Introductory Notes on the Structural and Dynamical Analysis of Networks

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Important Acknowledgement:

These notes are based (with edits/corrections) on an on-line "Complex Networks" slide presentation by

Changsong Zhou
AGNLD, Institute für Physik
Universität Potsdam

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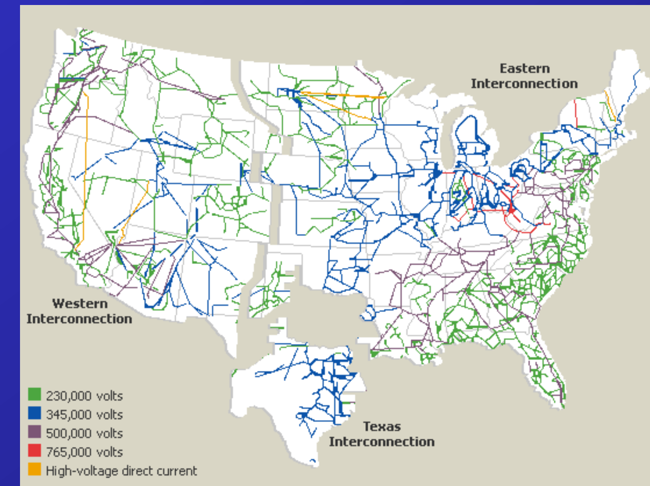
What is a Network?

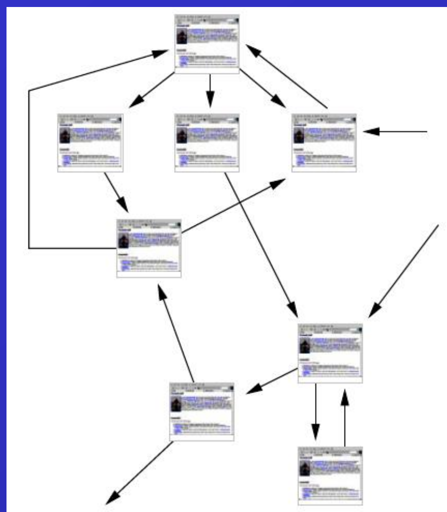
- A ***network*** is a (finite) collection of entities together with a specified pattern of relationships among these entities.
- Three main tools have been used for the quantitative study of networks:
  - graph theory;
  - statistical and probability theory;
  - algebraic models.

