we repeat to get

(21)
$$-\kappa_{i-1}(\sin\omega_i)^2+\frac{1}{\mathsf{f}_i(\sin\omega_i)^{n-i-2}}\frac{\partial}{\partial\omega_i}\left((\sin\omega_i)^{n-i}\frac{\partial\mathsf{f}_i}{\partial\omega_i}\right)+(n-2)\frac{(\cos\omega_i)(\sin\omega_i)}{\mathsf{f}_i}\frac{\partial\mathsf{f}_i}{\partial\omega_i} = -\kappa_i,$$

where each κ_i is a constant. Until the equation (16) becomes

$$\frac{-\lambda}{(\sin\omega_n)^2} = -\kappa_{n-1} + \frac{1}{\mathsf{f}_n}\left(\frac{\partial^2\mathsf{f}_n}{\partial\omega_n^2} + (n-2)\frac{\cos\omega_n}{\sin\omega_n}\frac{\partial\mathsf{f}_n}{\partial\omega_n}\right) \tag{22}$$

which looks very close to equation (4), except for the extra first degree term. Because ω_n is centered at $\pi/2$, $\sin\omega_n$ should be thought of as $\cos\varphi$. Also note that equations (20) and (22) are contained in the formulation of (21) so it suffices to study (21).

8.4. **Solving the ODEs.** Equation (22) can be transformed in such a way that the first-order term is eliminated. It will then be similar to equation (4).

The first step is to multiply (21) by f_i, move the term with κ_{i-1} to the right side and expand the derivative of the product to get

$$(\sin\omega_i)^2\frac{\partial^2\mathsf{f}_i}{\partial\omega_i^2} + (2n-2-i)(\cos\omega_i)(\sin\omega_i)\frac{\partial\mathsf{f}_i}{\partial\omega_i} = -\kappa_i\mathsf{f}_i + \kappa_{i-1}(\sin\omega_i)^2\mathsf{f}_i, \tag{23}$$

We look at (23) and want to combine all the derivatives as $\frac{\partial^2}{\partial\omega_i^2}((\sin\omega_i)^{\alpha_i}\mathsf{f}_i)$. In order to do so, the exponent α_i should be half of the constant in front of the term $(\cos\omega_i)(\sin\omega_i)\frac{\partial\mathsf{f}_i}{\partial\omega_i}$ and therefore $\alpha_i = n-1-\frac{i}{2}$. We multiply (23) by $(\sin\omega_i)^{\alpha_i-2}$ and obtain

(24)
$$(\sin\omega_i)^{\alpha_i}\frac{\partial^2\mathsf{f}_i}{\partial\omega_i^2} + 2\alpha_i(\cos\omega_i)(\sin\omega_i)^{\alpha_i-1}\frac{\partial\mathsf{f}_i}{\partial\omega_i} = -\kappa_i(\sin\omega_i)^{\alpha_i-2}\mathsf{f}_i + \kappa_{i-1}(\sin\omega_i)^{\alpha_2}\mathsf{f}_i.$$

The left-hand side is equal to

$$\frac{\partial^2}{\partial \omega_i^2}\left((\sin\omega_i)^{\alpha_i}\mathsf{f}_i\right) - \mathsf{f}_i\frac{\partial^2}{\partial \omega_i^2}\left((\sin\omega_i)^{\alpha_i}\right) = \frac{\partial^2 \mathsf{h}_i}{\partial \omega_i^2} - \alpha_i(\alpha_i - 1)\frac{\mathsf{h}_i}{(\sin\omega_i)^2} + \alpha_i^2\mathsf{h}_i,$$

where $\mathsf{h}_i = \mathsf{f}_i(\sin\omega_i)^{\alpha_i}$. Putting all of this into (24), we get

$$\frac{\partial^2 \mathsf{h}_i}{\partial \omega_i} + \frac{\kappa_i - \alpha_i(\alpha_i-1)}{(\sin\omega_i)^2}\mathsf{h}_i = (\kappa_{i-1} - \alpha_i^2)\mathsf{h}_i. \tag{25}$$

Note that (25) has the form of (4) under the change of variable $\varphi = \pi/2 - \omega_i$. For $i = n$, the change of variables transforms (22) into

$$\frac{\partial^2 \mathsf{h}_n}{\partial \omega_n} + \frac{\lambda - \alpha_n(\alpha_n-1)}{(\sin\omega_n)^2}\mathsf{h}_n = (\kappa_{n-1} - \alpha_n^2)\mathsf{h}_n, \tag{26}$$

where $\alpha_n = \frac{n}{2} - 1$.

8.5. **The proof in dimension $n \geq 3$.** The first eigenvalue of the Laplace operator on a domain $\Omega_{\sqrt{\mu},\delta_2,\ldots,\delta_{n-1},L}$ can be found in the following way. First, one computes the smallest κ_1 that is an eigenvalue of (17). Then one repeats the process for $i = 2, \ldots, n-1$: given κ_{i-1}, one takes κ_i to be the first eigenvalue of (25). Finally, with the knowledge of κ_{n-1}, the first eigenvalue for the Laplacian on the domain $\lambda_1(\Omega_{\sqrt{\mu},\delta_2,\ldots,\delta_{n-1},L})$ is the first eigenvalue of equation (26). With the same κ_{n-1}, the second eigenvalue of (26) is an eigenvalue of the Laplacian on the domain, but not necessarily the second one. Nevertheless, one can use this value as an upper bound for $\lambda_2(\Omega_{\sqrt{\mu},\delta_2,\ldots,\delta_{n-1},L})$. Therefore, as in the two-dimension case, to prove that $\lambda_2(\Omega_{\sqrt{\mu},\delta_2,\ldots,\delta_{n-1},L}) - \lambda_1(\Omega_{\sqrt{\mu},\delta_2,\ldots,\delta_{n-1},L}) \to 0$, it suffices to show that the difference between the first two eigenvalues of (26) go to zero. Using Theorem 7.1, one just needs the fact that $\kappa_{n-1} \to \infty$ when $\mu \to \infty$.

