

COMPLETE EMBEDDED SELF-TRANSLATING SURFACES UNDER MEAN CURVATURE FLOW.

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ABSTRACT. We describe a construction of complete embedded self-translating surfaces under mean curvature flow by desingularizing the intersection of a finite family of grim reapers in general position.

1. INTRODUCTION

Let $X(\cdot, t) : M^d \rightarrow \mathbf{R}^{d+1}$ be a one-parameter family of immersions of smooth hypersurfaces into \mathbf{R}^{d+1} . The family of hypersurfaces $M_t = X(M^d, t)$ is a *solution to the mean curvature flow* if

$$(1) \quad \begin{aligned} \frac{\partial}{\partial t} X(p, t) &= \mathbf{H}(p, t), \quad p \in M, t > 0, \\ X(p, 0) &= X_0(p), \end{aligned}$$

where $\mathbf{H}(p, t)$ is the mean curvature vector of the hypersurface M_t at the point $X(p, t)$ for some initial data given by the immersion X_0 . By the local existence theorem for parabolic equations, the flow can be continued past any time t as long as the norm of the second fundamental form $|A(p, t)|$ stays bounded on M_t . A singularity at time T is classified according to the rate at which $\max_{p \in M_t} |A(p, t)|$ blows up: if $\max_{p \in M_t} |A(p, t)|\sqrt{T-t} \leq C$, we say the singularity is fast forming, otherwise we say the singularity is slow forming. The behavior of the flow near fast forming singularities is modeled by self-similar surfaces, which are surfaces that are rescaled by the flow, while their shape is left unchanged [6]. The study of slow forming singularities is more complex because of the lack of control on the geometry of the solution.

In the present article, we work in dimension $d = 2$ and focus on surfaces that are translated by the mean curvature flow at constant speed. These surfaces are called self-translating surfaces (STS) and can give some insight in slow forming singularities. For example, if the initial hypersurface has nonnegative mean curvature and the blow up is slow, then the surfaces $\{M_t\}$ tend asymptotically to a strictly convex STS or $\mathbf{R}^{d-k} \times S^d$, where S^d is a lower dimensional strictly convex STS [7]. Detailed examples of asymptotic convergence are given in Angenent [2] and Angenent-Velázquez [3][4]. Although the study of STS and singularities of the mean curvature flow are linked, few examples are available. Besides the classic examples of a plane, a grim reaper cylinder, and a rotationally symmetric soliton, Altschuler and Wu [1] showed the existence of paraboloid type self-translating surfaces that are graphs over convex domains in \mathbf{R}^2 having a prescribed angle of contact with the

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boundary cylinder. In [14], we constructed STS by desingularizing the intersection of a grim reaper and a plane. We present here a more general result with a family of grim reapers as the initial configuration.

To find an equation for STS, we set $\frac{\partial}{\partial t} X = \mathbf{H} = \mathbf{a} + \mathbf{V}$, where \mathbf{a} is the constant velocity of the translation and \mathbf{V} is a vector field tangent to the surface M_t to account for possible reparametrizations of the surface. Without loss of generality, we can fix $\mathbf{a} = \vec{e}_y$ to be the second coordinate unit vector in \mathbf{R}^3 , and taking the inner product with the normal vector, we get

$$(2) \quad H - \vec{e}_y \cdot \nu = 0,$$

where H is the mean curvature of the surface considered and ν is the unit normal vector such that $\mathbf{H} = H\nu$. It is well known that a grim reaper is a self-translating curve, so the cylinder over a grim reaper shifted by (\tilde{b}, \tilde{c}) ,

$$\tilde{\Gamma} = \{(x, y, z) \in (-\frac{\pi}{2} + \tilde{b}, \frac{\pi}{2} + \tilde{b}) \times \mathbf{R}^2 \mid (y - \tilde{c}) = -\log \cos(x - \tilde{b})\}$$

is an example in dimension $d = 2$. With a slight abuse of language, we will also call $\tilde{\Gamma}$ a grim reaper throughout the article.

1.1. Main result. Let us consider a finite family of grim reapers, $\{\tilde{\Gamma}_n\}_{n=1}^{N_\Gamma}$. Such a family is said to be in general position if no three $\tilde{\Gamma}$'s intersect on the same line, and no two $\tilde{\Gamma}$'s have the same asymptotic plane. For a family in general position, let us denote by δ the minimum distance between two intersection lines and by δ_Γ the minimum of the measure of the angles formed by the $\tilde{\Gamma}$'s with the yz -plane at the intersection lines. Note that both δ and δ_Γ are positive.

To each intersection line l_k , we associate a positive integer m_k . The m_k 's allow us to take different scales at different intersection lines, making the construction as general as possible.

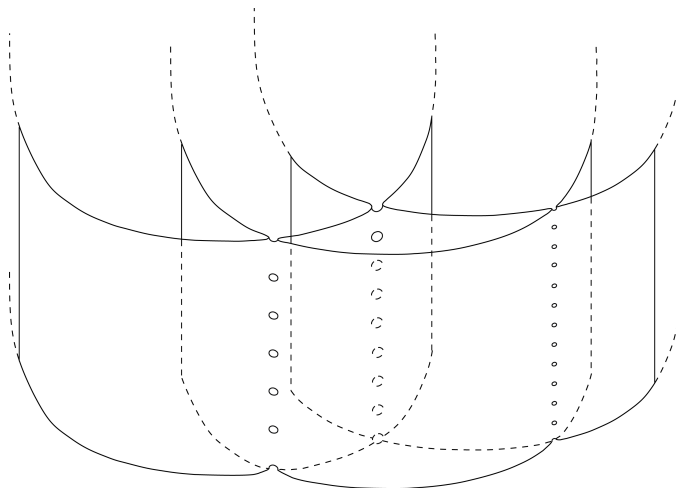


FIGURE 1. A self-translating surface $\tilde{\mathcal{M}}_\tau$

Theorem 1. *Suppose $\{\tilde{\Gamma}_n\}_{n=1}^{N_\Gamma}$ is a finite family of grim reapers in general position. There is a one parameter family of surfaces $\{\tilde{\mathcal{M}}_{\bar{\tau}}\}_{\bar{\tau} \in (0, \delta_{\bar{\tau}})}$, with $\delta_{\bar{\tau}}$ depending on $\max_k(m_k)$, N_Γ , δ and δ_Γ only, satisfying the following properties:*

- (i) $\tilde{\mathcal{M}}_{\bar{\tau}}$ is a complete embedded surface satisfying (2).
- (ii) $\tilde{\mathcal{M}}_{\bar{\tau}}$ is invariant under reflection with respect to the xy -plane.
- (iii) $\tilde{\mathcal{M}}_{\bar{\tau}}$ is singly periodic of period $2\pi\bar{\tau}$ in the z -direction.
- (iv) If U is a neighborhood in \mathbf{R}^2 such that $U \times \mathbf{R}$ contains no intersection line, then $\tilde{\mathcal{M}}_{\bar{\tau}} \cap (U \times \mathbf{R})$ converges uniformly in C^j norm, for any $j < \infty$, to $(\bigcup_{n=1}^{N_\Gamma} \tilde{\Gamma}_n) \cap (U \times \mathbf{R})$ as $\bar{\tau} \rightarrow 0$.
- (v) If T_k is the translation parallel to the xy -plane that moves the k th intersection line to the z -axis, then $\bar{\tau}^{-1}T_k(\tilde{\mathcal{M}}_{\bar{\tau}})$ converges uniformly in C^j norm, for any $j < \infty$, on any compact set of \mathbf{R}^3 to a Scherk surface with m_k periods between $z = 0$ and $z = 2\pi$ as $\bar{\tau} \rightarrow 0$.

We call a *Scherk surface* any singly periodic minimal surface with four ends asymptotic to planes.

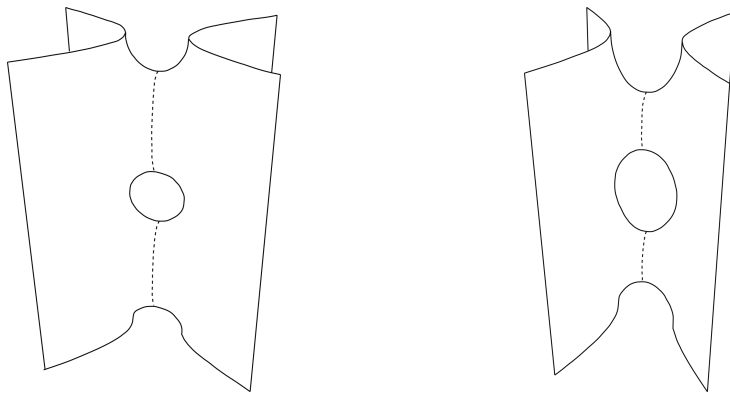


FIGURE 2. Two Scherk surfaces.

1.2. **Sketch of the proof.** We use a version of (2) rescaled by a factor $1/\bar{\tau}$, where $\bar{\tau}$ is a small constant to be determined:

$$(3) \quad H - \bar{\tau} \vec{e}_y \cdot \nu = 0.$$

Our result and its proof are inspired by Kapouleas' construction of minimal surfaces by desingularizing a family of coaxial catenoids [9]. Throughout the article, we treat equation (3) as a perturbation of $H = 0$ and show that the term $-\bar{\tau} \vec{e}_y \cdot \nu$ can be controlled at every step. Since the intersections we desingularize are lines, as opposed to circles, the construction of the initial approximate solution is simplified. We can therefore give explicit computations and more details in the proof overall. Kapouleas mapped the catenoids conformally to cylinders to estimate and control the asymptotic behavior of the minimal surfaces. This step is not necessary in our case since our surfaces are singly periodic and asymptotic to planes. The study of the linear operator is not simpler however (but not more difficult either). As in [9], we have to handle the presence of small eigenvalues for the linear operator and ensure exponential decay asymptotically.

The construction presented here can be easily adapted to desingularize the intersection of vertical planes to obtain minimal surfaces, provided no three planes intersect on the same line, and no two planes are parallel. Borrowing the idea from Kapouleas [9], we allow different scales at different intersections, so our result is a generalization of the construction of singly periodic minimal surfaces by Traizet [15].

Let us fix $\bar{\tau} > 0$ and denote the rescaled grim reapers by $\Gamma_n = \bar{\tau}^{-1}\tilde{\Gamma}_n$.

In a first attempt to construct an initial surface, we replace the intersection lines with Scherk surfaces with asymptotic planes parallel to the tangent planes at the intersection, then use cut-off functions to obtain a smooth surface. The surface obtained is a crude first approximation, and the discussion below explains why the actual construction has to be more subtle.

The initial surface is denoted by M , its position vector by X and the unit normal vector with positive \vec{e}_y component by ν . We are looking for a solution to (3) among graphs of small functions v with exponential decay over M , so we define $X_v = X + v\nu$, and denote by M_v the graph of v over M , by H_v its mean curvature and by ν_v its unit normal vector. We have

$$H_v - \tau\vec{e}_y \cdot \nu_v = H - \tau\vec{e}_y \cdot \nu + \Delta v + |A|^2 v + \tau\vec{e}_y \cdot \nabla v + Q_v,$$

where Q_v is at least quadratic in v , ∇v and $\nabla^2 v$, and A is the second fundamental form on M . The surface M_v is a STS if

$$(4) \quad \mathcal{L}v = -H + \tau\vec{e}_y \cdot \nu - Q_v,$$

where $\mathcal{L}v = \Delta v + |A|^2 v + \tau\vec{e}_y \cdot \nabla v$. An important part of the proof is dedicated to solving the differential equation $\mathcal{L}v = E$ on M . Once we can invert the linear operator \mathcal{L} , we expect the quadratic term to be small so the solution v to (4) could be obtained by iteration. Since M is a complicated surface, we study \mathcal{L} on different pieces first.

On a vertical ribbon joining two desingularizing surfaces, the Dirichlet problem $\mathcal{L}v = E$ with vanishing boundary conditions has a unique solution by the standard elliptic theory.

On a desingularizing piece of Scherk surface (Σ, g_Σ) , where g_Σ is the metric induced by the embedding into \mathbf{R}^3 , \mathcal{L} is a perturbation of the linear operator $L = \Delta + |A|^2$ associated to normal perturbations of the mean curvature. The mean curvature is invariant under translations, so the kernel of L contains the functions $\vec{e}_x \cdot \nu$, $\vec{e}_y \cdot \nu$ and $\vec{e}_z \cdot \nu$, where \vec{e}_x , \vec{e}_y and \vec{e}_z are the three coordinate vectors. By imposing the symmetry with respect to the xy -plane, we can eliminate $\vec{e}_z \cdot \nu$. The remaining functions $\vec{e}_x \cdot \nu$ and $\vec{e}_y \cdot \nu$ do not have the required exponential decay, nonetheless, they indicate the possible presence of small eigenvalues of L . The *approximate kernel* of L is defined to be the span of all the eigenfunctions of L corresponding to eigenvalues in $[-1, 1]$. We introduce two linearly independent functions w_1 and w_2 that have the important property of not being perpendicular to the approximate kernel (with respect to the inner product of $L^2(\Sigma, g_\Sigma)$). Given a function E , one can find constants θ_1 and θ_2 for which $E + \theta_1 w_1 + \theta_2 w_2$ is perpendicular to the approximate kernel and a function v satisfying $Lv = E + \theta_1 w_1 + \theta_2 w_2$. Since \mathcal{L} is a perturbation of L , a similar result is true for \mathcal{L} . For an exact solution, we must cancel any linear combination of w_1 and w_2 within the construction. This process is called *unbalancing* and consists in dislocating

the original Scherk surface so that the angles formed by its asymptotic planes are changed.

On a grim reaper end, we consider functions with exponential decay so that the asymptotic behavior of our solution matches our initial grim reaper closely. The difficulties do not arise from finding a solution to $\mathcal{L}v = E$ on the end per se, but from the fact that, if we hope to find a global solution with exponential decay, the solutions on the desingularizing surfaces Σ need to have exponential decay as well.

Let us call *wings* of Σ the four connected components obtained after removing a large enough cylinder from Σ . We can illustrate the behavior of a solution to the Dirichlet problem $\mathcal{L}v = E$ in Σ , $v = 0$ on $\partial\Sigma$ on a wing by considering the case of the Laplace operator on a cylinder. The surface is periodic in the z variable, so the behavior can be reduced to one coordinate, say s . Suppose we have a function E with exponential decay and a solution v to $v''(s) = E(s)$ that vanishes at the boundary $s = s_0$. Explicitly, the function v is given by

$$v(s) = \int_{s_0}^s \int_{s_0}^t E(r) dr dt + v'(s_0)(s - s_0), \quad s, t \leq s_0.$$

The first term on the right hand side has the right decay, but the second term is linear. From the point of view of the construction, we can cancel the term $v'(s_0)(s - s_0)$ by modifying the slope of Σ at the boundary of each wing. This involves bending each wing separately by an amount $\{\varphi_i\}_{i=1}^4$ to generate new functions $\{\bar{w}_i\}_{i=1}^4$ to achieve exponential decay along the wings.

We construct a global solution for $\mathcal{L}v = E$ on M by partitioning the support of the inhomogeneous term and solving on each piece inductively. The error generated is small thanks to the exponential decay, and a contraction principle gives us convergence. Finally, we use a fixed point theorem to find a self-translating surface.

1.3. Outline of the article. In Section 2, we study families of grim reapers and how the unbalancing at the intersection affects the initial configuration.

In Section 3, we turn our attention to Scherk surfaces and construct the desingularizing surfaces. We carefully study the interaction between the bending of the wings and the unbalancing, then introduce the functions w and \bar{w} and establish key estimates in Section 4.

In Section 5, we describe how to replace the lines of intersection by the desingularizing surfaces and construct smooth initial surfaces.

Section 6 is dedicated to the study of the linear operator, on each piece separately, then globally. In Sections 7 and 8, we finish the proof by estimating the quadratic term and applying the Schauder Fixed Point Theorem.

Some sections of this article follow the exposition of Kapouleas in [9], and we try to use the same notations, whenever possible.

1.4. Notations.

- E^3 is the Euclidean three space equipped with the usual metric.
- \vec{e}_x, \vec{e}_y and \vec{e}_z are the three coordinate vectors of E^3 .
- \mathbf{S}^2 refers to the standard unit sphere of dimension 2.
- Throughout this article, a surface with a tilde \tilde{S} is a surface in the original scale, with curvature comparable to 1. We use a notation without tilde for its rescaling $S = \frac{1}{\tau}\tilde{S} = \{(x, y, z) \in E^3 \mid (\tau x, \tau y, \tau z) \in \tilde{S}\}$, with curvature comparable to τ .

- We fix once and for all a smooth cut-off function ψ which is increasing, vanishes on $(-\infty, 1/3)$ and is equal to 1 on $(2/3, \infty)$. We define the functions $\psi[a, b] : \mathbf{R} \rightarrow [0, 1]$ which transition from 0 at a to 1 at b by

$$\psi[a, b](s) = \psi\left(\frac{s-a}{b-a}\right).$$

- We often have a function s defined on the surfaces with values in $\mathbf{R} \cup \{\infty\}$. If V is a subset of such a surface, we use the notations

$$(5) \quad V_{\leq a} := \{p \in V : s(p) \leq a\}, \quad V_{\geq a} := \{p \in V : s(p) \geq a\}.$$

- ν, g, A , and H denote respectively the oriented unit normal vector, the induced metric, the second fundamental form, and the mean curvature of an immersed surface in the Euclidean space E^3 . When we want to emphasize the surface S , we write these quantities with a subscript, for example g_S denotes the metric of S .
- Given a surface S in E^3 , which is immersed by $X : S \rightarrow E^3$, and a C^1 function $\sigma : S \rightarrow \mathbf{R}$, we call the graph of σ over S the surface given by the immersion $X + \sigma\nu$ and denote it by S_σ . We often use $X + \sigma\nu$ and its inverse to define projections from S to S_σ , or from S_σ to S respectively. When we refer to projections from S to S_σ or from S_σ to S , we always mean these projections.
- We work with the following weighted Hölder norms:

$$(6) \quad \|\phi : C^{k,\alpha}(\Omega, g, f)\| := \sup_{x \in \Omega} f(x)^{-1} \|\phi : C^{k,\alpha}(\Omega \cap B(x), g)\|,$$

where Ω is a domain, g is the metric with respect to which we take the $C^{k,\alpha}$ norm, f is the weight function, and $B(x)$ is the geodesic ball centered at x of radius 1.

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2. GRIM REAPERS AND INITIAL CONFIGURATION

We discuss dislocations at the intersection of the grim reapers, and how these affect the tangent vectors and the position of the grim reapers.

2.1. **Grim reapers.** A grim reaper $\tilde{\Gamma}_n$ is a self-translating solution to the mean curvature flow given by the equation

$$y = -\log(\cos(x - \tilde{b}_n)) + \tilde{c}_n, \quad |x - \tilde{b}_n| < \pi/2,$$

or, in arc length parametrization,

$$(7) \quad (\gamma_1(s) + \tilde{b}_n, \gamma_2(s) + \tilde{c}_n) := (\arctan(\sinh s) + \tilde{b}_n, \ln(\cosh s) + \tilde{c}_n), \quad s \in \mathbf{R}.$$

Since we will be using the word vertex in another context below, we call the point $(\tilde{b}_n, \tilde{c}_n)$ the *center* of the grim reaper to avoid confusion. With a slight abuse of language, we call any surface that is a rescaling of $\tilde{\Gamma}$, or a cylinder over $\tilde{\Gamma}$, a grim reaper also. The cylinders $\tilde{\Gamma}_n \times \mathbf{R}$ ($\Gamma_n \times \mathbf{R}$) will be denoted $\tilde{\Gamma}_n$ (Γ_n resp.) as well in this article. In this section however, we work in the xy -plane exclusively since the z -coordinate does not play any role.

Definition 2. We say that a finite family of grim reapers $\{\tilde{\Gamma}_n\}_{n=1}^{N_\Gamma}$ is in general position if it satisfies the following conditions

- (i) For some $\varepsilon > 0$, $|\tilde{b}_n - \tilde{b}_m - k\pi| > \varepsilon$ for $n \neq m$ and for all integers k .
- (ii) No three grim reapers intersect in one point.

The first condition ensures that two grim reapers do not share the same asymptote, therefore any two grim reapers that intersect do so transversally. Moreover, we have a lower and upper bound on the angle of intersection.

Since the result would be trivial otherwise, we assume that at least two grim reapers intersect. The two conditions (i) and (ii) above imply the following properties:

Lemma 3. Suppose $\{\tilde{\Gamma}_n\}_{n=1}^{N_\Gamma}$ is a family of grim reapers in general position. There are two real numbers $\delta_\Gamma > 0$ and $\delta > 0$ such that

- (i) the measure of the smallest of the two angles at the intersection of any two grim reapers is between $30\delta_\Gamma$ and $\pi/2 - 30\delta_\Gamma$.
- (ii) Any tangent vector to a grim reaper at an intersection forms an angle greater than $30\delta_\Gamma$ with \vec{e}_y .
- (iii) the arc length or distance on the grim reapers between any two intersection points is greater than 2δ .

Proof. The first result is immediate from the explicit formula for the grim reaper and Definition 2. We have the second and third properties since the number of grim reapers and therefore the number of intersection points is finite. \square

We now work on a larger scale: we fix τ a small positive constant and consider $\Gamma_n := \frac{1}{\tau}\tilde{\Gamma}_n$ with center $(b_n, c_n) := \frac{1}{\tau}(\tilde{b}_n, \tilde{c}_n)$. Let G be the graph of $\bigcup_{n=1}^{N_\Gamma} \Gamma_n$ and

- $V(G)$ the set of vertices of G (intersections of two grim reapers),
- $E(G)$ the set of edges of G (pieces of grim reapers connecting two intersections),
- $R(G)$ the set of rays of G (ends of grim reapers starting at an intersection),
- $R_l(G)$ the set of “left” rays of G (rays that start at an intersection and move toward the negative x -coordinate).

In what follows, we will also use G to keep track of the tangent unit vectors at the vertices. With this extra information, G is not just a graph; but we will call it such nonetheless for lack of a better word.

Unless otherwise specified, p denotes a vertex, e an edge and r a ray. For each vertex p , there are exactly four unit tangent vectors emanating from p , denoted by $v_{p1}, v_{p2}, v_{p3}, v_{p4}$, where the number refers to the order in which they appear as we rotate from \vec{e}_y counterclockwise. We say a graph is *balanced* if the four directing vectors cancel, $\sum_{i=1}^4 v_{pi} = 0$, at every vertex p . Note that the graph G is balanced.

