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Course Site for Econ 502 (M.S. Macro Theory)
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Introduction to Rational Expectations:
https://www2.econ.iastate.edu/tesfatsi/REINTRO.pdf
Notes on the Lucas Critique, Time Inconsistency, and Related Issues:
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Introductory Notes on Rational Expectations

1 Overview

The theory of rational expectations (RE) is a collection of assumptions regarding the manner in which economic agents exploit available information to form their expectations. In its stronger forms, RE operates as a coordination device that permits the construction of a "representative agent" having "representative expectations."

Section 2 of these introductory notes on RE provides several alternative RE definitions (collections of assumptions), both weak and strong.[1] Section 3 uses two illustrative examples to demonstrate standard solution techniques for the determination of strong-form RE solutions. The statistical implications of RE are discussed in Section 4, followed by a discussion in Section 5 of RE solutions as fixed-point solutions. A brief but rigorous discussion of probability spaces and the meaning of conditional expectation, which are essential theoretical underpinnings for RE, can be found in a series of appendices.

[1]For simplicity, these RE definitions focus solely on first moments, i.e., on mean (expected) values. However, once these basic definitions are grasped, it is rather straightforward to modify these definitions to cover the more general case of "rational expectations" regarding the entire probability distribution governing a stochastic variable.

2 Defining Rational Expectations

Since the publication of the seminal article on rational expectations (RE) by John Muth (1961), a variety of definitions have been proposed for this concept. Although a definition cannot be wrong, some ways of defining things can be more fruitful than others.

Listed below are two possible definitions for RE, one weak and one strong, together with some assessments regarding the usefulness of these definitions for economic purposes. The first definition of RE is independent of the content of agents' information sets.

WEAK-FORM RE:

> Suppose $I_{t-1,i}$ denotes the information set available to an agent i at the beginning of period $t = [t, t+1)$. Let $E_{t-1,i}v_{t+k}$ denote agent i's subjective (personal) expectation formed at the beginning of period t regarding the value that a variable v will take on in some period $t+k$ with $k \geq 0$. Also, let $E[v_{t+k}|I_{t-1,i}]$ denote the objectively-true expectation for v_{t+k} conditional on $I_{t-1,i}$.[2] Then agent i is said to have a *weak-form rational expectation* for v_{t+k} at the beginning of period t if the following condition holds:
>
> $$E_{t-1,i}v_{t+k} \;=\; E[v_{t+k}|I_{t-1,i}] \;+\; \mu_{t,i}, \tag{1}$$
>
> where $\mu_{t,i}$ is a time-t forecasting error satisfying $E[\mu_{t,i}|I_{t-1,i}] = 0$. It follows from (1) that
>
> $$E_{t-1,i}v_{t+k} \;=\; v_{t+k} + \left(E[v_{t+k}|I_{t-1,i}] - v_{t+k} + \mu_{t,i}\right) \;\equiv\; v_{t+k} + \epsilon_{t,i} \;\;, \tag{2}$$
>
> where $E[\epsilon_{t,i}|I_{t-1,i}] = 0$.

It is critically important to observe that the definition given above for a weak-form rational expectation assumes the existence of objectively-true probability assessments for the possible

[2] See the appendix at the end of these notes for a discussion of the distinction between conditional and unconditional expectations, and for a more careful discussion of the meaning of conditional expectations.

values that the variable v_{t+k} can take on.[3] Moreover, as discussed more carefully in Appendix A.6, the conditioning information set I_{t-1} must be a collection of assertions that are true for a (possibly empty) subset A of possible worlds to which an objectively true probability $P(A)$ can be assigned.

How restrictive is the requirement that objectively true probabilities can be assigned to the possible future realizations of v_{t+k}? A difficulty is that this requirement typically rules out considerations of behavioral uncertainty.[4]

Consider, for example, a situation in which $E_{t-1,i}v_{t+k}$ represents the expectation of agent i at the beginning of period t regarding the future price v_{t+k} that a rival agent j will set for a competing good in period $t+k$ as the result of a strategic calculation. Agent i might at most be able to form a crude subjective expectation regarding this price because agent j in period t has not yet determined a price-setting strategy but rather is in the process of learning by trial and error (along with agent i) how to set his price in the context of a sequential market game.

Under the weak-form definition of RE, the concept of RE essentially reduces to an assumption that agents make optimal use of whatever information they have to form their expectations. This is viewed by many economists as a natural extension of the usual postulate in economic theory that – unless there is good empirical evidence to the contrary – one should presume that agents will always strive to bring their expectations into consistency with their information.[5] Weak-form RE is also in accordance with John Taylor's idea

[3]An *objectively-true probability assessment* for a system variable at a given time t is a probability assessment that is a valid attribute of this variable at time t, independently of agent beliefs. An example would be the probability that a particular weather event will occur at time t.

[4]An economy is said to be characterized by *behavioral (or strategic) uncertainty* if decision-making agents are uncertain about the behavior or other agents whose actions affect their own outcomes. Contrast this case with the case in which all uncertainty in the economy arises from exogenous stochastic shock terms governed by objectively-true probability distributions that can in principle be learned. The latter situation is the usual presumption of rational expectations theorists.

[5]Note, however, that weak-form RE is *stronger* than this relatively uncontroversial postulate in that it assumes agent expectations always *are* perfectly consistent with their information. That is, further use of their information cannot improve their forecasts; and what they observe does not contradict their information.

of an *economically rational expectation* in which agents' information sets are the result of cost-benefit calculations by the agents regarding how much information to obtain.

One can also define a stronger form of RE in the sense of John Muth (1961) that places a strong restriction on the content of an agent's information sets. This definition of RE guarantees the existence of "objectively-true" conditional expectations but at the cost of transforming RE into an incredibly strong concept when compared to the form of expectations that real economic agents could reasonably be supposed to have.

STRONG-FORM RE:

> An agent i in a model of an economy is said to have *strong-form RE* if agent i has weak-form RE and if, in addition, for each time t of his existence, agent i's information set $I_{t-1,i}$ at the beginning of period t contains all the information known to the modeler at the beginning of period t. This information includes:
>
> **(a)** Equations, variable classification, and admissibility conditions for the model, *including the actual decision rules used by any* ***other*** *agent (private or public) appearing in the model to generate their actions and/or expectations*;
>
> **(b)** The true values for all deterministic exogenous variables for the model;
>
> **(c)** All properties of the probability distributions governing stochastic exogenous variables that are known by the modeler at the beginning of period t;
>
> **(d)** Realized values for all endogenous variables and stochastic exogenous variables as observed by the modeler through the beginning of period t.

