Elements of Dynamic Economic Modeling: Presentation and Analysis

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Abstract: The primary goal of these introductory notes is to promote the clear presentation and rigorous analysis of dynamic economic models, whether expressed in equation or agent-based form. A secondary goal is to promote the use of initial-value state space modeling with its regard for historical process, for cause leading to effect without the external imposition of global coordination constraints on agent actions. Economists who claim to respect individual rationality should not be doing for their modeled economic agents what in reality these agents must do for themselves.

Keywords: State space modeling, differential/difference equations, agent-based computational economics, presentation, analysis

JEL: A23; B4; C6

1 Overview

These notes on dynamic economic modeling are designed for self-study by graduate students of economics. The focus is on general presentation and analysis principles for dynamic economic models expressible by means of state space models in initial value form.[1]

The *state* of a modeled system at any given time is a characterization of system aspects deemed by the modeler to be relevant for a specified purpose. Typically the only aspects explicitly included in the state are aspects that can change over time; fixed aspects are suppressed for ease of notation. For an economic system, the state typically includes the asset holdings, information, and beliefs of economic entities such as firms, consumers, and government policy makers.

Hereafter, *state space model* will refer exclusively to a state space model in initial value form, that is, to a model that specifies how the state of a system changes over time, starting from a given state at an initial time t^0 [Aström and Murray, 2008, Ch. 2]. Roughly described, given any time $t \geq t^0$ and time increment $\Delta t > 0$, the state realized at time $t + \Delta t$ is postulated to be a function of the state realized at time t together with all inputs to the system between t and $t + \Delta t$, where the inputs can include controls and random event realizations. These functional relationships are conditioned on exogenously specified functional forms and parameter values.

State space models can be expressed in equation form or in agent-based form. In equation form, the models are represented either as continuous-time systems of ordinary differential equations or as discrete-time systems of difference equations. In agent-based form, the models are typically represented as software programs implementing successive discrete-time interactions among collections of persistent entities ("agents") in the system of interest.

State space modeling permits intertemporal planning and optimization as well as myopic decision-making. Modeled decision makers can form action plans and/or expectations for current and future time periods that take into account possible future states as long as these action plans and/or expectations are functions of the current state. Thus, modeled decision

[1] *Important Clarification.* These notes focus on the following theoretical question: How can dynamic systems be represented and analyzed using initial-value state space models? They do not address statistical smoothing, estimation, and/or forecasting issues, i.e., the approximate determination of past, current, and/or future system state values when system states are only imperfectly observable. Consequently, the initial-value state space models appearing in these notes are not augmented with measurement/observor equations postulating imperfect observability of system states.

makers can be as rational (or irrational) as real people.

Guidelines for the presentation of economic research supported by state space modeling are suggested in Section 2. The importance of variable classification is discussed in Section 3. Section 4 presents an illustrative continuous-time state space model in equation form, and Section 5 presents an illustrative discrete-time state space model in equation form that approximates this continuous-time model. Section 6 and Section 7 provide more detailed guidelines for the presentation and analysis of state space models in equation form. A general introduction to state space modeling in agent-based form is given in Section 8. Agent-based state space models specifically for economic study, referred to as *Agent-based Computational Economics (ACE)* models, are discussed in Section 9. The basic form of many ACE studies is outlined in Section 10, and key issues analyzed in ACE studies are discussed in Section 11. The final Section 12 briefly summarizes the ultimate goal of these notes.

Throughout these notes, pointers are given to on-line resources for more advanced discussion of topics. Extensive annotated pointers to additional materials on dynamic economic modeling in state space form, including specific forms of growth models (descriptive, optimal, overlapping-generations, dynamic stochastic general equilibrium, agent-based), expectation formation (adaptive, rational), and the constructive modeling of coordination processes for dynamic economic systems, can be found at the course website [Tesfatsion, 2016a].

2 General Presentation Considerations

The basic requirement for the effective presentation in written form of an economic study conducted by means of a state space model is to put yourself in the place of a potential reader. The first thing a reader will want to see is a brief but clear summary statement of your study's purpose. The second thing the reader will want to see is a brief but clear discussion clarifying the extent to which your study has new aspects relative to what has been studied before, and why these new aspects are important for the achievement of your study's purpose. This could constitute your Section 1.

Once you secure the reader's interest with this introductory overview, your next task is to explain to the reader why you are choosing to support your study's purpose by means of your model. You should first provide the reader with a big-picture overview of your model that establishes its relationship to your study's purpose. This overview should consist of a concise but clear verbal discussion, perhaps with an accompanying flow diagram indicating in simplified form the flow of activities at successive points in time (continuous-time modeling) or over successive time periods (discrete-time modeling). The objective here is to convey in general terms how your model captures aspects of a dynamic economic system important for your study's purpose. This could constitute your Section 2.

Once the reader has a big-picture understanding of your model, you need to provide a more detailed explanation of the structure of the model. As will be clarified in subsequent sections, this explanation should include:

(E1) a complete, consistent, and non-redundant specification of simultaneous and dynamic relationships detailing what system events occur at each time t (or during each time period t) and how the state of the system changes from one time point (or time period) t to the next;

(E2) a list of all variables and functions appearing in relationships (E1), together with their intended economic meaning;

(E3) a classification for each variable in (E2) as either endogenous (model determined) or exogenous (given from outside the model);

(E4) a unit of measurement for each variable in (E2);

(E5) a domain and range for each function in (E2);

(E6) admissibility restrictions on the exogenous variables and/or functions in (E2) to help ensure the empirical plausibility of the relationships (E1), both individually and as a whole, as well as the empirical plausibility of any resulting solution values for the endogenous variables.

The detailed explanations (E1)-(E6) of your model's structure would typically be given in a separate Section 3. It is useful to summarize explanations (E2) through (E6) in a nomenclature table for easy later reference.

You then need to explain carefully to the reader what type of analysis you intend to undertake with your model, in accordance with the purpose you have stated for your study. For example, do you intend to use the model to predict model outcomes under a particular empirically-determined specification of the exogenous variables and functions? In this case empirical input data will have to be given. Alternatively, do you intend to conduct sensitivity studies (analytical or computational) to determine model outcomes for a specified range of values for some subset of the exogenous variables? In this case you will need to explain to the reader the intended design for your sensitivity studies.

More precisely, to convey to the reader your *sensitivity design*,[2] you will first need to explain carefully to the reader which exogenous aspects of your model are *treatment factors*, in the sense that they will be systematically varied during your sensitivity studies, and which are *maintained factors* in the sense that they will be maintained at fixed specifications throughout your sensitivity studies. You will then need to report to the reader the particular range of specification configurations you plan to explore for your treatment factors, and the particular fixed specifications you plan to set for your maintained factors. If your model includes exogenous variables intended to represent realizations for random variables, you will also need to report the pseudo-random number seed values used in your sensitivity studies to generate multiple different runs for each studied treatment-factor configuration in order to analyze and control for random effects.[3]

After the form of your analysis is carefully explained to the reader, say in Section 4, the outcomes of your analysis would then typically be reported to the reader in a separate Section 5. The manner in which these outcomes are reported should be tailored to your

[2]Sensitivity design is sometimes referred to as experimental design, given the formal similarity between sensitivity studies and laboratory experimentation. However, some researchers argue that the term "experiment" should be reserved for a sensitivity study conducted on a natural system, not on a theoretical construct. The use of "experiment" is therefore avoided in these notes.

[3]See Tesfatsion [2016b] for a more extended discussion of sensitivity design for stochastic dynamic models.

study's purpose and form of analysis. For example, outcomes might be reported by means of verbal summaries, tables, charts, heat-maps, flow diagrams, phase diagrams, and/or various other types of figures.

The final section of your study should be a wrap-up section. Typically this section will start by summarizing the main accomplishments of your study as reported in previous sections. It will then identify interesting new questions or issues raised by the study that would be of interest to explore in subsequent studies.

3 Classification of Variables for State Space Models

A variable whose value is determined *outside* of a model is said to be *exogenous* relative to that model. A variable whose value is determined *within* a model is said to be *endogenous* relative to that model.