As $\mu \to \infty$, the first constant κ_1 goes to infinity by equation (19). Then for each $i = 2, \ldots, n-1$, (25) and Lemma 5.1 gives that

$$\kappa_i - \alpha_i(\alpha_i - 1) \geq (\cos\delta_i)^2(\kappa_{i-1} - \alpha_i^2),$$

where the α_i are constant. Consequently, $\kappa_i \to \infty$ for each i and applying Theorem 7.1 finishes the proof in higher dimensions.

References

[1] Ben Andrews and Julie Clutterbuck. Proof of the fundamental gap conjecture. *J. Amer. Math. Soc.*, 24(3):899–916, 2011.

[2] Sergei Artamoshin. Lower bounds for the first Dirichlet eigenvalue of the Laplacian for domains in hyperbolic space. *Math. Proc. Cambridge Philos. Soc.*, 160(2):191–208, 2016.

[3] Mark S. Ashbaugh and Rafael Benguria. Optimal lower bound for the gap between the first two eigenvalues of one-dimensional Schrödinger operators with symmetric single-well potentials. *Proc. Amer. Math. Soc.*, 105(2):419–424, 1989.

[4] Rafael D. Benguria and Helmut Linde. A second eigenvalue bound for the Dirichlet Laplacian in hyperbolic space. *Duke Math. J.*, 140(2):245–279, 2007.

[5] Theodora Bourni, Julie Clutterbuck, Xuan Hien Nguyen, Alina Stancu, Guofang Wei, and Valentina-Mira Wheeler. Explicit fundamental gap estimates for some convex domains in $\mathbb{H}^2$. To appear in *Mathematical Research Letters.* `arXiv:1911.12892`, 2019.

[6] Herm Jan Brascamp and Elliott H. Lieb. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Functional Analysis*, 22(4):366–389, 1976.

[7] Xianzhe Dai, Shoo Seto, and Guofang Wei. Fundamental gap estimate for convex domains on sphere– the case $n = 2$. *To appear in Comm. in Analysis and Geometry, arXiv:1803.01115*, 2018.

[8] Xianzhe Dai, Shoo Seto, and Guofang Wei. Fundamental gap comparison. In *Surveys in Geometric Analysis 2018*, pages 1–16, 2019.

[9] Manfredo Perdigão do Carmo. *Riemannian geometry.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992.

[10] Michael E. Gage. Upper bounds for the first eigenvalue of the Laplace-Beltrami operator. *Indiana Univ. Math. J.*, 29(6):897–912, 1980.

[11] Chenxu He, Guofang Wei, and Qi S. Zhang. Fundamental gap of convex domains in the spheres. *Amer. J. Math.*, 142(4):1161–1192, 2020.

[12] Antoine Henrot. *Extremum problems for eigenvalues of elliptic operators.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[13] Henry P McKean. An upper bound to the spectrum of Δ on a manifold of negative curvature. *J. Differential Geometry*, 4:359–366, 1970.

[14] Alessandro Savo. On the lowest eigenvalue of the Hodge Laplacian on compact, negatively curved domains. *Ann. Global Anal. Geom.*, 35(1):39–62, 2009.

[15] Shoo Seto, Lili Wang, and Guofang Wei. Sharp fundamental gap estimate on convex domains of sphere. *Journal of Differential Geometry*, 112(2):347–389, 2019.

[16] Ying Shih. A counterexample to the convexity property of the first eigenfunction on a convex domain of negative curvature. *Communications in Partial Differential Equations*, 14(7):867–876, 1989.

[17] M. van den Berg. On condensation in the free-boson gas and the spectrum of the Laplacian. *J. Statist. Phys.*, 31(3):623–637, 1983.

[18] Shing-Tung Yau. *Nonlinear analysis in geometry*, volume 33 of *Monographies de L'Enseignement Mathématique.* L'Enseignement Mathématique, Geneva, 1986. Série des Conférences de l'Union Mathématique Internationale, 8.

Department of Mathematics, University of Tennessee, 1403 Circle Dr, Knoxville, TN 37916 USA

Email address: tbourni@utk.edu

School of Mathematics, Monash University, 9 Rainforest Walk, VIC 3800 Australia

Email address: Julie.Clutterbuck@monash.edu

Department of Mathematics, Iowa State University, 411 Morrill Rd, Ames, IA 50011 USA

Email address: xhnguyen@iastate.edu

Department of Mathematics and Statistics, Concordia University, 1455 Blvd. de Maisonneuve Ouest, Montreal, QC, H3G 1M8 Canada

Email address: alina.stancu@concordia.ca

Department of Mathematics, UC Santa Barbara, Santa Barbara, CA 93106 USA

Email address: wei@math.ucsb.edu

Valentina-Mira Wheeler, Institute for Mathematics and its Applications, University of Wollongong, Northfields Avenue, Wollongong, NSW 2522 Australia

Email address: vwheeler@uow.edu.au