Strong-form RE has intuitive appeal in that it requires economists who construct theoretical models to assume that the agents they are attempting to model are as smart and as well informed about the economy as they are. Although the presumption that agents know *a priori* the actual decision rules used by each other agent is totally incredible, this type of assumption is familiar to economists from the definition of a Nash equilibrium: A configuration of decision rules currently adhered to by a collection of agents is a *Nash equilibrium* if, given the decision rules of all other agents, no individual agent has any incentive to deviate from his own decision rule. Strong-form RE can therefore be interpreted as an idealized Nash equilibrium benchmark for agents' expectations that agents may (or may not) eventually arrive at through some process of reasoning and/or learning.

In practice, theorists modeling economic systems generally assume that they have an extraordinary amount of information about the true workings of the economy. In consequence, under strong-form RE, economic agents are generally presumed to have a great deal more information than would actually be available to any econometrician who attempted to test these models against data. This point is stressed by Sargent (1993, Chapter 1).

Many economists are willing to assume that agents have weak-form RE, as a useful benchmark assumption consistent with the idea that agents are arbitrageurs who make optimal use of information. In contrast, many economists are uncomfortable with the more common assumption in the RE literature that agents have strong-form RE. Nevertheless, strong-form RE becomes more acceptable if it is viewed as a possible ideal limit point for the expectations of boundedly rational agents with limited information who engage in learning in successive time periods. Whether convergence to strong-form RE actually takes place can then be tested under alternative learning hypotheses. This is the approach taken by Sargent (1993).

Finally, it is interesting to consider the special case of perfect foresight RE.

PERFECT FORESIGHT RE:

An agent i is said to have *perfect foresight RE* if:

(a) Agent i has strong-form RE;

(b) No exogenous shock terms impinge on agent i's world, hence his expectations in each period t are correct. That is, $E_{t-1,i}v_{t+k} = v_{t+k}$ for all variables v.

It is important to point out here that perfect-foresight RE differs from the perfect foresight assumption often encountered in studies making use of "Walrasian general equilibrium" models. In the latter type of models, the decision problems of households and firms are linked only by prices and by dividend (profit) distributions from the firms to the households. Given prices and dividend distributions, each household perceives that it faces a budget-constrained utility maximization problem that is independent of the actions of any other household or firm; and, given prices, each firm perceives that it faces a technology-constrained profit maximization problem that is independent of the actions of any other household or firm. Perfect foresight in such contexts is the assumption that households and firms correctly foresee the market-clearing levels for these conditioning variables and solve their optimization problems conditional on these levels. However, since households and firms do not understand the true structure of the economy in which they reside – in particular, the fact that their quantity choices actually have an effect on market prices and dividend distributions – they do not have strong-form RE.

3 Illustrative Examples of Strong-Form RE

FIRST EXAMPLE:

Consider the following model of an economy over times $t \geq 1$:

Model Equations:

$$(1)^* \quad y_t = y_t^* + ap_t + bE_{t-1}p_t \ ; \tag{3}$$

$$(2)^* \quad p_t = m_t + \epsilon_t \ ; \tag{4}$$

$$(3)^* \quad E_t p_{t+1} = E[p_{t+1}|I_t] \ , \tag{5}$$

where:

- y_t^* notes the log of potential real GDP in period t.

- y_t denotes the log of actual real GDP in period t.
- p_t denotes the log of the general price level in period t.
- $E_t p_{t+1}$ denotes the subjective forward-looking expectation of a representative agent in period t regarding the price level in period $t + 1$.
- m_t denotes the log of the nominal money supply in period t.
- I_t denotes a period-t information set that is available to the representative agent at the end of period t (i.e., at the beginning of period t+1) and whose contents are consistent with strong-form RE. (*Note:* It is assumed the modeler is able to observe all past realized values for y_t, p_t, and ϵ_t.)

Classification of Variables and Admissibility Conditions:

> The three period-t endogenous variables are y_t, p_t, and $E_t p_{t+1}$. The only period-t predetermined variable is $E_{t-1}p_t$ for $t > 1$. The deterministic exogenous variables are: the exogenous constants a, b, $\{y_t^* \mid t = 1, 2, \ldots\}$, the monetary policy settings $\{m_t \mid t = 1, 2, \ldots\}$, and $E_0 p_1 = m_1$. The stochastic exogenous variables are the random error terms $\{\epsilon_t \mid t = 1, 2, \ldots\}$, which are assumed to constitute a serially independent process satisfying $E[\epsilon_t \mid I_{t-1}] = 0$ for all $t \geq 1$.

The model is incomplete as it stands. First, the "true conditional expectation" on the right hand side of equation (3)* needs to be determined in a manner consistent with strong-form RE. Second, this true conditional expectation — backdated to period $t - 1$ — needs to be substituted in for the subjective expectation $E_{t-1}p_t$ appearing in equation (1)*.

Once this true conditional expectation is substituted into model equation (1)*, the subsequent way in which the price level p_t for period t is actually determined by the model equations must, by construction, conform to this true conditional expectation in the sense that its objectively-true I_{t-1}-conditioned expectation must coincide with the expectation assumed for this price level in model equation (1)*. Similar arguments apply for the true conditional expectation of y_t given I_{t-1}. Roughly speaking, then, to complete this model with strong-form RE, we must solve a fixed point problem of the form $x_t = M(x_t)$ in each period t with $x_t = (E[y_t|I_{t-1}], E[p_t|I_{t-1}])$.[6]

For the simple linear model at hand, we can determine this strong-form RE in four steps, as follows.

Step 1: First, determine the specific types of information that must be included in the information set I_{t-1}. In accordance with the definition of strong-form RE, this information includes:

- Equations (1)*, (2)*, and (3)* plus classification of variables and admissibility conditions;
- True values for all of the deterministic exogenous variables: namely, a, b, $\{y_t^* \mid t = 1, 2, \ldots\}$, $\{m_t \mid t = 1, 2, \ldots\}$, and $E_0 p_1 = m_1$;
- All properties of the probability distribution governing the stochastic exogenous shock terms $\{\epsilon_t \mid t = 1, 2, \ldots\}$ that are known to the modeler at the beginning of period t;
- Values for all realized variables observed by the modeler through the beginning of period t, i.e. $\{p_{t-1}, p_{t-2}, \ldots, p_1; y_{t-1}, \ldots, y_1; \epsilon_{t-1}, \ldots, \epsilon_1\}$.

Step 2: Second, replace $E_{t-1}p_t$ in model equation (1)* by the as-yet undetermined expression for the strong-form RE: namely, $E[p_t|I_{t-1}]$.

