RESEARCH STATEMENT

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My main research interests lie in the field of geometric analysis. Specifically, I study the mean curvature flow and gluing constructions for elliptic and parabolic equations. More recently, I have become interested in free boundary problems and I enjoy problems involving partial differential equations and differential geometry in general.

In the first section of this research statement, I discuss some classic and current results regarding the mean curvature flow. The reader interested in going directly to my recent results and current research projects can jump to Sections 2 and 3.

1. Introduction

A family of surfaces $\{M_t\}$ in $\mathbf{R}^3$ evolves by mean curvature flow (MCF) if each point moves perpendicularly to the surface at a speed equal to the mean curvature at the point. One of the reasons why this flow is so important is because it is the flow that decreases the surface area the fastest, among all flows with the same norm. Since curvature can be considered as a second derivative of position, the MCF is a parabolic flow. In the same way that solutions to the heat equation are smooth, solutions to the MCF are smooth for almost all positive times.

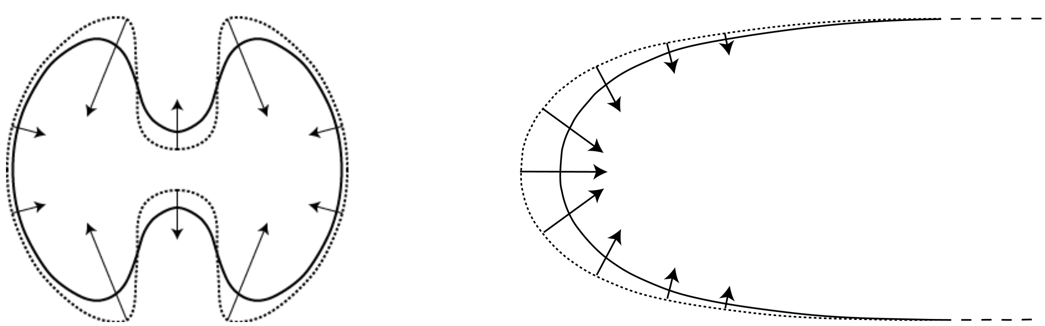

Figure 1. Curvature flow on curves. The solid line represents the flow at a later time.

In the case of curves in $\mathbf{R}^2$, the speed is just the curvature, so we can imagine the flow as the motion of the boundary of a melting thin sheet of ice floating on cold water. Grayson [Gra87] proved that any embedded closed curve in the plane will stay embedded and shrink to a point. In $\mathbf{R}^3$, the MCF shrinks spheres to their centers and cylinders to their axes.

The mean curvature flow and related geometric evolution equations have many applications. For example, they can be used for topological classifications of manifolds, and in

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the celebrated series of articles by Perelman, the Ricci flow was used to solve the Poincaré conjecture. Another application of flows can be found in image processing, where the MCF is an important tool because of its smoothing property.

In general, these evolution equations can develop singularities in finite time. In the case of the MCF, singularities will always develop if we start with a compact surface. Indeed, we can enclose such a surface in a large enough sphere and since two disjoint surfaces always remain disjoint, the flow of the compact surface can not exist longer than the flow of the shrinking sphere. In order to define a more general (weaker) version of the flows past their singularities, it is essential to have a good understanding of the behavior of the solutions near the singular/extinction time.

The second fundamental form $A(p,t)$ encodes all the information about the curvature of a surface. A singularity at time T is classified into one of two types according to the rate at which $\max_{p \in M_t} |A(p,t)|$ blows up. Roughly speaking, singularities are of type I if we have a good control of the geometry. For the MCF, Huisken [Hui90] showed that these singularities are modeled by *self-shrinking surfaces*, which are surfaces that are rescaled by the flow, while their shape is left unchanged. Currently, there are only four known examples of complete embedded self-shrinking surfaces in $\mathbf{R}^3$: spheres, cylinders, planes and shrinking doughnuts [Ang92]. However, there exist numerical evidence of many others [ACI95] [Cho94].