For a dynamic model in state space form, it is useful to highlight which particular exogenous and endogenous variables appear within the modeled relationships for each successive time (or time period) t. It is also useful to partition the endogenous variables appearing within time-t modeled relationships into those whose values are determined by these relationships and those whose values are determined by earlier modeled relationships. As will be seen in subsequent sections, this permits a slice-in-time representation of the modeled relationships at each successive time (or time period) that highlights their basic causal structure.

The following definitions clarify these variable distinctions. Although expressed for continuous-time state space models, they apply equally well for discrete-time state space models; simply replace "time t" with "time period t."

An exogenous variable appearing within the time-t modeled relationships for a state space model is said to be *time-t exogenous* relative to that model. Since exogenous variables appearing within modeled relationships for earlier times can be included among the exogenous variables appearing within modeled relationships for later times, this notational convention imposes no restrictions on model structure.

An endogenous variable appearing within the time-t modeled relationships for a state space model is said to be *time-t endogenous* (relative to that model) if its value is determined by means of these time-t modeled relationships. An endogenous variable appearing within the time-t modeled relationships for a state space model is said to be *time-t predetermined* (relative to that model) if its value is determined by means of modeled relationships for earlier times $s < t$.

The time-t predetermined variables for a state space model constitute the *time-t state variables* for this model, and the vector of these time-t state variables is referred to as the *time-t state*. For a state space model specified over times (or time periods) $t \geq t^0$, the state at the initial time t^0 needs to be exogenously given since there are no modeled relationships prior to this initial time.

4 Continuous-Time State Space Modeling: Equation Form

4.1 Model Description

A complete description for a continuous-time state space model in equation form over times $t \geq t^0$ consists of four parts: (i) a set of equations for each time t describing postulated functional relationships among a collection of variables; (ii) a complete list of variables and functional forms that identifies variable units and function domains and ranges; (iii) a classification of variables into time-t endogenous, time-t predetermined, and time-t exogenous; and (iv) admissibility restrictions on exogenous variables and/or functional forms to help ensure the empirical plausibility of equations and solutions.

An illustrative example of a continuous-time state space model is given below.

Continuous-time model equations for times $t \geq t^0$:

$$\text{Simultaneous equations:} \quad \mathbf{0} = H\big(\alpha(t), z(t), x(t)\big) \tag{1}$$

$$\text{Dynamic state equations:} \quad Dx(t) = S\big(\alpha(t), z(t), x(t)\big) \tag{2}$$

$$\text{Integral equations:} \quad x(t) = \int_{t^0}^{t} Dx(s)ds + x(t^0) \tag{3}$$

Variables and functional forms:

$\alpha(t) = \big(\alpha_1(t), \ldots, \alpha_k(t)\big) \in R^k$, for times $t \geq t^0$

$z(t) = \big(z_1(t), \ldots, z_m(t)\big) \in R^m$, for times $t \geq t^0$

$x(t) = \big(x_1(t), \ldots, x_n(t)\big) \in R^n$, for times $t \geq t^0$

$Dx(t) = \big(Dx_1(t), \ldots, Dx_n(t)\big) \in R^n$, for times $t \geq t^0$

$H{:}R^{k+m+n} \rightarrow R^m$

$S{:}R^{k+m+n} \rightarrow R^n$

Classification of variables:

Time-t endogenous variables for $t \geq t^0$: $Dx(t)$, $z(t)$

Time-t predetermined (state) variables for $t > t^0$: $x(t)$

Time-t exogenous variables for $t > t^0$: $\alpha(t)$

Time-t^0 exogenous variables: $x(t^0)$, $\alpha(t^0)$

Admissibility restrictions:

$x(t^0) \in X \subseteq R^n$

In the above illustrative model, the time-t endogenous variables $(z_1(t), \ldots, z_m(t))$ can include the actions of decision makers as well as other time-t system events. The time-t exogenous variables $(\alpha_1(t), \ldots, \alpha_k(t))$ can include both deterministic variables (e.g., parameter values) and stochastically generated variables (e.g., realizations for random shock terms).

Note that the model has $m+n$ equations (1) and (2) for the $m+n$ time-t endogenous variables $(z_1(t), \ldots, z_m(t))$ and $(Dx_1(t), \ldots, Dx_n(t))$, for each $t \geq t^0$, and n equations (3) for the n time-t predetermined variables $(x_1(t), \ldots, x_n(t))$ for each $t > t^0$. Consequently, the number of unknown (endogenous) variables for the model as a whole is equal to the number of equations.

As seen in this illustrative model, admissibility restrictions on deterministic exogenous variables, such as the components of the initial state $x(t^0)$, typically take the form of non-negativity constraints or other types of bounds. Admissibility restrictions can also be imposed on stochastically generated exogenous variables; for example, a modeler might require that random shock realizations approximate a sampling from a normal distribution. Admissibility restrictions on functional forms can include properties such as differentiability, monotonicity, and concavity.

4.2 Incorporation of Optimization Problems

As discussed and demonstrated in Sinitskaya and Tesfatsion [2015], state space modeling permits modeled decision makers to engage in intertemporal planning and optimization as well as myopic decision-making as long as each decision maker is *locally constructive*. This means that the decision process undertaken by each decision maker at each decision-time t can be expressed as a function of the state of that decision maker at time t, where this state includes the decision maker's physical state, knowledge state, and belief state at time t. In particular, any consideration of future events by the decision maker must take the form of *anticipations* regarding these future events, where these anticipations are determined as functions of the decision maker's current state.

If all modeled decision makers are locally constructive, their implemented decisions at each time t are functions of the time-t state. This functional dependence can be expressed either directly, in terms of state-conditioned decision rules, or indirectly in terms of state-conditioned binding necessary and/or sufficient conditions for optimization. In either case, the decision processes of the modeled decision makers are in state space form; that is, their time-t decisions as functions of the time-t state can be incorporated among the time-t simultaneous relationships of a state space model, such as relationships (1).

Economic models can (and typically do) include optimization problems for key types of decision makers, such as firms, consumers, and government entities. These optimization problems are often constrained to ensure the existence of unique solutions (or are assumed to have unique solutions) for which first-order necessary conditions are also sufficient. The solutions to these optimization problems are then represented by their binding first-order necessary conditions.

For simple growth models with utility-maximizing consumers and/or profit-maximizing firms, the binding first-order necessary conditions reduce to demand and supply functions for goods and services (Takayama [1985, Ch. 2], Tesfatsion [2016c]). For optimal growth models in calculus-of-variations form, expressed as intertemporal optimization problems for a welfare-maximizing social planner or an infinitely-lived utility-maximizing representative consumer, the binding first-order necessary conditions take the form of Euler-Lagrange equations constraining per-capita consumption and capital along an intertemporal solution path together with transversality conditions constraining the length of the planning horizon and the terminal per-capita capital stock (Takayama [1985, Ch. 5], Tesfatsion [2016d]).

For more general dynamic economic models incorporating consumer, firm, and/or social planner optimization problems, the binding first-order necessary conditions might take the form of binding Karush-Kuhn-Tucker conditions (Fletcher [1987], Takayama [1985, Ch. 1]), or of binding Hamiltonian conditions in accordance with variants of Pontryagin's optimal control theory [Takayama, 1985, Ch. 8]. Alternatively, they might take the form of recursive relationships among successive dynamic programming value functions in accordance with variants of Bellman's Principle of Optimality [Cooper, 2001; Powell, 2014; Violante, 2000].

It is important to note, however, that the necessary and/or sufficient conditions characterizing the solutions of intertemporal optimization problems can change over time as anticipations are replaced by realizations if decision makers are permitted to re-optimize their action plans as time proceeds. In particular, this re-optimization could result in *time-inconsistency* in the sense that the re-optimizing decision makers choose to deviate from their earlier determined action plans. The structural form of the state space model equations can then fundamentally change over time in ways that are difficult to specify *a priori*.[4]

[4] For this reason, many economic models either specify that decision makers engage in a single intertemporal optimization problem at an initial time, with no re-optimization permitted, or impose conditions on structural model aspects (e.g., the form of intertemporal discounting) such that time consistency is assured.

Moreover, (initial value) state space modeling rules out ahistorical representations in which the action plans and/or expectations of decision makers are externally coordinated by means of system restrictions requiring global solution methods; that is, solution methods that cannot be implemented, even in principle, by means of a *single forward pass* through the state space model relationships at successive points in time, starting from given conditions at an initial time point. An example of such a system restriction is a strong-form rational expectations requirement in the sense of Muth [1961]. As discussed in Tesfatsion [2016e, Sec. 5], economic models in which decision makers are constrained to have strong-form rational expectations require global fixed-point calculations for their solution.