Step 3: Third, take the I_{t-1}-conditional expectation of each side of the model equations (1)* and (2)* and use the specific information assumed to be contained in I_{t-1} to simplify

[6]See Section 5, below, for a more careful discussion of this point.

the form of these expressions.[7] These operations yield the following two equations in the two unknown strong-form RE terms $E[p_t|I_{t-1}]$ and $E[y_t|I_{t-1}]$.

$$E[y_t|I_{t-1}] \;=\; y_t^* + (a+b)E[p_t|I_{t-1}] \;\; ; \tag{6}$$

$$E[p_t|I_{t-1}] \;=\; m_t + E[\epsilon_t|I_{t-1}] \;=\; m_t + 0 \;\; . \tag{7}$$

Step 4: Fourth, solve for these two unknowns, obtaining

$$E[p_t|I_{t-1}] \;=\; m_t \tag{8}$$

and

$$E[y_t|I_{t-1}] \;=\; y_t^* + [a+b]m_t \;\; . \tag{9}$$

The strong-form RE solution (8) for expected price, together with model equations (1)* and (2)*, imply that the real GDP level y_t in each period t is given by

$$y_t \;=\; y_t^* + ap_t + bm_t \tag{10}$$

and the price level p_t in each period t is given by

$$p_t \;=\; m_t + \epsilon_t \;\; . \tag{11}$$

Equations (10) and (11) together imply that government can systematically affect the period-t real GDP level y_t by choice of its monetary policy instrument m_t if a is not equal to $-b$ in model equation (1)*. What happens when $a = -b$?

[7] More precisely, for model equation (3), use the linearity of the conditional expectations operator (see Appendix A.3) to argue that the conditional expectation of the right-hand side of (3) can be represented as the sum of three conditional expectations. Then use the fact that the exogenous variables y_t^*, a, and b in model equation (3) are assumed to be in the information set I_{t-1}, implying $E[y_t^*|I_{t-1}] = y_t^*$, $E[ap_t|I_{t-1}] = aE[p_t|I_{t-1}]$, and $E[bE[p_t|I_{t-1}]|I_{t-1}] = bE[E[p_t|I_{t-1}]|I_{t-1}] = bE[p_t|I_{t-1}]$. Apply similar reasoning to reduce down the conditional expectation for the right-hand side of model equation (4). Specifically, use the linearity of the conditional expectations operator and the fact that m_t and all of the modeler's presumptions about the distribution of the random shock term ϵ_t are assumed to be in the information set I_{t-1}.

When $a = -b$, model equation (1)* reduces to what is referred to as a *Lucas-Rapping supply curve*:

$$y_t \;=\; y_t^* + a[p_t - E_{t-1}p_t] \quad . \tag{12}$$

Equation (12) asserts that actual real GDP y_t deviates from potential real GDP y_t^* if and only if the expectation of the representative agent for the price p_t, formed at the end of period t-1, is not correct. Under strong-form RE, it follows from equations (11) and (12), together with the derived strong-form RE (8) for p_t, that

$$y_t \;=\; y_t^* + a\epsilon_t \quad . \tag{13}$$

Consequently, given *both* the Lucas-Rapping supply curve (12) *and* strong-form RE, y_t will deviate from y_t^* only by an unsystematic error term $a\epsilon_t$ that is independent of the government monetary policy variable m_t.

More generally, it follows directly from the form of the Lucas-Rapping supply curve (12) that government will not be able to systematically control y_t through the settings of its monetary policy variable m_t as long as the representative agent's subjective expectation $E_{t-1}p_t$ for the period-t price level p_t takes the form $E_{t-1}p_t = p_t$ - θ_t where the error term θ_t is independent of m_t. In this case, $y_t = y^* + a\theta_t$, regardless of m_t. Consequently, weaker expectational assumptions than strong-form RE produce monetary policy ineffectiveness in the presence of a Lucas-Rapping supply curve.

SECOND EXAMPLE:[8]

The following example illustrates a methodology called the *method of undetermined coefficients* that can sometimes be effectively used to solve for strong-form RE solutions. The basic idea is to postulate a parameterized form for the solution of the endogenous variables under strong-form RE and then try to determine what particular values the parameters must

[8] This example is adapted from Caplan (2000).

take on in order for strong-form RE to hold.[9]

Suppose an economy is described by the following six equations and classification of variables for a dynamic flexible-price IS-LM model with constant potential real GDP over periods $t \geq 1$.

Model Equations:

(1)*	(IS)	$y_t \;=\; -ar_t + u_t$;
(2)*	(LM)	$m_t - p_t \;=\; by_t - ci_t + v_t$;
(3)*	(Nominal and Real Interest Rates)	$i_t \;=\; r_t + E_t p_{t+1} - p_t$;
(4)*	(Lucas-Rapping Supply Curve)	$y_t \;=\; y^* + \alpha \cdot [p_t - E_{t-1}p_t]$;
(5)*	(Monetary Policy Rule)	$m_{t+1} \;=\; m_t + \phi_{t+1}$;
(6)*	(Strong-Form RE)	$E_t p_{t+1} \;=\; E[p_{t+1}\vert I_t]$,

where I_t denotes a period-$(t+1)$ predetermined information set that is available to the representative agent at the end of period t (i.e., at the beginning of time period t+1). The contents of I_t are assumed to be consistent with strong-form RE.

Classification of Variables and Admissibility Conditions:

All endogenous variables are in natural logarithms of their level values. The six period-t endogenous variables are as follows:

- y_t, which denotes the log of real GDP for period t;
- p_t, which denotes the log of the general price level for period t;
- m_{t+1}, which denotes the log of the nominal money supply for period $t+1$;

[9]Three problems can arise with this method. First, strong-form RE solutions taking the postulated parameterized form might not exist. Second, multiple solutions having the postulated parameterized form could exist. Third, strong-form RE solutions could exist that have a *different* form than postulated.

- r_t, which denotes the real interest rate for period t;
- i_t, which denotes the nominal interest rate for period t;
- $E_t p_{t+1}$, which denotes the subjective forward-looking expectation of a representative agent at time t regarding the log of the price level in period $t+1$.

The period-t predetermined variables are m_t and $E_{t-1}p_t$ for $t > 1$. The exogenous variables are: y^*, which denotes the log of potential real GDP; the random error terms u_t, v_t, and ϕ_t; the positive exogenous constants a, b, c, and α; an initial value $m_1 = m_0 + \phi_1$ for the period-1 money supply m_1, where m_0 is exogenously given, and an initial value for $E_0 p_1$. The random error terms are assumed to satisfy $E[u_t|I_{t-1}] = 0$, $E[v_t|I_{t-1}] = 0$, and $E[\phi_t|I_{t-1}] = 0$ for all $t \geq 1$.