Type II singularities are more complex because of the lack of control on the curvatures. The examples of convergence in [AV95] and [AV97] indicate that they are modeled by *self-translating surfaces* (STS), which are surfaces that are translated by the flow while their general shape is left unchanged. The second curve in Figure 1 is self-translating and is called the *grim reaper*. Huisken and Sinestrari [HS99] proved that if the initial surface has non-negative mean curvature, the family of evolving surfaces appropriately rescaled converges to a strictly convex STS or $\mathbf{R}^{d-k} \times \Sigma^k$, where Σ^k is a lower dimensional strictly convex STS. Until recently, few examples of this type were available. Besides the classic examples of a plane, a grim reaper cylinder, and a rotationally symmetric soliton [AW94], I constructed STSs by desingularizing the intersection of a grim reaper cylinder and a plane [Ngu09b]. Recently, I generalized the previous result and proved that the construction can be adapted to desingularize any generic finite family of grim reaper cylinders [Ngu10b].

A full classification of singularities is not possible. The most general studies to date use some properties preserved by the flow to restrict the families of evolving surfaces: convex [Hui84], mean convex [HS99], rotationally symmetric [KM], etc. Moreover, few examples of self-shrinking surfaces are stable. Indeed, Colding and Minicozzi recently proved that the only stable self-shrinkers in $\mathbf{R}^3$ are spheres, cylinders, and planes [CM]. In other words, any other self-shrinker can be deformed so that the family of evolving surfaces flows away from it. Unstable examples should be numerous but elusive and it is therefore essential to find successful methods to find them.

2. My research

Inspired by the works of Traizet [Tra96] and Kapouleas [Kap97] on minimal surfaces, I successfully constructed new examples of embedded self-translating surfaces in $\mathbf{R}^3$. The idea is to start with two STSs, for example two grim reaper cylinders, and cut out a neighborhood of the intersection (see Figure 2). I then fit a well-chosen surface in the gap using smooth transition functions to obtain an embedded approximate solution. The exact solution is found by solving a perturbation problem. This process, which transforms an immersed surface into an embedded one, is called *desingularization*.

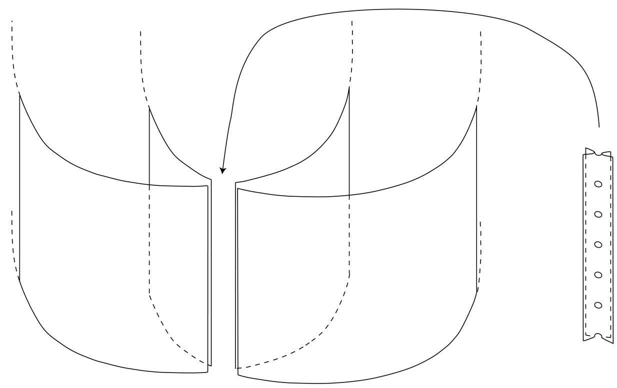

Figure 2. Construction of an approximate solution.

The surfaces used to replace the intersection line(s) are Scherk minimal surfaces, shown in Figure 3. They are ideal because they are embedded, asymptotic to four half-planes, and have many symmetries. Note that these surfaces are minimal surfaces (their mean curvature vanishes everywhere) so they won't move under MCF until we bend them.

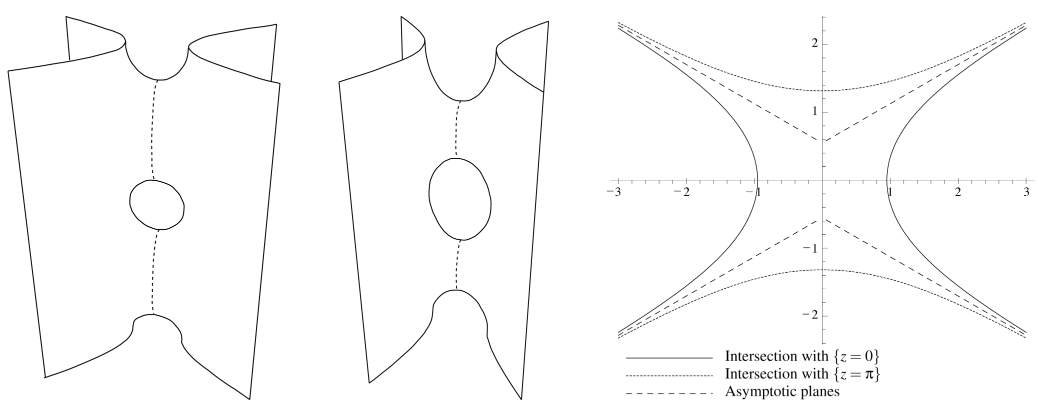

Figure 3. Two Scherk surfaces and the cross sections of one of them.