2.2. Construction of the initial configuration \overline{G} . Our goal is to perturb G into a graph \overline{G} for which the sum of the directing vectors at each vertex p is given by a small vector $\vec{\zeta}(p)$,

$$\sum_{i=1}^4 v_{pi} = \vec{\zeta}(p), \quad p \in V(\overline{G}).$$

The process is called *unbalancing* and is necessary for tackling the approximate kernel of the linear operator $L = \Delta + |A|^2$ on each desingularizing surface.

The equation (3) we set out to solve is a perturbation of $H = 0$. The mean curvature is invariant under translations, therefore the functions $\vec{e}_x \cdot \nu$, $\vec{e}_y \cdot \nu$ and $\vec{e}_z \cdot \nu$ are in the kernel of the linear operator L associated to normal perturbations of H . Imposing a symmetry (invariance of our surfaces with respect to reflection across the xy -plane), we can rule out $\vec{e}_z \cdot \nu$. The remaining functions $\vec{e}_x \cdot \nu$ and $\vec{e}_y \cdot \nu$ do not have the required exponential decay, however, they indicate that L has small eigenvalues. We call the span of eigenfunctions of L corresponding to these small eigenvalues the *approximate kernel* of L . One can only solve the differential equation $Lv = E$ with a reasonable estimate on v if E is perpendicular to the approximate kernel. We do not have such control over the inhomogeneous term, so we introduce two functions w_1 and w_2 to cancel any component parallel to the approximate kernel. Roughly speaking, w_1 has to be in the direction of $\vec{e}_x \cdot \nu$, in the sense that $\int w_1(\vec{e}_x \cdot \nu) \neq 0$, and similarly, $\int w_2(\vec{e}_y \cdot \nu) \neq 0$.

Let S be one period of the desingularizing surface Σ . According to the balancing formula from [12] (see also Lemma 28), the mean curvature of S satisfies

$$\int_S H \vec{e}_x \cdot \nu dg_S = 2\pi \sum_{i=1}^4 v_i \cdot \vec{e}_x, \quad \int_S H \vec{e}_y \cdot \nu dg_S = 2\pi \sum_{i=1}^4 v_i \cdot \vec{e}_y,$$

where each vector v_i is the direction of the half-plane asymptotic to the i th end of Σ . The idea is to define w_1 and w_2 as derivatives of H and use dislocations that move v_1 and v_3 , or v_2 and v_4 , away from being parallel to generate linear combinations of w_1 and w_2 . The role of the new graph \bar{G} is to determine the position of the vertices of the initial surface depending on an imposed unbalancing.

We define the notion of a tetrad to keep track of the different vectors and angles at the vertices. We use the notations

$$(8) \quad \vec{e}[\theta] = \cos \theta \vec{e}_x + \sin \theta \vec{e}_y, \quad \vec{e}'[\theta] = -\sin \theta \vec{e}_x + \cos \theta \vec{e}_y.$$

Definition 4. *An acceptable tetrad of vectors, or “tetrad” for short, is defined to be a tetrad of vectors $T = (v_1, v_2, v_3, v_4)$ such that $v_i = \vec{e}[\beta_i]$ for some $\beta_i \in \mathbf{R}$ satisfying*

$$0 < \beta_2 - \beta_1 < \beta_3 - \beta_1 < \beta_4 - \beta_1 < 2\pi.$$

Moreover, for such a tetrad T we define

$$\begin{aligned} \theta(T) &:= \frac{\beta_1 - \beta_2 + \beta_3 - \beta_4 + 2\pi}{4}, & \theta_1(T) &:= \frac{\beta_3 - \beta_1 - \pi}{2}, \\ \theta_r(T) &:= \frac{\beta_1 + \beta_2 + \beta_3 + \beta_4 - 4\pi}{4}, & \theta_2(T) &:= \frac{\beta_4 - \beta_2 - \pi}{2}, \end{aligned}$$

and require

$$(9) \quad \theta(T) \in [20\delta_\theta, \frac{\pi}{2} - 20\delta_\theta], \quad \theta_1(T), \theta_2(T) \in [-2\delta_\theta, 2\delta_\theta],$$

with δ_θ to be determined at the end of the section.

We characterize the unbalancing of a tetrad (v_1, v_2, v_3, v_4) with the angles θ_j , $j = 1, 2$, which measure how much the vectors v_j and v_{j+2} fail to point in opposite directions, instead of using the vector $\vec{\zeta} = \sum_{i=1}^4 v_i$. The angles $\theta(T)$ and $\theta_r(T)$ will not be used here but will come into play in Section 3.

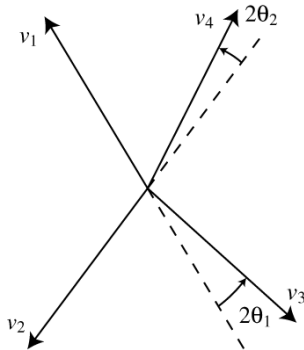


FIGURE 3. The angles θ_1 and θ_2 .

Let N_I be the number of intersection points, and recall that N_Γ is the number of grim reapers. The unbalancing restrictions do not determine \bar{G} completely, and we have enough degrees of freedom to impose a slight shift of each left ray. The graph \bar{G} constructed below is therefore a perturbation of G depending on $\{(\theta_{k,1}, \theta_{k,2})\}_{k=1}^{N_I}$ for the unbalancing and on $\{(b'_n, c'_n)\}_{n=1}^{N_\Gamma}$ for the position of the left rays.

The general construction does not depend on the first simpler two cases, but we describe them so the reader can understand the general case more easily.

1) We first consider the case of the intersection of two grim reapers Γ_1 and Γ_2 , where Γ_1 is the grim reaper that has a tangent at p in the v_{p1} direction. We translate Γ_1 to a grim reaper whose center is at $(b_1 + b'_1, c_1 + c'_1)$ and similarly, we translate Γ_2 to one centered at $(b_2 + b'_2, c_2 + c'_2)$. We denote by \bar{p} the intersection of the two perturbed grim reapers, and the four directing vectors $\{v_{\bar{p}i}\}$ by v_1, v_2, v_3 and v_4 , where the number refers to the order in which they appear as we rotate from \bar{e}_y counterclockwise. We fix $\bar{v}_1 = v_1$ and $\bar{v}_2 = v_2$, and we find \bar{v}_3 and \bar{v}_4 so that

$$\theta_1(\bar{T}) = \theta_{1,1}, \text{ and } \theta_2(\bar{T}) = \theta_{1,2}, \text{ for } \bar{T} = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)$$

We now join at \bar{p} two pieces of grim reapers with tangent vectors \bar{v}_3 and \bar{v}_4 respectively. Note that this joining is possible since by equation (7), the tangent unit vectors to a grim reaper map onto the set of unit vectors in \mathbf{R}^2 with positive x -coordinate. This completes the construction of an initial configuration for the simplest case.

2) The second case consists of three grim reapers intersecting in three points $p_k = (x_k, y_k)$, $k = 1, 2, 3$. We number the points so that $x_j < x_k$ for $j < k$, and we number the grim reapers so that Γ_1 has tangent at p_1 in the v_{p1} direction, Γ_2 is the other grim reaper through p_1 , and Γ_3 is the last grim reaper.

As in the previous case, we translate Γ_1 and Γ_2 by (b'_1, c'_1) and (b'_2, c'_2) respectively. We denote the intersection of the perturbed curves by \bar{p}_1 and the four directing vectors by $v_{\bar{p}1}, v_{\bar{p}2}, v_{\bar{p}3}$ and $v_{\bar{p}4}$ in the same order as before. We impose $\bar{v}_{11} = v_{\bar{p}1}$ and $\bar{v}_{12} = v_{\bar{p}2}$, and find \bar{v}_{13} and \bar{v}_{14} such that $\theta_1(\bar{T}) = \theta_{1,1}$, and $\theta_2(\bar{T}) = \theta_{1,2}$, for $\bar{T} = (\bar{v}_{11}, \bar{v}_{12}, \bar{v}_{13}, \bar{v}_{14})$.

At the point \bar{p}_1 , we consider the two rays (which are pieces of grim reapers) emanating from \bar{p}_1 with tangent directions \bar{v}_{13} and \bar{v}_{14} respectively. Both of these rays intersect the translation of the third grim reaper by (b'_3, c'_3) ; \bar{p}_2 is the leftmost

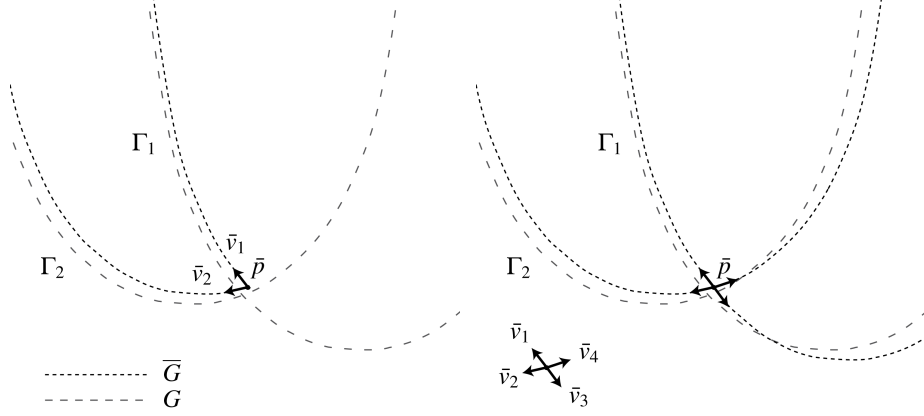


FIGURE 4. Construction of \overline{G} for two intersecting grim reapers.

intersection point. We impose $\bar{v}_{21} = v_{\bar{p}_{21}}$ and $\bar{v}_{22} = v_{\bar{p}_{22}}$. The two other directions \bar{v}_{23} and \bar{v}_{24} are determined by $(\theta_{2,1}, \theta_{2,2})$.

We now take the two rays emanating from \bar{p}_2 with tangent directions \bar{v}_{23} and \bar{v}_{24} respectively. One of these intersects one of the rays from \bar{p}_1 and we denote the intersection by \bar{p}_3 .

We fix the two left directions $\bar{v}_{31} = v_{\bar{p}_{31}}$ and $\bar{v}_{32} = v_{\bar{p}_{32}}$ and the two other directions \bar{v}_{33} and \bar{v}_{34} are determined by $(\theta_{3,1}, \theta_{3,2})$. We now take the two rays emanating from \bar{p}_3 with tangent directions \bar{v}_{33} and \bar{v}_{34} to complete the construction.

3) For the general case, let us denote the intersection points $p_k = (x_k, y_k)$, $k = 1, \dots, N_I$ where $x_j \leq x_k$ for $j \leq k$ and if $x_j = x_k$, $y_j > y_k$ for $j < k$. In other words, we number our intersection points from left to right, and if two points have the same abscissa, we take the one with higher ordinate first.

We now number the grim reapers. Γ_1 is the grim reaper through p_1 with tangent direction $v_{p_{1,1}}$, and Γ_2 is the grim reaper through p_1 with tangent in the $v_{p_{1,2}}$ direction. Note that the abscissa of the center of Γ_1 is greater than the abscissa of the center of Γ_2 , in other words, $b_1 > b_2$. We then proceed with the p_k 's by increasing k . If p_k is on an as yet unnumbered grim reaper, we just give the grim reaper the next available number. In the case p_k is on two unnumbered grim reapers, we number the one with the rightmost center first.

We translate Γ_1 by (b'_1, c'_1) and Γ_2 by (b'_2, c'_2) respectively, and denote the intersection by \bar{p}_1 . We impose $\bar{v}_{11} = v_{\bar{p}_{1,1}}$ and $\bar{v}_{12} = v_{\bar{p}_{1,2}}$ and determine the directions \bar{v}_{13} and \bar{v}_{14} using $(\theta_{1,1}, \theta_{1,2})$. We now modify the edges or rays on the right of \bar{p}_1 so that they have unit tangent vectors \bar{v}_{13} and \bar{v}_{14} respectively.

We suppose the point \bar{p}_{k-1} and the vectors $\{\bar{v}_{(k-1)i}\}_{i=1}^4$ are constructed and give a procedure for the k th intersection. Consider the two edges or rays on the left of p_k and intersecting at p_k . If both of the pieces on the left of p_k are edges, they have been modified already and their intersection gives us \bar{p}_k . If we have one edge and one ray, or two rays, then we modify the ray(s) in the following way. Each ray is supported on a grim reaper Γ_n . We translate each Γ_n by (b'_n, c'_n) to obtain a modified ray. The intersection of the edge and the modified ray, or of the two modified rays, gives us \bar{p}_k . We impose the two left vectors $\bar{v}_{k1} = v_{\bar{p}_{k,1}}$ and $\bar{v}_{k2} = v_{\bar{p}_{k,2}}$

and determine \bar{v}_{k3} and \bar{v}_{k4} from $(\theta_{k,1}, \theta_{k,2})$. We now modify the edges or rays on the right of \bar{p}_k so that their unit tangent vectors are \bar{v}_{k3} and \bar{v}_{k4} respectively.

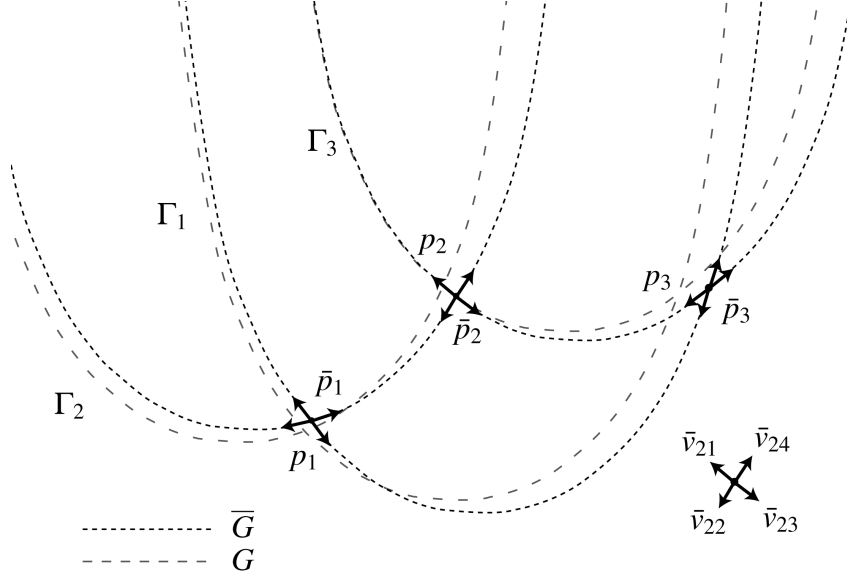


FIGURE 5. Construction of \bar{G} for three grim reapers, with $(b'_3, c'_3) = (0, 0)$.

Let us fix some notations for the new graph \bar{G} . To each vertex \bar{p}_k of \bar{G} corresponds the tetrad $(\bar{v}_{k1}, \bar{v}_{k2}, \bar{v}_{k3}, \bar{v}_{k4})$ of unit tangent vectors of rays or edges emanating from \bar{p}_k . Similarly, we have the tetrad $(v_{k1}, v_{k2}, v_{k3}, v_{k4})$ at each vertex p_k of the original graph G . When we want to emphasize the dependence of \bar{G} on its parameters, we write $\bar{G}(\underline{b}', \underline{c}', \underline{\theta}, \tau)$, where $\underline{b}' = \{b'_n\}_{n=1}^{N_\Gamma}$, $\underline{c}' = \{c'_n\}_{n=1}^{N_\Gamma}$, $\underline{\theta} = \{\theta_{k,1}, \theta_{k,2}\}_{k=1}^{N_\Gamma}$, and τ is related to the scaling.

Proposition 5. *There exist a $\delta'_\theta > 0$ and a constant C depending only on δ_Γ, δ and N_Γ with the following property: if $\max(\tau|(\underline{b}', \underline{c}')|, |\underline{\theta}|) \leq \delta'_\theta$, the vertices \bar{p}_k and the directing vectors \bar{v}_{ki} of the graph $\bar{G}(\underline{b}', \underline{c}', \underline{\theta}, \tau)$ satisfy*

$$\tau|p_k - \bar{p}_k| \leq C\tau|(\underline{b}', \underline{c}')| + C|\underline{\theta}|, \quad |\angle(v_{ki}, \bar{v}_{ki})| \leq C\tau|(\underline{b}', \underline{c}')| + C|\underline{\theta}|,$$

for $i = 1, \dots, 4$ and $k = 1, \dots, N_\Gamma$.

Proof. We start by studying how the location of the intersection changes as two grim reapers are translated by different vectors. Without loss of generality, we can assume that one of them stays centered at the origin.

Let Γ_0 and Γ_1 be two grim reapers given by the position vectors

$$\begin{aligned} \mathbf{r}_0(s) &= \frac{1}{\tau}(\gamma_1(\tau s), \gamma_2(\tau s)), \\ \mathbf{r}_1(t) &= \frac{1}{\tau}(\gamma_1(\tau t) + \gamma_1(\tau s_0) - \gamma_1(\tau s_1), \gamma_2(\tau t) + \gamma_2(\tau s_0) - \gamma_2(\tau s_1)), \end{aligned}$$

where

$$(10) \quad \gamma_1(s) = \arctan(\sinh s), \quad \gamma_2(s) = \ln(\cosh s),$$

Note that the functions γ_1 and γ_2 were already given implicitly in (7). The two grim reapers intersect at $\mathbf{r}_0(s_0) = \mathbf{r}_1(s_1)$ and the tangent vectors at the intersection are

$$(11) \quad \mathbf{r}'_i(s_i) = \left(\frac{1}{\cosh(\tau s_i)}, \tanh(\tau s_i) \right), \quad i = 0, 1.$$

The coordinates of the center of Γ_1 are

$$(12) \quad b = \tau^{-1}(\gamma_1(\tau s_0) - \gamma_1(\tau s_1)), \quad c = \tau^{-1}(\gamma_2(\tau s_0) - \gamma_2(\tau s_1)).$$

If the angles between the tangent vectors $\mathbf{r}'_i(s_i)$ and \vec{e}_y are larger than $20\delta_\Gamma$, then $|\tau s_0| + |\tau s_1| < C$, with C depending on δ_Γ . Therefore $\cosh(\tau s_0)$ and $\cosh(\tau s_1)$ are bounded above by a constant. In addition, if $|\tau b| = |\gamma_1(\tau s_0) - \gamma_1(\tau s_1)| > \varepsilon/2$, where ε is given in Definition 2, we have

$$\frac{\varepsilon}{2} < \left| \frac{1}{\cosh(\tau \bar{s})}(\tau s_0 - \tau s_1) \right| \leq \tau |s_0 - s_1|,$$

where \bar{s} is the number in (s_0, s_1) given by the Mean Value Theorem.

We define the function $F : \mathbf{R}^2 \rightarrow \mathbf{R}^2$, $F(s_0, s_1) = (b, c)$ using (12). According to the Inverse Function Theorem, F has an inverse if the determinant of its Jacobian does not vanish. Indeed,

$$|\det[DF]| = \frac{|\sinh(\tau s_1) - \sinh(\tau s_0)|}{\cosh(\tau s_0) \cosh(\tau s_1)} \geq \frac{\tau |s_1 - s_0|}{\cosh(\tau s_0) \cosh(\tau s_1)} \geq \frac{\varepsilon}{2C^2} > 0.$$

F^{-1} has bounded derivatives and the quantities τs_0 and τs_1 are bounded, therefore, if p and \bar{p} are the intersections of Γ_0 and grim reapers centered at (b, c) and (\bar{b}, \bar{c}) respectively, we have

$$(13) \quad |p - \bar{p}| \leq C|(b - \bar{b}, c - \bar{c})|$$

for (b, c) close enough to (\bar{b}, \bar{c}) .

Let us now fix the intersection point $\tau^{-1}(\gamma_1(\tau s_0), \gamma_2(\tau s_0))$ and study how a change in the tangent vector $\mathbf{r}'_1(s_1)$ at the intersection moves the center (b, c) . The angle α between $\mathbf{r}'_1(s_1)$ and \vec{e}_y satisfies

$$(14) \quad \tan \alpha = \sinh(\tau s_1).$$

From an earlier discussion, $|\tau s_0| + |\tau s_1|$ is bounded, so $|\sinh(\tau s_1)|$ is bounded. Hence,

$$(15) \quad C^{-1}\tau \leq \left| \frac{d\alpha}{ds_1} \right| = \left| \frac{\tau}{\cosh(\tau s_1)} \right| \leq C\tau$$

for some constant C .

Starting with our initial configuration where all the angles are bounded below by $30\delta_\Gamma$, the change of position of an intersection point and the changes in the tangent vectors are propagated to the next intersection points, but for $\tau|(\underline{b}', \underline{c}')|$ and $|\underline{\theta}|$ small enough depending on N_Γ , δ and δ_Γ , the perturbed configuration still has the properties of Lemma 3, with $20\delta_\Gamma$ instead of $30\delta_\Gamma$ in (i) and (ii), and δ instead of 2δ in (iii). The result follows from (13) and (15) for $\tau|(\underline{b}', \underline{c}')|$ and $|\underline{\theta}|$ small enough. \square

Definition 6. We fix $\delta_\theta = \min(C\delta'_\theta, \delta_\Gamma)$ for the rest of the article, where C and δ'_θ are as in the previous proposition.

3. DESINGULARIZING SURFACES

We now construct the surfaces that will replace the lines of intersection of the grim reapers. As mentioned in the introduction, we have to allow some flexibility so that opposite wings can fail to have opposite directions (unbalancing) and for each wing to be bent independently further along (bending).

3.1. Scherk surfaces. The one parameter family of Scherk surfaces $\Sigma(\theta)$ is a family of singly periodic minimal surfaces. Only the most symmetric of them, $\Sigma(\pi/4)$, is due to Scherk and the rest of the family was discovered by Karcher [11]. They are often called Scherk's fifth surfaces or Scherk's (saddle) towers but we will refer to them as Scherk surfaces for simplicity. $\Sigma(\theta)$ is given by the equation

$$\cos^2 \theta \cosh \frac{x}{\cos \theta} - \sin^2 \theta \cosh \frac{y}{\sin \theta} = \cos z.$$

The surfaces $\Sigma(\theta)$ become degenerate as $\theta \rightarrow 0$ or $\theta \rightarrow \pi/2$ so we will restrict ourselves to $\theta \in [10\delta_\theta, \frac{\pi}{2} - 10\delta_\theta]$ for δ_θ as in Definition 6.