The model equation (6)* is incomplete as it stands, in that the "true conditional expectation" on the right hand side needs to be determined in a manner consistent with strong-form RE. That is, given this expectation, the subsequent way in which the price level for period $t+1$ is actually determined by the model equations must conform to this expectation in the sense that the objectively-true I_t-conditioned expectation of the model-generated solution for the price level in period $t+1$ must coincide with the expectation assumed for this price level in model equation (6)*. Roughly speaking, then, to complete this model with strong-form RE, we must solve a fixed point problem of the form $f(x) = x$, where $x = E[p_{t+1}|I_t]$.

One possible approach to determining the needed expectational form on the right hand side of model equation (6)* is the *method of undetermined coefficients*. Conjecture a possible solution form for p_t as a parameterized function of other variables, where the parameter values are unknown. Then, determine values for these unknown parameters that ensure strong-form RE.

For example, assume for simplicity that $y^* = 0$. Combine model equations (1)* through

(4)* plus (6)* to obtain

$$p_t = \frac{1}{1+c}m_t + \frac{c}{1+c}E[p_{t+1}|I_t] - \beta\left[p_t - E[p_t|I_{t-1}]\right] + w_t \quad , \tag{14}$$

where

$$\beta = \alpha\left[\frac{b+c/a}{1+c}\right]; \quad w_t = \frac{1}{1+c}\left[\frac{c}{a}u_t - v_t\right] \quad . \tag{15}$$

To obtain equation (14), start with model equation (2)*, then use model equation (3)* to substitute out for i_t, model equation (1)* to substitute out for r_t, model equation (4)* to substitute out for y_t, and model equation (6)* to substitute out for the subjective price expectations.

Suppose it is conjectured that the solution for p_t takes the form

$$p_t = q_1 m_t + q_2 w_t + q_3 \phi_t \quad , t \geq 1 \quad , \tag{16}$$

where the coefficients q_1, q_2, and q_3 remain to be determined. Now apply the I_t-conditional expectations operator to each side of equation (16) bumped up one period to period $t+1$ and use the mean-zero property assumed for the random error terms together with the assumption that I_t is consistent with strong-form RE, to obtain

$$E[p_{t+1}|I_t] = q_1 E[m_{t+1}|I_t] \quad , t \geq 0 \quad . \tag{17}$$

Use model equation (5)*, together with the assumption that the contents of the information set I_t are consistent with the definition of strong-form RE (hence, in particular, $E[\phi_{t+1}|I_t] = 0$, and the value of the period-t predetermined variable m_t is in I_t), to conclude that $E[m_{t+1}|I_t] = m_t$, hence

$$E[p_{t+1}|I_t] = q_1 m_t \quad , t \geq 0 \quad . \tag{18}$$

Now consider equation (18) bumped down one period and use model equation (5)* bumped down one period to substitute $m_t - \phi_t$ in for m_{t-1}, thus obtaining

$$E[p_t|I_{t-1}] = q_1\left[m_t - \phi_t\right] \quad , t \geq 1 \quad . \tag{19}$$

Combining equations (16) and (19), one then has

$$p_t - E[p_t|I_{t-1}] \quad = \quad [q_1 + q_3]\,\phi_t + q_2 w_t \quad ,t \geq 1 \quad . \tag{20}$$

Now use equations (18) and (20) to substitute out for the expectations in the price equation (14) and combine terms. This gives

$$p_t \;=\; \left[\frac{1}{1+c} + \frac{c}{1+c}q_1\right] m_t + [1 - \beta q_2]\, w_t - \beta\,[q_1 + q_3]\,\phi_t \quad ,t \geq 1 \quad . \tag{21}$$

Notice that we now have two distinct equations — equations (16) and (21) — that express p_t as a linear function of m_t, w_t, and ϕ_t. To make these equations consistent, set the three coefficients in (16) equal to the three coefficients in (21). This yields three equations in the three unknowns q_1, q_2, and q_3. Solving for these q-values gives

$$q_1 \quad = \quad 1 \;\; ; \tag{22}$$

$$q_2 \quad = \quad \frac{1}{1+\beta} \;\; ; \tag{23}$$

$$q_3 \quad = \quad -\frac{\beta}{1+\beta} \;\; ; \tag{24}$$

It follows that *one* possible solution for p_t consistent with strong-form RE is

$$p_t \quad = \quad m_t + \frac{1}{1+\beta}w_t - \frac{\beta}{1+\beta}\phi_t \quad . \tag{25}$$

The corresponding strong-form RE for p_t, to be substituted in on the right hand side of model equation (6)*, is then found by taking the I_t-conditional expectation of each side of equation (25) bumped up one period. Using (5)*, and the information contained in I_t, this yields

$$E[p_{t+1}|I_t] \;=\; E[m_{t+1}|I_t] \;=\; m_t \;,\; t \geq 1 \quad . \tag{26}$$

Note, in particular, that the last equality in (26) follows from the fact that m_t is in the information set I_t available to a representative agent at the end of period t. Combining model equation (4)* (with $y^* = 0$) with (25) and (26), it follows that the solution for period-t real GDP consistent with strong-form RE is given by

$$y_t \;=\; \alpha \left[\left(\frac{1}{1+\beta} \right) \phi_t + \left(\frac{1}{1+\beta} \right) w_t \right] \quad . \tag{27}$$

Consequently, recalling that potential real GDP y^* equals 0 by assumption in this illustration, it follows from equation (27) that any deviations of actual real GDP y_t from potential real GDP $y^* = 0$ are entirely due to the random shocks ϕ_t to the money supply as well as to the random shocks u_t and v_t to the IS and LM curves. In particular, then, under strong-form RE, and assuming the "natural rate" form for aggregate supply in model equation (4)*, government has no ability through its monetary policy to *systematically* move actual real GDP away from potential real GDP.

An important caution is in order here. We have solved for one particular strong-form RE solution by conjecturing a solution form for the period-t price. We have by no means established that this is the *only* strong-form RE solution for the model at hand. Indeed, it is not. For example, it is easily seen that additional mean-zero error terms could be added to the conjectured form for p_t and the same type of derivation for the strong-form RE solution could then be carried out.

Also, as in the first example above, note that the monetary policy ineffectiveness result derived for this second example depends on *more* than just strong-form RE. In particular, it depends heavily on the "natural rate" assumption that aggregate supply in model equation (4)* takes the form of a Lucas-Rapping supply curve. Indeed, it is clear from equation (4)* that government will have no ability to systematically control real GDP y_t as long as the representative agent has an unbiased expectation for p_t, i.e., as long as $E_{t-1}p_t = p_t + \epsilon_t$ where the error term ϵ_t is a mean-zero random variable. Consequently, weaker expectational assumptions than strong-form RE would produce monetary policy ineffectiveness in the current model context. On the other hand, considering alternative model contexts, it is easy to show that strong-form RE alone is *not* sufficient in and of itself to ensure monetary policy ineffectiveness.