1. INTRODUCTION

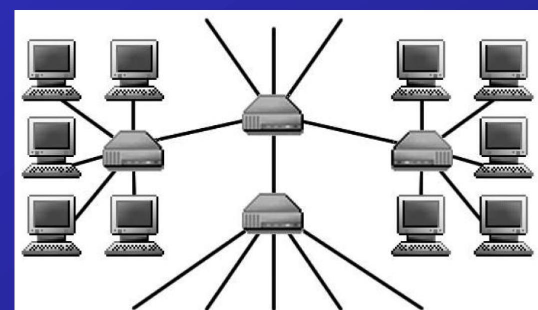
Technological Networks



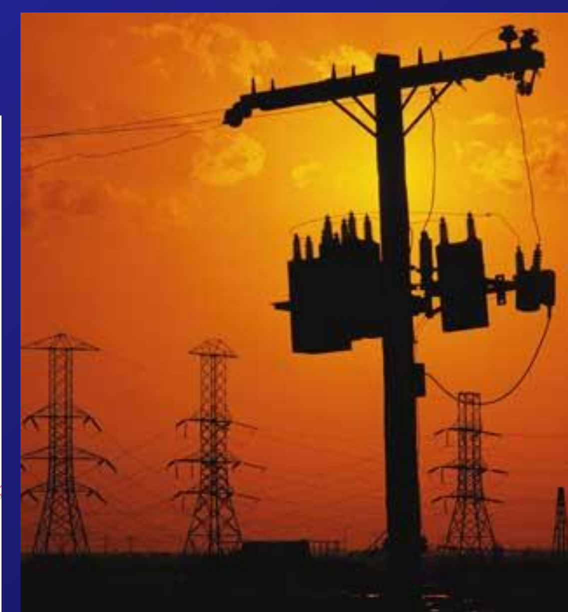
World-Wide Web

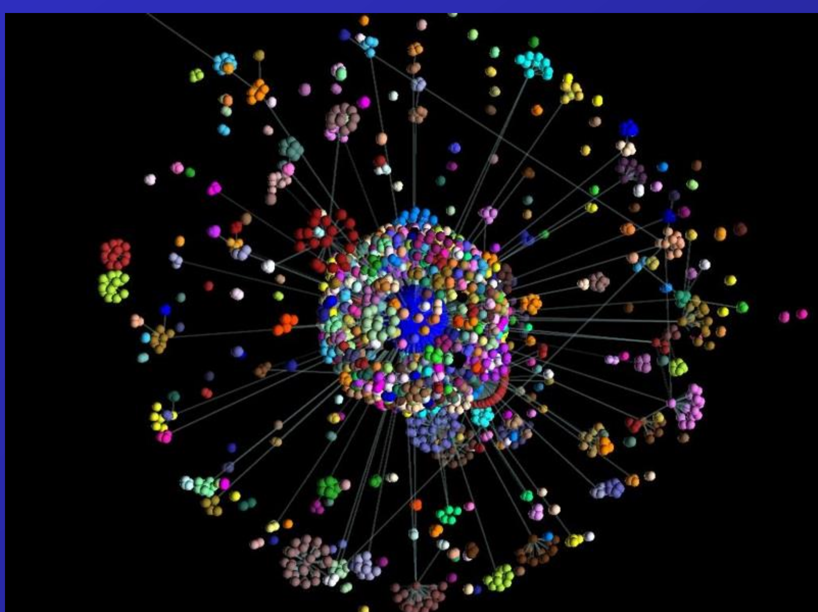


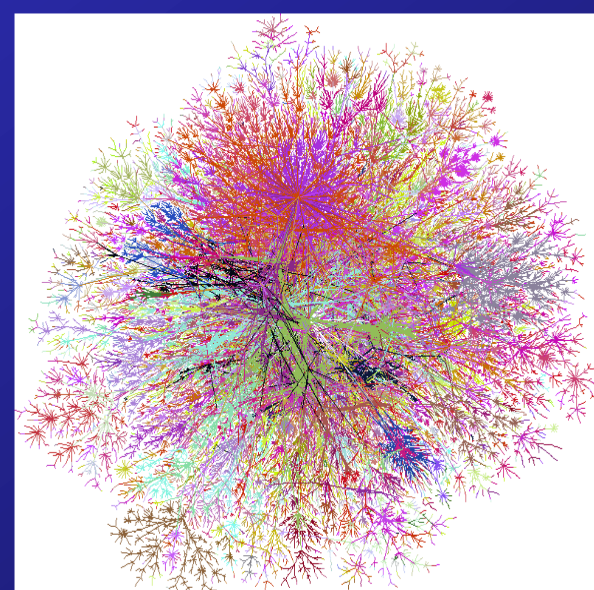
Internet



Power Grid



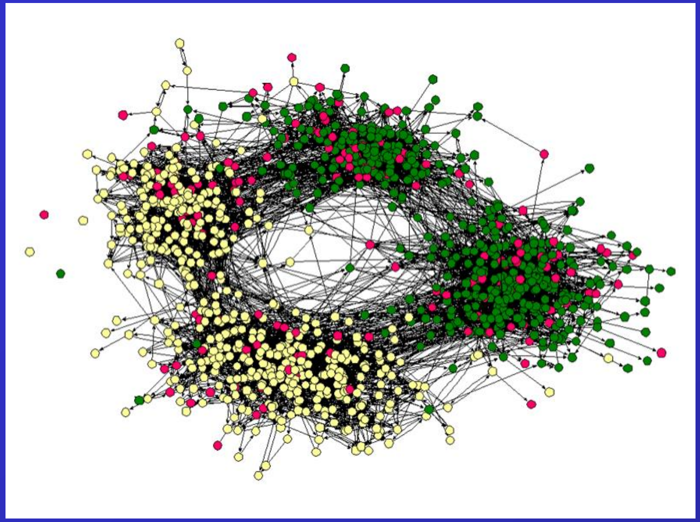




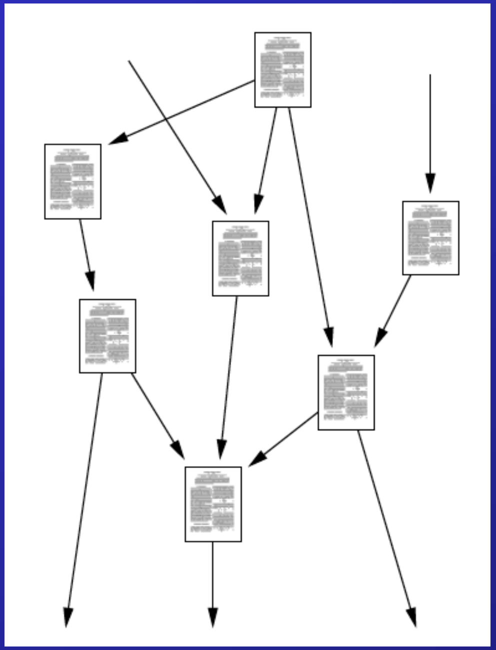
1. INTRODUCTION

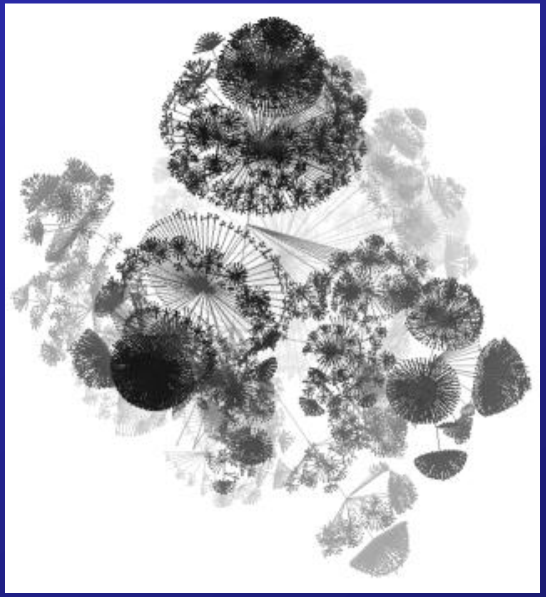
Social Networks

Friendship Net



Citation Networks





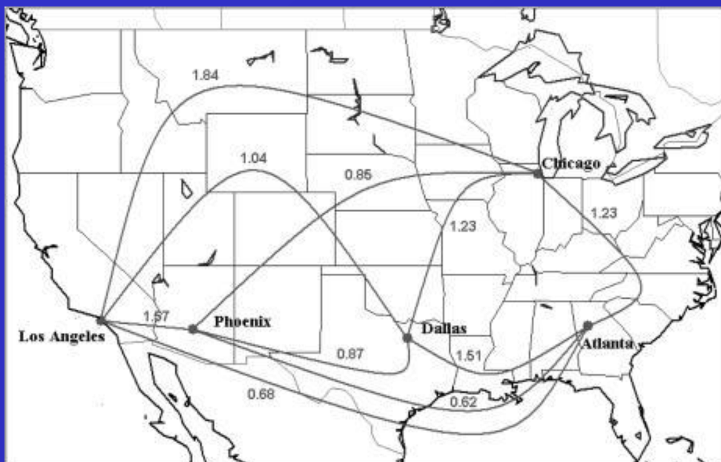
Movie Actors

Sexual Contacts

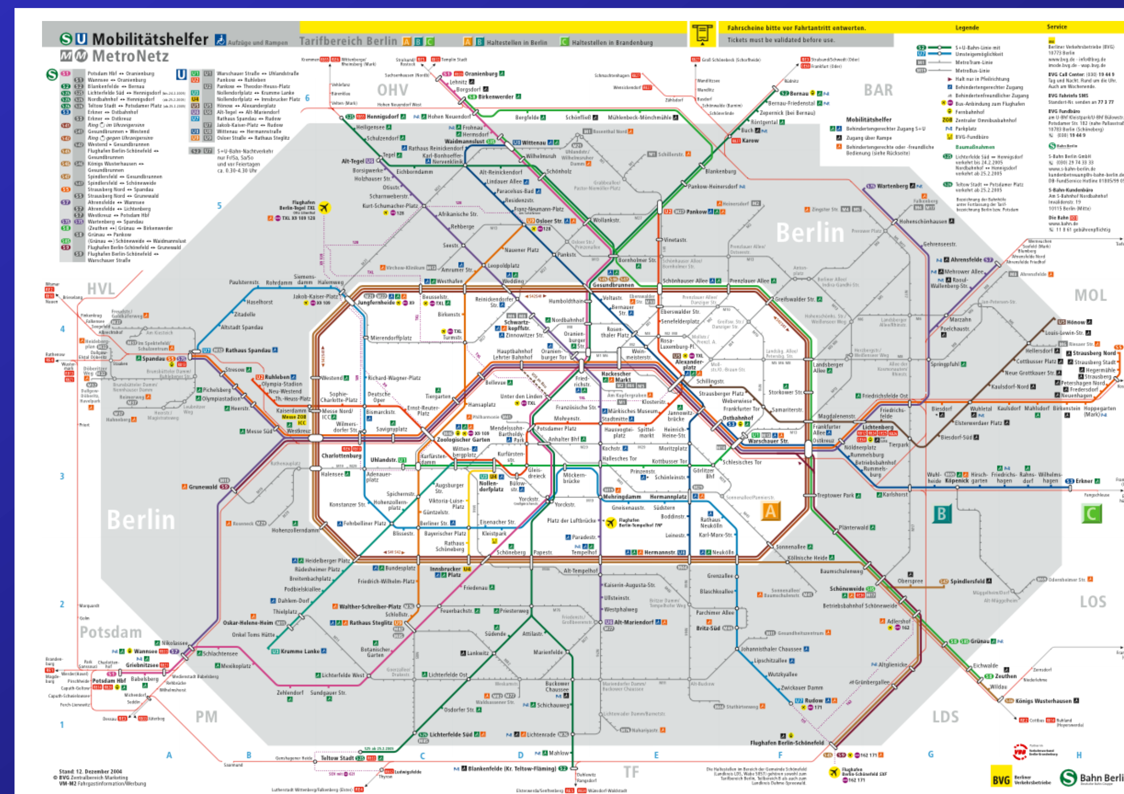
Collaboration Networks

Transportation Networks

Airport Networks



Local Transportation



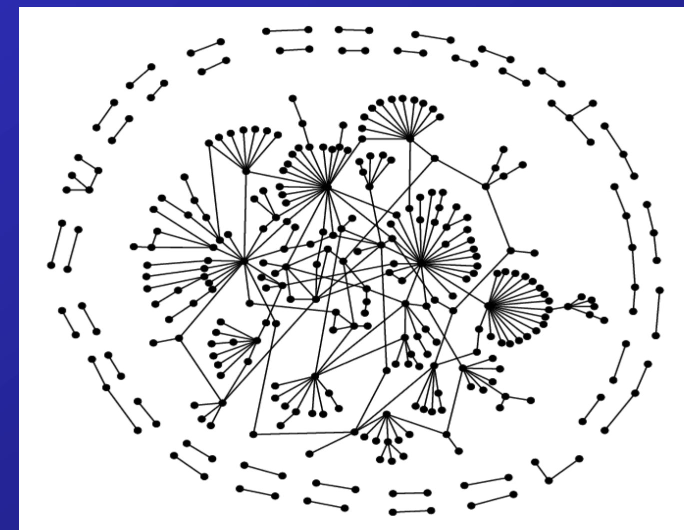
Road Maps

1. INTRODUCTION

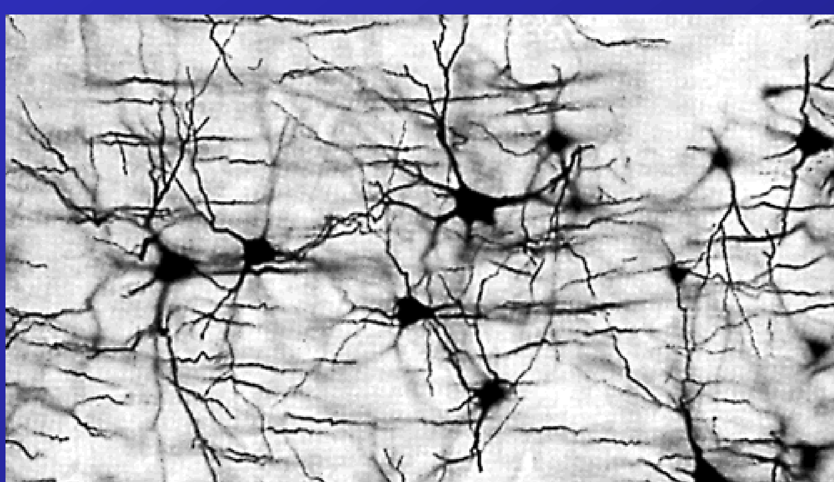
Biological Networks

Ecological Webs

Protein interaction



Neural Networks



Genetic Networks

Metabolic Networks

2. NETWORKS...

GOAL: A **unified approach**

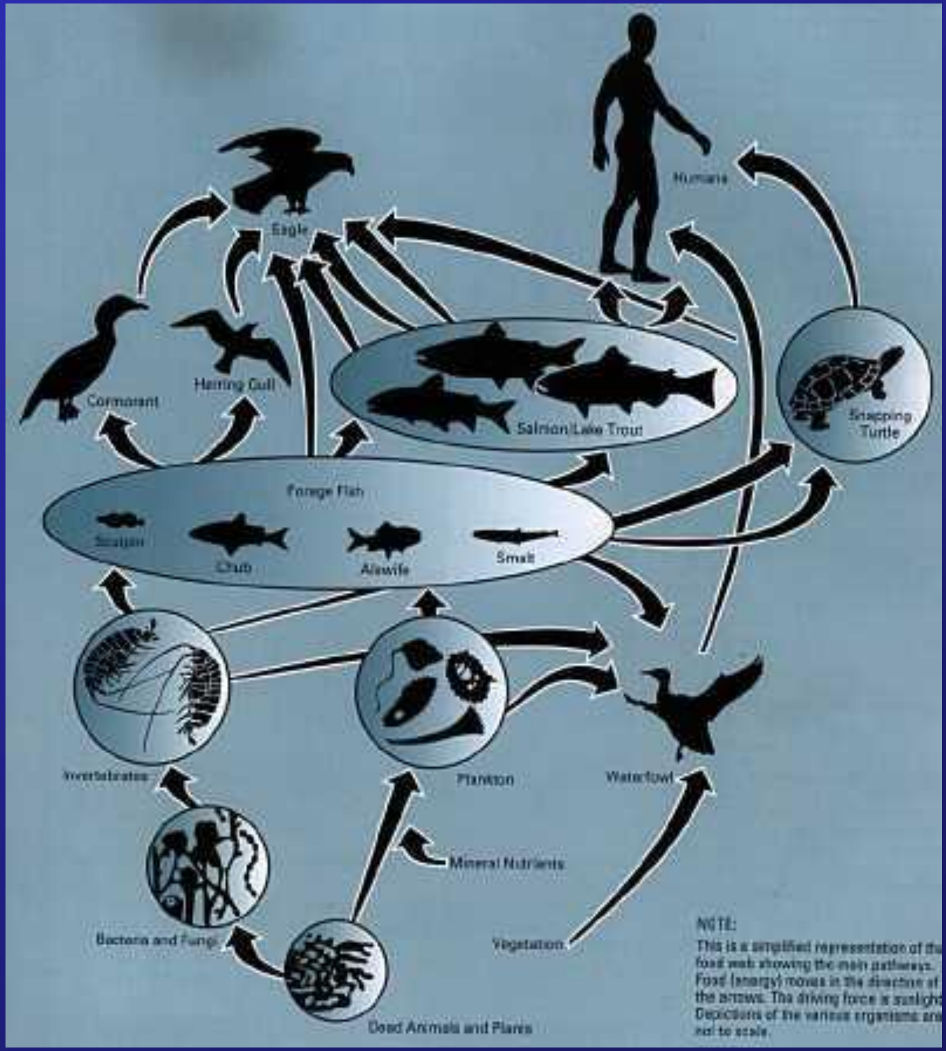
enabling analysis of the

connection topology

underlying a wide variety of

Complex Systems

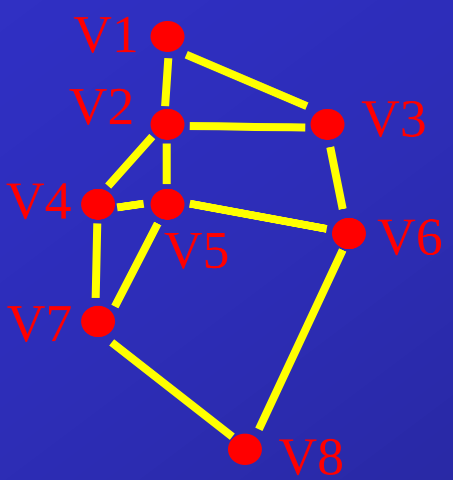
Example: Food Web

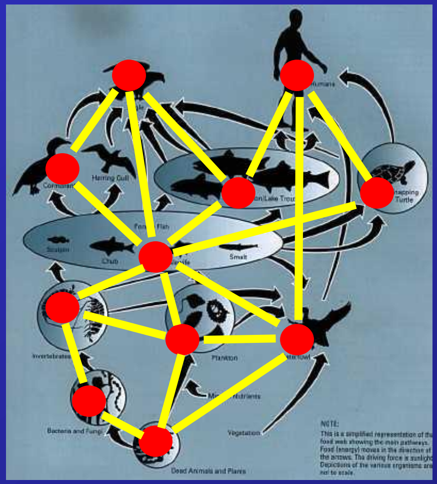


Graphical Approach: Vertices and Edges

Example: Simple graph G with
Vertex Set V(G)={V1,…,V8}

A_{ij} = 1 iff (i,j) is in the
Edge Set E(G)