Time inconsistency and the ahistorical nature of rational expectations assumptions are not addressed in these elementary notes. Introductory discussions of these topics can be found in Tesfatsion [2016e] and Tesfatsion [1986, 2016f].

4.3 Model Solutions

Consider the continuous-time state space model in equation form presented in Section 4.1. A *solution* for this model is a value for each endogenous variable determined as a function of exogenous variables, where these values satisfy the model equations starting from the exogenously given state $x(t^0)$ at the initial time t^0.

Since each time-t predetermined variable ($t > t^0$) is a time-s endogenous variable for some earlier time s, an equivalent definition is as follows: A *solution* for this model is a value for each time-t endogenous variable determined as a function of time-t exogenous variables and time-t predetermined (state) variables, where these values satisfy the time-t model equations for each time $t \geq t^0$, starting from the exogenously given state $x(t^0)$ at the initial time t^0.

In general, without further restrictions on model structure, there is no guarantee that a solution exists.[5] Even if a solution exists, there is no guarantee it is unique. Moreover, even

[5]The Cauchy-Peano Theorem demonstrates how difficult it is to find general conditions guaranteeing the existence of solutions for nonlinear systems of ordinary differential equations except locally, in an arbitrarily small neighborhood of the initial time. See Takayama [1985, Section 3.B] for a discussion of existence issues for such systems.

if a unique solution exists, there is no guarantee it can be derived in exact *closed form*, i.e., in the form of explicit functions mapping exogenous model elements into solution values for the endogenous variables. Rather, resort must be made to an approximate solution method [Judd, 1998; Miranda and Fackler, 2004; Kendrick et al., 2006].

Existence of solutions will not be addressed in these elementary notes. Instead, we outline the successive steps that could be taken, in principle, to derive a closed-form solution for the illustrative continuous-time state space model presented in Section 4.1 under the presumption that a derivable closed-form solution exists. First, for each time t, use equations (1) to solve for $z(t)$ as a function of $\alpha(t)$ and $x(t)$, say in the following form:

$$z(t) \;=\; h\big(\alpha(t), x(t)\big), \quad t \geq t^0 \; . \tag{4}$$

Next, substitute (4) into equations (2), thus eliminating z(t), which then gives Dx(t) as a function of $\alpha(t)$ and $x(t)$:

$$Dx(t) \;=\; S\big(\alpha(t), h\big(\alpha(t), x(t)\big), x(t)\big) \;\equiv\; f\big(\alpha(t), x(t)\big) \; , \quad t \geq t^0 \; . \tag{5}$$

Given equations (5), plus the exogenous initial state $x(t^0)$, use equations (3) to determine successive solution values for the time-t states $x(t)$, $t > t^0$, by successive integral calculations.

4.4 Remarks on the Need for the Integral Equations (3)

Suppose $Dx(s)$ in (3) is a continuous function of s over $s \geq t^0$. Then, using the fact that equations (3) hold for all $t \geq t^0$, it follows by the Fundamental Theorem of the Integral Calculus (found in any good calculus textbook) that

$$Dx(s) \;=\; \frac{\partial x(s)}{\partial t} \, , \quad s \geq t^0 \quad , \tag{6}$$

where $\partial x(s)/\partial t$ denotes the derivative[6] of the state x with respect to time, evaluated at the particular time point s. Consequently, equations (3) and (5) fully determine the motion of

[6] Only the right derivative of x is well-defined at $s = t^0$. See Section 4.5 for further discussion of state differentiability issues.

the state $x(t)$ over times $t \geq t^0$, conditional on the exogenous variables $\alpha(t)$ for $t \geq t^0$, the exogenous function $f(\cdot)$, and the exogenously given state $x(t^0)$ at the initial time t^0.

On the other hand, the crucial equations (6) expressing Dx as the time-derivative of x do not follow from model equations (1) and (2) alone. In particular, simply labeling a vector as "$Dx(t)$" in model equations (2) does not guarantee that this vector is indeed the time-t derivative of the state $x(t)$ in model equations (1). The relationship between $Dx(t)$ and $x(t)$ at each time t is entirely determined by the model structure. If the equations (3) are not included among the model equations, there is no reason to expect that $Dx(t)$ will be the time-derivative of $x(t)$ at any time $t \geq t^0$.

Many theoretical studies of continuous-time state space models do not explicitly include among the model equations the integral constraints (3) needed to ensure that Dx is indeed the time-derivative of x; these integral equations are implicitly assumed without comment. However, if a computer is being used to obtain an approximate numerical solution for the model, then all of the model's constraints (in appropriate approximate form) must be explicitly imposed on these variables. A computer will not automatically impose the integral constraints (3) that ensure Dx is the time-derivative of x if these integral constraints (in some appropriate approximate form) are not included in the coding.

4.5 Remarks on the Differentiability of the State

For many continuous-time state space models in equation form, it cannot be assured that solution values are continuous functions of time. In particular, for the illustrative model set out in Section 4.1, it cannot be assured that the solution values for $z(t)$, and $Dx(t)$ are continuous functions of t. This follows because the continuity of $z(t)$ and $Dx(t)$ would imply that these vectors, meant to be time-t endogenous, were in fact predetermined by their past realizations $z(s)$ and $Dx(s)$ for times $s < t$. In the latter case, the time-t model equations would be over-determined in the sense that there would be no degree of freedom left at time t to ensure that these equations could be satisfied.

Suppose $z(t)$ and $Dx(t)$ are only right continuous, meaning that $z(q)$ and $Dx(q)$ are only guaranteed to converge to $z(t)$ and $Dx(t)$, respectively, as long as $q \to t$ along a path for which $q > t$. This implies, in particular, that $Dx(s)$ in (3) can jump discontinuously at some points s as long as it satisfies

$$\lim_{q \to s,\ q > s} Dx(q) = Dx(s) \ . \tag{7}$$

In this case, equations (6) must be weakened to

$$Dx(s) \ = \ \frac{\partial x(s)}{\partial t}|_{+} \ = \ \lim_{q \to s,\ q > s} \Big[\frac{x(q) - x(s)}{q - s}\Big], \quad s \geq t^0 \ , \tag{8}$$

where the right-hand term in (8) defines the *right* time-derivative of x, evaluated at s. Intuitively, this means that, at each time $s \geq t^0$, $x(s)$ has a derivative approaching s "from the right" (i.e., from times $q > s$). However, at some times s' it could happen that $x(s')$ has a kink point ("sharp corner") or a discontinuity, implying that x is not differentiable at s'.

In the remainder of these notes it is assumed for simplicity of exposition that the ordinary time-derivative $\partial x(s)/\partial t$ exists at each time $s \geq t^0$.

4.6 Basic Causal System

Suppose the continuous-time state space model presented in Section 4.1 has a derivable closed-form solution. As explained in Section 4.3, this means that the model's time-t endogenous variables $z(t)$ and $Dx(t)$ can be solved for as explicit functions (4) and (5) of the time-t exogenous variables $\alpha(t)$ and the time-t state $x(t)$ for each $t \geq t^0$.