4 Statistical Implications of Expectations Modeling

The two examples in the previous section illustrate how the modeling of expectations can have a substantial effect on the predicted effectiveness of government policy. Using an extremely simple model for illustration, this section will demonstrate how the modeling of expectations can also have a substantial effect on the predicted statistical properties of key macro variables.

Consider the following simple model of an economy:

$$y_t = a + b \cdot E_{t-1}y_t + \epsilon_t \quad , t \geq 1 \quad . \tag{28}$$

The only period-t endogenous variable is period-t real GDP, y_t. It will be assumed that the solution value for y_t in each period t is observable at the end of period t. The terms a and b are assumed to be exogenously given constants satisfying $0 < a$ and $0 < b < 1$. The term ϵ_t is assumed to be an exogenous serially-independent stochastic shock term satisfying

$$E[\epsilon_t | I_{t-1}] \ = \ 0 \quad , \quad t \geq 1. \tag{29}$$

The period-t predetermined variable $E_{t-1}y_t$ is a "place-holder" for some specific modeling of the representative agent's subjective expectation for y_t based on the information set I_{t-1} available to the representative agent at the end of period t-1 (beginning of period t). It will be assumed that I_{t-1} contains all the information consistent with strong-form RE, which the representative agent might or might not optimally exploit. Consequently, I_{t-1} contains the model equation (28), the classification of variables for (28), all observations on past realized endogenous variables, all values for deterministic exogenous variables, and the nature of the probability distribution governing the stochastic shock terms ϵ_t, including property (29).

Case 1:

Suppose, first, that the subjective expectation in equation (28) takes the following simple adaptive expectations form:

$$E_{t-1}y_t \quad = \quad y_{t-1} \quad . \tag{30}$$

Substituting (30) into (28), and taking I_{t-1}-conditioned expectations of each side, one obtains

$$E[y_t|I_{t-1}] \quad = \quad E[a + by_{t-1} + \epsilon_t|I_{t-1}] \tag{31}$$

$$= \quad a + by_{t-1} \quad , \tag{32}$$

where the final equality in (32) follows from the fact that the values of a, b, and y_{t-1}, as well as the statistical properties of ϵ_t, are assumed to be included in the information set I_{t-1}.

Comparing (30) with (32), it is seen that the expectational error made by the representative agent in each period t is

$$E[y_t|I_{t-1}] - E_{t-1}y_t \quad = \quad a + [b-1]y_{t-1} \quad . \tag{33}$$

Since y_{t-1} is contained in I_{t-1}, this error is systematically correlated with the information contained in the representative agent's information set I_{t-1}. This indicates that the representative agent is not optimally exploiting this information to form his expectations.

Finally, substituting (30) back into (28), one obtains

$$y_t = a + b \cdot y_{t-1} + \epsilon_t \quad , t \geq 1 \quad . \tag{34}$$

Consequently, given the adaptive expectations form (30) for the representative agent's subjective expectation of y_t in period $t-1$, the resulting solution path (34) for real GDP y_t is such that successive y_t values are positively correlated – that is, y_t depends positively on y_{t-1}.

Case 2:

Now suppose, instead, that the representative agent forms his expectation for y_t in period $t-1$ in accordance with strong-form RE, that is,

$$E_{t-1}y_t \quad = \quad E[y_t|I_{t-1}] \quad . \tag{35}$$

In this case the y_t-generating process in (28) takes the form

$$y_t = a + b \cdot E[y_t|I_{t-1}] + \epsilon_t \quad , \quad t \geq 1. \tag{36}$$

Taking I_{t-1}-conditioned expectations of each side of (36), and using the assumption that I_{t-1} is a strong-form RE information set, it can be shown that the strong-form RE (35) has the specific analytic form

$$E[y_t|I_{t-1}] \quad = \quad a/[1-b] \quad . \tag{37}$$

Substituting (37) back into (36), it follows that

$$y_t = a + b \cdot a/[1-b] + \epsilon_t \quad , t \geq 1 \quad . \tag{38}$$

Consequently, if the representative agent's subjective expectation for y_t in period $t-1$ is given by the strong-form RE (37), the resulting solution path (38) for real GDP y_t is such that successive y_t values exhibit serial independence – that is, the realization of y_t is entirely independent of the realization of y_{t-1}.

5 RE Solutions as Fixed Point Solutions

Strong-form rational expectations solutions for linear economic systems can generally be represented as time-dated sequences of "fixed point problems" of the form $x = M(x)$.

For example, consider the following system of equations for each time $t \geq 1$, a generalization of the first illustrative model discussed in Section 3:

$$y_t \quad = \quad a \; + \; b \cdot E[y_t|I_{t-1}] \; + \; c \cdot E[p_t|I_{t-1}] \; + \; \epsilon_t \; ; \tag{39}$$

$$p_t \quad = \quad d \; + \; e \cdot E[y_t|I_{t-1}] \; + \; f \cdot E[p_t|I_{t-1}] \; + \; \phi_t \; , \tag{40}$$

where the stochastic error terms satisfy $E[\epsilon_t|I_{t-1}] = E[\phi_t|I_{t-1}] = 0$. Let the vectors x_t, z_t, μ_t and the function $g(\cdot)$ be defined as follows:

$$x_t \equiv (E[y_t|I_{t-1}], E[p_t|I_{t-1}]) \equiv g(I_{t-1}) \ ; \tag{41}$$

$$z_t \equiv (y_t, p_t)' \ ; \tag{42}$$

$$\mu_t \equiv (\epsilon_t, \phi_t)' \ . \tag{43}$$

Then equations (39) and (40) can be more compactly expressed in the form

$$z_t = M(x_t) + \mu_t = M(g(I_{t-1})) + \mu_t \ , \tag{44}$$

where the function $M{:}R^2 \rightarrow R^2$ denotes the deterministic portion of the right-hand sides of (39) and (40).

Now take the I_{t-1}-conditional expectation of each side of (44). The resulting expression has the form of a fixed point problem:

$$x_t = E[M(g(I_{t-1}))|I_{t-1}] + 0 = M(g(I_{t-1})) = M(x_t) \ . \tag{45}$$

Any solution $\bar{x}_t$ for (45) by construction gives an explicit rational expectations solution for the I_{t-1}-conditioned expectations for y_t and p_t in model (39)-(40).