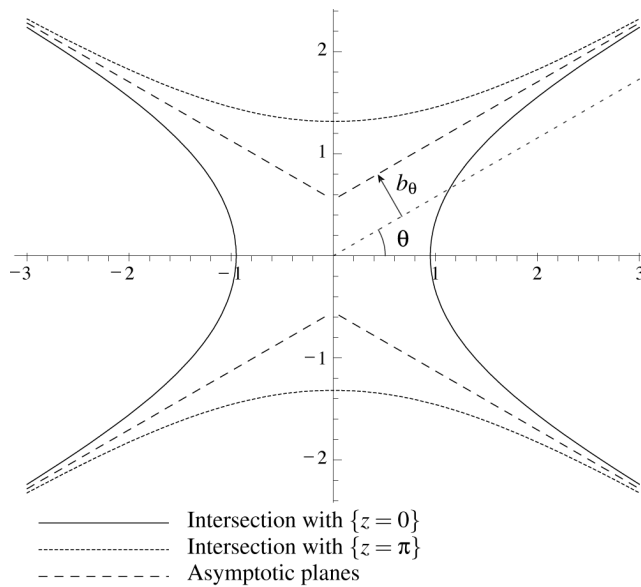


FIGURE 6. Sections of the Scherk surface $\Sigma(\theta)$.

Notations. The Scherk surface enjoys many symmetries. In order to refer to them easily in the future, we define the following isometries of the Euclidean space:

- \mathcal{R}_1 is the identity
- \mathcal{R}_2 is the reflection with respect to the yz -plane.
- \mathcal{R}_3 is the reflection with respect to the z -axis.
- \mathcal{R}_4 is the reflection with respect to the xz -plane.

We denote by H^+ the closed half-plane $H^+ = \{(s, z) \in \mathbf{R}^2 \mid s \geq 0\}$. The vectors $\vec{e}[\theta]$ and $\vec{e}'[\theta]$ are defined in equation (8).

We quote Proposition 2.4 from [9] for some properties of the Scherk surfaces.

Proposition 7. $\Sigma(\theta)$ is a singly periodic embedded complete minimal surface which depends smoothly on θ and has the following properties:

- (i) $\Sigma(\theta)$ is invariant under the \mathcal{R}_i 's above and also under reflection with respect to the planes $\{z = k\pi\}$ ($k \in \mathbf{Z}$).
- (ii) For given $\varepsilon \in (0, 10^{-3})$, there is a constant $a = a(\delta_\theta, \varepsilon) > 0$ and smooth functions $f_\theta : H^+ \rightarrow \mathbf{R}$, $A_\theta : H^+ \rightarrow E^3$, and $F_\theta : H^+ \rightarrow E^3$, such that $W_\theta := F_\theta(H^+) \subset \Sigma(\theta)$ and

$$A_\theta(s, z) = (a + s)\vec{e}[\theta] + z\vec{e}_z + b_\theta\vec{e}'[\theta],$$

$$F_\theta(s, z) = A_\theta(s, z) + f_\theta(s, z)\vec{e}'[\theta],$$

where $b_\theta = \sin(2\theta) \log(\cot \theta)$. Moreover f_θ and F_θ depend smoothly on $\theta \in [10\delta_\theta, \pi/2 - 10\delta_\theta]$ and (iii)-(vi) are satisfied.

- (iii) $\Sigma(\theta) \setminus \bigcup_{j=1}^4 \mathcal{R}_j(W_\theta)$ is connected and lies within distance $a + 1$ from the z -axis.
- (iv) $W_\theta \subset \{(r \cos \phi, r \sin \phi, z) : r > a, \phi \in [9\delta_\theta, \pi/2 - 9\delta_\theta]\}$.
- (v) $\|f_\theta : C^5(H^+, e^{-s})\| \leq \varepsilon$ and $\|df_\theta/d\theta : C^5(H^+, e^{-s})\| \leq \varepsilon$, where the weighted norms were defined in (6).
- (vi) $|b_\theta| + |db_\theta/d\theta| < \varepsilon a$.

For the rest of the article, ε is a fixed small constant so that a is fixed also.

W_θ is called the *first wing* of the Scherk surface, and the image of W_θ under \mathcal{R}_i is called the *i th wing*.

We consider as standard coordinates on the i th wing the coordinates (s, z) defined by $(s, z) = (\mathcal{R}_i \circ F_\theta)^{-1}(s, z)$ and extend the function s to be zero on the rest of $\Sigma(\theta)$. Using the notation (5), we call $\Sigma_{\leq 0}(\theta)$ the *core* of the Scherk surface. Note that the boundary of the core has four connected components, each of which is the boundary of a wing. The numbering of the wings can be reconciled with the numbering of the vectors v_1, v_2, v_3 and v_4 in Section 2.2 by taking β_1 from Definition 4 to be in the second quadrant.

3.2. Construction of the core. The goal of this section is to unbalance a Scherk surface so that its wings are tangent to asymptotic planes determined by a possibly unbalanced tetrad T . The dislocations are necessary for dealing with the approximate kernel, as discussed in Section 2.2.

Let us examine more closely the angles in Definition 4 when the tetrad is formed of the directing vectors of the planes asymptotic to $\Sigma(\theta)$, rotated by an angle β_r around the z -axis. We have $\beta_1 = \theta + \beta_r$, $\beta_2 = \pi - \theta + \beta_r$, $\beta_3 = \pi + \theta + \beta_r$, $\beta_4 = 2\pi - \theta + \beta_r$, $\theta(T) = \theta$, $\theta_1 = 0$, $\theta_2 = 0$, and $\theta_r = \beta_r$ is the angle of rotation.

Given a tetrad T for which $\theta_1 = 0$ and $\theta_2 = 0$, we can find a Scherk surface that has the vectors of T as directing vectors: it suffices to take $\Sigma(\theta(T))$ rotated around the z -axis by an angle θ_r . In general however, it is not enough to rotate one of the original Scherk surfaces $\Sigma(\theta)$, we need transformations Z_1 and Z_2 to change the respective directions of the vectors.

Definition 8. We define a family of diffeomorphisms $Z_1(\phi) : E^3 \rightarrow E^3$ parametrized by $\phi \in [-2\delta_\theta, 2\delta_\theta]$ such that:

- (i) Z_1 is the identity on the second and fourth quadrants $\{(x, y, z) \mid xy \leq 0\}$ and in the unit ball.
- (ii) on $\{(r \cos \theta', r \sin \theta', z) : r > 2, \theta' \in [9\delta_\theta, \pi/2 - 9\delta_\theta]\}$, Z_1 is a rotation of angle ϕ clockwise around the z -axis.

- (iii) on $\{(r \cos \theta', r \sin \theta', z) : r > 2, \theta' \in [\pi + 9\delta_\theta, 3\pi/2 - 9\delta_\theta]\}$, Z_1 is a rotation of angle ϕ counterclockwise around the z -axis.

Definition 9. The family of diffeomorphisms $Z_2(\phi) : E^3 \rightarrow E^3$ parametrized by $\phi \in [-2\delta_\theta, 2\delta_\theta]$ is defined by $Z_2(\phi) = \mathcal{R}_2 \circ Z_1(-\phi) \circ \mathcal{R}_2$.

The transformation Z_1 rotates points in the first and third quadrants by ϕ toward the fourth quadrant, and Z_2 rotates points in the second and fourth quadrants by ϕ toward the first quadrant.

For a tetrad T , we define

$$Z[T] := \mathcal{R} \circ Z_1(\theta_1) \circ Z_2(\theta_2),$$

where \mathcal{R} denotes the rotation around the z -axis with angle $\theta_r(T)$ counterclockwise. We define the surface

$$\Sigma[T] := Z[T](\Sigma(\theta(T))).$$

By construction, each plane asymptotic to the surface $\Sigma[T]$ is parallel to a vector in T . We will not touch the core $\Sigma_{\leq 0}[T]$ in the rest of the construction.

3.3. Construction of the wings. With dislocations, we can solve the linear operator on the desingularizing surface, but we do not control the asymptotic behavior of the solutions as we move away from the z -axis. The additional bending of each wing independently will help us achieve exponential decay. In addition, we use the bending to fit the wings smoothly in the construction of the initial surface in Section 5.

Let $\tau > 0$ be a small constant, $\{\varphi_i\}_{i=1}^4$ four real numbers such that $|\varphi_i| \leq \delta_\theta$, and T a tetrad as in the previous section. We describe the construction of the i th wing, with a bending of angle φ_i below.

Definition 10. We call the line $Z[T] \circ \mathcal{R}_i \circ A_{\theta(T)}(\partial H^+)$ the i th pivot and denote its intersection with the xy -plane by (x_i, y_i) . The i th pivot is the boundary of the core projected perpendicularly onto the i th asymptotic plane.

We define the map $\kappa[\tau, x_i, y_i, s_i] : H^+ \rightarrow E^3$ by

$$(16) \quad \begin{aligned} \kappa[\tau, x_i, y_i, s_i](s, z) \\ = \frac{1}{\tau} (\gamma_1(\tau(s + s_i)) - \gamma_1(\tau s_i) + \tau x_i, \gamma_2(\tau(s + s_i)) - \gamma_2(\tau s_i) + \tau y_i, \tau z), \end{aligned}$$

where γ_1 and γ_2 were defined in (10). Note that the line $\kappa[\tau, x_i, y_i, s_i](\partial H^+)$ is the i th pivot and the graph of κ is a piece of grim reaper. The constant s_i is chosen so that the conormal tangent vector to $\kappa[\tau, x_i, y_i, s_i]$ at $s = 0$ is $v_i = \bar{e}'[\beta_i]$ rotated by an angle φ_i counterclockwise around the z -axis, i. e. $\tan(\beta_i + \varphi_i) = \sinh(\tau s_i)$ by (11). We define the immersion of the asymptotic grim reaper to the i th wing by

$$A_i[T, \varphi_i, \tau] = \kappa[\tau, x_i, y_i, s_i]$$

and define $\nu_i[T, \varphi_i, \tau](s, z)$ to be the normal unit vector to $A_i[T, \varphi_i, \tau](H^+)$ at the point $A_i[T, \varphi_i, \tau](s, z)$ oriented such that $\nu_i[T, \varphi_i, \tau](0, 0) = (-1)^{i-1} \bar{e}'[\beta_i + \varphi_i]$.

Roughly speaking, the bent wing is the graph of $f_{\theta(T)}$ over the asymptotic grim reaper. For a smooth transition, we need to cut off $f_{\theta(T)}$ far enough so that the error generated is not too big while keeping the desingularizing surface small enough in the scale of the grim reapers. For this reason, we introduce a small constant δ_s that will be determined later. The function $F_i[T, \varphi_i, \tau]$ defined below is the immersion of the i th wing.

Definition 11. For given T and $\varphi_i \in [-\delta_\theta, \delta_\theta]$, we define $F_i[T, \varphi_i, \tau] : H^+ \rightarrow E^3$ by

$$F_i[T, \varphi_i, \tau](s, z) = \psi[1, 0](s)Z[T] \circ \mathcal{R}_i \circ F_{\theta(T)}(s, z) \\ + (1 - \psi[1, 0](s))(A_i[T, \varphi_i, \tau](s, z) + \psi_s(s)f_{\theta(T)}(s, z)\nu_i[T, \varphi_i, \tau](s, z))$$

where ψ_s is defined by $\psi_s(s) = \psi[4\delta_s/\tau, 3\delta_s/\tau](s)$.

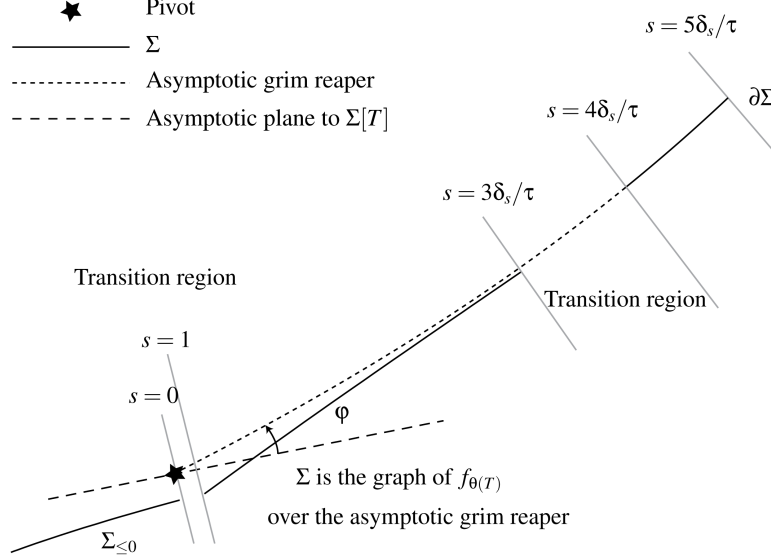


FIGURE 7. Construction of a wing.

3.4. The desingularizing surfaces $\Sigma[T, \underline{\varphi}, \tau]$.

Definition 12. For a tetrad T and $\underline{\varphi} = \{\varphi_i\}_{i=1}^4$ such that $|\underline{\varphi}| \leq \delta_\theta$, we define a map $Z[T, \underline{\varphi}, \tau] : \Sigma(\theta(T)) \rightarrow E^3$ to be $Z[T]$ on the core, and $F_i[T, \varphi_i, \tau] \circ F_\theta^{-1} \circ \mathcal{R}_i^{-1}$ on the i th wing of $\Sigma(\theta(T))$. The desingularizing surface $\Sigma[T, \underline{\varphi}, \tau]$ is given by

$$\Sigma = \Sigma[T, \underline{\varphi}, \tau] := Z[T, \underline{\varphi}, \tau](\Sigma_{\leq 5\delta_s/\tau}(\theta(T))).$$

The coordinates (s, z) on $\Sigma(\theta(T))$ are pushed forward by $Z[T, \underline{\varphi}, \tau]$ to coordinates on Σ . The desingularizing surface is divided in five regions:

- When $s \leq 0$, we are on the core of Σ . The surface is dislocated here but the bending related to $\underline{\varphi}$ does not affect this region.
- $s \in [0, 1]$ is a transition region.
- For $s \in [1, 3\delta_s/\tau]$, the wings are graphs of $f_{\theta(T)}$ on the asymptotic grim reapers; the i th asymptotic grim reaper makes an angle φ_i with the plane asymptotic to the i th wing of $\Sigma[T]$.
- $s \in [3\delta_s/\tau, 4\delta_s/\tau]$ is a second transition region where the function $f_{\theta(T)}$ is cut off.
- For $s \in [4\delta_s/\tau, 5\delta_s/\tau]$, the wings are just asymptotic grim reapers.

Note that the desingularizing surfaces are truncated at $s = 5\delta_s/\tau$. The next proposition collects the properties of $\Sigma[T, \underline{\varphi}, \tau]$.

Proposition 13. *There is a constant $\delta'_\tau = \delta'_\tau(\delta_\theta) > 0$ such that for T satisfying (9), $|\underline{\varphi}| \leq \delta_\theta$, and $\tau \in (0, \delta'(\tau))$, the function $Z[T, \underline{\varphi}, \tau]$ satisfies the following properties:*

- (i) $Z[T, \underline{\varphi}, \tau]$ is a smooth embedding depending smoothly on its parameters.
- (ii) For each $n \in \mathbf{Z}$, the map $Z[T, \underline{\varphi}, \tau]$ is invariant under reflections of the domain and range with respect to the plane $\{z = n\pi\}$.

4. ESTIMATES ON THE DESINGULARIZING SURFACES

In the previous section, the dislocations and the bending are constructed independently. It is in our interest to keep the dislocations and the bending small to control the error generated in the core and the first transition region. If a dislocation is moving a wing in one direction, and the bending moves it back by some amount, we can let the bending cancel part of the dislocation by changing our angles from the onset. This process is called *straightening* and is studied in Lemma 14.

The main part of this section is dedicated to estimating $H - \tau \vec{e}_y \cdot \nu$ on the desingularizing surface and studying the impact of the dislocations, bending and straightening. For this, we define functions $\{w_j\}_{j=1}^2$ generated by the dislocations, and functions $\{\bar{w}_i\}_{i=1}^4$ generated by the straightening. In Lemma 27, we describe how the functions $\{w_j\}_{j=1}^2$ can be used to cancel any linear combination of eigenfunctions $\vec{e}_x \cdot \nu$ and $\vec{e}_y \cdot \nu$. Finally, we approximate $H - \tau \vec{e}_y \cdot \nu$ by a linear combination of the w 's and the \bar{w} 's with an error of second order in θ_1 , θ_2 and φ in Proposition 29. We follow the exposition of [9] and adapt all the proofs to our case, in particular, we show that the term $-\tau \vec{e}_y \cdot \nu$ can be controlled throughout.

4.1. Straightening.

Lemma 14. *There are constants $\delta_\varphi = \delta_\varphi(\delta_\theta) \in (0, \delta_\theta)$ and $\delta_\tau = \delta_\tau(\delta_\theta) \in (0, \delta'_\tau)$ such that for a given tetrad T as in Definition 4, $\underline{\varphi} \in \mathbf{R}^4$, and $\tau \in (0, \delta_\tau]$ satisfying*

$$\theta(T) \in [30\delta_\theta, \frac{\pi}{2} - 30\delta_\theta], \quad \theta_1(T), \theta_2(T) \in [-\delta_\theta, \delta_\theta], \quad |\underline{\varphi}| \leq \delta_\varphi,$$

we have for each $\underline{\varphi}' = \{\varphi'_i\}_{i=1}^4$ with $|\underline{\varphi}'| \leq \delta_\varphi$, a tetrad T' which depends smoothly on $T, \underline{\varphi}, \tau$ and $\underline{\varphi}'$, and is characterized by the following properties:

- (i) $(T', \underline{\varphi} - \underline{\varphi}', \tau)$ satisfies the conditions of Proposition 13.
- (ii) $T' = T$ when $\underline{\varphi}' = 0$.
- (iii) $T' = \{\vec{e}[\beta'_i]\}_{i=1}^4$ where each β'_i depends smoothly on $\underline{\varphi}'$ and

$$\left| \frac{\partial \beta'_i}{\partial \varphi'_j} - \delta_{ij} \right| \leq C\tau.$$

- (iv) *There is a smooth function $f_{\underline{\varphi}'}$ on $\Sigma[T, \underline{\varphi}, \tau]$ which depends smoothly on $T, \underline{\varphi}, \tau$ and $\underline{\varphi}'$, satisfies $f_{\underline{\varphi}'} \equiv 0$ on $\partial\Sigma[T, \underline{\varphi}, \tau]$, and whose graph over $\Sigma[T, \underline{\varphi}, \tau]$ is contained in the image of $Z[T', \underline{\varphi} - \underline{\varphi}', \tau]$.*

Proof. We fix $T, \underline{\varphi}$ and τ . Without loss of generality, we can assume that $\theta_r(T) = 0$, otherwise we rotate the whole configuration by $-\theta_r(T)$. For small variations $|\beta_i - \beta'_i| \leq \delta_\theta$ and $|\varphi'_i| \leq \delta_\theta$, the image of $Z[T', \underline{\varphi} - \underline{\varphi}', \tau]$ is the graph of a function f over $\Sigma[T, \underline{\varphi}, \tau]$. On the component of $\partial\Sigma[T, \underline{\varphi}, \tau]$ on the i th wing, the function

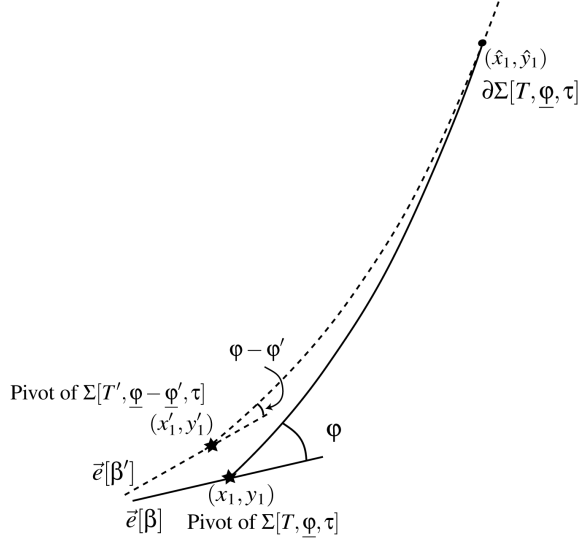


FIGURE 8. The image of $Z[T', \underline{\varphi} - \underline{\varphi}', \tau]$ passes through $\partial\Sigma[T, \underline{\varphi}, \tau]$.

f is a constant, which we denote by $f_i(\underline{\varphi}', T')$. Clearly, if $T' = T$ and $\underline{\varphi}' = 0$, $(f_1, f_2, f_3, f_4) = 0$. The Implicit Function Theorem on $(f_1, f_2, f_3, f_4)(\underline{\varphi}', T') = 0$ will give us T' as a function of $\underline{\varphi}'$ for $\underline{\varphi}'$ small enough, provided we show the matrix $[\partial f_i / \partial \beta'_j]_{i,j=1,\dots,4}$ is invertible.