Symmetrical Adjacency Matrix A
for the Simple Graph G

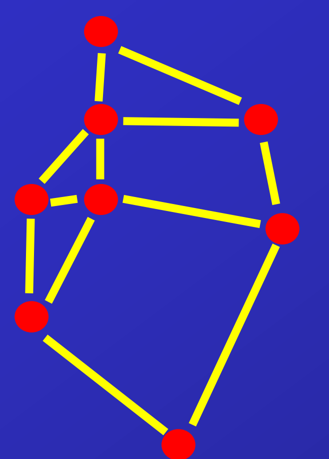
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	0	1	0	0
4	0	1	0	0	1	0	1	0
5	0	1	0	1	0	1	1	0
6	0	0	1	0	1	0	0	1
7	0	0	0	1	1	0	0	1
8	0	0	0	0	0	1	1	0

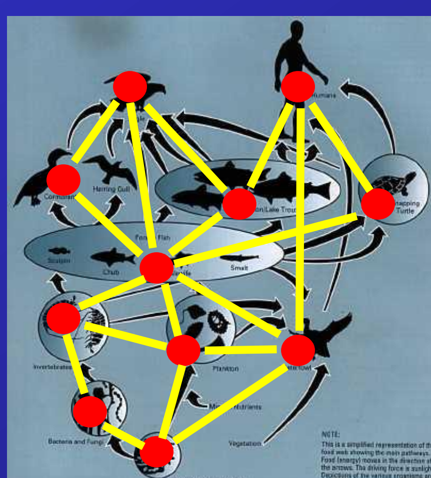
2. NETWORKS...

Graphical Approach…

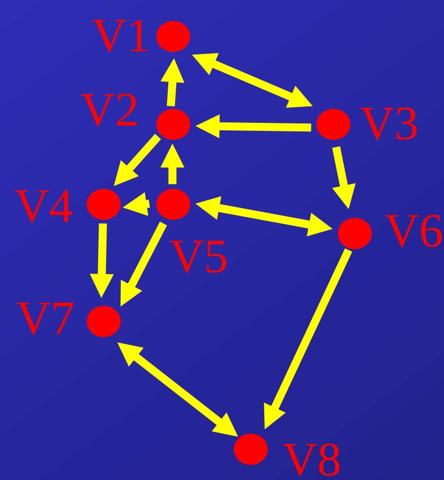
$A_{ij} = 1$ iff (i,j) is in the edge set E(G)

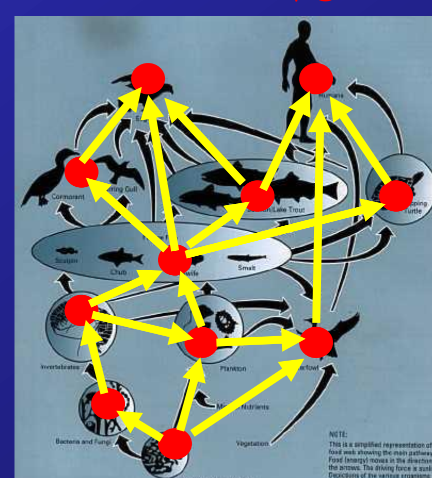
Simple Graph





Directed Graph G (DiGraph)





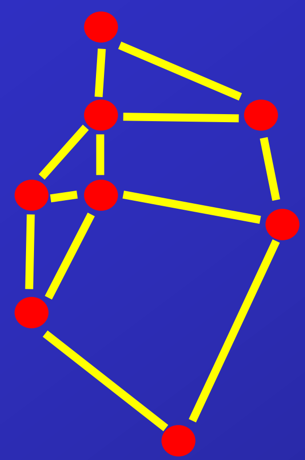
Non-Symmetrical Adjacency Matrix A for the DiGraph

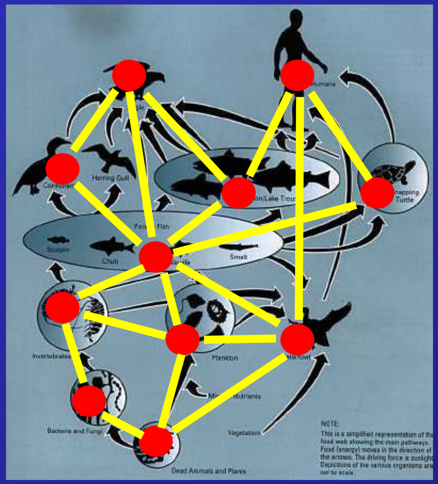
	1	2	3	4	5	6	7	8
1	0	0	1	0	0	0	0	0
2	1	0	0	1	0	0	0	0
3	1	1	0	0	0	1	0	0
4	0	0	0	0	0	0	1	0
5	0	1	0	1	0	1	1	0
6	0	0	0	0	1	0	0	1
7	0	0	0	0	0	0	0	1
8	0	0	0	0	0	0	1	0