One major problem for RE highlighted by the fixed point representation (45) is the possible nonuniqueness of solutions to (45). Nonuniqueness occurs for the linear model (39)-(40) if $a = d$, $b = e$, and $c = f$. More generally, nonuniqueness is a common occurrence when the model equations $M(x)$ are nonlinear functions of x. Nonuniqueness throws into question the "rationality" of RE solutions.

Suppose, for example, that the RE solution $x_t = (E[y_t|I_{t-1}], E[p_t|I_{t-1}])$ for the real GDP y_t and price level p_t of a model economy in period t satisfy a fixed point problem having the general form (45) and that two distinct solutions x_t' and x_t'' exist – that is, $M(x_t') = x_t'$ and $M(x_t'') = x_t''$. Thus, if all agents in the economy at the end of period $t-1$ *expect* x_t' for period t, then the objectively-true expectation for (y_t, p_t) conditional on I_{t-1} *will in fact* be x_t'. And if, instead, all agents in the economy at the end of period $t-1$ *expect* x_t'', then the objectively-true expectation for (y_t, p_t) conditonal on I_{t-1} *will in fact* be x''. What, then, constitutes a *rational* expectation for (y_t, p_t) at the end of period $t-1$?

This nonuniqueness issue is more carefully taken up in Tesfatsion (2017).

TECHNICAL APPENDIX

A.1 Definition of a Probability Space

A collection $\mathcal{F}$ of subsets of a space Ω is said to be a *σ-field* of Ω if $\mathcal{F}$ is closed under complementation, countable intersections, and countable unions. It follows from closure under complementation and countable unions that every σ-field of Ω contains both the entire space Ω and the complement of Ω given by the empty set $\emptyset$.

A function $P{:}\mathcal{F} \to [0,1]$ defined on a σ-field $\mathcal{F}$ of a space Ω is said to be a *probability measure* on $(\Omega, \mathcal{F})$ that assigns a *probability* $P(A)$ to each element A in $\mathcal{F}$ if P satisfies the following two properties:

- (unit normalization): $P(\Omega) = 1$;
- (countable additivity): Given any finite or countable collection $\{A_k\}$ of elements in $\mathcal{F}$ such that $A_{k''}$ is disjoint from $A_{k'}$ for $k' \neq k''$,

$$P(\bigcup_k A_k) = \sum_k P(A_k) \tag{46}$$

A triplet $(\Omega, \mathcal{F}, P)$ is said to be a *probability space* if:

- Ω is a space of points ω, called the *sample space* and *sample points.*
- $\mathcal{F}$ is a *σ-field* of subsets of Ω, called *events*;
- $P{:}\mathcal{F} \to [0,1]$ is a probability measure on $(\Omega, \mathcal{F})$.

A.2 Definition of a Random Variable and a Stochastic Process

A subset $E \subseteq R$ of the real line R is said to be *open* if for every $y \in E$ there exists some $\epsilon > 0$ (depending on y) such that the interval $(y-\epsilon, y+\epsilon)$ is contained in E. A subset $B \subseteq R$ is said to be a *Borel subset of* R if it is derivable from the open subsets of R by means of complementation, countable intersection, and countable union.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $Y\colon\Omega \rightarrow R$ is said to be a *random variable* on $(\Omega, \mathcal{F}, P)$ if, for every Borel subset $B \subseteq R$, the set $\{\omega \in \Omega \mid Y(\omega) \in B\}$ is an element of the σ-field $\mathcal{F}$. Given any index set T, a collection (Y_t) of random variables Y_t, $t \in T$, is said to be a *stochastic process* on $(\Omega, \mathcal{F}, P)$ if each Y_t is a random variable on $(\Omega, \mathcal{F}, P)$. In applications, the index set T is often taken to be a subset of the real line R, and the parameter t is commonly taken to represent time.

The following implication is noted for later purposes. Suppose $Y\colon\Omega \rightarrow R$ is a random variable on $(\Omega, \mathcal{F}, P)$. For any $y \in R$, the subset $B_y = \{r \in R \mid r < y\}$ is an open subset of R. Thus, the subset of Ω given by $A(y) = \{\omega \in \Omega \mid Y(\omega) \in B_y\} = \{\omega \in \Omega \mid Y(\omega) < y\}$ is an element of $\mathcal{F}$, and the probability of $A(y)$ is given by $P(A(y))$.

A.3 The Expected Value of a Random Variable

Let Y denote a random variable on a probability space $(\Omega, \mathcal{F}, P)$. As explained in Section A.2, for each $y \in R$ the set $A(y) = \{\omega \in \Omega \mid Y(\omega) < y\}$ is an element of $\mathcal{F}$ that has probability $\hat{P}(y) \equiv P(A(y))$. The function $\hat{P}\colon R \rightarrow [0, 1]$ is called the *(cumulative) distribution function* for Y. Assuming certain technical regularity conditions regarding the integrability of Y with respect to P,[10] the *(unconditional) expectation* of Y is given by

$$EY = \int_{\Omega} Y(\omega)P(d\omega) = \int_R y\hat{P}(dy) \quad , \tag{47}$$

where

$$\hat{P}(dy) \;\equiv\; P(\{\omega \in \Omega \mid Y(\omega) \in dy\}) \tag{48}$$

Given various additional regularity conditions (see Chung (2000)), the distribution function $\hat{P}$ can be expressed in terms of a continuous *probability density function* $f{:}R \to R$ as follows:

$$\hat{P}(y) = \int_{-\infty}^{y} f(z)dz \quad , \quad \text{for all } y \in R \quad . \tag{49}$$

In this case EY in (47) can alternatively be expressed as

$$EY = \int_R yf(y)dy \quad . \tag{50}$$

One way to think about EY is by means of the following frequency interpretation. Suppose N realizations $Y(\omega_n), n = 1, \ldots N$, could be obtained for the random variable Y corresponding to N possible sample points $\omega_n \in \Omega$, generated as N independent draws from the distribution P. Then the *average* value

$$\bar{Y}(N) \;=\; \frac{\sum_{n=1}^{N} Y(\omega_n)}{N} \tag{51}$$

obtained for Y would "almost surely" approach EY as the number N of sample points becomes arbitrarily large. *Almost surely (a.s.)* means that the collection of all sequences $(\omega_n)_{n=1}^{\infty}$ for which this convergence does not hold has P-probability zero.