We study how changes in the tetrad and in the bending affect the first wing in detail. The variation of the three other wings can be obtained similarly. Given $T' = \{\underline{e}[\beta'_i]\}_{i=1}^4$ and $\underline{\varphi}'$ with $|\beta'_i - \beta_i|$ and $|\underline{\varphi}'|$ small, the asymptotic grim reapers to the first wings of $\Sigma[T, \underline{\varphi}, \tau]$ and $\Sigma[T', \underline{\varphi} - \underline{\varphi}', \tau]$ are parametrized by $\kappa[\tau, x_1, y_1, s_1](s, z)$ and $\kappa[\tau, x'_1, y'_1, s'_1](s, z)$ respectively, where $\kappa[\tau, x_1, y_1, s_1](s, z)$ is as in (16), which is the version of the equation below without the primes,

$$\begin{aligned} \kappa'[\tau, x'_1, y'_1, s'_1](s, z) \\ = \frac{1}{\tau}(\gamma_1(\tau(s + s'_1)) - \gamma_1(\tau s'_1) + \tau x'_1, \gamma_2(\tau(s + s'_1)) - \gamma_2(\tau s'_1) + \tau y'_1, \tau z) \end{aligned}$$

with γ_1 and γ_2 as in (10), and

$$\begin{aligned} (x_1, y_1) &= (a \cos \beta_1, a \sin \beta_1) + b_{\theta(T)}(-\sin \beta_1, \cos \beta_1), \\ (x'_1, y'_1) &= (a \cos \beta'_1, a \sin \beta'_1) + b_{\theta(T')}(-\sin \beta'_1, \cos \beta'_1), \\ (17) \quad \sinh(\tau s_1) &= \tan(\beta_1 + \varphi_1), \quad \sinh(\tau s'_1) = \tan(\beta'_1 + \varphi_1 - \varphi'_1). \end{aligned}$$

As in the proof of Proposition 5, the angles $(\beta_1 + \varphi_1)$ and $(\beta_1 + \varphi_1 - \varphi'_1)$ stay away from $\pi/2 + k\pi$, $k \in \mathbf{Z}$, by a fixed amount, so $|\tau s_1|$ and $|\tau s'_1|$ are bounded. Using Proposition 7, we bound the derivatives of x'_1, y'_1 , and s'_1 with respect to β'_i

by

$$(18) \quad \left| \frac{\partial x'_1}{\partial \beta'_i} \right| \leq C, \quad \left| \frac{\partial y'_1}{\partial \beta'_i} \right| \leq C,$$

$$(19) \quad \frac{1}{C\tau} \delta_{1i} \leq \left| \frac{\partial s'_1}{\partial \beta'_i} \right| = \left| -\frac{\partial s'_1}{\partial \varphi'_i} \right| = \left| \frac{1}{\tau \cosh(\tau s'_1)} \delta_{1i} \right| \leq \frac{C}{\tau} \delta_{1i}.$$

To simplify notations, let us work in the xy -plane. The boundary $\partial\Sigma[T, \underline{\varphi}, \tau]$ is at the point $(\hat{x}_1, \hat{y}_1) := \kappa[\tau, x_1, y_1, s_1](5\delta_s/\tau)$. The line orthogonal to $\Sigma[T, \underline{\varphi}, \tau]$ at the point (\hat{x}_1, \hat{y}_1) intersects the surface $\Sigma[T', \underline{\varphi} - \underline{\varphi}', \tau]$ at a distance

$$(20) \quad f_1(\underline{\varphi}', T') = t = \frac{\gamma_2(\tau(s + s'_1)) - \gamma_2(\tau s'_1) + \tau y'_1 - \tau \hat{y}_1}{\tau \gamma'_1(5\delta_s + \tau s_1)},$$

where γ'_1 and γ'_2 are the derivatives of γ_1 and γ_2 respectively, and s is the coordinate on $\Sigma[T', \underline{\varphi} - \underline{\varphi}', \tau]$ of the intersection point. Note that since $|\tau s_1|$ is bounded, $\gamma'_1(5\delta_s + \tau s_1)$ is bounded away from 0. The value of s is given implicitly by the equation

$$(21) \quad (\gamma_1(\tau s'_1) - \tau x'_1 + \tau \hat{x}_1) \gamma'_1(5\delta_s + \tau s_1) + (\gamma_2(\tau s'_1) - \tau y'_1 + \tau \hat{y}_1) \gamma'_2(5\delta_s + \tau s_1) \\ = \gamma_1(\tau(s + s'_1)) \gamma'_1(5\delta_s + \tau s_1) + \gamma_2(\tau(s + s'_1)) \gamma'_2(5\delta_s + \tau s_1).$$

Hence, given $\underline{\varphi}'$ and T' , we can find x_1, y_1, s_1 and x'_1, y'_1, s'_1 . Solving the system of equations (20) (21) above, we get s and t . We first study the dependence of s on x'_1, y'_1, s'_1 and estimate its derivatives with respect to each of the x'_1, y'_1 and s'_1 .

Consider x'_1, y'_1, s'_1 as independent variables and define the function F below related to (21),

$$F(x'_1, y'_1, s'_1, \sigma) \\ = (\gamma_1(\tau s'_1) - \tau x'_1 + \tau \hat{x}_1) \gamma'_1(5\delta_s + \tau s_1) + (\gamma_2(\tau s'_1) - \tau y'_1 + \tau \hat{y}_1) \gamma'_2(5\delta_s + \tau s_1) \\ - \gamma_1(\tau \sigma) \gamma'_1(5\delta_s + \tau s_1) - \gamma_2(\tau \sigma) \gamma'_2(5\delta_s + \tau s_1).$$

Note that $F(x_1, y_1, s_1, 5\delta_s/\tau + s_1) = 0$. Its derivative with respect to σ satisfies

$$\frac{\partial F}{\partial \sigma}(x_1, y_1, s_1, 5\delta_s/\tau + s_1) = -\tau \gamma_1'^2(5\delta_s + \tau s_1) - \tau \gamma_2'^2(5\delta_s + \tau s_1) = -\tau.$$

Let us denote by $B_r(x_1, y_1, s_1)$ the ball of radius r centered at (x_1, y_1, s_1) . By the Implicit Function Theorem, there are constants $r > 0$, $\tilde{r} > 0$ and a function $h : B_r(x_1, y_1, s_1) \rightarrow (5\delta_s + \tau s_1 - \tilde{r}, 5\delta_s + \tau s_1 + \tilde{r})$ such that $F(x'_1, y'_1, s'_1, h(x'_1, y'_1, s'_1)) = 0$. Since $s = h - s'_1$,

$$(22) \quad \left| \frac{\partial s}{\partial x'_1}(x'_1, y'_1, s'_1) \right| \leq C, \quad \left| \frac{\partial s}{\partial y'_1}(x'_1, y'_1, s'_1) \right| \leq C,$$

$$(23) \quad \frac{\partial s}{\partial s'_1} = \frac{\gamma'_1(\tau s'_1) \gamma'_1(5\delta_s + \tau s_1) + \gamma'_2(\tau s'_1) \gamma'_2(5\delta_s + \tau s_1)}{\gamma'_1(\tau(s + s'_1)) \gamma'_1(5\delta_s + \tau s_1) + \gamma'_2(\tau(s + s'_1)) \gamma'_2(5\delta_s + \tau s_1)} - 1.$$

We now show that the determinant of $\left[\frac{\partial f_i}{\partial \beta'_j} \right]_{i,j=1,\dots,4}$ does not vanish. Since the functions $\{f_i\}_{i=1}^4$ play a similar role, it suffices to study one of them, say f_1 , in detail. We have

$$\frac{\partial s}{\partial \beta'_i} = \frac{\partial s}{\partial x'_1} \frac{\partial x'_1}{\partial \beta'_i} + \frac{\partial s}{\partial y'_1} \frac{\partial y'_1}{\partial \beta'_i} + \frac{\partial s}{\partial s'_1} \frac{\partial s'_1}{\partial \beta'_i},$$

and, from (20),

$$(24) \quad \gamma'_1(5\delta_s + \tau s_1) \frac{\partial f_1}{\partial \beta'_i} = I + II,$$

where

$$I := \left(\gamma'_2(\tau(s + s'_1)) \left(\frac{\partial s}{\partial s'_1} + 1 \right) - \gamma'_2(\tau s'_1) \right) \frac{\partial s'_1}{\partial \beta'_i},$$

$$II := \gamma'_2(\tau(s + s'_1)) \left(\frac{\partial s}{\partial x'_1} \frac{\partial x'_1}{\partial \beta'_i} + \frac{\partial s}{\partial y'_1} \frac{\partial y'_1}{\partial \beta'_i} \right) + \frac{\partial y'_1}{\partial \beta'_i}.$$

From a previous discussion, $\gamma'_1(5\delta_s + \tau s_1)$ is bounded away from 0, so it is enough to estimate I and II . The bounds (18) and (22) imply $|II| \leq C$. For I , we use the explicit formula (23),

$$(25) \quad I = \frac{\gamma'_1(5\delta_s + \tau s_1) (\gamma'_1(\tau s'_1) \gamma'_2(\tau(s + s'_1)) - \gamma'_1(\tau(s + s'_1)) \gamma'_2(\tau s'_1))}{\gamma'_1(\tau(s + s'_1)) \gamma'_1(5\delta_s + \tau s_1) + \gamma'_2(\tau(s + s'_1)) \gamma'_2(5\delta_s + \tau s_1)} \frac{\partial s'_1}{\partial \beta'_i}.$$

We can assume without loss of generality that $\tilde{r} \leq \delta_s$ so the denominator is close to 1 since $\tau(s + s'_1) = \tau\sigma \in (4\delta_s + \tau s_1, 6\delta_s + \tau s_1)$. The second factor of the numerator is equal to

$$(26) \quad \frac{\sinh(\tau(s + s'_1)) - \sinh(\tau s'_1)}{\cosh(\tau(s + s'_1)) \cosh(\tau s'_1)} = \frac{\cosh(\tau \bar{s})}{\cosh(\tau(s + s'_1)) \cosh(\tau s'_1)} \tau \bar{s}$$

for some $\bar{s} \in (s'_1, s'_1 + s_1)$ given by the Mean Value Theorem. Combining (19), (25) and (26), we have

$$(27) \quad \frac{1}{C\tau} (4\delta_s + \tau(s_1 - s'_1)) \delta_{i1} \leq |I| \leq \frac{C}{\tau} (6\delta_s + \tau(s_1 - s'_1)) \delta_{i1}.$$

Choosing δ_φ small enough, we can ensure that $4\delta_s - \tau|s_1 - s'_1| > 0$ by (17). From (24), (27), the fact that $|II| \leq C$, and similar estimates for f_2, f_3 and f_4 , we have for τ small enough,

$$(28) \quad \det \left(\left[\frac{\partial f}{\partial \beta'} \right] \right) \geq \frac{1}{C\tau},$$

therefore $\partial f / \partial \beta'$ has an inverse with norm bounded by $C\tau$. By the Implicit Function Theorem, for every $\underline{\varphi}'$ small enough, there is a tetrad T' such that $(f_1, f_2, f_3, f_4)(T', \underline{\varphi}') = 0$. To get the estimate (iii), we write $\frac{\partial f}{\partial \beta'} \frac{\partial \beta'}{\partial \underline{\varphi}'} + \frac{\partial f}{\partial \underline{\varphi}'} = 0$, or equivalently,

$$(29) \quad \frac{\partial f}{\partial \beta'} \left(\frac{\partial \beta'}{\partial \underline{\varphi}'} - Id \right) = -\frac{\partial f}{\partial \underline{\varphi}'} - \frac{\partial f}{\partial \beta'}.$$

To estimate $\partial f / \partial \beta' + \partial f / \partial \underline{\varphi}'$, note that the contribution of order τ^{-1} in $\partial f / \partial \beta'$ comes from the derivative $\partial s'_1 / \partial \beta'_i$. Adding the derivative with respect to φ' , we get $\partial s'_1 / \partial \beta'_i + \partial s'_1 / \partial \varphi'_i = 0$, by (17). Therefore $\left| \frac{\partial f}{\partial \beta'} + \frac{\partial f}{\partial \underline{\varphi}'} \right| \leq C$, and the result (iii) follows from (28) and (29). \square

We may need the values of δ_φ and δ_τ to be smaller than the ones given in Lemma 14 for later estimates. When we write “for τ small enough” in the rest of the article, we mean that the value of δ_τ has to be adjusted accordingly, and similarly for δ_φ .

4.2. Graphs of functions on a surface. The wings are the graphs of small functions over the asymptotic grim reapers. In order to estimate the mean curvature and the second fundamental form, we take a brief detour and discuss some standard facts about normal perturbations of surfaces (see Appendix C of [8], or Appendix B of [9]).

Suppose we have a surface M in E^3 , immersed by a C^2 map $X : M \rightarrow E^3$. We write g, A, H and ν for the first and second fundamental forms, the mean curvature and the Gauss map of M respectively. For a C^2 function σ on M , we define $X_\sigma : M \rightarrow \mathbf{R}^3$ by $X_\sigma \equiv X + \sigma\nu$. When X_σ is an immersion, we denote by M_σ the graph of σ over M and by $g_\sigma, A_\sigma, H_\sigma$, and ν_σ the first and second fundamental forms, the mean curvature, and the Gauss map of $X_\sigma(M)$ pulled back to M .

We use Φ to denote a term which can be either σA or $\nabla\sigma$. We use an $*$ to denote a contraction with respect to g . Also, all the G 's below stand for linear combinations with universal coefficients of terms which are contractions with respect to g of at least two Φ 's. The \hat{G} 's denote linear combinations (with universal coefficients) of terms which are contractions of a number of - possibly none - Φ 's with one of the following:

- (i) $A * \Phi * \Phi$,
- (ii) $\sigma \nabla A * \Phi$,
- (iii) $\sigma A * \nabla^2 \sigma$,
- (iv) $\nabla^2 \sigma * \Phi * \Phi$.

Let e_1, e_2, ν be a local orthonormal frame of E^3 whose restriction to M has e_1 and e_2 tangent to M . If $|\sigma A| < 1$, then X_σ is an immersion and we have

$$(30) \quad g_{\sigma ij} = g_{ij} - 2\sigma A_{ij} + G_{ij},$$

$$(31) \quad \nu_\sigma = \nu - \nabla\sigma + \mathbf{Q}_\sigma^\nu,$$

where $\mathbf{Q}_\sigma^\nu = G_1 e_1 + G_2 e_2 + G_3 \nu + \frac{G_5 e_1 + G_6 e_2 + G_7 \nu}{1 + G_4 + \sqrt{1 + G_4}}$.

$$(32) \quad H_\sigma = H + (\Delta\sigma + |A|^2\sigma) + Q_\sigma,$$

where $Q_\sigma = \frac{\hat{G}_1}{\sqrt{1+G_8}} + \frac{\hat{G}_2}{1+G_8+\sqrt{1+G_8}}$. Therefore

$$(33) \quad H_\sigma - \tau \vec{e}_y \cdot \nu_\sigma = H - \tau \vec{e}_y \cdot \nu + \Delta_g \sigma + |A|^2 \sigma + \tau \vec{e}_y \cdot \nabla \sigma + Q_\sigma + \tau \vec{e}_y \cdot \mathbf{Q}_\sigma^\nu.$$

4.3. Notations. We will use the same notation for functions, tensors, and operators on the asymptotic grim reaper and their pushforwards by $F_i \circ A_i^{-1}$ to $\Sigma_{\geq 1}[T, \underline{\varphi}, \tau]$, and vice versa. To avoid confusion, we use symbols without subscripts for the geometric quantities considered on the asymptotic grim reapers; we use symbols with subscripts Σ for their counterparts on $\Sigma_{\geq 1}$. For example, g denotes the metric on the asymptotic grim reaper (induced by its immersion in E^3) and it also denotes the pushforward of this metric to $\Sigma_{\geq 1}$, while g_Σ denotes the metric on $\Sigma_{\geq 1}$ induced by the metric in E^3 or its pullback to the asymptotic grim reaper.

For $\underline{\varphi}'$ as in Lemma 14 and a fixed $i \in \{1, \dots, 4\}$, we use a dot \cdot to denote the differentiation $\partial/\partial\varphi'_i|_{\varphi'=0}$.

4.4. Estimates on the desingularizing surface $\Sigma[T, \underline{\varphi}, \tau]$.

Lemma 15. $|(\theta(T'))^\cdot| \leq C$ and the following are valid on $\Sigma_{\geq 1}[T, \underline{\varphi}, \tau]$:

- (i) $\|\dot{\kappa} : C^k(g)\| \leq C$,

- (ii) $\|A : C^k(g)\| \leq C\tau,$
- (iii) $\|\dot{A} : C^k(g)\| \leq C\tau^2,$
- (iv) $\|\nu : C^k(g)\| \leq C,$
- (v) $\|\dot{\nu} : C^k(g)\| \leq C\tau,$

where κ is as in (16), A and ν are as in Section 1.4 and Notations 4.3, and all the constants C depend only on k .

Proof. From (16), κ is an isometry of H^+ to an asymptotic grim reaper. Keeping in mind that $\dot{f} = \sum_{j=1}^4 \frac{\partial f}{\partial \beta_j'} \frac{\partial \beta_j'}{\partial \varphi_i'} + \frac{\partial f}{\partial \varphi_i'}|_{\varphi'=0}$ for any function $f(T', \varphi')$, the lemma follows immediately from the explicit formula for the position κ and the estimates (18) and (iii) of Lemma 14. \square

Corollary 16. *The following estimates are valid on $\Sigma_{\geq 1}[T, \varphi, \tau]$, where $l = 5\delta_s/\tau$:*

$$\begin{aligned} \|g_\Sigma - g : C^3(\Sigma_{\geq 1}, g, e^{-s})\| &\leq C\varepsilon, \\ \||A_\Sigma|^2 - |A|^2 : C^3(\Sigma_{\geq 1}, g, e^{-s})\| &\leq C\varepsilon, \\ \||A_\Sigma|^2 : C^3(\Sigma_{\geq 1}, g, e^{-s} + l^{-2})\| &\leq C\varepsilon + C\delta_s^2. \end{aligned}$$

In particular, g and g_Σ are uniformly equivalent on $\Sigma_{\geq 1}[T, \varphi, \tau]$ by assuming without loss of generality that the constant ε from Proposition 7 is small enough.

Proof. The variation of a metric under a normal perturbation $\sigma = \psi_s f_{\theta(T)}$ is given by (30). The first estimate follows from (ii) in Lemma 15 and Proposition 7.

Similarly, we prove the bound on $\||A_\Sigma|^2 - |A|^2\|$ using the fact that the perturbation of $|A|^2$ is at least linear in σ , and the second fundamental form is controlled by the previous lemma.

For the last estimate, we write

$$\||A_\Sigma|^2 : C^3(g, e^{-s} + l^{-2})\| \leq \||A_\Sigma^2 - |A|^2 : C^3(g, e^{-s})\| + l^{-2} \||A|^2 : C^3(g)\|.$$

The first term on the right hand side is controlled by $C\varepsilon$, while the second term is bounded by $C\delta_s^2$ using Lemma 15. \square

Lemma 17. *Given $\gamma \in (0, 1)$, we have*

$$\|H_\Sigma - \tau \vec{e}_y \cdot \nu_\Sigma : C^2(\Sigma_{\geq 1}[T, \varphi, \tau], g, e^{-\gamma s})\| \leq C\tau,$$

where H_Σ is the mean curvature of $\Sigma[T, \varphi, \tau]$.

The proof is the same as the proof of Lemma 10 in [14]. We reproduce it here for the reader's convenience.

Proof. First, note that the estimate is true for $s \geq 4\delta_s/\tau$. Let us now work in the region $s \in [3\delta_s/\tau, 4\delta_s/\tau]$. We have $H - \tau \vec{e}_y \cdot \nu \equiv 0$ on the grim reaper cylinder, so by the variation formulas in Section 4.2, $H_\Sigma - \tau \vec{e}_y \cdot \nu_\Sigma$ has terms at least linear involving σA_{ij} , $\nabla \sigma$ and $\nabla_{ij}^2 \sigma$ (with $\sigma = \psi_s f_{\theta(T)}$). We are on the support of the derivative of the cut-off function ψ_s , so these terms, their first and second derivatives behave like $(\frac{\tau}{\delta_s})^k e^{-s}$, $0 \leq k \leq 4$. For $s \geq 3\delta_s/\tau$, we can arrange $(\frac{\tau}{\delta_s})^k e^{-s} \leq e^{-s} \leq \tau e^{-\gamma s}$ to be true by taking τ small enough in terms of γ .

In the region $s \leq 3\delta_s/\tau$, we have $\psi_s \equiv 1$ and $\sigma = f_{\theta(T)}$. The plane and the original Scherk surface are minimal surfaces, so

$$0 = \Delta_{g_{\mathbb{R}^2}} \sigma + Q',$$

where Q' is the Q_σ defined by equation (32), and every geometric quantity and tensor in Q' is taken with respect to the flat metric on the asymptotic plane (to the original Scherk surface). Since $H - \tau \vec{e}_y \cdot \nu$ vanishes on the grim reaper cylinder,

$$H_\Sigma - \tau \vec{e}_y \cdot \nu_\Sigma = \Delta_g \sigma + |A|^2 \sigma + Q + \tau \vec{e}_y \cdot \nabla \sigma + \tau \vec{e}_y \cdot \mathbf{Q}^\nu,$$

where Q is Q_σ from equation (32) and where every geometric quantity and tensor in Q is taken with respect to the metric on the asymptotic grim reaper. The term \mathbf{Q}^ν is an expression at least quadratic in $\nabla \sigma$ and σA given by (31). Since the asymptotic grim reaper is isometric to a plane with flat metric,

$$H_\Sigma - \tau \vec{e}_y \cdot \nu_\Sigma = -Q' + |A|^2 \sigma + Q + \tau \vec{e}_y \cdot \nabla \sigma + \tau \vec{e}_y \cdot \mathbf{Q}^\nu.$$

By Lemma 15, the fact that $\|\sigma : C^5(\Sigma_{\geq 1}, g, e^{-s})\| \leq \varepsilon$ and equation (31), we have

$$\| |A|^2 \sigma + \tau \vec{e}_y \cdot \nabla \sigma + \tau \vec{e}_y \cdot \mathbf{Q}^\nu : C^2(\Sigma_{\geq 1}, g, e^{-\gamma s}) \| \leq C\tau.$$

We are left with

$$Q - Q' = \frac{\hat{G}'_1}{\sqrt{1 + G'_8}} - \frac{\hat{G}_1}{\sqrt{1 + G_8}} + \frac{\hat{G}'_2}{1 + G'_8 + \sqrt{1 + G'_8}} - \frac{\hat{G}_2}{1 + G_8 + \sqrt{1 + G_8}}.$$

We can reduce the fractions to the same denominator and expand the numerators and the square roots in Taylor series. The expressions for G 's, G' 's, \hat{G} 's and \hat{G}' 's comprise terms involving σ , $\nabla \sigma$, $\nabla^2 \sigma$, A , σA and $\sigma \nabla^2 A$ and at least quadratic in σ . If a term in the expansion of the numerators has either A , σA or $\sigma \nabla^2 A$, it can be bounded by $C\tau$ thanks to Lemma 15. We now claim that there is no term involving only $\nabla \sigma$ and $\nabla^2 \sigma$. The grim reaper is isometric to a flat plane, so \hat{G}'_1 only differs from \hat{G}_1 by terms involving the second fundamental form A . In other words, setting $A = 0$ in the expression for \hat{G}_1 gives us \hat{G}'_1 . The same property is true for any \hat{G} (G) and its corresponding \hat{G}' (G' respectively). Therefore, they contain exactly the same terms involving only $\nabla \sigma$ and $\nabla^2 \sigma$, and these terms can be paired and cancelled. \square

Lemma 18.

$$\left\| \frac{\partial}{\partial \varphi'_i} \Big|_{\varphi'=0} (H_\Sigma - \tau \vec{e}_y \cdot \nu_\Sigma) : C^1(\Sigma_{\geq 1}[T, \underline{\varphi}, \tau], e^{-\gamma s}) \right\| \leq C\tau.$$

Proof. From equation (33), we have

$$H_\Sigma - \tau \vec{e}_y \cdot \nu_\Sigma = \Delta_g \sigma + |A|^2 \sigma + Q + \tau \vec{e}_y \cdot \nabla \sigma + \tau \vec{e}_y \cdot \mathbf{Q}^\nu,$$

where $\sigma = \psi_s f_\theta$. Therefore

$$(H_\Sigma - \tau \vec{e}_y \cdot \nu_\Sigma)' = (|A|^2)' \sigma + |A|^2 \psi_s \frac{\partial f_\theta}{\partial \theta} (\theta(T'))' + \dot{Q} + \tau \psi_s \vec{e}_y \cdot \frac{\partial \nabla f_\theta}{\partial \theta} (\theta(T'))' + \tau \vec{e}_y \cdot \dot{\mathbf{Q}}^\nu$$

and we have the result from the estimates in Proposition 7, Lemma 15 and the definitions of Q' and \mathbf{Q}^ν given in Section 4.2. \square

4.5. The functions \bar{u}_i and \bar{w}_i .

Definition 19. Let Y be the variation vector field on $\Sigma_{\geq 1}[T, \underline{\varphi}, \tau]$ due to changing φ'_i that is on the component contained in the j th wing,

$$Y = (F_j[T', \underline{\varphi} - \varphi', \tau])',$$

and let $Y_{\parallel} := Y - (Y \cdot \nu_\Sigma) \nu_\Sigma$ be the tangential component of Y .