2. NETWORKS...

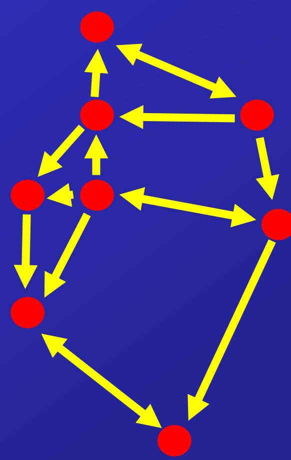
Graphical Approach…

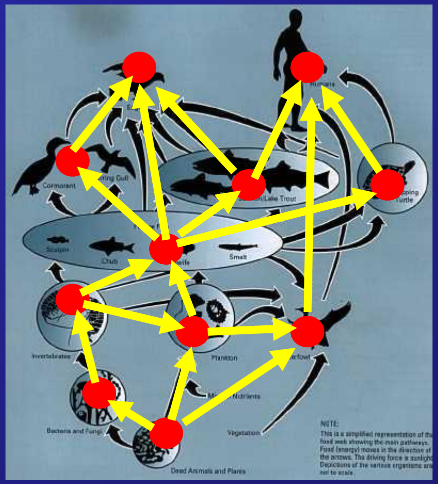
Simple Graph



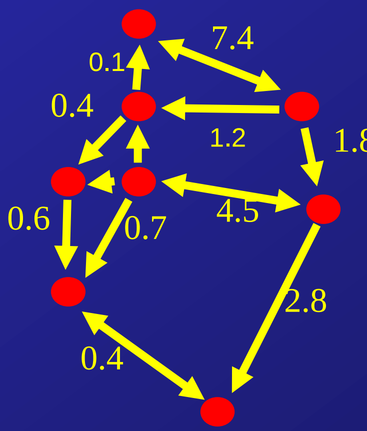


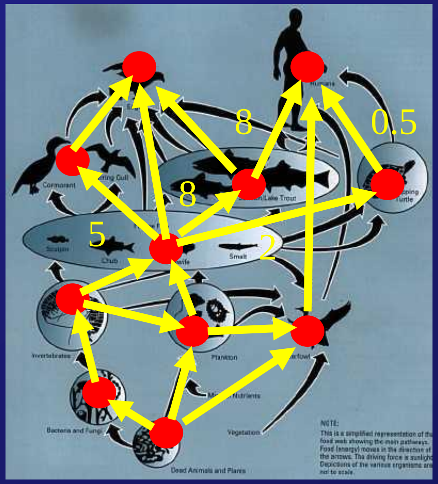
DiGraph





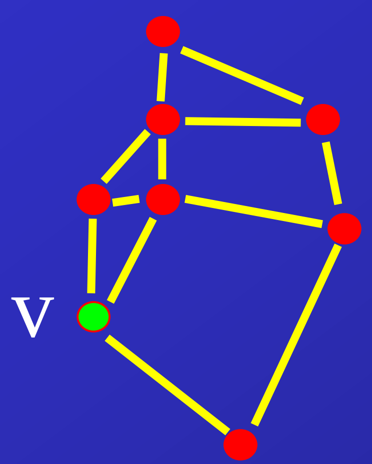
Weighted DiGraph





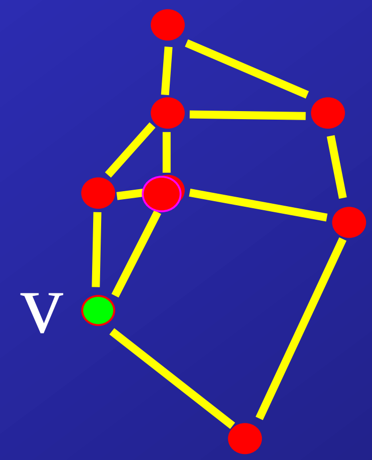
Structural Characterization

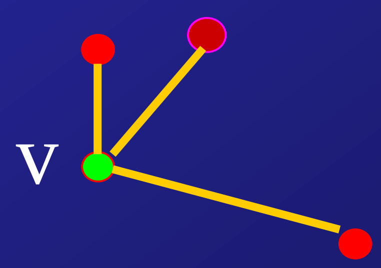
Simple Graph



e.g., Trade Network

Vertex Degree: *k(v)*



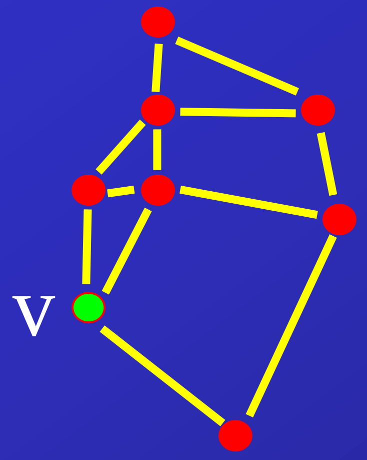


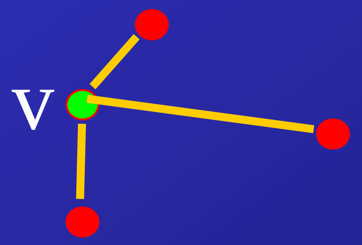
$$k(\cdot) = 3$$

Structural Characterization…

Clustering Coefficient: *C(v)*

Simple Graph



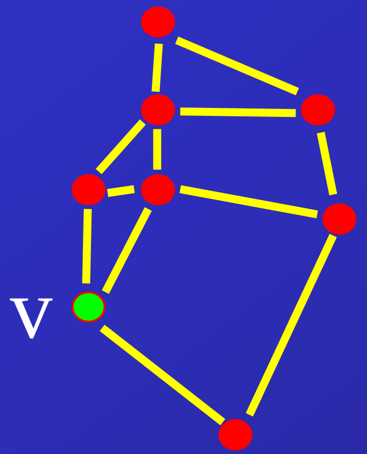


e.g., Trade Network

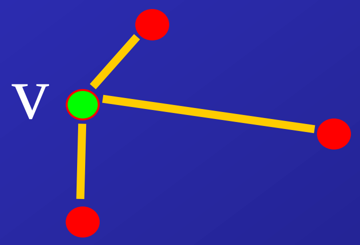
Structural Characterization…

Clustering Coefficient: *C(v)*

Simple Graph



e.g., Trade Network

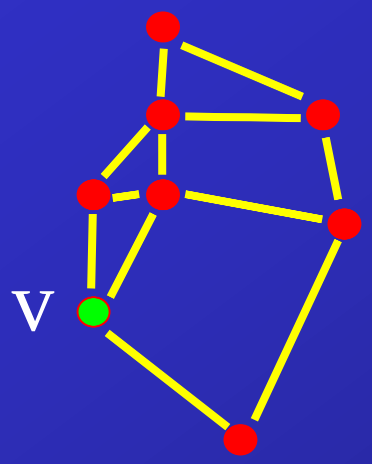


- Degree of vertex (number of directly connected vertices): k( ) = 3

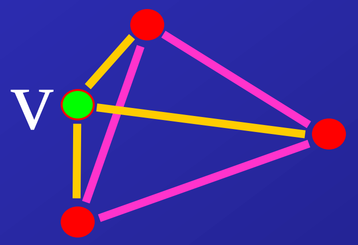
Structural Characterization…

Clustering Coeficient: *C(v)*

Simple Graph



e.g., Trade Network



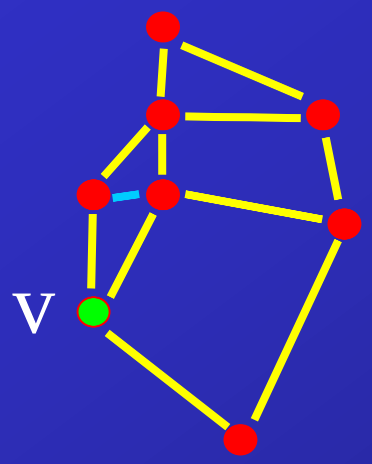
- Degree of vertex : k( ) = 3
- Total number of possible connections among these 3 neighbors:

$$½·k(v)·[k(v)-1] = ½·[3·2] = 3$$

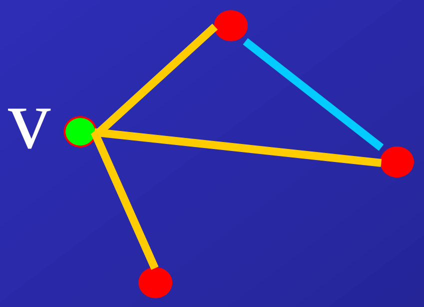
Structural Characterization…

Clustering Coefficient: *C(v)*

Simple Graph



e.g. Trade Network

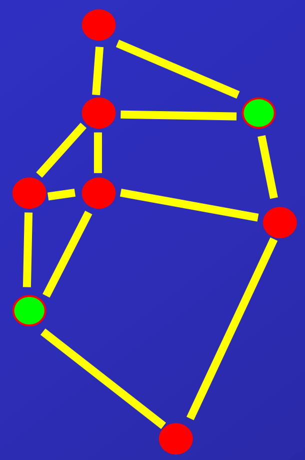


- Number of actual connections among the three neighbors = 1
- Total number of possible connections:

  $\frac{1}{2} \cdot k(v) \cdot [k(v)-1] = \frac{1}{2} \cdot [3 \cdot 2] = 3$
- ***C(v)*** = 1 / 3 = 0.33333
- Measures how well my neighbors are connected to each other!

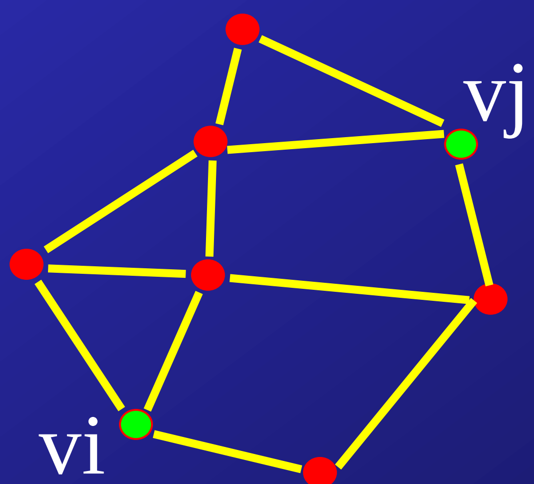
Structural Characterization…

Simple ***Connected*** Graph



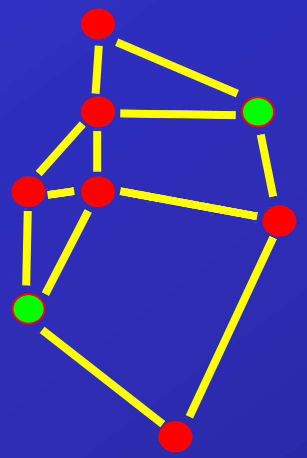
e.g., Trade Network

"Distance" vi to vj?



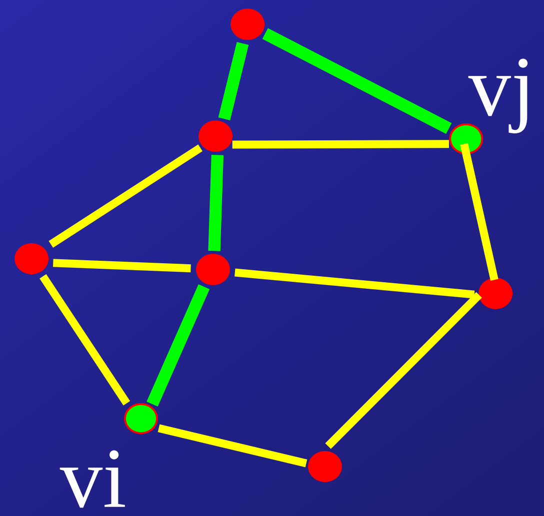
Structural Characterization …

Simple Connected Graph



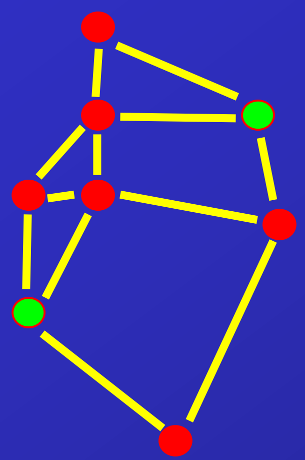
e.g., Trade Network

Length of this path vi to vj = 4



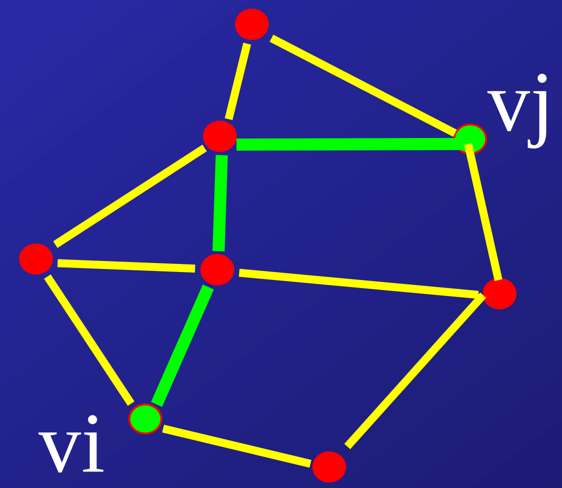
Structural Characterization…

Simple Connected Graph



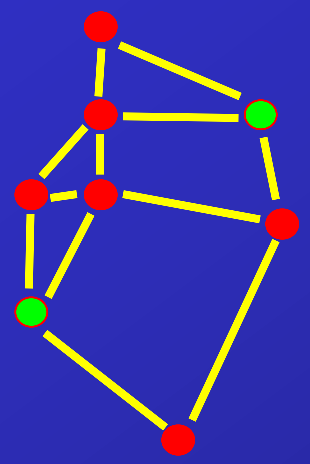
e.g., Trade Network

Length of this path vi to vj = 3



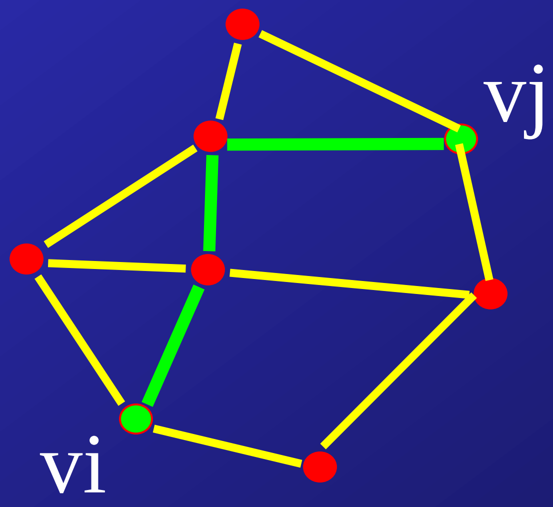
Structural Characterization…

Simple Connected Graph



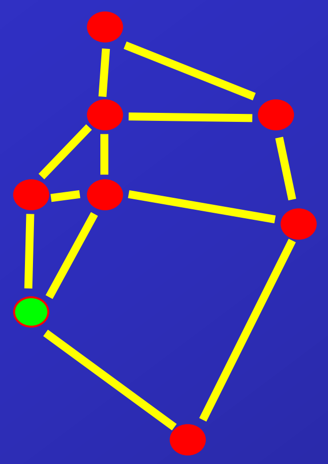
e.g., Trade Network

Distance vi to vj **=** ***Shortest*** path length vi to vj, here equal to 3



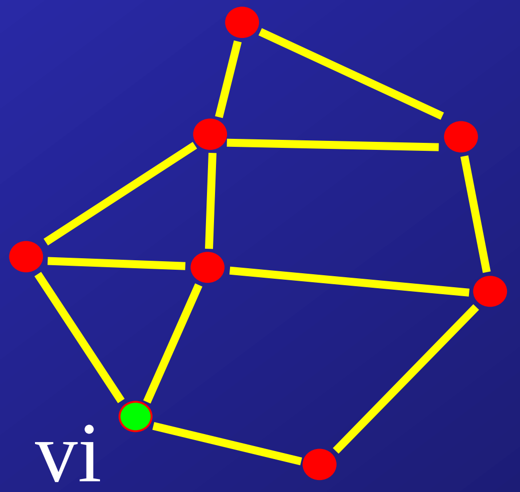
Structural Characterization...