As in Section 4.3, let the functional relationships (4) be used to substitute out the endogenous variables $(z(t))_{t \geq t^0}$. Using (4) together with (5), the resulting reduced-form model, hereafter referred to as the model's *Basic Causal System (BCS)*, takes the following form:

BCS model equations for times $t \geq t^0$:

$$\text{Dynamic state equations:} \quad Dx(t) \ = \ f\big(\alpha(t), x(t)\big) \tag{9}$$

$$\text{Integral equations:} \quad x(t) \ = \ \int_{t^0}^{t} Dx(s)ds + x(t^0) \tag{10}$$

BCS variables and functional forms:

$\alpha(t) = \big(\alpha_1(t), \ldots, \alpha_k(t)\big) \in R^k$, for times $t \geq t^0$

$x(t) = \big(x_1(t), \ldots, x_n(t)\big) \in R^n$, for times $t \geq t^0$

$Dx(t) = \big(Dx_1(t), \ldots, Dx_n(t)\big) \in R^n$, for times $t \geq t^0$

$f{:}R^{k+n} \rightarrow R^n$

BCS classification of variables:

Time-t endogenous variables for $t \geq t^0$: $Dx(t)$

Time-t predetermined (state) variables for $t > t^0$: $x(t)$

Time-t exogenous variables for $t > t^0$: $\alpha(t)$

Time-t^0 exogenous variables: $x(t^0)$, $\alpha(t^0)$

BCS admissibility restrictions:

$x(t^0) \in X \subseteq R^n$

The dynamic properties of the BCS state solution $(x^*(t))_{t \geq t^0}$ as a function of the initial state $x(t^0)$ can be explored using a variety of methods, such as phase diagram techniques, frequency domain analysis, and computer explorations [Aström and Murray, 2008; Flake, 2013]. The dynamic properties of the BCS solution $(z^*(t))_{t \geq t^0}$ for the remaining endogenous variables can then be explored, in turn, using the functional relationships (4) with the state variables replaced by their solution values.

5 Discrete-Time State Space Modeling: Equation Form

Since computer implementations of state-space models are limited to finitely many calculations at finitely many time points, many researchers choose to represent real-time systems of interest directly in terms of discrete-time state space models in equation form. One danger in doing so is that researchers can lose sight of the degree to which this form of modeling induces ad hoc synchronization across modeled system events, which in turn can result in spurious regularities in model outputs.

As discussed in subsequent sections, agent-based modeling permits flexible asynchronization across modeled system events. Another approach that can be taken is to start with a continuous-time state space model in equation form and then approximate this model in discrete-time form, taking care to select empirically meaningful step sizes for the discretization of the timeline. This section illustrates the widely-used finite-difference method for the discrete-time approximation of a continuous-time state space model in equation form (Judd [1998, Ch. 10], Miranda and Fackler [2004, Ch. 5]).

Consider the continuous-time state space model presented in Section 4.1. Let $t \geq t^0$ be given, and let Δt denote a positive time increment whose length is measured in some given time unit (e.g., hours). Let the derivative $Dx(t)$ at time t be approximated by the following finite-difference expression:

$$Dx(t) \;\approx\; \frac{x(t+\Delta t) - x(t)}{\Delta t} \tag{11}$$

Substituting (11) in place of $Dx(t)$ in (2), and manipulating terms, one obtains

$$x(t+\Delta t) \;\approx\; S(\alpha(t), z(t), x(t)) \cdot \Delta t \;+\; x(t) \tag{12}$$

Note the important appearance of the time increment Δt on the right-hand side of (12). Even if Δt is set to one unit of time (e.g., one hour), it cannot be omitted from this equation. The expressions $x(t+\Delta t)$ and $x(t)$ are measured in x units; however, $S(\alpha(t), z(t), x(t))$ is measured in x per t, and it must be multiplied by Δt in order to obtain a commensurable expression in x units.

For each $j = 0, 1, \cdots$, let *period* j denote the time interval $\big[t^0 + j\Delta t,\, t^0 + (j+1)\Delta t\big)$. Also, define

$$F(\alpha_j, z_j, x_j) \;\equiv\; S(\alpha_j, z_j, x_j) \cdot \Delta t \;+\; x_j \tag{13}$$

where

$$\alpha_j \;=\; \alpha(t^0 + j\Delta t) \tag{14}$$

$$z_j = z(t^0 + j\Delta t) \tag{15}$$

$$x_j = x(t^0 + j\Delta t) \tag{16}$$

Then the original continuous-time state space model over times $t \geq t^0$ can be expressed in discrete-time approximate form over periods $j = 0, 1, \ldots$, as follows:

Discrete-time approximation equations for periods $j \geq 0$:

$$\text{Simultaneous equations:} \quad \mathbf{0} = H(\alpha_j, z_j, x_j) \tag{17}$$

$$\text{Dynamic state equations:} \quad x_{j+1} = F(\alpha_j, z_j, x_j) \tag{18}$$

Variables and functional forms:

$\alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,k}) \in R^k$, for periods $j \geq 0$

$z_j = (z_{j,1}, \ldots, z_{j,m}) \in R^m$, for periods $j \geq 0$

$x_j = (x_{j,1}, \ldots, x_{j,n}) \in R^n$, for periods $j \geq 0$

$H{:}R^{k+m+n} \rightarrow R^m$ and $F{:}R^{k+m+n} \rightarrow R^n$

Classification of variables:

Period-j endogenous variables for $j \geq 0$: x_{j+1}, z_j

Period-j predetermined (state) variables for $j > 0$: x_j

Period-j exogenous variables for $j > 0$: α_j

Period-0 exogenous variables: x_0, α_0

Admissibility restrictions:

$x_0 \in X \subseteq R^n$

By construction, the above discrete-time approximation converges to the original continuous-time state space model as the period-length Δt converges to 0. Convergence to a well-posed continuous-time state space model as the period length approaches zero is an important check on the basic logical consistency of a discrete-time state space model.

6 Presentation of Dynamic Economic Models in State Space Equation Form

Building on the materials in Sections 3 through 5, it is now possible to offer more specific guidelines for the presentation of dynamic economic models in state space equation form. Six key steps are outlined below.

Step 1: Provide a complete, consistent, and non-redundant set of model equations. By *complete* is meant that the model equations provide enough information to permit (in principle) the determination of a solution value for each endogenous variable. By *consistent* is meant that the model equations do not contradict each other. By *non-redundant* is meant that each model equation constitutes a new restriction on the endogenous variables, hence no model equation can be derived from the remaining model equations.

Step 2: Provide a classification of the variables appearing in your model equations. Specify the time-t endogenous variables, time-t predetermined variables, and time-t exogenous variables for each time (or time period) t. For example, if your model equations include a relationship of the form $y(t) = a(t)F(\ell(t), k(t))$, you might specify that $y(t)$, $\ell(t)$, and $k(t)$ are time-t endogenous variables and $a(t)$ is a time-t exogenous variable.

Step 3: Explain the intended economic meaning of each variable and function appearing in your model equations. For example, if your model equations include a relationship of the form $y(t) = a(t)F(\ell(t), k(t))$, as described in Step 2, you might explain that $y(t)$ denotes a particular firm's time-t output, $a(t)$ denotes the firm's time-t total factor productivity, $\ell(t)$ and $k(t)$ denote the firm's time-t labor services and time-t capital services, and $F: R^2_+ \rightarrow R$ denotes the firm's production function.

Step 4: Specify admissibility restrictions on exogenous variables and functions. Admissibility restrictions are conditions imposed on exogenous variables and functions to help ensure the empirical plausibility of a model's structure and solution values. For example, if your model equations include a particular firm's production relationship $y(t) = a(t)F(\ell(t), k(t))$, as described in Steps 2 and 3, you should impose empirically plausible restrictions on the form of the production function F, such as monotonicity, and you should require the exogenous total factor productivity variable $a(t)$ to be non-negative, or to be generated by a probability distribution with a non-negative support. Moreover, you should try to find additional empirically plausible restrictions to impose on your exogenous variables and/or functional forms to ensure that solution values for the time-t endogenous variables $y(t)$, $\ell(t)$, and $k(t)$ are non-negative.[7]

Step 5: Provide an economic interpretation for each of your model equations. For each model equation, explain whether it is an identity, an assumed form of behavior, an imposed coordination condition, or some other form of relationship. For example, if your model equations include a particular firm's production relationship $y(t) = a(t)F(\ell(t), k(t))$, as described in Steps 2 through 4, you should explain that this relationship guarantees that the firm's time-t production is efficient in the sense that its time-t output $y(t)$ is the maximum possible output obtainable from its time-t inputs $\ell(t)$ and $k(t)$, given $a(t)$.

Step 6: Provide a complete description of any modeled optimization problem. Explain what entity is undertaking the optimization, and provide complete careful descriptions of the objective function, decision variables, feasible decision set, and constraints for this optimization problem. Also, explain how this optimization problem is represented among your model equations. For example, if a relationship $y(t) = a(t)F(\ell(t), k(t))$ appearing among your model equations is derived as a binding first-necessary condition for a firm's profit maximization problem, carefully explain this derivation.