Note from (47) that the expectation operator is a linear operator in the following sense. For any exogenously given real-valued constants, say 2 and 10, the expectation for $2Y + 10$

[10] See the appendix of Breiman (1992) for a discussion of these regularity conditions. Note that the "unconditional" expectation of Y is in fact conditional on the underlying probability space; that is, the expectation is taken with respect to this probability space. This dependence on the probability space is customarily suppressed.

is given by

$$E[2Y + 10] = 2EY + 10. \tag{52}$$

A.4 Conditional Expectation

The concept of a "conditional expectation" is actually quite subtle, and it is not easy to give an introductory treatment that is both clear and rigorous. For a rigorous treatment, see any basic graduate-level text on probability theory, such as Breiman (1992) or Chung (2000). For a more intuitive introduction limited to random variables with discrete range spaces, see Fellner (2000).

To give some idea of the form that a rigorous definition for conditional expectation would take, consider the following definitions expressed for different forms of conditioning events.

Let $\mathcal{B}$ denote the σ-field consisting of all Borel sets $B \subseteq R$. Let $Y{:}\Omega \rightarrow R$ and $X{:}\Omega \rightarrow R$ denote random variables on a probability space $(\Omega, \mathcal{F}, P)$, with $\mathrm{E}\,|Y| < \infty$. Also, let $\mathcal{F}(X) \subseteq \mathcal{F}$ denote the σ-field of all sets of the form $\{\omega \in \Omega \mid X(\omega) \in B\}$ for $B \in \mathcal{B}$.

Let $C \equiv \{\omega \in \Omega \mid X(\omega) \in B\}$ for some particular $B \in \mathcal{B}$ for which $P(C) > 0$. Then, given certain additional regularity conditions, the expectation of Y conditional on $\{X \in B\}$ can be expressed as

$$E[Y|X \in B] \;=\; E[Y|C] \;=\; \frac{\int_C Y(\omega)P(d\omega)}{P(C)} \quad . \tag{53}$$

Note that (53) is a real-valued function of $B \in \mathcal{B}$, or equivalently, a real-valued function of $C \in \mathcal{F}(X)$, subject to the restriction $P(X \in B) = P(C) > 0$.

One special case of (53) is when B is a singleton set $\{x\}$; the conditional expectation (53) is then typically expressed as $\mathrm{E}[Y|X = x]$, a real-valued function of x.[11] Another special

[11] More precisely, $\mathrm{E}[Y|X = x]$ is a random variable on $(R,\mathcal{B},\hat{P})$, assuming $P(X = x) > 0$ for all $x \in R$. However, many random variables X of interest have the property that $P(X = x) = 0$ for all $x \in R$, e.g., any random variable with a non-atomistic distribution function, such as a normally distributed random variable. Consequently, condition (53) does not provide a sufficiently general definition of conditional expectation for handling all conditioning events of interest.

case of (53) is when Y is the 0-1 indicator function for a set $A \in \mathcal{F}$. In this case definition (53) reduces to the well known Bayes' Rule formula

$$P(A \mid C) \;=\; \frac{P(A \bigcap C)}{P(C)} \quad . \tag{54}$$

More generally, $\mathrm{E}[Y|X]$ is any random variable on $(\Omega,\mathcal{F}(X),P)$ that satisfies

$$\int_A E[Y|X](\omega)P(d\omega) \;=\; \int_A Y(\omega)P(d\omega) \;\text{ for each } A \in \mathcal{F}(X) \quad . \tag{55}$$

Note that Y itself does not typically satisfy property (55) for $\mathrm{E}[Y|X]$ because Y is not necessarily measurable with respect to $\mathcal{F}(X)$. That is, for some Borel set $B \in \mathcal{B}$, the set $\{\omega \in \Omega \mid Y(\omega) \in B\}$ could fail to be an element of $\mathcal{F}(X) \subseteq \mathcal{F}$ even though it is necessarily an element of $\mathcal{F}$.

Finally, let $\mathcal{H}$ denote any σ-field satisfying $\mathcal{H} \subseteq \mathcal{F}$. Then the conditional expectation $\mathrm{E}[Y|\mathcal{H}]$ is defined to be any random variable on $(\Omega, \mathcal{H}, P)$ that satisfies

$$\int_H E[Y|\mathcal{H}](\omega)P(d\omega) \;=\; \int_H Y(\omega)P(d\omega) \;\text{ for each } H \in \mathcal{H} \quad . \tag{56}$$

It can be shown that any two versions of $\mathrm{E}[Y|\mathcal{H}]$ differ on a set of probability zero.

Finally, given random variables Y and X on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, the following important properties can be shown to hold [see, e.g., Breiman (1992)]:

- Suppose $EY^2 < \infty$. Let Z^o denote the a.s. unique best predictor of Y among the collection $\mathcal{Z}$ of all random variables Z on $(\Omega,\mathcal{F}(X),P)$, in the sense that $Z^o \in \mathcal{Z}$ minimizes $E[Y - Z]^2$ over $Z \in \mathcal{Z}$. Then $Z^o(\omega) = \mathrm{E}[Y|X](\omega)$ a.s.

- If $\mathcal{F}(Y) \subseteq \mathcal{F}(X)$, then $\mathrm{E}[Y|X](\omega) = Y(\omega)$ a.s.

- If Y and X are independent, then $\mathrm{E}[Y|X](\omega) = \mathrm{E}Y$ a.s.

- If $\mathcal{D}$ and $\mathcal{H}$ are σ-fields, and $\mathcal{D} \subseteq \mathcal{H} \subseteq \mathcal{F}$, then $\mathrm{E}\Big[\ \mathrm{E}[Y|\mathcal{H}] \mid \mathcal{D}\Big](\omega) = E[Y|\mathcal{D}](\omega)$ a.s.

- If $\mathcal{H}$ is a σ-field, and $\mathcal{H} \subseteq \mathcal{F}$, then $\mathrm{E}\Big[\ \mathrm{E}[Y|\mathcal{H}]\ \Big] = EY$.

In the statement of the above properties, "a.s. (almost surely)" means the property holds for all $\omega \in \Omega$ except for a set $A \in \mathcal{F}$ with $P(A) = 0$.

Finally, it is important to note ways of representing conditional expectations at different levels of specificity. Let $Y{:}\Omega \rightarrow R$ and $X{:}\Omega \rightarrow R$ denote random variables on a probability space $(\Omega, \mathcal{F}, P)$, with $\mathrm{E}\,|Y| < \infty$. The expression $\mathrm{E}[Y|X = x] \equiv h(x)$ defined for any $x \in R$ represents a conditional expectation of Y given $X = x$ as a function $h{:}R \rightarrow R$. In particular, then, an expression such as $\mathrm{E}[Y|X = 2]$ represents a conditional expectation of Y given $X = 2$ as a specific numerical value in R. Finally, the expression $E[Y|X]$ represents the conditional expectation of Y given X as a random variable Z on $(\Omega,\mathcal{F}(X),P)$ of the form $Z{:}\Omega \rightarrow R$ with $Z(\omega) = E[Y|X](\omega)$; cf. (55).