Simple Connected Graph



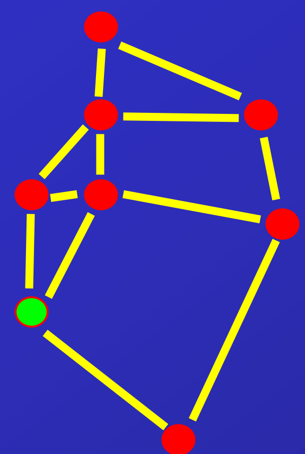
e.g., Trade Network

Distance from vertex vi to each other vertex v?

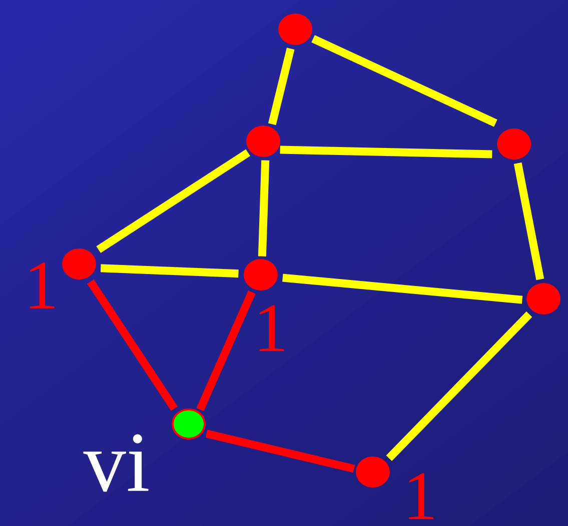


Structural Characterization…

Simple Connected Graph



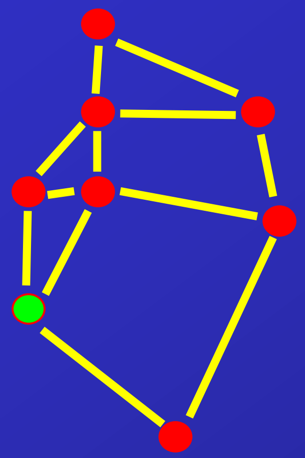
Distance-1 Vertices from Vertex vi



e.g., Trade Network

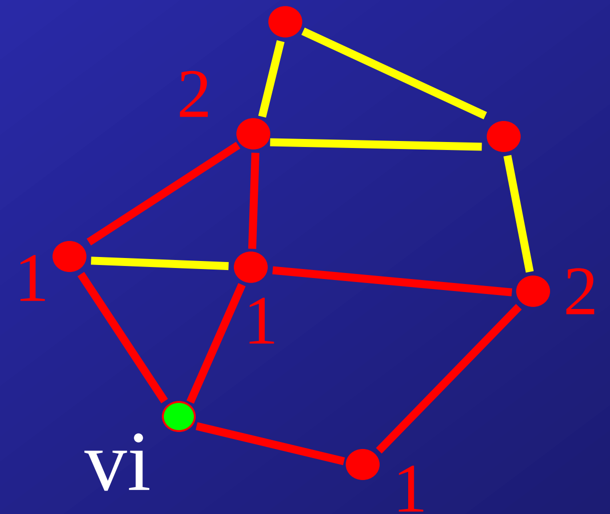
Structural Characterization…

Simple Connected Graph



e.g. Trade Network

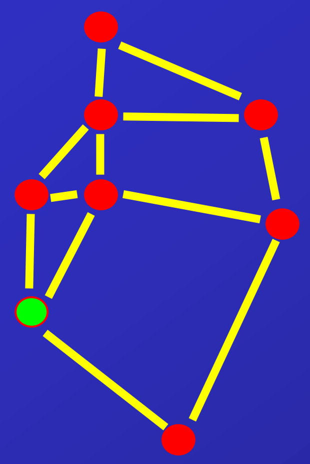
Distance-2 Vertices from Vertex vi

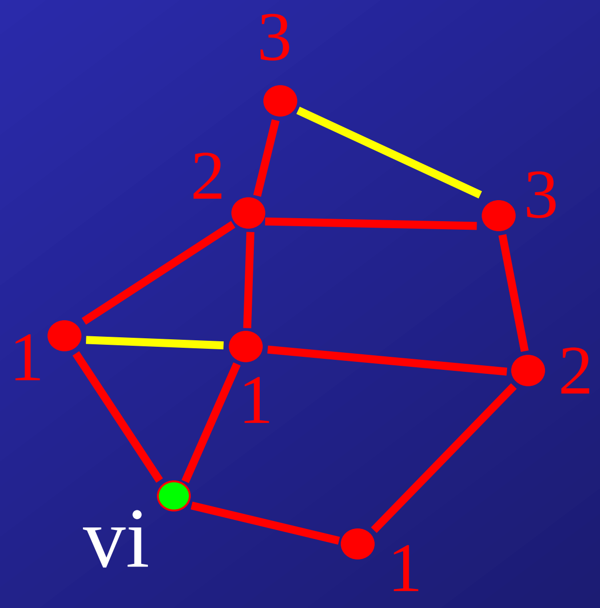


Characterization

Distance 3-Vertices from Vertex vi

Simple Connected Graph





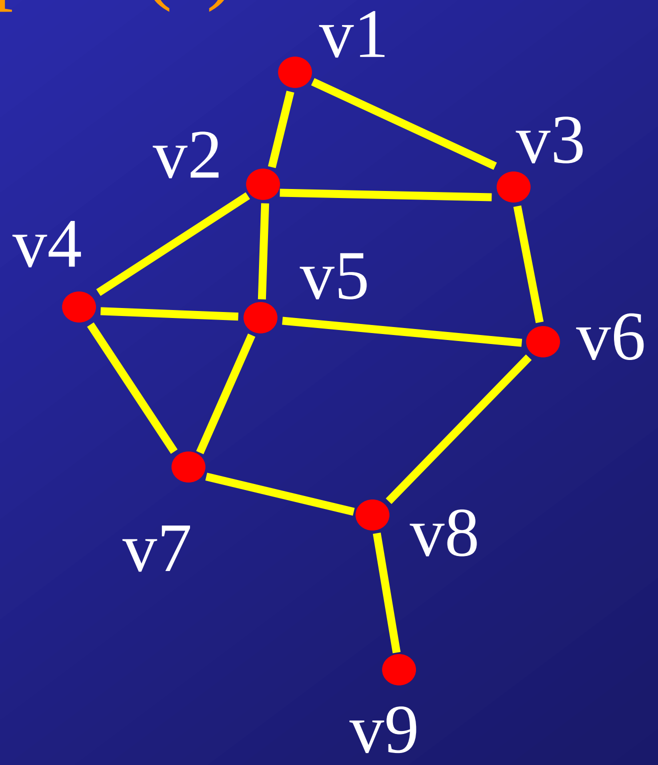
Distance L_{ij} : Length of the ***shortest*** path(s) from vi to vj

L(G) = Characteristic Path Length of Graph G

- All-to-all distance matrix:

L_{ij} Length of the shortest path(s)

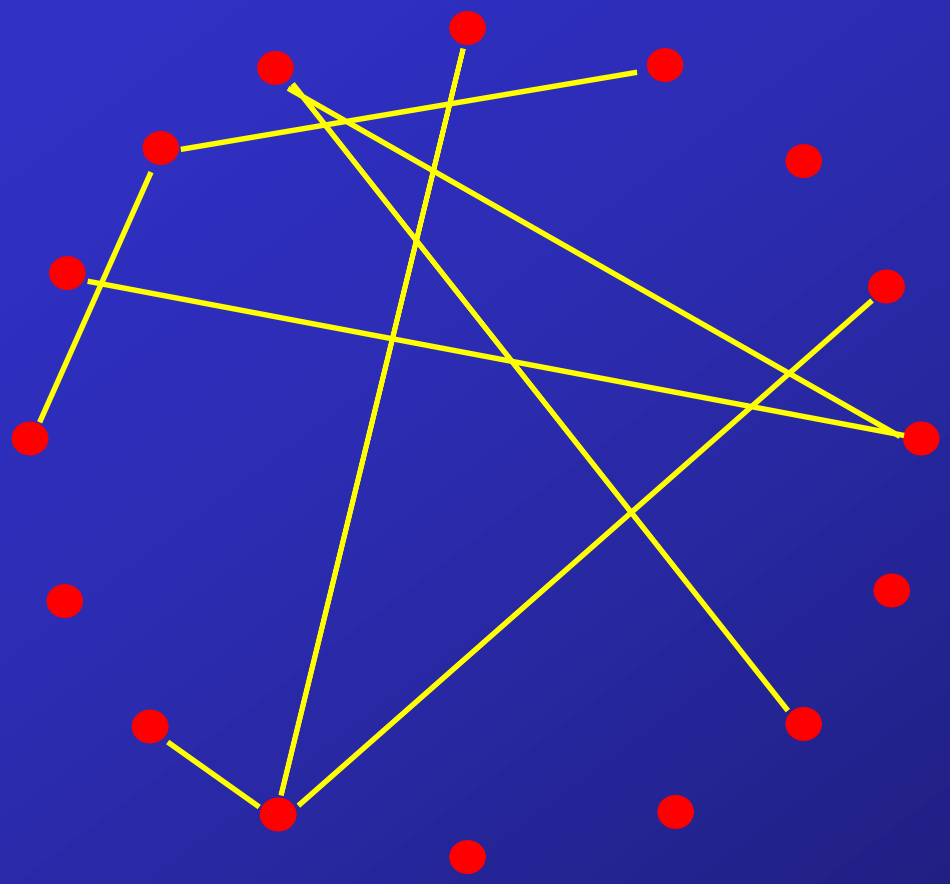
	1	2	3	4	5	6	7	8	9
1	0	1	1	2	2	2	3	3	4
2	1	0	1	1	1	2	2	3	4
3	1	1	0	2	2	1	3	2	3
4	2	1	2	0	1	2	1	2	3
5	2	1	2	1	0	1	1	2	3
6	2	2	1	2	1	0	2	1	2
7	3	2	3	1	1	2	0	1	2
8	3	3	2	2	2	1	1	0	1
9	4	4	3	3	3	2	2	1	0