Lemma 20. $\|Y_{\parallel} : C^1(\Sigma_{\geq 1}[T, \underline{\varphi}, \tau])\| \leq C$ and $\|Y \cdot \nu_{\Sigma} : C^1(\Sigma_{\geq 1}[T, \underline{\varphi}, \tau])\| \leq C$.

Proof. We have

$$Y = (\kappa + \psi_s f_{\theta} \nu) \cdot = \dot{\kappa} + \psi_s \frac{\partial f_{\theta}}{\partial \theta} (\theta(T')) \cdot + \psi_s \sigma_{\theta} \dot{\nu}$$

and the result follows from Proposition 7 and Lemma 15. \square

Definition 21. We define the functions \bar{u}'_i for $i = 1, \dots, 4$ on $\Sigma[T, \underline{\varphi}, \tau]$ by

$$\bar{u}'_i := \left. \frac{\partial}{\partial \varphi'_i} \right|_{\varphi'=0} f_{\varphi'},$$

where $f_{\varphi'}$ is as in Lemma 14.

Definition 22. Given any smooth surface S in E^3 , we define the linear differential operators

$$\begin{aligned} L_S &:= \Delta_S + |A_S|^2, \\ \mathcal{L}_S &:= \Delta_S + |A_S|^2 + \tau \bar{e}_y \cdot \nabla, \end{aligned}$$

where g_S , $|A_S|^2$, and Δ_S denote the first fundamental form of S , the square norm of the second fundamental form of S , and the Laplacian with respect to g_S on S .

Lemma 23. On $\Sigma_{\geq 1}[T, \underline{\varphi}, \tau]$, $\bar{u}' = Y \cdot \nu_{\Sigma}$ and

$$(H_{\Sigma} - \tau \bar{e}_y \cdot \nu_{\Sigma}) \cdot = Y_{\parallel} (H_{\Sigma} - \tau \bar{e}_y \cdot \nu_{\Sigma}) + \mathcal{L}_{\Sigma} \bar{u}'_i.$$

Proof. As φ' changes, one can see the change in the surface as a reparametrization of the surface followed by $X + f_{\varphi'} \nu_{\Sigma}$. The derivative \cdot of the variation of the surface is $Y = Y_{\parallel} + \bar{u}'_i \nu_{\Sigma}$. Using the fact that differentiation is linear, we get

$$(H_{\Sigma} - \tau \bar{e}_y \cdot \nu_{\Sigma}) \cdot = Y_{\parallel} (H_{\Sigma} - \tau \bar{e}_y \cdot \nu_{\Sigma}) + Y_{\perp} (H_{\Sigma} - \tau \bar{e}_y \cdot \nu_{\Sigma}),$$

and the derivative of the normal variation is $\mathcal{L}_{\Sigma} \bar{u}'$, from Section 4.2. \square

Corollary 24. $\|\mathcal{L}_{\Sigma} \bar{u}'_i : C^1(\Sigma_{\geq 1}[T, \underline{\varphi}, \tau], e^{-\gamma s})\| \leq C\tau$.

Proof. By Lemma 23, $\mathcal{L}_{\Sigma} \bar{u}'_i = Y_{\perp} (H_{\Sigma} - \tau \bar{e}_y \cdot \nu_{\Sigma})$, so the estimate follows from Lemmas 17 and 20. \square

We correct \bar{u}'_i to \bar{u}_i so that $\mathcal{L}_{\Sigma} \bar{u}_i = 0$ on $\Sigma_{\geq 2}$ by solving a differential equation on each asymptotic grim reaper $\Gamma^j = A_j[T, \varphi_j, \tau](H^+)$, $j = 1, \dots, 4$. The region $\Gamma^j_{\geq 1}$ is isometric to a cylinder $\Omega = [1, 5\delta_s/\tau]/G'$, where G' is the group generated by $(s, z) \rightarrow (s, z + 2\pi)$. We consider the linear operator \mathcal{L}_{Σ} on the asymptotic grim reaper, where g_{Σ} and A_{Σ} are the metric and the second fundamental form on Σ pulled back to the asymptotic grim reaper, following the Notations 4.3. By Corollary 16, we can choose ε and δ_s small enough so that Proposition 37 applies to \mathcal{L}_{Σ} . Hence, we obtain a solution v_i to the linear equation $\mathcal{L}_{\Sigma} v_i = -\mathcal{L}_{\Sigma} \bar{u}'_i$ that vanishes on $s = 5\delta_s/\tau$, is given up to a constant on $s = 1$, and has exponential decay on each asymptotic grim reaper.

We fix now once and for all $\alpha \in (0, 1)$, and take v_i to be the solution described above pushed forward to Σ . More precisely, $v_i := \underline{\mathcal{R}}(0, -\mathcal{L}_{\Sigma} \bar{u}'_i)$, with $\underline{\mathcal{R}}$ as in Proposition 37, on each component of $\Sigma_{\geq 1}[T, \underline{\varphi}, \tau]$. We define

$$\bar{u}_i = \bar{u}'_i + (\psi[1, 2] \circ s) v_i, \quad \bar{w}_i = \mathcal{L}_{\Sigma} \bar{u}_i.$$

Lemma 25. *The functions \bar{u}_i and \bar{w}_i satisfy the following properties:*

- (i) *They depend continuously on $(T, \underline{\varphi}, \tau)$.*
- (ii) *\bar{w}_i is supported on $\Sigma_{\leq 2}$, $\|\bar{w}_i : C^{0,\alpha}(\Sigma)\| \leq C$, and $\|\bar{w}_i : C^{0,\alpha}(\Sigma_{\geq 1})\| \leq C\tau$.*
- (iii) *$\|\bar{u}_i : C^{2,\alpha}(\Sigma)\| \leq C$.*
- (iv) *$\mathcal{L}_\Sigma \bar{u}_i = 0$ on $\Sigma_{\geq 2}[T, \underline{\varphi}, \tau]$, $\bar{u}_i = 0$ on $\partial\Sigma$ and $|\bar{u}_i - (a+2)\delta_{ij}| \leq Ca\varepsilon$ on the component of $\partial\Sigma_{\leq 2}$ contained in the i th wing.*

Proof. By definition of \bar{u}_i and \bar{w}_i , it is immediate that \bar{w}_i is supported on $\Sigma_{\leq 2}$, and (i) follows from the construction. From Corollary 24 and Proposition 37, we have $\|v_i : C^{2,\alpha}(\Sigma_{\geq 1}[T, \underline{\varphi}, \tau], g, e^{-\gamma s})\| \leq C\tau$. Since the metrics g and g_Σ are equivalent,

$$(34) \quad \|(\psi[1, 2] \circ s)v_i : C^{2,\alpha}(\Sigma_{\geq 1}[T, \underline{\varphi}, \tau], g_\Sigma, e^{-\gamma s})\| \leq C\tau.$$

Corollary 24 and $\bar{w}_i = \mathcal{L}_\Sigma \bar{u}'_i + \mathcal{L}_\Sigma((\psi[1, 2] \circ s)v_i)$ imply $\|\bar{w}_i : C^{0,\alpha}(\Sigma_{\geq 1})\| \leq C\tau$.

The function $\bar{u}'_i = Y \cdot \nu_\Sigma$ is bounded in C^1 norm from Lemma 20. The rest of part (ii) and part (iii) follow from (34).

The function \bar{u}_i vanishes on the boundary $\partial\Sigma$ because both \bar{u}'_i and v_i are zero on the boundary by Lemma 14 and by the definition of v_i respectively, so the first assertion of (iv) is proved.

Finally, we estimate $|\bar{u}_i - (a+2)\delta_{ij}|$ at $s = 2$. The function $(\psi[1, 2] \circ s)v_i$ brings at most a contribution of order τ , which can be chosen to be smaller than ε , so we only need to study $|\bar{u}'_i - (a+2)\delta_{ij}|$ at $s = 2$. The curvature of the grim reaper is of order τ , therefore, it suffices to approximate the behavior of \bar{u}'_i up to first order, which means we can consider planes instead of grim reapers. Given $(T', \underline{\varphi}', \tau)$, the position of the point on the asymptotic plane corresponding to $s = 2$ is given by $(x'_j, y'_j) = ((a+2)\cos\beta'_j, (a+2)\sin\beta'_j) + b_{\theta(T')}(-\sin\beta'_j, \cos\beta'_j)$. Differentiating with respect to φ'_i , using (iii) in Lemma 14 and (vi) from Proposition 7, we get

$$\left| \frac{\partial}{\partial \varphi'_i}(x'_j, y'_j) - (a+2)(-\sin\beta'_j, \cos\beta'_j)\delta_{ij} \right| \leq C\tau + C\varepsilon a.$$

We can approximate the position $F_j(T', \underline{\varphi} - \underline{\varphi}', \tau)|_{s=2}$ by (x'_j, y'_j) and the normal ν_Σ by $(-\sin\beta'_j, \cos\beta'_j)$ committing an error of order ε so we have the desired result for τ small enough. \square

4.6. The functions w_1 and w_2 . In this section, we describe how the unbalancing generates functions w_1 and w_2 close to being in the approximate kernel of \mathcal{L}_Σ .

Definition 26. *Let H_ϕ be the mean curvature on the surface $Z_1[\phi](\Sigma(\theta(T)))$ and let $w_1 : \Sigma(\theta(T)) \rightarrow \mathbf{R}$ be defined by*

$$w_1 := \left. \frac{d}{d\phi} \right|_{\phi=0} H_\phi \circ Z_1[\phi].$$

We also denote by w_1 the pushforward to $\Sigma[T, \underline{\varphi}, \tau]$ by $Z[T, \underline{\varphi}, \tau]$ of the function above. Similarly, we define w_2 to be $\left. \frac{d}{d\phi} \right|_{\phi=0} H_\phi \circ Z_2[\phi]$ and its pushforward by $Z[T, \underline{\varphi}, \tau]$ to $\Sigma[T, \underline{\varphi}, \tau]$.

Lemma 27. *The functions w_1 and w_2 depend continuously on $(T, \underline{\varphi}, \tau)$. They are supported on $\Sigma_{\leq 0}$ and $\|w_j : C^0(\Sigma)\| \leq C$, for $j = 1, 2$.*

Let V be the span of the pushforwards by $Z[T, \underline{\varphi}, \tau]$ of the functions $\vec{e}_x \cdot \nu$ and $\vec{e}_y \cdot \nu$ on $\Sigma(\theta(T))$ and let us define $P : L^2(\Sigma[T, \underline{\varphi}, \tau], |A_\Sigma|^2 g_\Sigma/2) \rightarrow V$ to be the orthogonal projection of functions onto V . Every function in V is the projection of a linear

combination of $w_1/|A_\Sigma|^2$ and $w_2/|A_\Sigma|^2$. More precisely, for every $(\mu_1, \mu_2) \in \mathbf{R}^2$, there is a pair $(\eta_1, \eta_2) \in \mathbf{R}^2$ such that

$$P((\eta_1 w_1 + \eta_2 w_2)/|A_\Sigma|^2) = \mu_1 \vec{e}_x \cdot \nu + \mu_2 \vec{e}_y \cdot \nu,$$

with $|(\eta_1, \eta_2)| \leq C(|(\mu_1, \mu_2)|)$.

Proof. The assertions about the continuous dependence and the supports of w_1, w_2 are easily derived from the definitions of Z_1, Z_2, w_1 and w_2 .

To prove the second part, we recall the following balancing formula from [12] and sketch its proof.

Lemma 28. *If S is a surface with four Scherk ends, then $\int_S H \nu d\mu = \int_{\partial S} \vec{n}$, where \vec{n} is the unit outward conormal vector at the boundary ∂S .*

If S is a surface with boundary embedded in E^3 and Y is a vector field that perturbs S , the change in area of S is given by

$$\begin{aligned} \delta(\text{area}(S)) &= \int_S \text{div} Y = \int_S \text{div}(Y^\parallel) + \text{div}(Y^\perp), \\ &= \int_{\partial S} \vec{n} \cdot Y - \int_S H \nu \cdot Y \end{aligned}$$

where $Y^\perp = Y \cdot \nu$ and $Y^\parallel = Y - Y^\perp$ are the orthogonal and parallel components of Y , and \vec{n} is the unit outward conormal at ∂S . If $Y = \vec{e}_x, \vec{e}_y$, or \vec{e}_z , the area of S does not change, so $\int_S H \nu = \int_{\partial S} \vec{n}$.

If the surface S is one period of $\Sigma_{\leq 0}$, we have $\int_{\partial S} \vec{n} = 2\pi \sum_{i=1}^4 v_i$, where the v_i 's are the vectors of the tetrad T . Differentiating the balancing formula, we obtain

$$\frac{\partial}{\partial \phi} \int_{\Sigma_{\leq 0}/G'} H_\phi \circ Z_1[\phi] \nu d\mu = 2\pi \frac{\partial}{\partial \phi} \sum_{i=1}^4 v_i.$$

Under $Z_1(\phi)$, the first and third wings rotate by an angle ϕ towards the fourth quadrant, so $\frac{\partial}{\partial \phi} \sum_{i=1}^4 v_i = (2 \sin \theta(T), -2 \cos \theta(T))$. Similarly, the rate of change of the sum of the conormals under $Z_2[\phi]$ is $(2 \sin \theta(T), 2 \cos \theta(T))$. The two vectors $(2 \sin \theta(T), \pm 2 \cos \theta(T))$ are linearly independent since $\theta(T) \in (20\delta_\theta, \pi/2 - 20\delta_\theta)$ in (9), hence the norm of P^{-1} is bounded by a constant C depending on δ_θ . \square

4.7. The decomposition of $H_\Sigma - \tau \vec{e}_y \cdot \nu_\Sigma$.

Proposition 29. *For $T, \underline{\varphi}, \tau$ as in Lemma 14,*

$$\|H_\Sigma - \tau \vec{e}_y \cdot \nu_\Sigma - \sum_{j=1}^2 \theta_{j,\Sigma} w_j - \sum_{i=1}^4 \varphi_i \bar{w}_i : C^{0,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s})\| \leq C(\tau + |\underline{\theta}_\Sigma|^2 + |\underline{\varphi}|^2),$$

where $\underline{\theta}_\Sigma = (\theta_1(T) + \frac{\varphi_3}{2} - \frac{\varphi_1}{2}, \theta_2(T) + \frac{\varphi_4}{2} - \frac{\varphi_2}{2})$.

Proof. Without loss of generality, we can assume that $\theta_r(T) = 0$. On $\Sigma_{\geq 1}$, $w_1, w_2 = 0$ and the result follows from Lemmas 17 and 25.

Let T' be the tetrad given by Lemma 14 when $\underline{\varphi}' = \underline{\varphi}$, $\hat{T} := \{\vec{e}(\beta_i + \varphi_i)\}_{i=1}^4$, and let T_0 be the balanced tetrad of the directing vectors of the Scherk surface $\Sigma(\theta(T))$. We have

$$H_{\Sigma[T, \underline{\varphi}, \tau]} - \tau \vec{e}_y \cdot \nu_{\Sigma[T, \underline{\varphi}, \tau]} = I + II + III,$$

with

$$\begin{aligned} I &:= H_{\Sigma[T, \varphi, \tau]} - \tau \vec{e}_y \cdot \nu_{\Sigma[T, \varphi, \tau]} - (H_{\Sigma[T', 0, \tau]} - \tau \vec{e}_y \cdot \nu_{\Sigma[T', 0, \tau]}), \\ II &:= H_{\Sigma[T', 0, \tau]} - \tau \vec{e}_y \cdot \nu_{\Sigma[T', 0, \tau]} - (H_{\Sigma[\hat{T}, 0, \tau]} - \tau \vec{e}_y \cdot \nu_{\Sigma[\hat{T}, 0, \tau]}), \\ III &:= H_{\Sigma[\hat{T}, 0, \tau]} - \tau \vec{e}_y \cdot \nu_{\Sigma[\hat{T}, 0, \tau]}. \end{aligned}$$

On $\Sigma_{<1}$, $\bar{w}_i = \mathcal{L}_{\Sigma} \bar{u}'_i$, so $I - \sum_{i=1}^4 \varphi_i \bar{w}_i$ is of order $|\varphi|^2$ by definition of \bar{u}'_i . For II , Lemma 14 implies $\beta'_i \sim \beta_i + \varphi_i$, with an error of order τ . The smooth dependence of the construction on T gives $\|II : C^{0, \alpha}(\Sigma_{\leq 1})\| \leq C\tau$. For III , note that $\theta_{j, \Sigma} = \theta_j(\hat{T})$, $j = 1, 2$ and, by the definition of w_1, w_2 ,

$$\|H_{\Sigma[\hat{T}, 0, \tau]} - H_{[T_0, 0, \tau]} - \sum_{j=1}^2 \theta_{j, \Sigma} w_j : C^{0, \alpha}(\Sigma_{\leq 1})\| \leq C|\underline{\theta}_{\Sigma}|^2.$$

Using the facts that $H_{[T_0, 0, \tau]} \equiv 0$ and $\|\tau \vec{e}_y \cdot \nu_{\Sigma[\hat{T}, 0, \tau]} : C^{0, \alpha}(\Sigma_{\leq 1})\| \leq C\tau$, we finish the proof. \square

5. INITIAL SURFACES

We construct smooth initial surfaces by fitting the desingularizing surfaces at the vertices of the flexible graph \bar{G} from Section 2. The ends of the grim reapers have to be slightly adjusted, and the wings of the desingularizing surfaces bent to attach them smoothly; we keep track of these variations with the variables (\tilde{b}, \tilde{c}) and $\tilde{\varphi}$ respectively.

Let $\{\tilde{\Gamma}_n\}_{n=1}^{N_{\Gamma}}$ be a finite family of grim reapers in general position as in Section 2.1 and recall that N_I is the number of intersection points. To each intersection point p_k , $k = 1, \dots, N_I$, we assign a positive integer m_k and define

$$(35) \quad \tau_k := \frac{\bar{\tau}}{m_k}, \quad \underline{m} := \{m_k\}_{k=1}^{N_I}.$$

The parameter $\bar{\tau}$ controls the overall scaling for the surface and the m_k 's allow for different scalings at different intersection points: the desingularizing surface for the k th intersection will have a period of $2\pi/m_k$ in the z -direction. We choose $\bar{\tau} \in (0, \delta_{\tau})$ small enough so that Proposition 13 applies and the desingularizing surfaces are embedded. Unless otherwise stated, our constants C will depend on $\delta_{\Gamma}, N_{\Gamma}, \delta$ from Lemma 3.

To each intersection point corresponds some unbalancing and bending of the wings measured by $\theta_{k,1}, \theta_{k,2}$ and $\{\varphi_{k,i}\}_{i=1}^4$ respectively. We collect all of the angles for each intersection, along with the perturbations of the left rays for an initial configuration in the next definition.

Definition 30. Let $\mathcal{V}_{\varphi} := (\mathbf{R}^4)^{N_I}$, $\mathcal{V}_{\theta} := (\mathbf{R}^2)^{N_I}$, $\mathcal{V}_{b,c} := (\mathbf{R}^2)^{N_{\Gamma}}$, and $\mathcal{V} := \mathcal{V}_{\theta} \times \mathcal{V}_{b,c} \times \mathcal{V}_{\varphi}$. We fix $\zeta > 0$, which will be determined later in the proof of Theorem 48 and define

$$\Xi_{\mathcal{V}} := \{\xi \in \mathcal{V} : |\xi| \leq \zeta \bar{\tau}\}.$$

We fix now $\xi = (\underline{\theta}, \bar{\tau} \underline{b}', \bar{\tau} \underline{c}', \varphi) \in \Xi_{\mathcal{V}}$, where

- $\underline{\theta} = \{\theta_{k,1}, \theta_{k,2}\}_{k=1}^{N_I}$ and $(\theta_{k,1}, \theta_{k,2})$ determines the dislocation of the tetrad at the k th intersection.
- $\underline{b}' := \{b'_n\}_{n=1}^{N_{\Gamma}}$ and $\underline{c}' := \{c'_n\}_{n=1}^{N_{\Gamma}}$. The point $(b_n + b'_n, c_n + c'_n)$ is the center of the grim reaper on which the n th left ray of \bar{G} lies (see Proposition 5).

- $\varphi := \{\{\varphi_{k,i}\}_{i=1}^4\}_{k=1}^{N_I}$ relates to the bending of the wings at each intersection.