A.5 Unconditional vs. Conditional Expectation: Numerical Example

Consider a system described by the following three equations

$$y_1 = y_0 + \epsilon_1 \ ; \tag{57}$$

$$y_2 = y_1 + \epsilon_2 \ ; \tag{58}$$

$$y_3 = y_2 + \epsilon_3 \ , \tag{59}$$

where the ϵ_i terms are shock terms. Suppose the set Ω^* of all possible states of the world ω for this system is given by

$$\Omega^* = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\} \ , \tag{60}$$

where:

$$\omega_1 = (y_0 = 2, \epsilon_1 = -1; \epsilon_2 = -1; \epsilon_3 = -1) \ ; \tag{61}$$

$$\omega_2 = (y_0 = 2, \epsilon_1 = +1; \epsilon_2 = -1; \epsilon_3 = -1) \ ; \tag{62}$$

$$\omega_3 = (y_0 = 2, \epsilon_1 = -1; \epsilon_2 = +1; \epsilon_3 = -1) \;\; ; \tag{63}$$

$$\omega_4 = (y_0 = 2, \epsilon_1 = +1; \epsilon_2 = +1; \epsilon_3 = -1) \;\; ; \tag{64}$$

$$\omega_5 = (y_0 = 2, \epsilon_1 = -1; \epsilon_2 = -1; \epsilon_3 = +1) \;\; ; \tag{65}$$

$$\omega_6 = (y_0 = 2, \epsilon_1 = +1; \epsilon_2 = -1; \epsilon_3 = +1) \;\; ; \tag{66}$$

$$\omega_7 = (y_0 = 2, \epsilon_1 = -1; \epsilon_2 = +1; \epsilon_3 = +1) \;\; ; \tag{67}$$

$$\omega_8 = (y_0 = 2, \epsilon_1 = +1; \epsilon_2 = +1; \epsilon_3 = +1) \;\; . \tag{68}$$

Let $\mathcal{F}^*$ denote the collection of all subsets of Ω^*, including the empty set ϕ. Also, define a probability measure P^*:$\mathcal{F}^* \rightarrow$ [0,1] by $P^*(\omega_s) = 1/8$, $s = 1, \ldots 8$, and

$$P^*(A \cup B) \;=\; P^*(A) + P^*(B) - P^*(A \cap B) \tag{69}$$

for each A and B in $\mathcal{F}^*$.

Given the probability space $(\Omega^*, \mathcal{F}^*, P^*)$, *marginal* probability distributions can be separately derived for each of the variables $(y_0, y_1, y_2, y_3, \epsilon_1, \epsilon_2, \epsilon_3)$. For example, it follows from (61) through (68) that $P^*(y_0 = 2) = P^*(\omega_1, \ldots, \omega_8) = P^*(\Omega^*) = 1$. Similarly, if follows from (61) through (68) that $P^*(\epsilon_i =$ -1) $= P^*(\epsilon_i = 1) = 1/2$ for $i = 1, 2, 3$, which implies E$[\epsilon_i] =$ 0 for $i = 1, 2, 3$.

Now let $A = \{\omega_1, \omega_5\}$, $B = \{\omega_2, \omega_3, \omega_6, \omega_7\}$, and $C = \{\omega_4, \omega_8\}$. Then the following probability distribution can be deduced for y_2:

$$P^*(y_2 \;=\; 0) \;=\; P^*(A) \;=\; 1/4; \tag{70}$$

$$P^*(y_2 \;=\; 2) \;=\; P^*(B) \;=\; 1/2; \tag{71}$$

$$P^*(y_2 \;=\; 4) \;=\; P^*(C) \;=\; 1/4; \tag{72}$$

$$P^*(y_2 = x) \;=\; P^*(\phi) \;=\; 0 \quad \text{for } x \neq 0, 2, \text{ or } 4. \tag{73}$$

Probability distributions can similarly be derived for y_1 and y_3. Finally, applying formula

(53) from Section A.1, note for later use that

$$
\begin{aligned}
E[\epsilon_3|C] &= \frac{(P^*(\omega_4)\cdot[-1] + P^*(\omega_8)\cdot[+1])}{P^*(C)} && (74)\\
&= \frac{-1/8 + 1/8}{1/4} && (75)\\
&= 0 \ . && (76)
\end{aligned}
$$

Similar calculations show that $E[\epsilon_1|C] = E[\epsilon_2|C] = 1$.

Now consider the following expectations for y_3, the first one conditional and the second one unconditional.

$$
\begin{aligned}
E[y_3|y_2 = 4] &= E[y_3|C]\\
&= E[y_2 + \epsilon_3|C]\\
&= E[y_2|C] + E[\epsilon_3|C]\\
&= 4 + 0 && (77)\\
&= 4 && (78)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
E[y_3] &= E[y_0] + E[\epsilon_1] + E[\epsilon_2] + E[\epsilon_3]\\
&= 2 \ + \ 0 \ + \ 0 \ + \ 0 && (79)\\
&= 2 \ . && (80)
\end{aligned}
$$

Clearly (78) differs from (80) in that (78) makes use of the fact, derivable from the conditioning information set C, that $y_2 = 4$.

A.6 Information Sets as Conditioning Subsets of Ω

Suppose the possible states-of-the-world for a decision-making agent at some time t are described by a probability space $(\Omega, \mathcal{F}, P)$, and suppose I_{t-1} denotes the information available to this agent at time t. The information appearing in I_{t-1} could be very meager; for example, it could consist only of the past realized values for a small number of endogenous variables. Alternatively, it could be relatively extensive, containing detailed information about the physical and institutional aspects of the agent's world in addition to past realized values for endogenous variables and exogenous shock terms.

Let I_{t-1} be identified with a subset of Ω, as follows:

$$I_{t-1} \equiv \{\omega \in \Omega \mid I_{t-1} \text{ is true for } \omega\} \tag{81}$$

Assuming $I_{t-1} \in \mathcal{F}$, the agent's rational expectation regarding a period-t random variable Y_t, conditional on the information set I_{t-1}, can then be defined as in Appendix A.4 to be $E[Y_t|I_{t-1}]$. As discussed more carefully in Appendix A.4, this means that $E[Y_t|I_{t-1}]$ is a "best possible predictor" (in a mean squared error sense) of the value that Y_t will take on in period t, conditional on knowing the information in I_{t-1} and only this information.

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