L(G) = Average of L_{ij} over all vertices vi and vj ($i \neq j$) in V(G) = 1.94

E-R Random Graph Model

Paul Erdös & Alfréd Rényi **(*Hungarian Academy of Sciences*, 1960):**

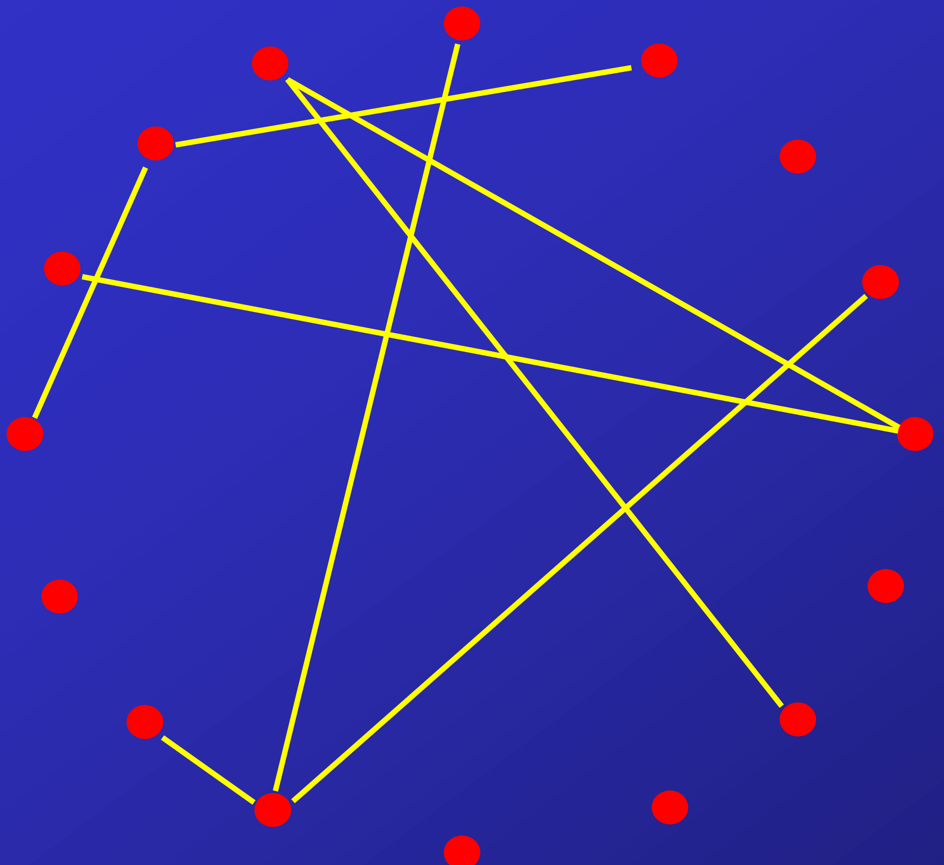


Start with a collection
of N unconnected vertices.

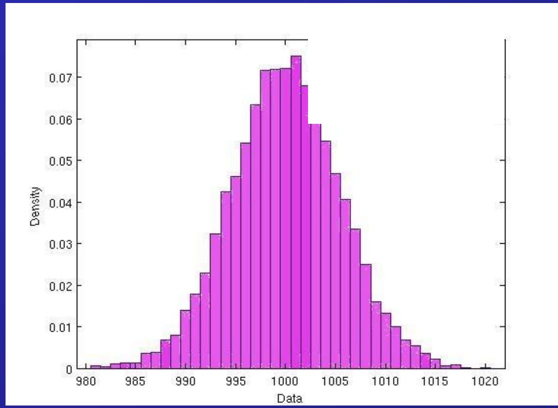
Then, for each distinct pair
of vertices, connect them
by an edge with probability p.

Denote the resulting graph as
G = G(N,p)

E-R Random Graph Model…Continued



- Degree distribution: $P_G(k)$



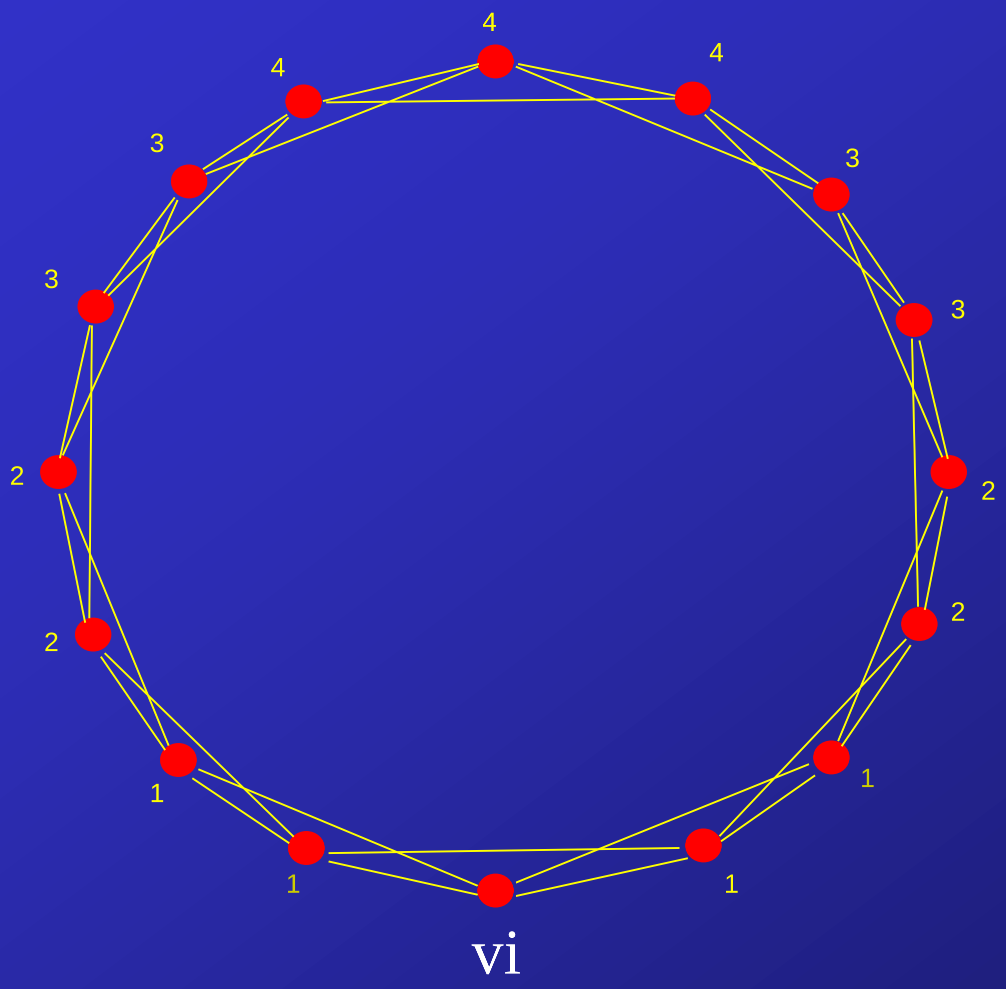
N=1020
p = 0.2

Poissonian!

$P_G(k)$ = Probability that a randomly selected vertex in G will have degree k

$P_G(k) \sim [\ e(-z)\ z^k\ ]/k!$ for $G=G(N,p)$
where z = mean k (depends on N,p)

Graph G for a Regular Ring Lattice

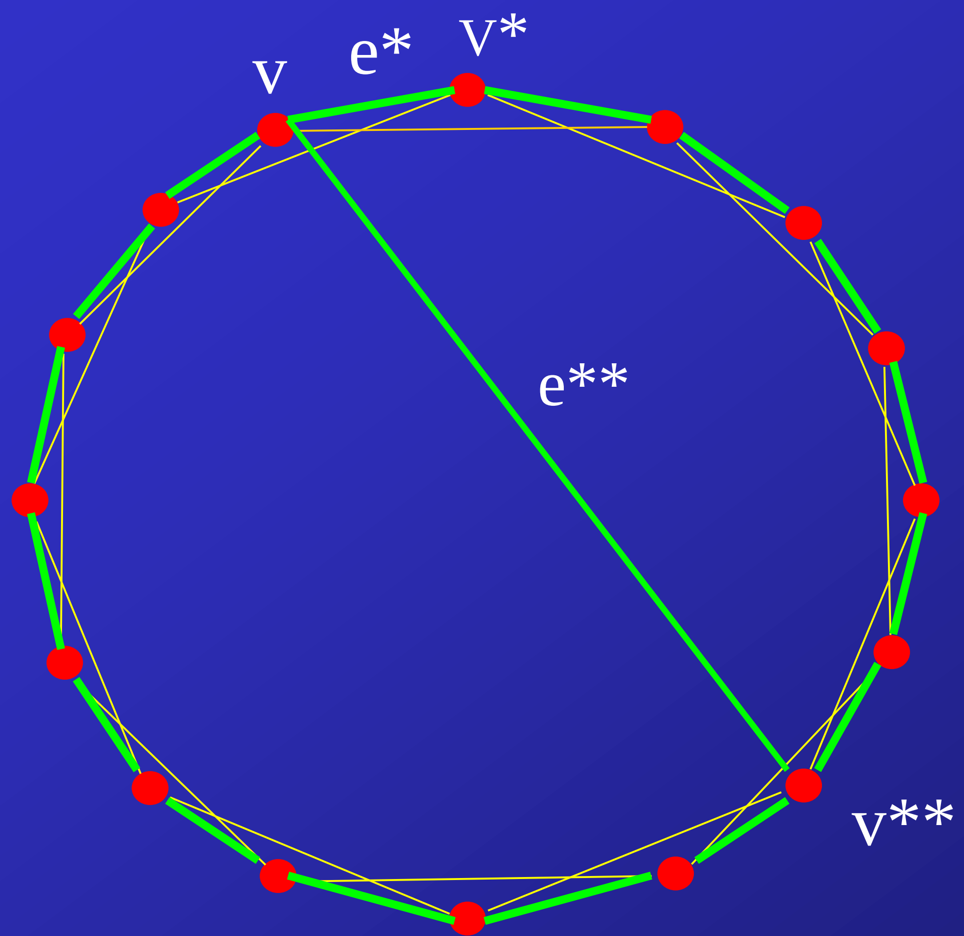


- Regular = Every vertex has the same degree
- $|V(G)|$ = No. of Vertices = 16
- Degree k = 4
- Clustering: C(G) = 1/2
- Characteristic Path Length:

  L(G) = 36/15 = 12/5

Small-World Network (SWN) Models

Duncan Watts & Steven Strogatz (*Nature*, 1998):