My first result about STSs concerns the desingularization of a plane and a grim reaper cylinder. One can think of the initial configuration as a trident times $\mathbf{R}$.

Theorem 1 ([Ngu09b]). *There exists a complete embedded self-translating surface that desingularizes the intersection of a grim reaper cylinder and a plane, provided the plane is positioned exactly half-way between the asymptotic planes of the grim reaper cylinder and is parallel to them.*

The proof gives the existence of not only one STS, but a whole family characterized by a small parameter τ. Roughly speaking, τ controls the size of the neighborhood that is cut out and the scaling of the Scherk surface (the scale factor is $\sim \tau$). A smaller τ means working in a smaller neighborhood and achieving a better approximate solution in the first step. As long as this parameter is small enough, it will be possible to solve the perturbation problem and the construction will work.

The previous result relies heavily on the many symmetries of the initial configuration. Recently, I proved that this theorem can be generalized to the setting of a finite family of grim reaper cylinders in general position, which only has symmetries in the z-direction.

Theorem 2 ([Ngu10b])**.** *If $\{\Gamma_n\}_{n=1}^N$ is a finite family of grim reaper cylinders moving at the same velocity under MCF and such that*

- *no three grim reapers intersect on the same line,*
- *no two grim reapers share the same asymptotic plane,*

then there exist a constant $\delta_\tau > 0$ and a one parameter family of self-translating surfaces $\{\mathcal{M}_\tau\}_{\tau\in(0,\delta_\tau)}$ desingularizing $\bigcup_{n=1}^N \Gamma_n$. The surfaces $\mathcal{M}_\tau$ converge uniformly to the union of the grim reapers as $\tau \to 0$ on any neighborhood of $\mathbf{R}^3$ that does not contain the lines of intersection.

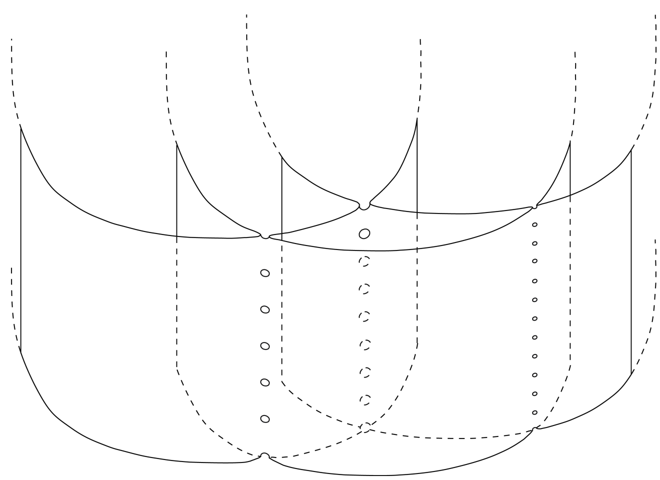

FIGURE 4. A self-translating surface $\mathcal{M}_{\bar{\tau}}$

Desingularizations are special cases of more general gluing constructions, where one simply fits together adequate pieces to construct the approximate solution. These techniques have been applied to diverse settings, such as constructing constant mean curvature surfaces [Kap90] and gluing wormholes into 3-manifolds in relativity [IMP02]. Gluing constructions are very technical, however the overarching principle is the same. First, one constructs an approximate solution by attaching appropriate pieces with smooth cut-off functions. Then, in what is typically a much harder step, one adjusts this initial guess to obtain an exact solution.