Given $\bar{\tau}$ small and extracting $\underline{\theta}$ and $(\underline{b}', \underline{c}')$ from ξ , we construct the initial configuration $\bar{G}(\underline{b}', \underline{c}', \underline{\theta}, \bar{\tau})$ as in Section 2.

Let us recall that $\{\bar{v}_{ki}\}_{i=1}^4$ are the directing vectors of the edges or rays emanating from the k th vertex of \bar{G} . We define the angles β_{ki} by the relation $\bar{v}_{ki} = \bar{e}[\beta_{ki}]$ and the tetrad $T_k(\xi)$ by

$$(36) \quad T_k(\xi) = \{\bar{e}[\beta_{ki} + \varphi_{k,i}]\}_{i=1}^4.$$

We take the mapping \mathcal{H}_k to be the scaling by $1/m_k$ followed by the translation that sends the origin to \bar{p}_k , the k th vertex of \bar{G} . The surface $\mathcal{S}_k(\bar{\tau}, \xi)$, which desingularizes the k th intersection, is defined by

$$\mathcal{S}_k(\bar{\tau}, \xi) := \mathcal{H}_k(\Sigma[T_k(\xi), \tilde{\varphi}_k, \tau_k])$$

with $\tilde{\varphi}_k$ to be determined later and τ_k as in (35). The pivots of \mathcal{S}_k are defined to be the image under \mathcal{H}_k of the pivots given in Definition 10. Note that they do not depend on $\tilde{\varphi}_k$.

Let us now construct the edges and the rays of the initial surface. For an edge in \bar{G} connecting the two vertices \bar{p}_k and $\bar{p}_{k'}$, we consider the pivots of \mathcal{S}_k and $\mathcal{S}_{k'}$ on the appropriate wings, say the i th wing of \mathcal{S}_k and the i' th wing of $\mathcal{S}_{k'}$. There is a unique grim reaper $\Gamma_{kk'}$ passing through the two pivots. We choose $\tilde{\varphi}_{k,i}$ so that the grim reaper asymptotic to the i th wing of \mathcal{S}_k matches $\Gamma_{kk'}$ and define $\tilde{\varphi}_{k',i'}$ similarly; in other words, $\tilde{\varphi}_{k,i}$ is the angle formed by $\Gamma_{kk'}$ with the grim reaper asymptotic to the i th wing of $\mathcal{H}_k(\Sigma[T_k(\xi), \underline{0}, \tau_k])$ at the pivot. Now that the relevant $\tilde{\varphi}$'s are fixed, let us denote the piece of $\Gamma_{kk'}$ between the two pivots by \mathcal{E}' and the piece between the boundaries $\partial\mathcal{S}_k$ and $\partial\mathcal{S}_{k'}$ by \mathcal{E}'' . We do not assign any index to the notation \mathcal{E}' or \mathcal{E}'' to differentiate the various edges because it is not important for what follows.

If we have a ray emanating from \bar{p}_k and corresponding to the i th wing of \mathcal{S}_k , we just translate the ray so that its boundary matches the corresponding pivot. $\tilde{\varphi}_{k,i}$ is taken so that the asymptotic grim reaper to the i th wing matches the translated ray. We denote the translated ray by \mathcal{N}' and the piece of the ray starting at $\partial\mathcal{S}_k$ by \mathcal{N}'' .

Definition 31. We define the initial surface $M_{\bar{\tau}}$, the union of the asymptotic surfaces $M'_{\bar{\tau}}$, and the union of the desingularizing pieces $\mathcal{S}_{\bar{\tau}}$:

$$\begin{aligned} M &= M_{\bar{\tau}} = M(\bar{\tau}, \xi) := \bigcup_{k=1}^{N_I} \mathcal{S}_k(\bar{\tau}, \xi) \cup \bigcup \mathcal{E}'' \cup \bigcup \mathcal{N}'', \\ M' &= M'_{\bar{\tau}} = M'(\bar{\tau}, \xi) := \bigcup \mathcal{E}' \cup \bigcup \mathcal{N}', \\ \mathcal{S} &= \mathcal{S}_{\bar{\tau}} = \mathcal{S}(\bar{\tau}, \xi) := \bigcup_{k=1}^{N_I} \mathcal{S}_k(\bar{\tau}, \xi). \end{aligned}$$

The metric g_M on $M_{\bar{\tau}}$ is the metric induced by the immersion of $M_{\bar{\tau}}$ in E^3 .

Definition 32. We push forward the function s from Section 3.4 by \mathcal{H}_k to each \mathcal{S}_k and extend it to M in the following way:

- $s = \max_k(5\delta_s/\tau_k)$ on \mathcal{E}'' .

- The coordinate s of a point $p \in \mathcal{N}''$ is $d(p, \partial\mathcal{N}'') + 5\delta_s/\tau_k$, where d denotes the distance measured with respect to arclength on \mathcal{N}'' , and k is such that $\mathcal{S}_k \cap \mathcal{N}'' \neq \emptyset$.

The function s is continuous at $\mathcal{S}_\tau \cap \mathcal{N}''$ but may be discontinuous at $\mathcal{S}_\tau \cap \mathcal{E}''$.

To keep track of the unbalancing of M , the displacement of the asymptotic grim reapers, and the bending of M , we define

- $\tilde{\theta} = \{\tilde{\theta}_{k,1}, \tilde{\theta}_{k,2}\}_{k=1}^{N_I} \in \mathcal{V}_\theta$ where $\tilde{\theta}_{k,1} = \theta_1(T_k(\xi))$ and $\tilde{\theta}_{k,2} = \theta_2(T_k(\xi))$,
- $(\tilde{b}, \tilde{c}) \in \mathcal{V}_{b,c}$ where $(b_n + b'_n + \tilde{b}_n, c_n + c'_n + \tilde{c}_n)$ is the position of the center of the grim reaper supporting the n th left ray.
- $\tilde{\varphi} \in \mathcal{V}_\varphi$ as described earlier.

The proposition below follows from the construction and Proposition 5.

Proposition 33. *The surface $M_{\tilde{\tau}}$ constructed above is well defined and has the following properties:*

- $M_{\tilde{\tau}}$ is a complete smooth surface which depends smoothly on ξ .
- $M_{\tilde{\tau}}$ is periodic in z ; more precisely, it is invariant under the group generated by the translation $z \rightarrow z + 2\pi$.
- $M_{\tilde{\tau}}$ is invariant under reflections across the xy -plane.
- There is a large ball $B \subset \mathbf{R}^2$ such that $M_{\tilde{\tau}} \setminus (B \times \mathbf{R})$ is the union of grim reaper ends that are in one-to-one correspondence with the ends of our rescaled initial family $\{\Gamma_n\}_{n=1}^{N_I}$ and such that corresponding centers of the left rays differ by $(b'_n + \tilde{b}_n, c'_n + \tilde{c}_n)$.
- We have $\tilde{\theta}_{k,j} = \theta_{k,j} + (\varphi_{k,j+2} - \varphi_{k,j})/2$ for $k = 1, \dots, N_I$ and $j = 1, 2$.
- $|(\tilde{b}, \tilde{c})| \leq C$ and $|\tilde{\varphi} + \varphi| \leq C\tilde{\tau}$.
- $\mathcal{H}_k^{-1}(M_{\tilde{\tau}})$ converges uniformly in C^j norm, for any $j < \infty$, on any compact subset of E^3 to a Scherk surface as $\tilde{\tau} \rightarrow 0$, for $k = 1, \dots, N_I$.

Corollary 34. *For $\tilde{\tau}$ small enough so that $\zeta\tilde{\tau} < \delta_\theta$, the parameters of $\Sigma[T_k, \tilde{\varphi}_k, \tau_k] = \mathcal{H}_k^{-1}(\mathcal{S}_k)$ satisfy the following estimates,*

- $25\delta_\theta \leq \theta(T_k) \leq \frac{\pi}{2} - 25\delta_\theta$.
- $|\tilde{\theta}_{k,j}| = |\theta_j(T_k)| \leq 3\zeta\tilde{\tau}$, $j = 1, 2$.
- $|\tilde{\varphi}_k| \leq (\zeta + C)\tilde{\tau}$.

Proof. This corollary follows from the construction, Definition 6, Definition 30, and the proof of Proposition 5. \square

As mentioned in the introduction, we divide the study of the linear operator to various pieces of the initial surface, then use cut-off functions and an iteration to solve the equation $\mathcal{L}v = f$ on the whole surface $M_{\tilde{\tau}}$. In the definition of $M_{\tilde{\tau}}$, the \mathcal{S}_k 's, \mathcal{E}'' 's, and \mathcal{N}'' 's intersect only on the boundary of the \mathcal{S}_k 's. In order to use cut-off functions, we define neighborhoods \mathcal{E} and \mathcal{N} of \mathcal{E}'' and \mathcal{N}'' to have some overlap with \mathcal{S}_k .

Definition 35. *Let $\underline{a} := 8|\log \tilde{\tau}|$. We consider*

- \mathcal{N} to be the connected component of $M(\tilde{\tau}, \xi)_{\geq \underline{a}}$ that contains \mathcal{N}'' .
- \mathcal{E} to be the connected component of $M(\tilde{\tau}, \xi)_{\geq \underline{a}}$ that contains \mathcal{E}'' .

There are $2N_I$ grim reaper ends, and we denote each of them by \mathcal{N}'' , without any index to differentiate them because they are treated in the same manner in the

rest of the article. Therefore, \mathcal{N}'' is a generic end and \mathcal{N} is a neighborhood of \mathcal{N}'' . The same remark applies to an edge \mathcal{E}'' and its neighborhood \mathcal{E} as well.

6. LINEAR OPERATOR

We study the linear operator $\mathcal{L} = \mathcal{L}_M := \Delta_{g_M} + |A_M|^2 + \bar{\tau}\bar{e}_y \cdot \nabla$ associated to normal perturbations of $H - \bar{\tau}\bar{e}_y \cdot \nu$. The strategy is to first solve the Dirichlet problem associated to \mathcal{L} with vanishing boundary conditions on the various pieces S_k , \mathcal{E} and \mathcal{N} . Since \mathcal{E} and \mathcal{N} are close to being flat, standard theory for the Laplace operator gives us insight into the Dirichlet problems on \mathcal{E} and \mathcal{N} . On a desingularizing surface Σ , Proposition 40 shows the existence of an exponentially decaying solution to $\mathcal{L}v = E$ with vanishing boundary conditions if E is exponentially decaying. We use the functions w_1 and w_2 to handle the approximate kernel in Lemma 43, then the functions $\{\bar{w}_i\}_{i=1}^4$ for the asymptotic decay.

We are then ready to solve the equation $\mathcal{L}v = E$ on the whole surface M . We first restrict the support of E to the desingularizing surfaces using a cut-off function ψ and find a solution u to $\mathcal{L}u = \psi E$ on each S_k . Extending u by zero to the rest of M may not produce a smooth function, therefore we need to cut off u also. We then solve the Dirichlet problem $\mathcal{L}u'' = E - \mathcal{L}(\psi u)$ on \mathcal{N} and \mathcal{E} with appropriate boundary conditions. Adding ψu to a cut-off version of u'' gives us a first approximate solution. The error created by the cut-off functions is small compared to the inhomogeneous term therefore we can iterate to obtain a sequence of functions converging to an exact solution. The exponential decay of the solutions on the desingularizing surfaces plays an essential role in controlling the error.

6.1. Linear operator on \mathcal{E} and \mathcal{N} . The domains \mathcal{E} and \mathcal{N} are graphs of small functions over grim reapers. The grim reapers are isometric to planes, so \mathcal{E} and \mathcal{N} are close to being isometric to planes with an error controlled by Corollary 16. Moreover, \mathcal{E}/G' is bounded, where G' is the group generated by $(s, z) \rightarrow (s, z + 2\pi)$, so the existence of a unique solution to the Dirichlet problem with vanishing boundary conditions on $\partial\mathcal{E}/G'$ is a standard result from elliptic theory.

The operator \mathcal{L} on \mathcal{N} is a perturbation of the Laplace operator on flat cylinders, which has been well studied. Any result for \mathcal{L} on \mathcal{N} follows directly from a similar one for the Laplace operator on cylinders.

Definition 36. We define (Ω, g_0) to be the cylinder $\Omega = H_{\leq l}^+/G'$ equipped with the standard metric $g_0 = ds^2 + dz^2$, where G' is the group generated by $(s, z) \rightarrow (s, z + 2\pi)$, and $l \in (10, \infty)$ is called the length of the cylinder. We have $\partial\Omega = \partial_0 \cup \partial_l$ where ∂_0 and ∂_l are the boundary circles $\{s = 0\}$ and $\{s = l\}$ respectively.

Let \mathcal{L} denote an operator on Ω of the form

$$\mathcal{L}v = \Delta_\chi v + \mathbf{V} \cdot \nabla v + dv,$$

where χ is a C^2 Riemannian metric, \mathbf{V} a C^1 vector field, and d a C^1 function on Ω . For $\underline{c} > 0$, we define

$$N(\mathcal{L}) := \|\chi - g_0 : C^2(\Omega, g_0, e^{-\underline{c}s})\| + \|\mathbf{V} : C^1(\Omega, g_0, l^{-1})\| + \|d : C^1(\Omega, g_0, e^{-\underline{c}s} + l^{-2})\|.$$

If the inhomogeneous term E is exponentially decaying, there is an exponentially decaying solution to the Dirichlet problem $\mathcal{L}v = E$ in Ω , $v = 0$ on ∂_l , and given up to a constant on ∂_0 . The following result is a generalization of Proposition A.3 in [9], with an added term $\mathbf{V} \cdot \nabla$. Its proof is postponed to the Appendix A.

Proposition 37. *Given $\gamma \in (0, 1)$ and $\varepsilon > 0$, if $N(\mathcal{L})$ is small enough in terms of \underline{c} , α , γ and ε (but independently of l), there is a bounded linear map*

$$\underline{\mathcal{R}} : C^{2,\alpha}(\partial_0, g_0) \times C^{0,\alpha}(\Omega, g_0, e^{-\gamma s}) \rightarrow C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})$$

such that for (f, E) in $\underline{\mathcal{R}}$'s domain and $v = \underline{\mathcal{R}}(f, E)$, the following properties are true, where the constants C depend only on α and γ :

- (i) $\mathcal{L}v = E$ on Ω .
- (ii) $v = f - \text{avg}_{\partial_0} f + B(f, E)$ on ∂_0 , where $B(f, E)$ is a constant on ∂_0 and $\text{avg}_{\partial_0} f$ denotes the average of f over ∂_0 .
- (iii) $v \equiv 0$ on ∂_l .
- (iv) $\|v : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})\| \leq C\|f - \text{avg}_{\partial_0} f : C^{2,\alpha}(\partial_0, g_0)\| + C\|E : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\|$.
- (v) $|B(f, E)| \leq \varepsilon\|f - \text{avg}_{\partial_0} f : C^{2,\alpha}(\partial_0, g_0)\| + C\|E : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\|$.
- (vi) *If $v' \in C^2(\Omega)$ satisfies $\mathcal{L}v' = E$ on Ω , and $v = v'$ on $\partial\Omega$, then $v' = v$ on Ω . Moreover, if E vanishes, then*

$$\|v : C^0(\Omega)\| \leq 2\|v : C^0(\partial_0)\|.$$

A similar result is true for infinite cylinders, once the terms involving l^{-1} and l^{-2} are removed from the definition of $N(\mathcal{L})$.

Proposition 38. *Given $\gamma \in (0, 1)$ and $\varepsilon > 0$, if $N(\mathcal{L})$ is small enough in terms of \underline{c} , α , γ and ε (but independently of l), there is a bounded linear map*

$$\underline{\mathcal{R}} : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s}) \rightarrow C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})$$

such that for E in $\underline{\mathcal{R}}$'s domain and $v = \underline{\mathcal{R}}(E)$, the following properties are true, where the constants C depend only on α and γ :

- (i) $\mathcal{L}v = E$ on Ω .
- (ii) $v = B(E)$ on $\partial\Omega$, where $B(E)$ is a constant.
- (iii) $\|v : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})\| \leq C\|E : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\|$.

Corollary 39. *Given $E \in C^{0,\alpha}(\mathcal{N}, g_M, e^{-\gamma s})$, there is a unique function $v \in C^{2,\alpha}(\mathcal{N}, g_M, e^{-\gamma s})$ such that $\mathcal{L}v = E$, v is a constant on $\partial\mathcal{N}$ and*

$$\|v : C^{2,\alpha}(\mathcal{N}, g_M, e^{-\gamma s})\| \leq C\|E : C^{0,\alpha}(\mathcal{N}, g_M, e^{-\gamma s})\|.$$

Proof. By Corollary 16, we can apply Proposition 38 to $\mathcal{L} = \mathcal{L}_M$ on \mathcal{N} if ε and δ_s are small enough. The uniqueness of the solution $v \in C^{2,\alpha}(\mathcal{N}, g_M, e^{-\gamma s})$ follows from (iii) in Proposition 38. \square

6.2. Linear operator on \mathcal{S}_k . We now prove that we can solve a Dirichlet problem with vanishing boundary data on \mathcal{S}_k , modulo linear combinations of w 's and \bar{w} 's. Let us fix \mathcal{S}_k , $\tau = \tau_k$, and consider the surface $\Sigma = \Sigma[T_k, \tilde{\varphi}_k, \tau] = \mathcal{H}_k^{-1}(\mathcal{S}_k)$, the parameters of which are controlled by Corollary 34. In this section, the linear operator is $\mathcal{L}_\Sigma := \Delta_{g_\Sigma} + |A_\Sigma|^2 + \tau_k \vec{e}_y \cdot \nabla$, where τ_k replaces $\bar{\tau}$.

Proposition 40. *Given $E \in C^{0,\alpha}(\Sigma)$, there are $\underline{\theta}_E := \{\theta_{E,j}\}_{j=1}^2 \in \mathbf{R}^2$, $\underline{\varphi}_E := \{\varphi_{E,i}\}_{i=1}^4 \in \mathbf{R}^4$ and $v_E \in C^{2,\alpha}(\Sigma)$ such that:*

- (i) $\underline{\theta}_E, \underline{\varphi}_E$ and v_E are uniquely determined by the construction below.
- (ii) $\mathcal{L}_\Sigma v_E = E + \sum_{j=1}^2 \theta_{E,j} w_j + \sum_{i=1}^4 \varphi_{E,i} \bar{w}_i$ on Σ and $v_E = 0$ on $\partial\Sigma$.
- (iii) $|\underline{\theta}_E| \leq C\|E\|$, where $\|E : C^{0,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s/m_k})\|$.
- (iv) $|\underline{\varphi}_E| \leq C\|E\|$.

$$(v) \quad \|v_E : C^{2,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s/m_k})\| \leq C\|E\|.$$

We first reduce to the case when E is supported on $\Sigma_{\leq 2}$.

Lemma 41. *If Proposition 40 is valid when E is supported on $\Sigma_{\leq 2}$, it is valid in general.*

Proof. We start on the first wing. Let $\Omega = [1, 5\delta_s/\tau] \times \mathbf{R}/G'$, where G' is the group generated by $(s, z) \rightarrow (s, z + 2\pi)$. By equation (16), the asymptotic grim reaper $\kappa(H_{\geq 1}^+)$ is isometric to Ω . We consider the operator $\mathcal{L}_\Sigma = \Delta_{g_\Sigma} + |A_\Sigma|^2 + \tau \vec{e}_y \cdot \nabla$ on $\kappa(H_{\geq 1}^+)$, where g_Σ and A_Σ are as in Notations 4.3.

By Corollary 16, we can apply Proposition 37 to \mathcal{L}_Σ to obtain $v_1 = \underline{\mathcal{R}}(0, E)$ and extend v_1 , possibly discontinuously, by 0 on the rest of Σ . The functions v_2, v_3 and v_4 are defined similarly for the three other wings. With the usual abuse of notation, we denote their pushforwards on Σ by v_i as well. Using the estimates of Proposition 37 and the fact that g and g_Σ are equivalent metrics, we get

$$(37) \quad \|v_i : C^{2,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s/m_k})\| \leq C\|E\|.$$

On Σ , we define

$$E' = E - \mathcal{L}_\Sigma(\psi[1, 2] \circ s(v_1 + v_2 + v_3 + v_4)).$$

Clearly, E' is supported on $\Sigma_{\leq 2}$. We can apply Proposition 40 with E' to obtain $v_{E'}, \underline{\theta}_{E'}$ and $\underline{\varphi}_{E'}$, then define $\underline{\theta}_E = \underline{\theta}_{E'}$, $\underline{\varphi}_E = \underline{\varphi}_{E'}$, and $v_E = v_{E'} + (\psi[1, 2] \circ s)(v_1 + v_2 + v_3 + v_4)$. The required estimates are valid thanks to (37). \square

In the definition below, the sole purpose of the small constant ε_h is to ensure that the metric h is non degenerate.

Definition 42. *Let us define a metric $h := (\frac{1}{2}|A_\Sigma|^2 + \varepsilon_h)g_\Sigma$, where $\varepsilon_h > 0$ is small depending on τ and δ_θ , and will be determined in the proof of Lemma 43. We also define the c -approximate kernel of $L_h := \Delta_h + 2|A_\Sigma|^2/(|A_\Sigma|^2 + 2\varepsilon_h)$ to be the span of the eigenfunctions of L_h with corresponding eigenvalues in $[-c, c]$.*

Lemma 43. *There are positive constants C and c such that given $E \in L^2(\Sigma/G', h)$, there is $\underline{\theta}_E = (\theta_{E,1}, \theta_{E,2})$ such that $(E - \sum_{j=1}^2 \theta_{E,j} w_j)/(|A_\Sigma|^2 + 2\varepsilon_h)$ is $L^2(\Sigma/G', h)$ -orthogonal to the c -approximate kernel and*

$$|\underline{\theta}_E| \leq C\|E/(|A_\Sigma|^2 + 2\varepsilon_h) : L^2(\Sigma/G', h)\|.$$

The reader familiar with the reference can see that this is Lemma 7.4 from [9]. We sketch the main ideas, paraphrasing the proof from [9] in Appendix B for the sake of completeness.