Construction of SWN G(p), $0 \leq p \leq 1$

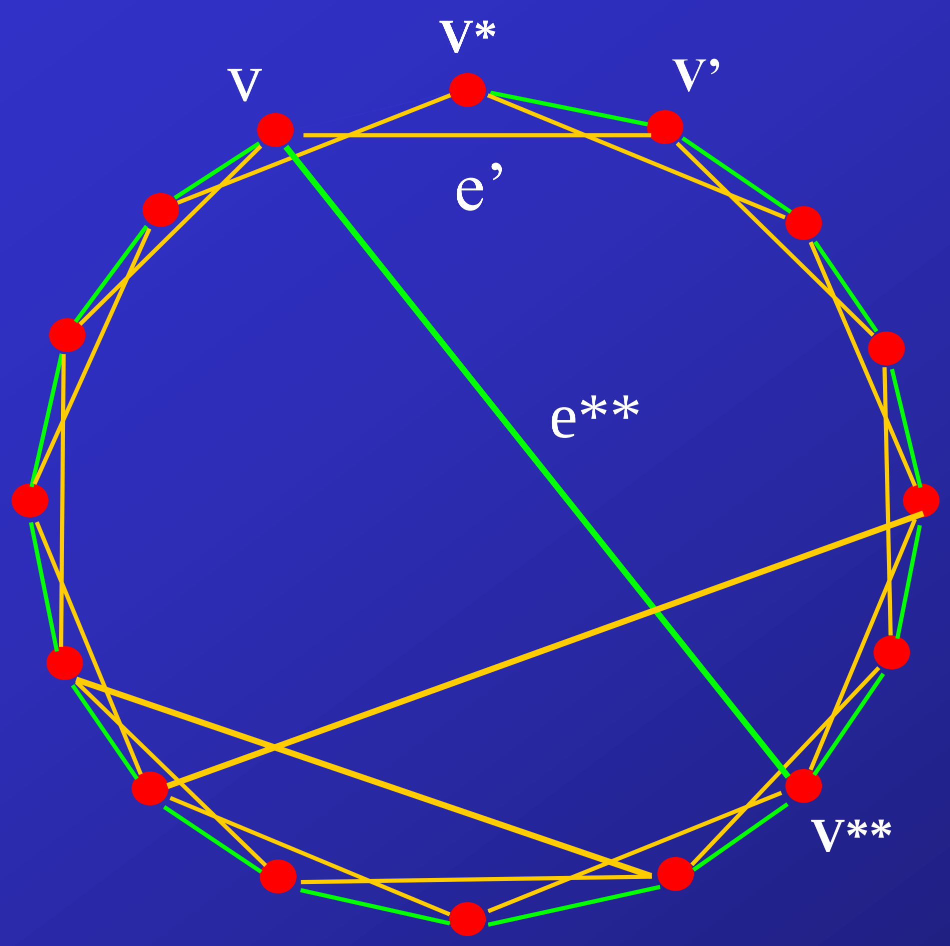
Choose a vertex v and edge e* that connects v to its nearest neighbor v* in clockwise direction.

With probability p, reconnect edge to a vertex v** chosen uniformly at random over the ring but with duplicate edges forbidden.

Continue process clockwise around ring until 1 lap is complete.

SWN Models…Continued

Watts-Strogatz 1998: Construction of Small-World Network G(p)



Next consider edges e' at distance 2 from from each v in clockwise direction, and randomly rewire with probability p.

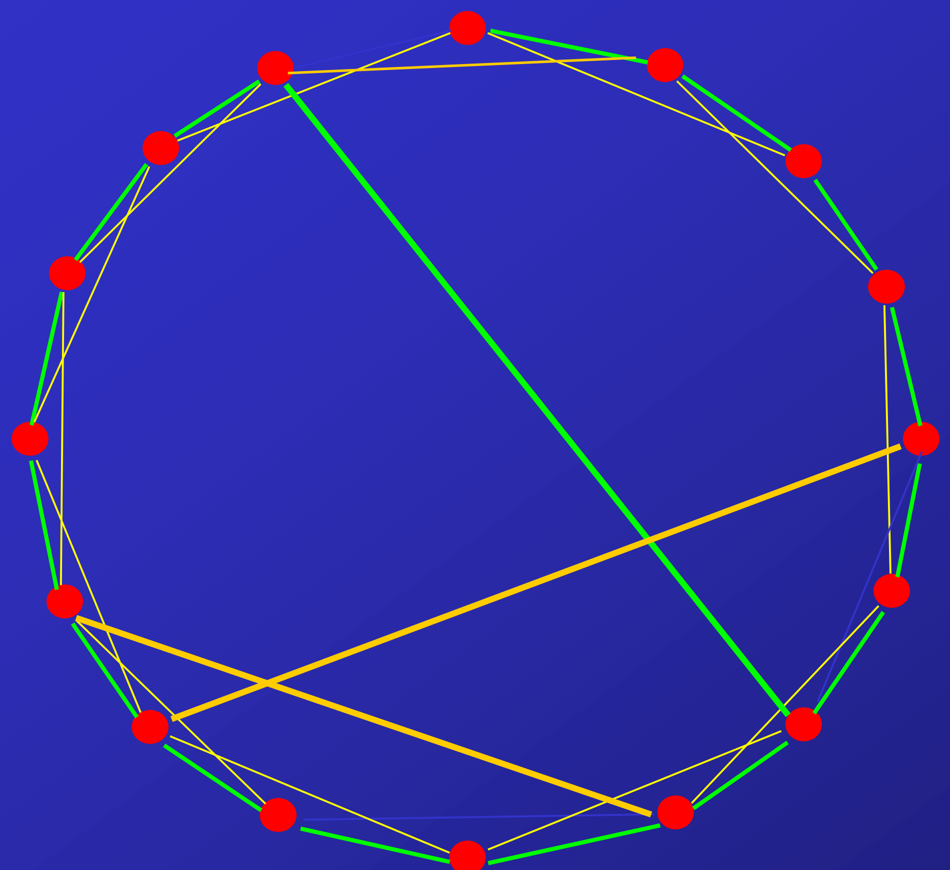
Moving clockwise, complete a full lap of distance-2 rewiring.

In general, for a ring of any even degree k, successively rewire ALL edges with probability p by completing k/2 laps around ring.

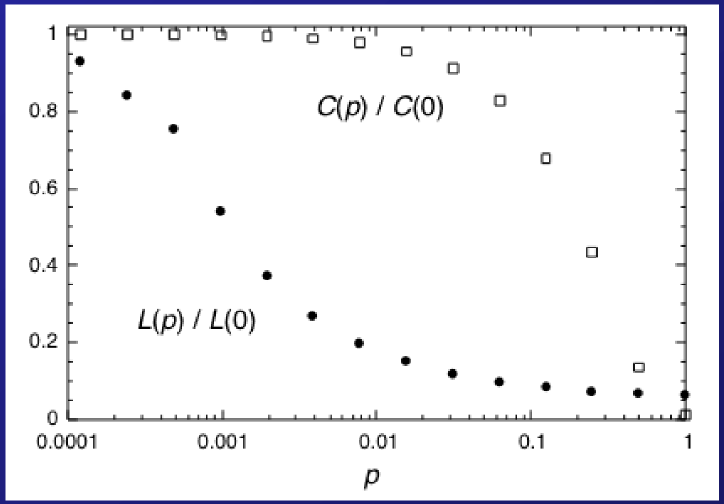
Rewired edges are called "SHORT-CUTS"

SWN Models…Continued

Watts-Strogatz 1998: Construction of Small-World Network G(p)



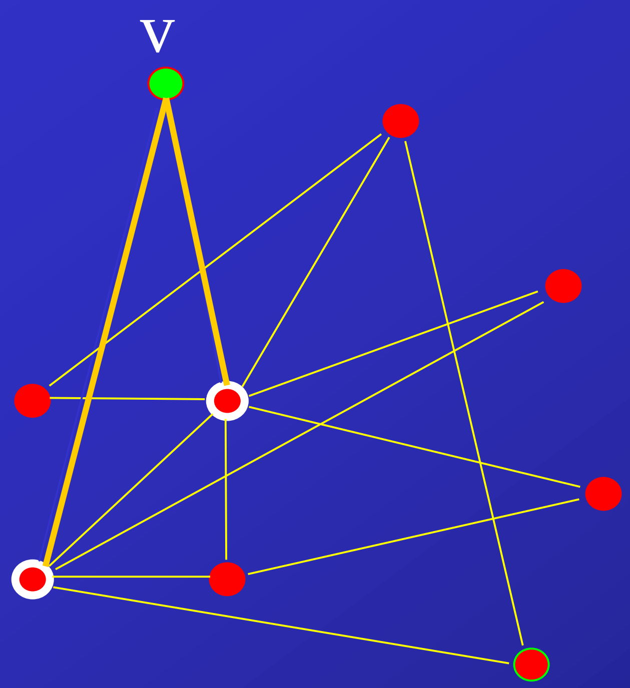
- For a range of p's with 0 < p < 1, the SWN G(p) is characterized by
  - High clustering C(p)/C(0)
  - Short path length L(p)/L(0)



Watts, Strogatz. Nature 393/4, 1998

SWN Models…Continued

Albert-Lázló Barabási (A-B) **Scale-Free Network** (*Science*, 1999):

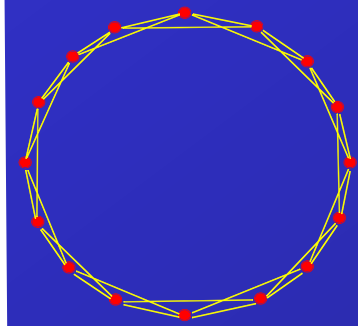
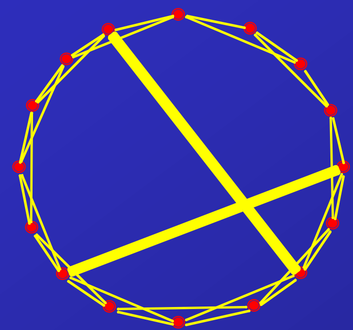
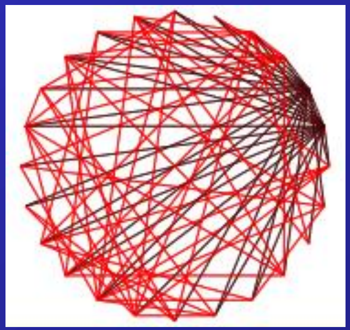
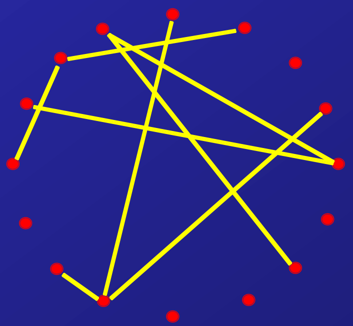


- At each step add new vertex v to graph and connect it to 2 randomly selected existing vertices v_i using "preferential attachment" prob's

$$p_i = \frac{k_i}{\sum_j k_j} = \mathrm{Prob}(v_i)$$

- Results:
  - "Richer-Get-Richer"
  - $P_G(k) \sim k^{-3}$ **(Power Law =Scale Free)**