7 Analysis of Dynamic Economic Models in State Space Equation Form

The range of techniques that have been developed for the analysis of dynamic economic models in state space equation form would take multiple volumes to convey with clarity and

[7]Note that admissibility restrictions should not be imposed directly on the endogenous variables $y(t)$, $\ell(t)$, and $k(t)$, as augmentations to your complete, consistent, and non-redundant model equations, since this could over-determine your model and result in non-existence of solutions. That is, you would be imposing too many restrictions on too few endogenous variables.

care. The goal here is much more modest: namely, to give readers a summary description of seven types of issues that have traditionally been analyzed by economic researchers making use of such models.

Issue 1: Existence and Uniqueness of Solutions

(a) Does a model have at least one solution?

(b) If a solution exists, is it unique?

(c) If a unique solution exists, can it be derived in exact closed form? If not, can it be approximated to a degree of accuracy sufficient for the modeler's purpose?

Issue 2: Equilibrium Properties of Solutions

(a) Do markets clear?

(b) Are plans realized?

(c) Are expectations fulfilled?

(d) Do solutions display other types of coordination? For example, are solutions necessarily Nash equilibria or core equilibria?

(e) Can solutions display *emergent* coordination, i.e., coordination not directly and deliberately imposed *a priori* by the modeler through structural restrictions?

(f) Can solutions display coordination *failure* in some sense? For example, can a solution be a Pareto-dominated Nash equilibrium?

Issue 3: Dynamic Properties of Solutions

(a) Can a model be reduced to a *Basic Causal System (BCS)*?[8]

[8] In the older economic growth literature, a dynamic model represented as a system of ordinary differential equations is said to be *non-causal* if there exists at least one reachable state $x(t)$ at which $Dx(t)$ is not uniquely determined. For non-causal models, some form of external selection principle would be needed to resolve the ambiguity in $Dx(t)$ and hence in the continuation of the state solution. It follows that non-causal dynamic models do not have a BCS. For example, the classic continuous-time two-sector growth model developed by Uzawa [1963] is known to be non-causal under some parameter specifications.

(b) *If yes,* can the dynamic properties of the model solution be determined from the dynamic properties of the BCS solution?

(c) *If not,* can selection principles, phase diagrams, computer simulations, and/or other methods be used to understand the potential dynamic properties of the model solution?

Issue 4: Stability Properties of Solutions

(a) Does a solution $\boldsymbol{s^*}$ starting from a given state $x^*(t^0)$ at the initial time t^0 exhibit *global stability with respect to perturbations in its initial state*, in the sense that any solution $\boldsymbol{s'}$ starting from $x'(t^0) \neq x^*(t^0)$ at the initial time t^0 eventually converges to $\boldsymbol{s^*}$?

(b) Does a solution $\boldsymbol{s^*}$ starting from a given state $x^*(t^0)$ at the initial time t^0 exhibit *local stability with respect to perturbations in its initial state*, in the sense that any solution $\boldsymbol{s'}$ starting from a state $x'(t^0)$ sufficiently close to $x^*(t^0)$ at the initial time t^0 eventually converges to $\boldsymbol{s^*}$?

(c) Does there exist a *stable attractor* (e.g., a stable stationary point or limit cycle) for all solutions starting from a specified state set X at the initial time t^0, in the sense that all such solutions ultimately approach and remain near (or within) this attractor?

Issue 5: Optimality Properties of Solutions

(a) Are solutions *efficient*, in the sense that no physical resources are wasted?

(b) Are solutions *Pareto-efficient*, in the sense that there is no wastage of opportunity to improve the welfare of any one modeled human agent in a way that does not diminish the welfare of any other modeled human agent?

(c) Are solutions *socially optimal* in the sense of maximizing some meaningful measure of welfare for society as a whole?

Issue 6: Sensitivity of Solutions to Changes in Exogenous Conditions

(a) *Robustness analysis:* How do solutions vary in response to changes in parameters or functional forms?

(b) *Basin boundaries and attractors:* How do attractors and basins of attraction for solutions vary in response to changes in parameters or functional forms?

(c) *Scenario analysis:* How sensitive are solutions to changes in realizations for stochastically determined exogenous variables?

Issue 7: Empirical Validation

(a) *Input validation:* Are the exogenous inputs for the model (e.g., functional forms, random shock realizations, data-based parameter estimates, and/or parameter values imported from other studies) empirically meaningful and appropriate for the purpose at hand?

(b) *Process validation:* How well do the structural conditions, institutional arrangements, and human behaviors represented within the model reflect real-world aspects important for the purpose at hand?

(c) *Descriptive output validation:* How well are model-generated outputs able to capture the salient features of the sample data used for model identification? (in-sample validation)

(d) *Predictive output validation:* How well are model-generated outputs able to forecast distributions, or distribution moments, for sample data withheld from model identification or for new data acquired at a later time? (out-of-sample validation)

8 Discrete-Time State Space Modeling: Agent-Based Form

8.1 Overview of Agent-Based Modeling

An *agent-based model (ABM)* is a modeling of a system populated by agents whose successive interactions drive all system events over time.[9] The agent-based modeler begins

[9]Some of the materials in this section are adapted from Axelrod and Tesfatsion [2006], Borrill and Tesfatsion [2011], and Tesfatsion [2016g].

with assumptions about the agents and their potential interactions and then uses computer simulations to generate histories that reveal the dynamic consequences of these assumptions.

The agents represented in an ABM consist of entities of interest for the modeler's purpose. These entities can represent: (i) physical features such as buildings, geographical regions, and weather; (ii) institutions such as markets and legal systems; (iii) social groupings such as families and communities; (iv) biological lifeforms such as insects, forests, and crops; and (v) human decision makers such as workers, managers, and government regulators.

ABM researchers investigate how large-scale effects arise from the micro-level interactions of agents, starting from initial conditions, much as a biologist might study the dynamic properties of a culture in a petri dish. The first step is to construct a model of a system that is suitable for the purpose at hand. The second step is to specify the initial state of the modeled system, which consists of the initial state of each constituent agent. The final step is to permit the modeled system to change over time driven solely by agent interactions, with no further intervention from the modeler.

Agent-based modeling is well suited for the study of dynamic systems in which complexity arises from the interactions of natural and human systems [Tesfatsion, 2016h]. Multidisciplinary teams of ABM researchers can develop empirically-based frameworks capturing physical, institutional, biological, and social aspects of real-world systems salient for their specific research objectives without concern for analytical tractability.[10] Simplification is a considered choice (simple but not too simple) rather than an analytical necessity.

Agent-based modeling is also well-suited for the study of dynamic systems in which complexity arises from the interactions of decision-making agents [Tesfatsion, 2016i]. At one extreme, each decision-making agent in an ABM might have a simple if-then decision rule resulting in a relatively small range of individually expressible behaviors. However, just as the simple fixed rules of a chess game can produce an enormously large space of different games through player interactions, so too can the simple fixed decision rules of ABM agents

[10] An added benefit of multidisciplinary ABM research is that it keeps you humble because you are always working with people who know more than you do.

produce surprisingly intricate global system behaviors through agent interactions.

At the other extreme, each decision-making agent in an ABM might have a decision mode involving sophisticated data gathering and calculations. For example, an agent might engage in intertemporal optimization conditional on anticipated future states, where these anticipations reflect both individual learning based on own experiences and social learning based on social communication and observations.

In between these extremes lie many other possibilities. A decision-making agent might be a reactive learner who asks "if this event happens, what should I do?" A decision-making agent might be an anticipatory learner who asks "if I do this, what will happen?" A decision-making agent might be a strategic learner who asks "what should I do to get him to do that?" Or a decision-making agent might be a goal-directed learner who asks "what should I do differently in order to achieve this objective?"

Moreover, if agent birth and death are permitted in accordance with some form of fitness criterion, the composition of the agent population will evolve over time. In this case, even if each decision-making agent has a fixed decision mode unaffected by learning, the "ecology" of decision modes present in the population will evolve over time as well.

An ABM is typically implemented as a software program that determines the motion of the modeled system's state over successive time periods, starting from a user-specified initial state [Tesfatsion, 2016j]. Although any ABM can, in principle, be expressed in equation form, this representation would be extremely complicated. Instead, ABMs are usually motivated and explained by means of figures, verbal descriptions, Unified Modeling Language (UML) diagrams, flow diagrams, and/or pseudo-code expressing structural model aspects and the logical flow of agent processes and interactions over time. These communication aids should be accompanied by access to the software, itself, in either binary or source code form.