Proof of Proposition 40. We can assume that E has support in $\Sigma_{\leq 2}$ and, by the smooth dependence on the parameters in Proposition 13, we have uniform control over the geometry of $\Sigma_{\leq 2}$, hence

$$\|E/(|A_\Sigma|^2 + 2\varepsilon_h) : L^2(\Sigma/G', h)\| \leq C\|E\|.$$

We apply Lemma 43 to obtain c and $(\theta'_1, \theta'_2) = (\theta_{E,1}, \theta_{E,2})$. The Dirichlet problem

$$(38) \quad L_h v_E'' = (E - \sum_{j=1}^2 \theta'_j w_j)/(|A_\Sigma|^2/2 + \varepsilon_h) \text{ in } \Sigma, \quad v_E'' = 0 \text{ on } \partial\Sigma,$$

can be solved by using the Lax-Milgram Theorem [5]. Equation (38) is equivalent to

$$L_\Sigma v''_E = \Delta_\Sigma v''_E + |A_\Sigma|^2 v''_E = E - \sum_{j=1}^2 \theta'_j w_j.$$

The non-homogeneous term is $C^{0,\alpha}$, so elliptic regularity, Lemma 43 and the control of the geometry of $\Sigma_{\leq 2}$ imply that the solution v''_E is unique and satisfies

$$\|v''_E : C^{2,\alpha}(\Sigma_{\leq 2}, g_\Sigma)\| \leq C\|E\|.$$

The operator \mathcal{L}_Σ is a perturbation of L_Σ so we can use an iteration process similar to the one in Section 4.1 [14] to obtain v'_E and $\underline{\theta}_E = \{\theta_{E,j}\}_{j=1}^2$ satisfying

$$\begin{aligned} \mathcal{L}_\Sigma v'_E &= E - \sum_{j=1}^2 \theta_{E,j} w_j, \\ \|v'_E : C^{2,\alpha}(\Sigma_{\leq 2}, g_\Sigma)\| &\leq C\|E\|, \quad |\underline{\theta}_E| \leq C\|E\|. \end{aligned}$$

The solutions v'_E and $\underline{\theta}_E$ are unique by construction.

We now arrange the exponential decay of v'_E . Define

$$v_E = v'_E + \sum_{i=1}^4 \varphi_{E,i} \bar{u}_i,$$

where the \bar{u}_i 's are as in Section 4.4 and the constants $\varphi_{E,i}$'s will be chosen below. We set up exactly as in Lemma 41 and consider the restrictions of v'_E and \bar{u}_i 's to the component of $\Sigma_{\geq 2}$ on the j th wing pulled back to the asymptotic grim reaper. Let Ω be a cylinder of length $(5\delta_s/\tau - 2)$. For $i, j = 1, \dots, 4$, we define

$$a_j = \text{avg}_{\partial_0} v'_E - B(v'_E, 0), \quad a_{i,j} = \text{avg}_{\partial_0} \bar{u}_i - B(\bar{u}_i, 0),$$

where ∂_0 , B , and avg_{∂_0} are as in Proposition 37. From the uniqueness of the solution in (vi) of Proposition 37, we have $v_E = \underline{\mathcal{R}}(v_E, 0)$ if and only if

$$a_j + \sum_{i=1}^4 \varphi_{E,i} a_{i,j} = 0, \quad j = 1, \dots, 4.$$

By (v) in Proposition 37 and (iv) in Lemma 25, we can solve the above system and find unique solutions $\varphi_{E,i}$'s. With such a choice of $\varphi_{E,i}$'s, the function v_E has exponential decay and

$$\begin{aligned} |\varphi_E| &\leq C\|v'_E : C^{0,\alpha}(\Sigma_{\leq 2}, g_\Sigma)\| \leq C\|E\|, \\ \|v_E : C^{2,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s/m_k})\| &\leq C\|E\|. \end{aligned} \quad \square$$

6.3. Linear operator on M . Let us recall that we defined $M = M_{\bar{\tau}}$ in the larger scale, where the mean curvature of the asymptotic grim reapers is of order $\bar{\tau}$.

Definition 44. *Given $v \in C^{r,\alpha}(M)$, $r = 0, 2$, we define the norms*

$$\|v\|_r := \|v : C^{r,\alpha}(M, g_M, e^{-\gamma s})\|,$$

where s is the coordinate given in Definition 32.

As in Section 5, we denote the scaling used to fit the desingularizing surface in place of the k th intersection by

$$(39) \quad \mathcal{H}_k : \Sigma[T_k, \tilde{\varphi}_k, \tau_k] \rightarrow \mathcal{S}_k.$$

Note that $\mathcal{H}_k^* g_M = \frac{1}{m_k^2} g_\Sigma$. Moreover, for any function $v \in C^{r,\alpha}(M)$ supported on \mathcal{S}_k , we have

$$\frac{1}{C} \|v\|_r \leq \|\mathcal{H}_k^* v : C^{r,\alpha}(\Sigma[T_k, \tilde{\varphi}_k, \tau_k], g_\Sigma, e^{-\gamma s/m_k})\| \leq C \|v\|_r,$$

where the constant C depends only on r, α and m_k . Since $\{m_k\}_{k=1}^{N_I}$ is a finite set, we will stop mentioning the explicit dependence in m_k and incorporate it in the constants C in the rest of the article.

Let us fix $k \in \{1, \dots, N_I\}$. Given a function E on M with support in \mathcal{S}_k , we define $E' = \frac{1}{m_k} E \circ \mathcal{H}_k$. By Proposition 40, there exist a function v' and constants $\{\theta_{E',j}\}_{j=1}^2, \{\varphi_{E',i}\}_{i=1}^4$ for which $\Delta_{g_\Sigma} v' + |A_\Sigma|^2 v' + \tau_k \bar{e}_y \cdot \nabla_M v' = E' + \sum_{j=1}^2 \theta_{E',j} w_j + \sum_{i=1}^4 \varphi_{E',i} \bar{w}_i$. The function $v = \frac{1}{m_k} v' \circ \mathcal{H}_k^{-1} = \frac{1}{m_k} \mathcal{H}_{k*} v'$ therefore satisfies

$$\mathcal{L}_M v = \Delta_{g_M} v + |A_M|^2 v + \bar{\tau} \bar{e}_y \cdot \nabla_{g_M} v = E + m_k \mathcal{H}_{k*} \left(\sum_{j=1}^2 \theta_{E',j} w_j + \sum_{i=1}^4 \varphi_{E',i} \bar{w}_i \right).$$

To simplify notations, we define the linear map $\Theta : \mathcal{V}_\theta \times \mathcal{V}_\varphi \rightarrow C^\infty(M)$ by

$$\Theta(\underline{\theta}', \underline{\varphi}') = \sum_{k=1}^{N_I} m_k \mathcal{H}_{k*} \left(\sum_{j=1}^2 \theta'_{k,j} w_j + \sum_{i=1}^4 \varphi'_{k,i} \bar{w}_i \right),$$

where $\underline{\theta}' = \{\{\theta'_{k,j}\}_{j=1}^2\}_{k=1}^{N_I} \in \mathcal{V}_\theta$ and $\underline{\varphi}' = \{\{\varphi'_{k,i}\}_{i=1}^4\}_{k=1}^{N_I} \in \mathcal{V}_\varphi$.

Theorem 45. *Given $E \in C^{0,\alpha}(M)$ with finite norm $\|E\|_0$, there exist $v \in C^{2,\alpha}(M)$, $\underline{\theta}_E \in \mathcal{V}_\theta$, and $\underline{\varphi}_E \in \mathcal{V}_\varphi$ uniquely determined by the construction below, such that*

$$\mathcal{L}_M v = E + \Theta(\underline{\theta}_E, \underline{\varphi}_E),$$

and

$$\|v\|_2 \leq C \|E\|_0, \quad \|\underline{\theta}_E\| \leq C \|E\|_0, \quad \|\underline{\varphi}_E\| \leq C \|E\|_0.$$

Proof. The proof uses an iteration: at the n th step, we define the functions v_n and E_n ; u, u' , and u'' are just intermediate functions and are reset after every step.

We define two cut-off functions on M by $\psi = \psi[5\delta_s/\tau, 5\delta_s/\tau - 1] \circ s$ and by $\psi' = \psi[\underline{a}, \underline{a} + 1] \circ s$, where $\underline{a} = 8|\log \bar{\tau}|$ as in Definition 35.

We take $E_0 := E$ and proceed by induction. Given E_{n-1} , we define $E_n, v_n, \underline{\theta}_n$ and $\underline{\varphi}_n$ in the following way.

For each $k = 1, \dots, N_I$, we consider the function $E'_k = \frac{1}{m_k} (\psi E_{n-1}) \circ \mathcal{H}_k$ on $\Sigma_k = \mathcal{H}_k^{-1}(\mathcal{S}_k)$ and apply Proposition 40 to get $v_{E'_k}, \underline{\theta}_{E'_k}$ and $\underline{\varphi}_{E'_k}$. We take $\underline{\theta}_n = \{\{\theta_{E'_k,j}\}_{j=1}^2\}_{k=1}^{N_I}$, and $\underline{\varphi}_n = \{\{\varphi_{E'_k,i}\}_{i=1}^4\}_{k=1}^{N_I}$. From the construction, $u = \sum_{k=1}^{N_I} \frac{1}{m_k} \mathcal{H}_{k*}(u_k)$ satisfies on \mathcal{S}

$$\begin{aligned} \mathcal{L}_M u &= \psi E_{n-1} + \Theta(\underline{\theta}_n, \underline{\varphi}_n), \\ \|u : C^{2,\alpha}(\mathcal{S}, g_M, e^{-\gamma s})\| &\leq C \|E_{n-1}\|_0. \end{aligned}$$

Note that $\mathcal{L}_M(\psi u) = \psi^2 E_{n-1} + [\mathcal{L}, \psi]u + \Theta(\underline{\theta}_n, \underline{\varphi}_n)$, where we used the notation $[\mathcal{L}_M, \psi]u := \mathcal{L}_M(\psi u) - \psi(\mathcal{L}_M u)$.

From the discussion in Section 6.1, on every edge \mathcal{E} , there exists a solution u' to the Dirichlet problem $\mathcal{L}_M u' = E_{n-1} - \psi^2 E_{n-1} - [\mathcal{L}, \psi]u$ with zero boundary conditions. The function $E_{n-1} - \psi^2 E_{n-1} - [\mathcal{L}, \psi]u$ is supported on $s \geq \frac{5\delta_s}{\bar{\tau}} - 1$, therefore

$$(40) \quad \begin{aligned} \|u' : C^{2,\alpha}(\mathcal{E})\| &\leq C \|E_{n-1} - \psi^2 E_{n-1} - [\mathcal{L}, \psi]u : C^{0,\alpha}(\mathcal{E})\|, \\ &\leq C e^{-\gamma(5\delta_s \max_k(m_k)/\bar{\tau}-1)} \|E_{n-1}\|_0 \leq C e^{-2\delta_s/\bar{\tau}} \|E_{n-1}\|_0. \end{aligned}$$

We will also denote by u' the sum of the solutions u' 's on all the edges extended by zero to the rest of M .

On every end \mathcal{N} , there is a solution u'' to $\mathcal{L}_M u'' = E_{n-1} - \psi^2 E_{n-1} - [\mathcal{L}, \psi]u$ with exponential decay and given up to a constant on $\partial\mathcal{N}$ by Corollary 39. As before, we also denote by u'' the sum of the solutions on the ends, extended by zero to the rest of M . Since $E \in C^{0,\alpha}(\mathcal{N}, g, e^{-\gamma s/2})$, u'' satisfies

$$(41) \quad \begin{aligned} \|u'' : C^{2,\alpha}(\mathcal{N}, g, e^{-\gamma s/2})\| &\leq C \|E_{n-1} - \psi^2 E_{n-1} - [\mathcal{L}, \psi]u : C^{0,\alpha}(\mathcal{N}, g, e^{-\gamma s/2})\|, \\ &\leq C e^{-\gamma(5\delta_s/\bar{\tau}-1)/2} \|E_{n-1}\|_0 \leq C e^{-2\delta_s/\bar{\tau}} \|E_{n-1}\|_0. \end{aligned}$$

We choose to focus on the decay of order $e^{-\gamma s/2}$, as opposed to $e^{-\gamma s}$, for the iteration process. The function u'' does have a decay of order $e^{-\gamma s}$ and the estimate $\|u'' : C^{2,\alpha}(\mathcal{N}, g, e^{-\gamma s})\| \leq \|E_{n-1} - \psi^2 E_{n-1} - [\mathcal{L}, \psi]u\|_0 \leq C \|E_{n-1}\|_0$ is true, although it does not help us with the iteration.

We define $v_n = \psi u + \psi'(u' + u'')$. Since the supports of ψ' and $1 - \psi^2$ are disjoint, as well as the supports of ψ' and $[\mathcal{L}, \psi]$, we have

$$\mathcal{L}v_n = E_{n-1} + [\mathcal{L}, \psi'](u' + u'') + \Theta(\underline{\theta}_n, \underline{\varphi}_n).$$

Define $E_n = -[\mathcal{L}, \psi'](u' + u'')$. By (40), (41), and the fact that $[\mathcal{L}, \psi']$ is supported on $[a, a+1]$, we have, for $\bar{\tau}$ small enough,

$$(42) \quad \begin{aligned} \|E_n\|_0 &\leq C e^{\gamma(a+1)} \|[\mathcal{L}, \psi'](u' + u'') : C^{0,\alpha}(M, g_M)\| \\ &\leq C e^{\gamma(a+1)} (\|u' : C^{2,\alpha}(\mathcal{E})\| + \|u'' : C^{2,\alpha}(\mathcal{N}_{\leq(a+1)})\|) \\ &\leq C e^{\gamma(a+1)} (1 + e^{\gamma(a+1)}) e^{-2\delta_s/\bar{\tau}} \|E_{n-1}\|_0 \\ &\leq e^{-\delta_s/\bar{\tau}} \|E_{n-1}\|_0. \end{aligned}$$

We define $v_E := \sum_{n=1}^{\infty} v_n$, $\underline{\theta}_E := \sum_{n=1}^{\infty} \underline{\theta}_n$, and $\underline{\varphi}_E := \sum_{n=1}^{\infty} \underline{\varphi}_n$. The three series converge and we have the desired estimates from (42) and Proposition 40. The function v_E is uniquely determined from the construction and satisfies $\mathcal{L}_M v_E = E + \Theta(\underline{\theta}_E, \underline{\varphi}_E)$. \square

Corollary 46. *There are $v_H \in C^{2,\alpha}(M)$ and $\underline{\theta}_H, \underline{\varphi}_H$ such that*

$$\begin{aligned} \mathcal{L}_M v_H &= H_M - \bar{\tau} \bar{e}_y \cdot \nu_M + \Theta(\underline{\theta}_H, \underline{\varphi}_H), \\ |\underline{\theta}_H - \underline{\theta}| &\leq C\bar{\tau}, |\underline{\varphi}_H - \underline{\varphi}| \leq C\bar{\tau}, \|v_H\|_2 \leq C\bar{\tau}, \end{aligned}$$

where $M = M(\bar{\tau}, \xi) = M(\bar{\tau}, \underline{\theta}, \bar{\tau} \underline{b}', \bar{\tau} \underline{c}', \varphi)$.

Proof. From the smooth dependence of M on its parameters, the uniform control of the geometry of $M_{\leq 2} = \mathcal{S}_{\leq 2}$, and $|\xi| \leq \zeta\bar{\tau}$, we have

$$\|H_M - \bar{\tau} \bar{e}_y \cdot \nu_M : C^{0,\alpha}(\mathcal{S}_{\leq 2}, g_M)\| \leq C\bar{\tau}.$$

On the regions \mathcal{E}'' and \mathcal{N}'' , $H_M - \bar{\tau} \vec{e}_y \cdot \nu_M \equiv 0$. We use Proposition 29 and Corollary 34 on $\mathcal{S}_{\geq 1}$ to get

$$\|H_M - \bar{\tau} \vec{e}_y \cdot \nu_M + \Theta(\underline{\theta} + \underline{\theta}', \underline{\tilde{\varphi}})\|_0 \leq C\bar{\tau},$$

where $\underline{\tilde{\varphi}}$ is as in Section 5 and $\underline{\theta}' = \{\{\theta'_{k,j}\}_{j=1}^{N_\Gamma}\}_{k=1}^{N_\Gamma}$, with $\theta'_{k,j} = (\tilde{\varphi}_{k,j+2} - \tilde{\varphi}_{k,j} + \varphi_{k,j+2} - \varphi_{k,j})/2$. We apply Theorem 45 with $E = H_M - \bar{\tau} \vec{e}_y \cdot \nu_M + \Theta(\underline{\theta} + \underline{\theta}', \underline{\tilde{\varphi}})$ to obtain v_E , $\underline{\theta}_E$ and $\underline{\varphi}_E$ and define $v_H = v_E$, $\theta_H = \underline{\theta} + \underline{\theta}' + \underline{\theta}_E$, and $\varphi_H = \underline{\tilde{\varphi}} + \underline{\varphi}_E$. The estimates follow from Theorem 45, Proposition 33 (vi) and Corollary 34. \square

7. QUADRATIC TERM

Proposition 47. *Given $v \in C^{2,\alpha}(M)$ with $\|v\|_2$ smaller than a suitable constant, the graph M_v of v over M is a smooth immersion, moreover*

$$\|H_v - \bar{\tau} \vec{e}_y \cdot \nu_v - (H - \bar{\tau} \vec{e}_y \cdot \nu) - \mathcal{L}_M v\|_0 \leq C\|v\|_2^2,$$

where H and H_v are the mean curvature of M and M_v pulled back to M respectively, and similarly, ν and ν_v are the oriented unit normal of M and M_v pulled back to M .

Proof. The result follows immediately from equation (33) since we have uniform control of the second fundamental form A_M . \square

8. FIXED POINT ARGUMENT

We are now ready to prove the main result of this paper. Theorem 1 in the introduction is a rescaled version of the theorem below, where $\tilde{\mathcal{M}} = \bar{\tau} M_v$. The larger scale avoids singularities when $\bar{\tau} \rightarrow 0$.

Theorem 48. *Given a finite family of grim reapers $\{\tilde{\Gamma}_n\}_{n=1}^{N_\Gamma}$ in general position as in Section 2, let us denote the rescaled family by $\{\Gamma_n\}_{n=1}^{N_\Gamma}$, where $\Gamma_n = \bar{\tau}^{-1} \tilde{\Gamma}_n$. There is a $\delta_{\bar{\tau}}$ depending only on $\max_k \{m_k\}$, N_Γ , δ , and δ_Γ from Lemma 3 such that for every $\bar{\tau} \in (0, \delta_{\bar{\tau}})$, there is a $\xi_{\bar{\tau}} \in \mathcal{V}$ and a smooth function v on the smooth initial surface $M = M(\bar{\tau}, \xi_{\bar{\tau}})$ with the following properties:*

- (i) *The graph M_v of v over M is a complete embedded surface in \mathbf{R}^3 that is self-translating under mean curvature flow.*
- (ii) *M_v is invariant under reflection with respect to the xy -plane.*
- (iii) *M_v is singly periodic with period 2π in the z -direction.*
- (iv) *There is a large ball $B \subset \mathbf{R}^2$ such that $M_v \setminus (B \times \mathbf{R})$ is the union of ends in one-to-one correspondence with the ends of $\{\Gamma_n\}_{n=1}^{N_\Gamma}$. The left ends of M_v are exponentially decaying to grim reaper ends in $\{\Gamma_n\}_{n=1}^{N_\Gamma}$ (without any change in the position of the grim reapers). For the right ends of M_v , the difference between the center of a grim reaper asymptotic to M_v and the center of its corresponding end in $\{\Gamma_n\}_{n=1}^{N_\Gamma}$ is at most a constant $C(N_\Gamma, \delta, \delta_\Gamma, \max_k \{m_k\})$.*
- (v) *If $T_k^{\bar{\tau}}$ is the translation that moves the k th intersection line of $\{\Gamma_n\}_{n=1}^{N_\Gamma}$ to the z -axis, then $T_k^{\bar{\tau}}(M_v)$ converges uniformly in C^j norm, for any $j < \infty$, on any compact set of E^3 to a Scherk surface of period $2\pi/m_k$ as $\bar{\tau} \rightarrow 0$.*

Proof. Let us fix $\alpha' \in (0, \alpha)$, and define the Banach space

$$\chi = C^{2,\alpha'}(M(\bar{\tau}, \mathbf{0})).$$

Denote by $D_{\bar{\tau}, \xi} : M(\bar{\tau}, \underline{0}) \rightarrow M(\bar{\tau}, \xi)$ a family of smooth diffeomorphisms which depend smoothly on ξ and satisfy the following conditions: for every $f \in C^{2,\alpha}(M(\bar{\tau}, \underline{0}))$ and $f' \in C^{2,\alpha}(M(\bar{\tau}, \xi))$, we have

$$\|f \circ D_{\bar{\tau}, \xi}^{-1}\|_2 \leq C\|f\|_2, \quad \|f' \circ D_{\bar{\tau}, \xi}\|_2 \leq C\|f'\|_2.$$

The diffeomorphisms $D_{\bar{\tau}, \xi}$ are used to pull back functions and norms from $M(\bar{\tau}, \xi)$ to $M(\bar{\tau}, \underline{0})$.