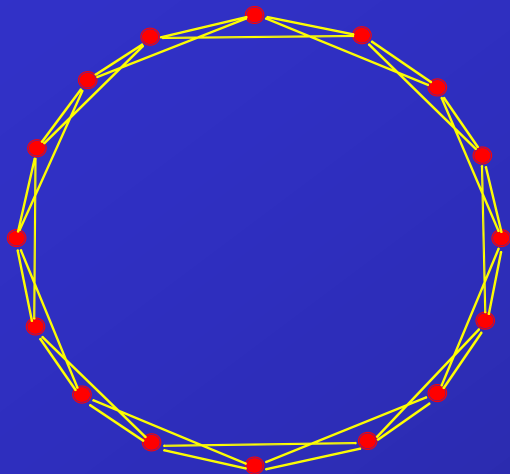
Properties of the Network Models

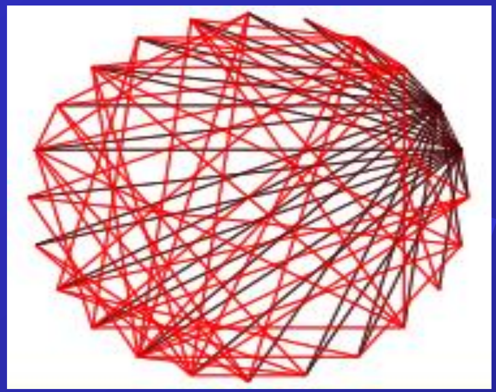
	Regular	WS Small-World	AB Small-World	E-R Random
				
Path length	Long	Short	Short	Short
Clustering	Large	Large	Large	Small

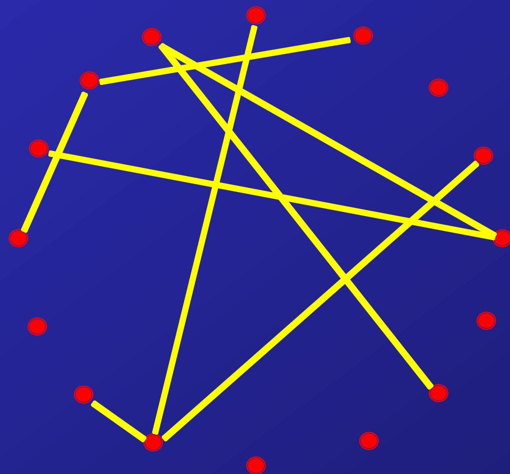
Small-world networks fall "between" regular and E-R random networks!

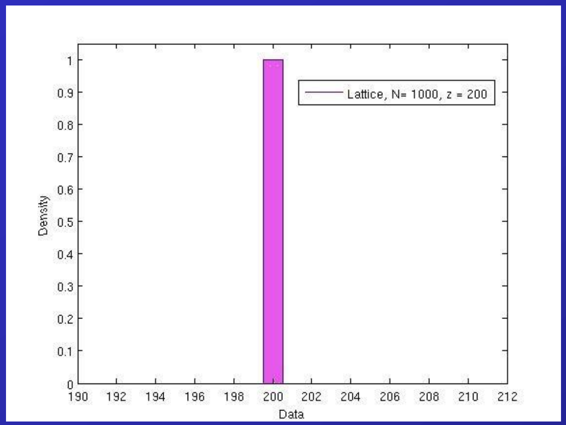
Properties of the Network Models…

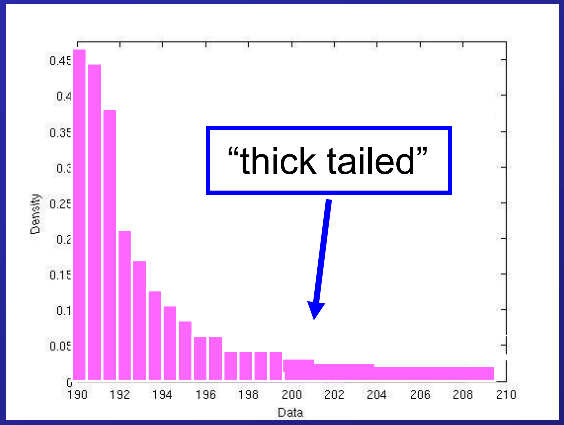
Regular Lattice | A-B Scale-Free SWN | E-R Random Graph

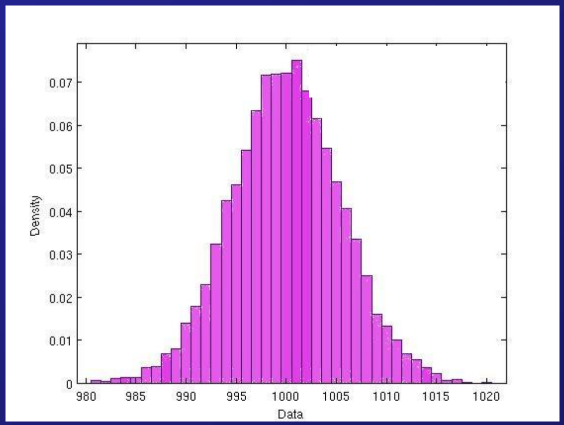












$P_G(k) = \delta(k\ ; kTrue)$

where $\delta(k\ ; kTrue)$ equals 1 if $k = kTrue$ and 0 for all other k

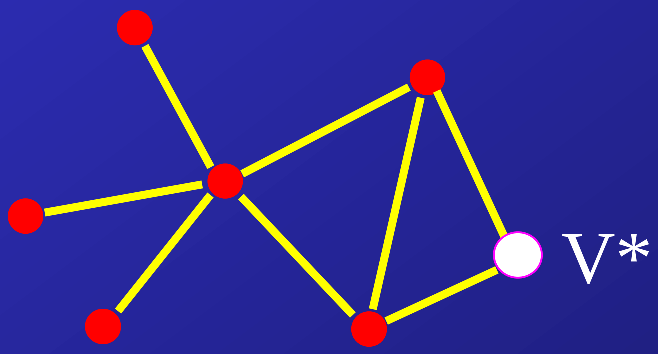
$P_G(k) \sim k^{-3}$

power law

$P_G(k) \sim [\, e(-z) z^{k} \,]/k!$

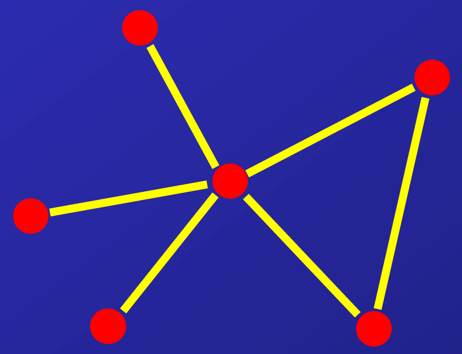
$z =:$ mean k

Small-World Nets: Robustness to Shocks



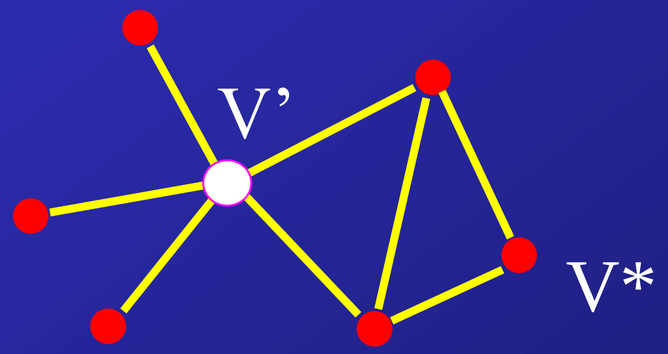
- **Network Resilience:**
  - Highly robust against **RANDOM** failures of vertices, e.g., vertex v* shown above

Small-World Nets: Significant Impacts



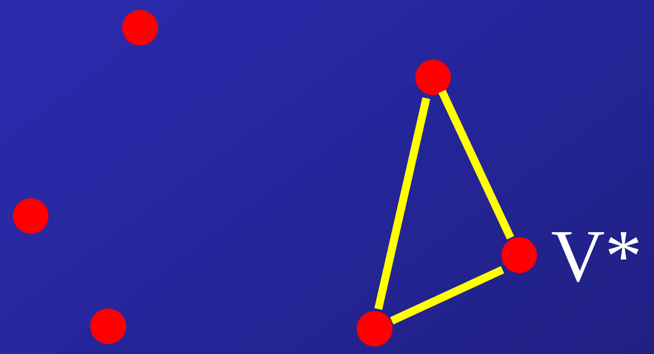
- **Network Resilience:**
  - Highly robust against **RANDOM** failures of vertices, e.g., vertex v* shown on previous slide
  - **However, …**

Small-World Nets: Significant Impacts



- **Network Resilience:**
  - Highly robust against **<u>RANDOM</u>** failures of vertices, such as vertex v* shown above
  - **However,** highly vulnerable to deliberate attacks on **<u>HUBS</u>** (i.e., *vertices having a relatively high degree k*), such as vertex v' shown above.

Small-World Nets: Significant Impacts



- **Network Resilience:**
  - Highly robust against **RANDOM** failures of vertices, such as vertex v* shown above
  - However, highly vulnerable to deliberate attacks on **HUBS**, e.g., vertex v' shown on previous slide

So how well do YOU know Kevin Bacon?

- **Small-World Effect** = Hypothesis that every two people in the world are connected by a surprisingly short chain of social acquaintances.

- ***Example:*** The trivia game **Six Degrees of Kevin Bacon**

Six Degrees of Kevin Bacon…

- Name taken from 1990 stage play by American playright John Guare: *Six Degrees of Separation*
- Play loosely based on 1967 small-world experiment by Stanley Milgrom suggesting random pairs of U.S. citizens were connected on average by a chain of six social acquaintances (people on a first-name basis).
- Pick any **film actor A**, then try to link this actor to Bacon via a chain of films.
- Actor set for first film in chain must include A, each successive film must include an actor from previous film, and final film must include Bacon among its actors.

Six Degrees of Kevin Bacon….

Example: (from Wikipedia, accessed 4/8/07)

https://en.wikipedia.org/wiki/Six_Degrees_of_Kevin_Bacon

- **Elvis Presley** was in *Change of Habit* (1969) with Edward Asner
- Edward Asner was in *JFK* (1991) with Kevin Bacon
- Therefore, Elvis Presley has a Bacon Number = 2.

2. NETWORKS…

What's the average distance between Kevin Bacon and all other actors?
(from Albert-Lázló Barabási, https://barabasi.com/book/network-science)

Kevin Bacon

No. of movies : 46 No. of actors : 1811
Average separation: 2.79

Is Kevin Bacon the most connected actor?

NO!

Rank	Name	Average distance	# of movies	# of links
1	Rod Steiger	2.537527	112	2562
2	Donald Pleasence	2.542376	180	2874
3	Martin Sheen	2.551210	136	3501
4	Christopher Lee	2.552497	201	2993
5	Robert Mitchum	2.557181	136	2905
6	Charlton Heston	2.566284	104	2552
7	Eddie Albert	2.567036	112	3333
8	Robert Vaughn	2.570193	126	2761
9	Donald Sutherland	2.577880	107	2865
10	John Gielgud	2.578980	122	2942
11	Anthony Quinn	2.579750	146	2978
12	James Earl Jones	2.584440	112	3787
…				
876	**Kevin Bacon**	**2.786981**	**46**	**1811**
…				