For example, in Tesfatsion et al. [2017] an agent-based software platform is developed for the study of watersheds as coupled natural and human systems. Figure 1 is displayed to convey to readers that the platform can be used to construct ABMs permitting the study of interactions among hydrology, climate, and strategic human decision-making in a watershed over time.

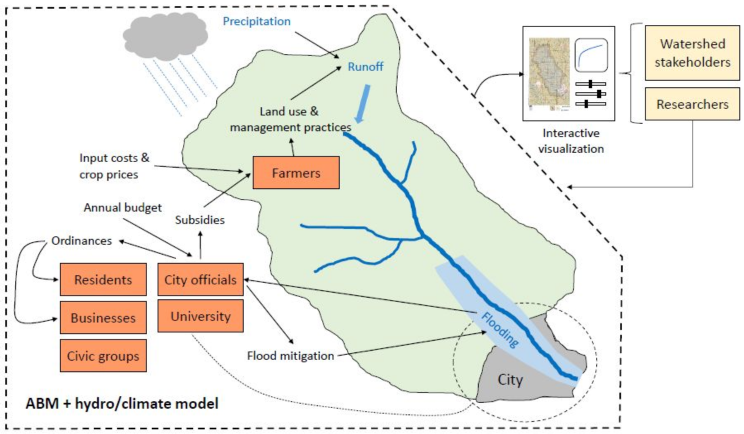

Figure 1: An agent-based software framework for the study of watersheds as coupled natural and human systems. Source: Tesfatsion et al. [2017]

A diagram highlighting key platform components is next presented and discussed. To demonstrate the capabilities of the platform, a sensitivity study is then reported for a test-case ABM capturing, in simplified form, the structural attributes of the Squaw Creek watershed in central Iowa. Attention is focused on the alignment of farmer and city manager welfare outcomes as a prerequisite for effective watershed governance. Verbal descriptions, tables, and figures are used to report key findings. Code and data for the test-case application are provided at an on-line repository.

8.2 Distinctive Attributes of Agent-Based Modeling

Listed below are seven distinctive attributes of many ABMs that, taken together, distinguish ABMs from standard state space models in equation form. These distinctive attributes reflect the fundamental goal of many agent-based modelers: namely, to be able to study real-world dynamic systems as historical processes unfolding through time, driven solely by their own internal dynamics.

(D1): The state of an agent at any given time consists of the agent's internal data, attributes, and methods at that time.

(D2): Each decision-making agent is locally constructive. That is, the decision-making process undertaken by a decision-making agent at any given time is entirely expressible as a function of the agent's state at that time.

(D3): Heterogeneity across agent states can change endogenously over time as a result of successive agent interactions.

(D4): The state of the modeled system at any given time consists of the collection of agent states at that time.

(D5): Coordination of agent interactions is not externally imposed by means of free-floating restrictions, i.e., restrictions not embodied within agent states.

(D6): Given initial agent states, all subsequent events in the modeled system are determined solely by agent interactions.

(D7): The role of the modeler is limited to the setting of initial agent states, and to the non-perturbational observation of model outcomes.

The key attributes (D5) through (D7) require more discussion. As seen in previous sections, (initial value) state space models in equation form rule out the imposition of intertemporal restrictions requiring global solution methods. However, as standardly formulated, these models do not encapsulate data, attributes, and methods into separate autonomous interacting agents. Thus, the time-t simultaneous equations can be used to impose *external* restrictions that coordinate system events at each time t. The important qualifier "external" means that the coordination restrictions do not arise from the data, attributes, and methods of the agents (physical, institutional, biological, and/or social) constituting the modeled system but instead represent the *a priori* beliefs or desires of the modeler regarding the way the modeled system should behave over time.

In contrast, the only chance an ABM researcher has to influence the dynamics of his modeled system is through his specification of initial agent states. All subsequent events in his modeled system are then determined solely by agent interactions; recall the analogy of a culture developing in a petri dish. Consequently, ABM researchers can hypothesize and test for the existence of equilibria requiring coordinated agent interactions at successive times, but they cannot externally impose these forms of equilibria on their modeled agents.

9 Agent-Based Computational Economics

Agent-Based Computational Economics (ACE) is the computational modeling of economic processes (including whole economies) as open-ended dynamic systems of interacting agents.[11] ACE is a specialization of agent-based modeling to the study of economic systems. Thus, the discussion of agent-based modeling in Section 8 applies equally well to ACE.

In this section illustrative ACE studies are used to discuss the potential usefulness of agent-based modeling specifically for the study of *economic* systems. The following three aspects are stressed.

1. *Agent heterogeneity:* The range of agent types that can be considered.

2. *Agent autonomy:* The degree to which agents are in control of their own behaviors, including the manner in which these behaviors change over time.

3. *Asynchronicity:* The degree to which system events can be modeled as occurring at asynchronous times to better match the flow of events in real-world systems of interest.

Aspect 1: Agent Heterogeneity

ACE models permit a wide range of agents to be included in a flexible, modular, plug-and-play manner. For example, Fig. 2 depicts a nested hierarchy of agents for an ACE study of a decentralized market economy with three basic agent types: namely, Decision-Making Agent (DMAgent), Asset, and Market.

[11] Annotated pointers to ACE tutorials, publications, demos, software, research groups, and research area sites are posted at the ACE website [Tesfatsion, 2016g]. For broad overviews, see Arthur [2015], Chen [2016], and Kirman [2011].

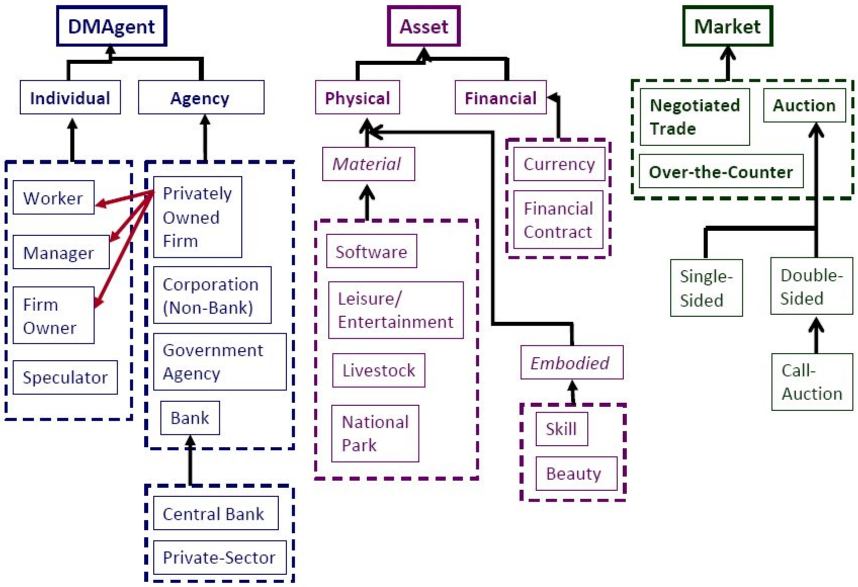

Figure 2: Partial hierarchy of agents for an ACE modeling of a decentralized market economy. Upward-pointing (black) arrows denote "is a" relationships and downward-pointing (red) arrows denote "has a" relationships. Source: Borrill and Tesfatsion [2011]

The state of each basic agent type consists of data, attributes, and methods common to all agents of its type. Each basic agent type can be used to instantiate (construct in software form) further agents of its type that are distinguished by the inclusion of additional data, attributes, and/or methods in their states. For example, as depicted in Fig. 2, the state of a DMAgent can be augmented in order to instantiate either "individual" agents representing individual decision makers or "agency" agents representing groups of decision makers who must arrive at collective decisions. The state of an "agency" agent can be augmented to instantiate "privately-owned firm" agents, "corporation (non-bank)" agents, "government agency" agents, and/or "bank" agents. And the state of a "bank" agent can be augmented to instantiate a "central bank" agent or "private-sector" bank agents.

Moreover, the agents in ACE models can include other agents as data members. For example, as depicted in Fig. 2, a privately-owned firm agent can include worker agents, manager agents, and a firm-owner agent among its data members. Thus, ACE models can be used to study the formation and evolution of hierarchical organizations.