We fix $\bar{\tau}$ for now and omit the dependence in $\bar{\tau}$ in our notations of maps and surfaces from now on. Let

$$\Xi = \{(\xi, u) \in \mathcal{V} \times \mathcal{X} : |\xi| \leq \zeta\bar{\tau}, \|u\|_2 \leq \zeta\bar{\tau}\},$$

where ζ is a large constant to be determined below. The map $\mathcal{I} : \Xi \rightarrow \mathcal{V} \times \mathcal{X}$ is defined as follows. Given $(\xi, u) \in \Xi$, let $v = u \circ D_{\xi}^{-1}$, $M = M(\xi)$ and let M_v be the graph of v over M . We define the function $\mathcal{F} : \mathcal{V} \times C^{2,\alpha}(M, g_M, e^{-\gamma s}) \rightarrow \mathbf{R}$ by

$$\mathcal{F}(\xi, v) = H_v - \bar{\tau}\bar{e}_y \cdot \nu_v,$$

where H_v and ν_v are the mean curvature and the oriented unit normal of M_v respectively pulled back to M . Proposition 47 asserts that

$$\|\mathcal{F}(\xi, v) - \mathcal{F}(\xi, 0) - \mathcal{L}_M v\|_0 \leq C\zeta^2\bar{\tau}^2.$$

Applying Theorem 45 with $E = \mathcal{F}(\xi, v) - \mathcal{F}(\xi, 0) - \mathcal{L}_M v$, we obtain v_E , $\underline{\theta}_E$, and $\underline{\varphi}_E$ such that

$$\begin{aligned} \mathcal{L}_M v_E &= E + \Theta(\underline{\theta}_E, \underline{\varphi}_E), \\ \|v_E\|_2 &\leq C\zeta^2\bar{\tau}^2, \quad |\underline{\theta}_E| \leq C\zeta^2\bar{\tau}^2, \quad |\underline{\varphi}_E| \leq C\zeta^2\bar{\tau}^2. \end{aligned}$$

Hence,

$$\mathcal{F}(\xi, v) = \mathcal{F}(\xi, 0) + \mathcal{L}_M v + \mathcal{L}_M v_E - \Theta(\underline{\theta}_E, \underline{\varphi}_E).$$

Corollary 46 gives us v_H , $\underline{\theta}_H$ and $\underline{\varphi}_H$ satisfying $\mathcal{L}_M v_H = \mathcal{F}(\xi, 0) + \Theta(\underline{\theta}_H, \underline{\varphi}_H)$, so

$$\mathcal{F}(\xi, v) = \mathcal{L}_M v + \mathcal{L}_M v_H + \mathcal{L}_M v_E - \Theta(\underline{\theta}_E + \underline{\theta}_H, \underline{\varphi}_E + \underline{\varphi}_H).$$

We define the map $\mathcal{I} : \Xi \rightarrow \mathcal{V} \times \mathcal{X}$ by

$$\mathcal{I}(\xi, u) = ((\underline{\theta} - \underline{\theta}_E - \underline{\theta}_H, -\bar{\tau}\tilde{b}, -\bar{\tau}\tilde{c}, \underline{\varphi} - \underline{\varphi}_E - \underline{\varphi}_H), (-v_E - v_H) \circ D_{\xi}).$$

Note that we arrange for $\mathcal{I}(\Xi) \subset \Xi$ since

$$\begin{aligned} \|-v_E - v_H\|_2 &\leq C(\bar{\tau} + \zeta^2\bar{\tau}^2), \\ |\underline{\theta} - \underline{\theta}_E - \underline{\theta}_H| &\leq C(\bar{\tau} + \zeta^2\bar{\tau}^2), \\ |\underline{\varphi} - \underline{\varphi}_E - \underline{\varphi}_H| &\leq C(\bar{\tau} + \zeta^2\bar{\tau}^2), \end{aligned}$$

and choosing $\zeta > 2C$ and $\bar{\tau} < \zeta^{-2}$, we get $C(\bar{\tau} + \zeta^2\bar{\tau}^2) < \zeta\bar{\tau}$.

The set Ξ is clearly convex. It is a compact set of $\mathcal{V} \times \mathcal{X}$ from the choice of the Hölder exponent $\alpha' < \alpha$ and the imposed exponential decay. The map \mathcal{I} is continuous by construction, therefore we can apply the Schauder Fixed Point Theorem (p. 279 in [5]) to obtain a fixed point $(\xi_{\bar{\tau}}, u_{\bar{\tau}})$ of \mathcal{I} for every $\bar{\tau} \in (0, \delta_{\bar{\tau}})$ with $\delta_{\bar{\tau}}$ small enough. The graph of $v = u_{\bar{\tau}} \circ D_{\bar{\tau}, \xi}^{-1}$ over the surface $M(\bar{\tau}, \xi_{\bar{\tau}})$ is then a self-translating surface. It is a smooth surface by the regularity theory for elliptic equations. The properties (ii) and (iii) follow from the construction. \square

Part (iv) of the theorem quantifies how much the asymptotic grim reapers are perturbed. The construction is slightly more general than advertised since we can impose the position of the left asymptotic grim reapers to be any slight perturbation of the initial family, although we have to surrender precise control of the ones on the right (see also Kapouleas [9]).

APPENDIX A. LINEAR OPERATOR ON LONG CYLINDERS

We give here the proof of Proposition 37 from Section 6. Kapouleas has a similar result, without the gradient term $\mathbf{V} \cdot \nabla$ in Appendix A of [9]. His reasoning applies readily here as well, except we have to work a little more for the uniqueness in (vi), as shown below.

Let us recall that \mathcal{L} denotes an operator on the cylinder (Ω, g_0) of the form

$$\mathcal{L}v = \Delta_\chi v + \mathbf{V} \cdot \nabla v + dv,$$

where χ is a C^2 Riemannian metric, \mathbf{V} a C^1 vector field, and d a C^1 function on Ω . For $\underline{c} > 0$, we define

$$N(\mathcal{L}) := \|\chi - g_0 : C^2(\Omega, g_0, e^{-\underline{c}s})\| + \|\mathbf{V} : C^1(\Omega, g_0, l^{-1})\| + \|d : C^1(\Omega, g_0, e^{-\underline{c}s} + l^{-2})\|,$$

where l is the length of the cylinder (see Definition 36).

Proposition 37. *Given $\gamma \in (0, 1)$ and $\varepsilon > 0$, if $N(\mathcal{L})$ is small enough in terms of \underline{c} , α , γ and ε (but independently of l), there is a bounded linear map*

$$\underline{\mathcal{R}} : C^{2,\alpha}(\partial_0, g_0) \times C^{0,\alpha}(\Omega, g_0, e^{-\gamma s}) \rightarrow C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})$$

such that for (f, E) in $\underline{\mathcal{R}}$'s domain and $v = \underline{\mathcal{R}}(f, E)$, the following properties are true, where the constants C depend only on α and γ :

- (i) $\mathcal{L}v = E$ on Ω .
- (ii) $v = f - \text{avg}_{\partial_0} f + B(f, E)$ on ∂_0 , where $B(f, E)$ is a constant on ∂_0 and $\text{avg}_{\partial_0} f$ denotes the average of f over ∂_0 .
- (iii) $v \equiv 0$ on ∂_l .
- (iv) $\|v : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})\| \leq C\|f - \text{avg}_{\partial_0} f : C^{2,\alpha}(\partial_0, g_0)\| + C\|E : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\|$.
- (v) $|B(f, E)| \leq \varepsilon\|f - \text{avg}_{\partial_0} f : C^{2,\alpha}(\partial_0, g_0)\| + C\|E : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\|$.
- (vi) If $v' \in C^2(\Omega)$ satisfies $\mathcal{L}v' = E$ on Ω , and $v = v'$ on $\partial\Omega$, then $v' = v$ on Ω . Moreover, if E vanishes, then

$$\|v : C^0(\Omega)\| \leq 2\|v : C^0(\partial_0)\|.$$

Proof. The proposition is valid for the standard Laplacian with respect to the flat metric, with a vanishing ε in (v). One can prove this fact by separating variables, and using Fourier series with coefficients depending on $s \in (0, l)$. In the case of vanishing boundary conditions, the operator $\mathcal{L} : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s}) \rightarrow C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})$ is a perturbation of Δ , therefore the statements (i)-(v) follow. If the boundary conditions are not zero, we can find the solution u to the Laplace equation with the given boundary conditions, then solve $\mathcal{L}v = E - \mathcal{L}u$ with vanishing boundary data.

We now prove uniqueness of the solution by showing that the smallest eigenvalue of \mathcal{L} is bounded away from 0. Let ϕ be a $C^{2,\alpha}$ function that vanishes on $\partial\Omega$. We denote the average of ϕ by $\bar{\phi}(s) = (2\pi)^{-1} \int_0^{2\pi} \phi(s, \theta) d\theta$. From the Poincaré

inequality $\int_0^{2\pi} |\phi(s, \theta) - \bar{\phi}(s)|^2 d\theta \leq 4\pi^2 \int_0^{2\pi} |\partial_\theta \phi(s, \theta)|^2 d\theta$, we get

$$\begin{aligned} |\bar{\phi}(s)|^2 &= \left(\int_0^s \bar{\phi}'(t) dt \right)^2 \leq s \int_0^s |\bar{\phi}'(t)|^2 dt \leq s \int_0^s \int_0^{2\pi} |\nabla \phi|^2 d\theta dt \\ (43) \quad &\leq s \int_0^l \int_0^{2\pi} |\nabla \phi|^2 d\theta dt. \end{aligned}$$

Since $\int_0^{2\pi} (\phi - \bar{\phi})^2 d\theta = \int_0^{2\pi} \phi^2 d\theta - 2\pi \bar{\phi}^2$, we have

$$\begin{aligned} \int_0^l \int_0^{2\pi} |\phi|^2 d\theta ds &= \int_0^l \int_0^{2\pi} (\phi - \bar{\phi})^2 d\theta ds + 2\pi \int_0^l |\bar{\phi}|^2 ds \\ &\leq (4\pi^2 + \pi l^2) \int_0^l \int_0^{2\pi} |\nabla \phi|^2 d\theta ds. \end{aligned}$$

The inequality above proves uniqueness for Δ . For \mathcal{L} , we write

$$\int_0^l \int_0^{2\pi} e^{-\underline{c}s} \phi^2(s, \theta) d\theta ds = \int_0^l \int_0^{2\pi} e^{-\underline{c}s} (\phi - \bar{\phi})^2 d\theta ds + 2\pi \int_0^l e^{-\underline{c}s} \bar{\phi}^2 ds =: I + II,$$

where \underline{c} is as in the definition of $N(\mathcal{L})$. I can be estimated using Poincaré's inequality,

$$\begin{aligned} \int_0^l \int_0^{2\pi} e^{-\underline{c}s} (\phi - \bar{\phi})^2 d\theta ds &\leq C \int_0^l \left(\int_0^{2\pi} |\nabla \phi|^2 d\theta \right) e^{-\underline{c}s} ds \\ &\leq C \|e^{-\underline{c}s}\|_{L^\infty} \int_0^l \int_0^{2\pi} |\nabla \phi|^2 d\theta ds = C \int_0^l \int_0^{2\pi} |\nabla \phi|^2 d\theta ds. \end{aligned}$$

II is estimated using (43),

$$II \leq 2\pi \left(\int_0^l e^{-\underline{c}s} s ds \right) \int_0^l \int_0^{2\pi} |\nabla \phi|^2 d\theta ds \leq \frac{2\pi}{\underline{c}^2} \int_0^l \int_0^{2\pi} |\nabla \phi|^2 d\theta ds.$$

Hence, we have $\int_\Omega e^{-\underline{c}s} \phi^2 \leq C(\underline{c}) \int_\Omega |\nabla \phi|^2$, and

$$\begin{aligned} - \int_\Omega (\mathcal{L}\phi)\phi &= - \int_\Omega (\Delta_{g_0}\phi)\phi - \int_\Omega d\phi^2 - \int_\Omega \phi \mathbf{V} \cdot \nabla_{g_0}\phi \\ &\quad + \int_\Omega (\Delta_{g_0}\phi - \Delta_\chi\phi)\phi + \int_\Omega \phi \mathbf{V} \cdot (\nabla_{g_0}\phi - \nabla_\chi\phi) \\ &\geq \int_\Omega |\nabla \phi|^2 - CN(\mathcal{L}) \left(\int_\Omega (C(\underline{c}) + 1) |\nabla \phi|^2 + \frac{1}{l^2} \int_\Omega \phi^2 \right) \\ &\geq \frac{1}{2(4\pi^2 + \pi l^2)} \int_\Omega \phi^2, \end{aligned}$$

if $N(\mathcal{L})$ is small enough depending on $C(\underline{c})$, but independently of l . Therefore, if $\mathcal{L}\phi = 0$ and ϕ vanishes on the boundary of Ω , $\phi \equiv 0$.

For the estimate in (iv), we quote Kapouleas (Appendix A of [9]): The desired estimate reduces to the case where $v = f \equiv 1$ on ∂_0 , because otherwise we can produce a subdomain of Ω with a vanishing eigenvalue. If $\mathcal{L} = \Delta$, the solution is $v = (l - s)/l$. This can be corrected to the solution to \mathcal{L} of the form $\Delta + \mathbf{V} \cdot \nabla + d$ with $\|\mathbf{V} : C^1(\Omega, l^{-1})\|$ and $\|d : C^1(\Omega, l^{-2})\|$ appropriately small, by scaling the length of the cylinder to unit, while leaving the meridian unchanged. We thus can have the estimate for v with a constant $3/2$ for example instead of 2. By using now

the earlier proven parts of the proposition we can correct this v to a v for a general \mathcal{L} while establishing the estimate at the same time. \square

Proposition 38 is proved by treating \mathcal{L} as a perturbation of the Laplace operator Δ , for which the results are standard.

APPENDIX B. PROOF OF LEMMA 43

Let us recall that $h := (\frac{1}{2}|A_\Sigma|^2 + \varepsilon_h)g_\Sigma$, where $\varepsilon_h > 0$ is small depending on τ and δ_θ . The c -approximate kernel of $L_h := \Delta_h + 2|A_\Sigma|^2/(|A_\Sigma|^2 + 2\varepsilon_h)$ is the span of the eigenfunctions of L_h with corresponding eigenvalues in $[-c, c]$, and G' is the group generated by $(s, z) \rightarrow (s, z + 2\pi)$.

Lemma 43. *There are positive constants C and c such that given $E \in L^2(\Sigma/G', h)$ there is $\underline{\theta}_E = (\theta_{E,1}, \theta_{E,2})$ such that $(E - \sum_{j=1}^2 \theta_{E,j} w_j)/(|A_\Sigma|^2 + 2\varepsilon_h)$ is $L^2(\Sigma/G', h)$ -orthogonal to the c -approximate kernel and*

$$|\underline{\theta}_E| \leq C \|E/(|A_\Sigma|^2 + 2\varepsilon_h) : L^2(\Sigma/G', h)\|.$$

Proof. First, we prove that all eigenfunctions f of L_h on Σ corresponding to low eigenvalues, say less than 10, satisfy

$$(44) \quad \|f : C^0(\Sigma)\| \leq C \|f : L^2(\Sigma/G', h)\|.$$

Indeed, we control the geometry of $\Sigma_{\leq 2}$ uniformly so we can use standard elliptic theory to bound $\|f : C^0(\Sigma_{\leq 2})\|$ in terms of $\|f : L^2(\Sigma_{\leq 2}/G', h)\|$. In the region $s \geq 1$, we consider f as a solution to the equation $\Delta_\Sigma f + |A_\Sigma|^2 f + \frac{\lambda}{2}(|A_\Sigma|^2 + 2\varepsilon_h)f = 0$. Using Corollary 16 and the fact that ε_h is small, we can apply the result (vi) of Proposition 37 to obtain $\|f : C^0(\Sigma_{\geq 1})\| \leq 2\|f : C^0(\partial\Sigma_{\leq 1})\|$, which establishes (44).

Since $\Sigma(\theta(T))$ is minimal without umbilical points, the Gauss map is conformal. It maps $\Sigma(\theta(T))/G'$ to a sphere minus the four points $(\pm \sin \theta(T), \pm \cos \theta(T), 0)$. By the Weierstrass-Enneper representation of Scherk surfaces from [11], the standard metric on the sphere pulled back to $\Sigma(\theta(T))$ is $h_0 = \frac{1}{2}|A|^2 g_{\Sigma(\theta(T))}$.

Let $\rho : \mathbf{S}^2 \rightarrow \mathbf{R}$ denote the distance from $\{(\pm \sin \theta(T), \pm \cos \theta(T), 0)\}$. We define a logarithmic function $\psi_{\mathbf{S}^2} : \mathbf{S}^2 \rightarrow [0, 1]$ by

$$\psi_{\mathbf{S}^2}(p) = \psi[2, 1](\log \rho(p) / \log \delta_h),$$

where δ_h is a small positive constant to be determined in the course of the proof. Notice that $\psi_{\mathbf{S}^2}$ vanishes at distance $\leq \delta_h^2$ from these points, and $\psi_{\mathbf{S}^2} \equiv 1$ at distance $\geq \delta_h$.

For a function f on $\Sigma(\theta(T))/G'$, we define $\mathcal{F}_1(f)$ to be the pushforward to Σ/G' by $Z = Z[T, \underline{\varphi}, \tau]$ of the function $f\psi_{\mathbf{S}^2} \circ \nu$. Similarly, for a function f' on Σ/G' , $\mathcal{F}_2(f')$ is a function on $\Sigma(\theta(T))/G'$ defined by $\mathcal{F}_2(f') = (f' \circ Z)(\psi_{\mathbf{S}^2} \circ \nu)$. The region on $\Sigma(\theta(T))/G'$ where $\psi_{\mathbf{S}^2} \circ \nu \neq 1$ is contained in four disks of radius δ_h , therefore its h -area is small for δ_h small. On Σ/G' , the region where $\psi_{\mathbf{S}^2} \circ \nu \neq 1$ has small area also. Indeed, the points at distance δ_h of $(\pm \sin \theta(T), \pm \cos \theta(T), 0)$ all have to same s -coordinate on $\Sigma(\theta(T))$ (see [13] Section 4.3). Let us call this value s_0 . We use Lemma 15 and Corollary 16 to bound the area by

$$\begin{aligned} \int_{(\Sigma_{\geq s_0})/G'} dV_h &= \frac{1}{2} \int_{(\Sigma_{\geq s_0})/G'} (|A_\Sigma|^2 + 2\varepsilon_h) dV_{g_\Sigma} \\ &\leq C \int_{s_0}^l (\varepsilon e^{-s} + \tau^2 + \varepsilon_h) ds \leq C(e^{-s_0} + (\tau^2 + \varepsilon_h)(l - s_0)), \end{aligned}$$

where dV_g is the volume form associated to the metric g and $l = 5\delta_s/\tau$. Taking ε_h small with respect to $\sqrt{\tau}$ so that A_Σ is not singular, and taking δ_h small enough so that s_0 is large, we obtain a small h -area for the region where $\psi_g^2 \circ \nu \neq 1$ on Σ .

Using Rayleigh quotients, one can prove that if two manifolds (M_1, h_1) and (M_2, h_2) are close to being isometric, except on a set of small area, and if there exist $\varepsilon_1 > 0$ and maps $\mathcal{F}_1 : C_0^\infty(M_1) \rightarrow C_0^\infty(M_2)$ and $\mathcal{F}_2 : C_0^\infty(M_2) \rightarrow C_0^\infty(M_1)$ such that for all $f, g \in C_0^\infty(M_i)$, $i, j = 1, 2$, $j \neq i$,

$$\begin{aligned} \|\mathcal{F}_i f\|_\infty &\leq 2\|f\|_\infty, \\ |\langle f, g \rangle - \langle \mathcal{F}_i(f), \mathcal{F}_i(g) \rangle| &\leq \varepsilon_1 \|f\|_\infty \|g\|_\infty, \\ \|\nabla(\mathcal{F}_i(f))\|_{L^2} &\leq (1 + \varepsilon_1) \|\nabla f\|_{L^2} + \varepsilon_1 \|f\|_\infty, \\ \|f - \mathcal{F}_j \circ \mathcal{F}_i(f)\|_{L^2} &\leq \varepsilon_1 \|f\|_\infty, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ stands for the L^2 inner product and $\|\cdot\|_\infty$ is the L^∞ norm, then the eigenvalues of the operator L_{h_1} on M_1 and L_{h_2} on M_2 are close. Moreover, if f is an eigenfunction of L_{h_i} on M_i with corresponding eigenvalue λ , $\mathcal{F}_i(f)$ is close to a linear combination of eigenfunctions of corresponding eigenvalues close to λ on M_j . All the ‘‘closeness’’ depends on ε_1 and can be estimated. The reader can find more detail in Appendix B of [8] and note that the first condition in B.1.6 was corrected by Kapouleas (see [10] page 281) and should be replaced by the one given here.

One can generate functions close the approximate kernel of L_h on Σ by cutting off functions in the approximate kernel of $\Delta_{h_0} + 2$ on $\Sigma(\theta(T))$. The eigenfunctions corresponding to eigenvalues less than 1 of the operator $\Delta_{h_0} + 2$ on $\Sigma(\theta(T))$ are exactly $\vec{e}_x \cdot \nu$ and $\vec{e}_y \cdot \nu$. In Lemma 27, we proved that we can generate any combination of $\vec{e}_x \cdot \nu$ and $\vec{e}_y \cdot \nu$ with a linear combination of w_1 and w_2 . The bound on $|\underline{\theta}_E|$ follows from the fact that P^{-1} in Lemma 27 is bounded. \square

REFERENCES

- [1] S. J. ALTSCHULER AND L. F. WU, *Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle*, Calc. Var. Partial Differential Equations, 2 (1994), pp. 101–111.
- [2] S. ANGENENT, *On the formation of singularities in the curve shortening flow*, J. Differential Geom., 33 (1991), pp. 601–633.
- [3] S. B. ANGENENT AND J. J. L. VELÁZQUEZ, *Asymptotic shape of cusp singularities in curve shortening*, Duke Math. J., 77 (1995), pp. 71–110.
- [4] ———, *Degenerate neckpinches in mean curvature flow*, J. Reine Angew. Math., 482 (1997), pp. 15–66.
- [5] D. GILBARG AND N. S. TRUDINGER, *Elliptic partial differential equations of second order*, vol. 224 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, second ed., 1983.
- [6] G. HUISKEN, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom., 31 (1990), pp. 285–299.
- [7] G. HUISKEN AND C. SINISTRARI, *Mean curvature flow singularities for mean convex surfaces*, Calc. Var. Partial Differential Equations, 8 (1999), pp. 1–14.
- [8] N. KAPOULEAS, *Constant mean curvature surfaces in Euclidean spaces*, in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Basel, 1995, Birkhäuser, pp. 481–490.
- [9] ———, *Complete embedded minimal surfaces of finite total curvature*, J. Differential Geom., 47 (1997), pp. 95–169.
- [10] N. KAPOULEAS AND S.-D. YANG, *Minimal surfaces in the three-sphere by doubling the Clifford torus*, Amer. J. Math., 132 (2010), pp. 257–295.
- [11] H. KARCHER, *Embedded minimal surfaces derived from Scherk’s examples*, Manuscripta Math., 62 (1988), pp. 83–114.

- [12] N. J. KOREVAAR, R. KUSNER, AND B. SOLOMON, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom., 30 (1989), pp. 465–503.
- [13] X. H. NGUYEN, *Construction of complete embedded self-similar surfaces under mean curvature flow. Part I*, Trans. Amer. Math. Soc., 361 (2009), pp. 1683–1701.
- [14] ———, *Translating tridents*, Comm. Partial Differential Equations, 34 (2009), pp. 257–280.
- [15] M. TRAZET, *Construction de surfaces minimales en recollant des surfaces de Scherk*, Ann. Inst. Fourier (Grenoble), 46 (1996), pp. 1385–1442.

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