In our case, we perturb a surface by adding the graph of a small function in the normal direction. More precisely, the position vector X becomes $X + f\nu$, where f is the function, and ν is the oriented unit normal vector. We then determine how the function f affects the geometric quantities in the equation we wish to solve. Finding a self-translating surface amounts to finding a solution f to a nonlinear partial differential equation and relies on a good understanding of the associated linear operator $\mathcal{L}$. In most cases, the linear operator has a non trivial kernel, so solving $\mathcal{L}f = E$ directly is not possible. But this problem is not fatal.

One way to deal with the presence of a kernel is to restrict the class of possible perturbations by imposing symmetries on all the surfaces considered, and hopefully rule out functions in the kernel. This method only works if the initial configuration has the imposed symmetries. Another approach is to invert the linear operator modulo eigenfunctions corresponding to small or vanishing eigenvalues. In other words, one can add or subtract a linear combination of eigenfunctions to the inhomogeneous term E in order to land in the space perpendicular to the kernel, where the operator is invertible. The key to a successful construction is to be able to generate these linear combinations by slight adjustment of the approximate solution. The initial configuration therefore has to be flexible. Once the study of the linear case is completed, one uses a fixed point theorem to finish the proof.

3. Current and future projects

3.1. **Desingularization for self-shrinking surfaces under MCF.** I have made significant progress in desingularizing the intersection of a sphere and a plane to obtain a new example of a self-shrinking surface [Ngu09a] [Ngu10a]. The construction is more delicate, because of the lack of flexibility. Although spheres shrink to their centers no matter where they are in $\mathbf{R}^3$, in order to find the equation for self-similar surfaces one has to choose the center of homothety, thereby fixing the origin. As a result, the equation is not translation invariant. Despite these difficulties, the desingularization of a sphere and a plane through its center should be possible. It would be interesting to figure out if all combinations of the known examples (sphere, cylinder, plane, and doughnut) can be desingularized. If not, under what conditions would the construction work?

3.2. **Stability of the grim reaper.** Colding and Minicozzi have shown that spheres, cylinders and planes are the only stable self-shinking surfaces in $\mathbf{R}^3$. The same question concerning self-translating surfaces is open. I have started a collaboration with Ivan Blank to prove that the grim reaper is stable under small perturbations. One related question would be to determine how much we can open the U shape of the grim reaper before the family of evolving surfaces moves away from it.

3.3. **Gluing constructions in relativity.** Inspired by the work of Isenberg, Mazzeo and Pollack [IMP02], I am working on a gluing construction along hyperbolic cusps for initial data to Einstein equations, in collaboration with David Auckly. This problem is related to Thurston's geometrization conjecture, which was solved by Perelman and states that any 3-manifold has a natural decomposition into geometric pieces each having one of eight possible geometries. Perleman's proof shows that the parabolic Ricci flow uncovers the geometric decomposition of a 3-manifold. The Einstein equations can be considered as a hyperbolic Ricci flow, so it is natural to wonder how the evolution of the vacuum Einstein equations relates to the geometric decomposition of the 3-manifold. In particular we ask if it is possible to glue initial data that is close to geometric data along tori.

3.4. **Traveling wave Allen-Cahn equation.** Let u be an entire solution of the traveling wave Allen Cahn equation

$$\Delta u + c\partial_{x_{n+1}} u + u(1-u^2) = 0, \text{ in } \mathbf{R}^{n+1}, \tag{1}$$

$$|u(x)| \leq 1, \quad \partial_{x_{n+1}} u > 0, \quad \lim_{x_{n+1}\to\pm\infty} u(x', x_{n+1}) = \pm 1. \tag{2}$$

Del Pino, Kowalczyk, and Wei [dPKW] claim that given any self-translating surface that separates $\mathbf{R}^3$ into two connected components, they can construct a solution to (1) whose zero level set is the original self-translating surface. I plan on studying whether the zero level sets of all solutions to (1) and (2) are self-translating hypersurfaces for the MCF. The problem is related to the de Giorgi conjecture, which states that the zero level sets of entire solutions to $\Delta u + u(1-u^2) = 0$ and (2) are minimal surfaces. The conjecture has been solved with a positive answer in dimensions $n \leq 7$ and with a negative answer in dimensions $n \geq 8$.

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