Aspect 2: Agent Autonomy

ACE models allow agents to have more autonomy than is typically permitted for agents in standard economic models. This increased autonomy arises from *agent encapsulation*, i.e., from the ability of agents to hide their state from other agents. Encapsulation can make agents unpredictable to other agents.

More precisely, an ACE agent can self-activate and self-determine its actions on the basis of internal data, attributes, and methods that are hidden from other agents, or whose access is restricted to certain other agents. The agent's methods can include *pseudo-random number generators (PRNGs)*[12] permitting randomizations of behaviors and decisions. For example, the agent might use "coin flips" to decide among equally preferred actions or action delays, mixed strategies in game situations to avoid exploitable predictability, and mutations (random perturbations) of previous behaviors to explore new possibilities. Moreover, the agent's hidden data, attributes, and methods can change over time as it interacts within its world, making it difficult for other agents to predict its future behavior [Tesfatsion, 2016i].

For example, Fig. 3 depicts the flow of events in an ACE modeling of a goods market organized as a double auction, adapted from Nicolaisen et al. [2001]. The decision-making agents participating in this market consist of profit-seeking buyers and sellers with learning capabilities who submit strategic bids and asks for goods in successive trading rounds, plus an auctioneer who clears the market in each trading round by matching bids with asks.

Each bid takes the form of a descending schedule depicting a buyer's willingness to pay for each successive unit of good, and each ask takes the form of an ascending schedule depicting a seller's minimum acceptable sales price for each successive unit of good. The auctioneer identifies and clears all inframarginal units, that is, all units that can be sold because their maximum bid price exceeds their minimum ask price. The auctioneer sets the market prices for these inframarginal units using one of two pricing rules, either uniform or discriminatory.[13] Buyer-seller trades then take place at these market prices, resulting in a

[12] Alternatively, "true" random data from the real world can be directly streamed into an ACE model in place of PRNG-generated data, a possibility that raises interesting philosophical questions.

[13] Under the uniform pricing rule, the market price for each inframarginal unit is set at the midpoint

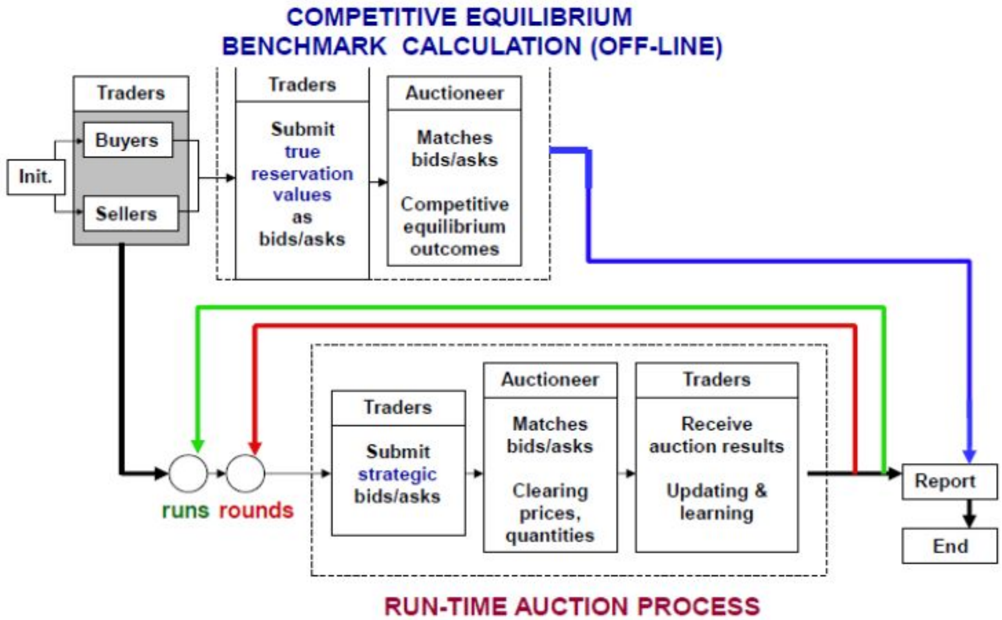

Figure 3: Flow diagram for an ACE double-auction market study.

(possibly zero) profit outcome for each trader. At the end of each trading round, each trader uses its learning method and its profit outcome to update its bid or ask for the next trading round. Thus, bid and ask prices can deviate from true reservation values.

Figure 3 indicates the form of the sensitivity design used by Nicolaisen et al. [2001] to study the performance of this double auction under different trader learning methods and market pricing rules. Multiple runs of the double auction were conducted for each learning/pricing treatment, where each run consisted of a specified number of trading rounds conditional on a specified seed for the PRNG used in the learning method. The average profit outcomes resulting for each treatment were then compared against competitive equilibrium profit outcomes calculated off-line for the same market pricing rule but using the traders' true reservation values as their bid and ask prices. These comparisons were used to determine the degree to which different treatments permitted buyers or sellers to achieve strategic market power, i.e., to learn bid/ask behaviors resulting in sustained profit advantages relative to the

between the maximum bid price and the minimum ask price for the marginally cleared inframarginal unit. Under the discriminatory pricing rule, the market price for each inframarginal unit is set at the midpoint between the maximum bid price and the minimum ask price for this particular unit.

profit outcomes they would achieve under competitive equilibrium.

The greater autonomy of decision-making agents in ACE models relative to standard economic models is also highlighted in an ACE study by Sinitskaya and Tesfatsion [2015]. In standard dynamic macroeconomic models, such as *Dynamic Stochastic General Equilibrium (DSGE)* models, the decisions of consumers and firms with intertemporal utility and profit objectives are coordinated by externally imposed equilibrium conditions. Typically these equilibrium conditions include the requirement that all consumers and firms exhibit strong-form rational expectations in the sense of Muth [1961].

In contrast, Sinitskaya and Tesfatsion transform a standard dynamic macroeconomic model into an ACE macroeconomic model by requiring consumers and firms with intertemporal utility and profit objectives to be locally constructive learning agents. Tested learning methods for the consumers and firms range from reactive reinforcement learning to adaptive dynamic programming. Sensitivity studies are used to explore macroeconomic performance under alternative learning treatments relative to a social planner benchmark solution.

Aspect 3: Asynchronicity

Careless treatments of timing issues in state space models can induce spurious regularities in model outcomes, which in turn can result in misleading or incorrect inferences [Borrill and Tesfatsion, 2011, Sec. 4.2]. The flexible modular architecture of ACE models gives researchers wide latitude with regard to the timing of agent actions. In principle, each agent could be permitted to proceed forward on its own action thread, taking actions in response to experienced events instead of in response to an externally clocked time.[14]

To date, however, despite the availability of agent-based toolkits permitting event-driven time advance [Meyer, 2015], most ACE researchers still use discrete-time models that impose a great deal of artificial synchronicity (simultaneity) on agent actions. When asynchronicity is permitted, it is often implemented mechanically by assuming either a fixed or randomized ordering for agent action updating. Another approach is to assume that certain agents take

[14]That is, ACE models can be structured as discrete-event state space models rather than discrete-time state space models. In a discrete-event state space model, the evolution of the state over time is driven entirely by the occurrence of discrete events.

actions (or not) at successive times in accordance with a binary probability distribution.[15]

In two recent ABM/ACE studies [Tesfatsion et al., 2017; Sinitskaya and Tesfatsion, 2015] a method is introduced that permits a more flexible timing of agent actions without resort to a full-blown discrete-event modeling. The timeline is divided into discrete time periods $t = [t, t+1)$, as in standard discrete-time state space models, and agent states are calculated as usual at each time t. However, each time period t is further subdivided into subperiods $t_1, \ldots, t_K$ during which either some random event is realized or an action is taken by some agent. The proximate effects of this realization or action on other system variables within the subperiod are then calculated and carried over to the next subperiod.

10 Basic Form of ACE Theoretical Studies

Many ACE researchers interested in theoretical issues follow a particular sequence of steps to conduct their studies. These steps are outlined below, followed by additional discussion.[16]

Step [1]: Model Construction. Construct an ACE model consisting of a collection of agents suitable for the study of the theoretical issue of interest.

Step [2]: Sensitivity Design. Determine appropriate treatment factors for the issue of interest, and specify which configurations of treatment factors will be tested. Designate all remaining aspects of the model as maintained factors.

Step [3]: Model Configuration. Set specific values for all maintained factors.

Step [4]: Sensitivity Testing. Conduct tests to implement the sensitivity design, and record outcomes of interest. If the model includes exogenous random elements, test each treatment-factor configuration multiple times for multiple possible random realizations to control for random effects.

[15] The (in)famous Calvo fairy pricing mechanism in many DSGE models is an example of this approach.

[16] The five steps below are adapted from Borrill and Tesfatsion [2011, Sec. 3.2]. See Tesfatsion et al. [2017] for a concrete illustration in which these five steps are sequentially implemented, with careful accompanying explanations, in order to develop an agent-based watershed management model.

Step [5]: Outcome Analysis. Report the outcomes resulting from the sensitivity tests, and analyze their implications for the issue of interest.

To illustrate Step [1], suppose an ACE researcher wishes to study the relationship between learning methods and stock price volatility. Suppose he hypothesizes that stock prices are less likely to exhibit volatility over time when traders have sophisticated learning capabilities, assuming price volatility is calculated using a particular measure PV.

To examine this issue, the researcher constructs an ACE model populated by K stock traders with bid/ask learning capabilities. The K traders engage in a sequence of stock trades over T successive trading periods. The stock trades takes place within a stock market organized as an automated double auction. In each trading period, stock market outcomes are perturbed by a random shock that affects trader confidence levels. The constituent agents for this ACE model thus include: the stock market (institution); the shock process (environment); and the K traders (decision makers).

For Step [2], the researcher decides his two treatment factors will be bid/ask learning methods and random shock realizations. He specifies admissible bid/ask learning methods ranging from simple decision rules to sophisticated intertemporal optimization based on sequentially updated expectations. He chooses N distinct learning treatments, where each learning treatment n denotes a particular configuration of admissible bid/ask learning methods to be assigned to the K traders over the T trading periods. He also specifies a finite number S of shock scenarios s with associated probabilities $\mathrm{Prob}(s)$, where each s is a distinct sequence of possible shock realizations over the T trading periods.

The researcher's sensitivity design then consists of N *price volatility tests*, where each test n is conditioned on a particular learning treatment n. For each test n the researcher will conduct S different runs, one run for each shock scenario s. The researcher designates all remaining aspects of his model to be maintained factors.

For Step [3], the researcher configures all maintained factors appearing within each agent's initial state. Specifically, he sets the stock market's initial data (number of participant traders), attributes (double auction market type), and methods (rules of operation). He sets the maintained attributes of the shock process (timing of shock realizations). He specifies each trader's initial data (information about other traders), fixed attributes (preferences), and initial values for time-varying attributes (money holdings, information, and beliefs). He also specifies each trader's maintained (non-learning) methods, such as information-collection methods and methods for submitting bids/asks into the stock market.

For Step [4], the researcher carries out each price volatility test n, as follows. The researcher sets the learning treatment n. He then conducts S different runs for this learning treatment, where a different shock scenario s is set for each run. For each run the modeled system is permitted to develop over the T trading periods, driven solely by agent interactions; and the researcher records the stock market price $\pi(t)_{s,n}$ that is observed in each trading period $t = 1, \ldots, T$. The researcher then uses these recorded price outcomes to calculate price volatility for test n in accordance with his price volatility measure PV.

For Step [5], the researcher analyzes the price volatility outcomes obtained from his N price volatility tests. He then uses this analysis to update his prior hypothesis into one or more refined posterior hypotheses.

The above five steps presume that ACE researchers conduct their studies without stakeholder participation. However, the flexible modular architecture of ACE models makes them particularly well suited for an alternative approach, called *Iterative Participatory Modeling (IPM)* [Tesfatsion, 2016k; Voinov and Bousquet, 2010].

The IPM approach envisions multidisciplinary researchers and stakeholders engaging together in an ongoing study of a real-world system of mutual interest. This ongoing study involves a repeated looping through four stages: field study and data analysis; role-playing games; agent-based model development; and intensive systematic sensitivity studies. For example, the agent-based watershed platform developed in Tesfatsion et al. [2017] and depicted in Fig. 1 is currently being used as an initial modeling platform for an IPM process whose purpose is improved local governance for the Squaw Creek watershed in central Iowa.

However, attempts to develop agent-based models through IPM processes have highlighted a key problem. Currently there is no consensus among agent-based researchers regarding how best to present agent-based models and model findings to external parties for evaluation and possible use. Some researchers advocate for standardized presentation protocols while others argue that protocols must be specialized for the purpose at hand.

Useful discussions of this issue can be found in [An, 2012; Grimm et al., 2010; Müller et al., 2014; Tesfatsion, 2016p]. See, also, Wallace et al. [2015] for an exceptionally thoughtful attempt to develop presentation and evaluation protocols for agent-based models designed for policy purposes.

11 Issues Analyzed in ACE Studies

All of the issues identified in Section 7 as traditional topics analyzed by economic researchers using state space models in equation form can also be analyzed using ACE models. This section focuses, instead, on four issues addressed in ACE studies that exploit the distinctive aspects of ACE modeling.[17]

One key issue stressed in ACE research is empirical understanding: Why have particular observed empirical regularities evolved and persisted despite the absence of top-down planning and control? Examples of such regularities include social norms, socially accepted monies, market protocols, business cycles, persistent wealth inequality, and the common adoption of technological innovations. ACE researchers seek possible explanations grounded in the repeated interactions of agents operating in realistically rendered ACE models. Specifically, they try to understand whether particular types of observed empirical regularities can be reliably generated within these ACE models [Epstein, 2006]. Agent-based macroeconomics and agent-based financial economics are particularly active ACE research areas along these lines; see Tesfatsion [2016l] and Tesfatsion [2016m].

A second key issue is normative understanding: How can ACE models be used as computational laboratories for the discovery of good economic designs? As discussed in Tesfatsion [2011], the typical ACE approach to normative economic design is akin to filling a bucket with water to determine if it leaks. An ACE model is constructed that captures the salient

[17] Materials for this section are adapted from Tesfatsion [2016g].

aspects of an economic system operating under the design. The ACE model is populated with decision-making agents with learning capabilities and allowed to develop over time. One concern is the extent to which the resulting model outcomes are efficient, fair, and orderly, despite attempts by decision-making agents to gain individual advantage through strategic behavior. A second concern is the possibility of adverse unintended consequences.

The double auction study depicted in Fig. 3 is an example of ACE research directed towards a normative objective: namely, good market design. Annotated pointers to additional ACE research along these lines can be found at the resource sites [Tesfatsion, 2016n,o].

A third key issue is qualitative insight and theory generation: How can ACE models be used to gain a better understanding of dynamic economic systems through a better understanding of their full range of potential behaviors over time (equilibria plus basins of attraction)? Such understanding would help to clarify not only why certain types of regularities have evolved and persisted but also why others have not.

A quintessential example of this third line of ACE research is the desire to resolve a long-standing issue raised by economists such as Adam Smith (1723-1790), Ludwig von Mises (1881-1973), John Maynard Keynes (1883-1946), Joseph Schumpeter (1883-1950), and Friedrich von Hayek (1899-1992): namely, what are the self-organizing capabilities of decentralized market economies? As evidenced by the extensive materials posted at the resource site Tesfatsion [2016l], this is a particularly active ACE research area.

A fourth key issue is methodological advancement: How best to provide ACE researchers with the methods and tools they need to undertake theoretical studies of dynamic economic systems through systematic sensitivity studies, and to examine the compatibility of sensitivity-generated theories with real-world data? As documented at the resource site Tesfatsion [2016k], ACE researchers are exploring a variety of ways to address this fourth issue ranging from careful considerations of methodological principles to the practical development of programming, visualization, and empirical validation tools.

12 Concluding Remarks

These introductory notes have covered general presentation and analysis principles for economic state space models, whether in equation or agent-based form. The primary intended readership is graduate students of economics, early in their careers, who plan to support their thesis research by some form of dynamic modeling effort. The ultimate purpose of these notes is to provide support and encouragement for young economists who dare to take a road less traveled in hopes of reaching a finer place.

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