

WELFARE IMPLICATIONS OF NET SOCIAL SECURITY WEALTH

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Real net social wealth (NSSW), the real present value of social security benefits received minus social security taxes paid, is frequently used as a direct proxy measure for the impact of a social security system on generation welfare. The present paper establishes to the contrary, for a class of overlapping generation economies, that NSSW can be simultaneously *negatively* correlated with welfare for every agent in every generation. More generally, the paper determines the extent to which social security is needed in these economies to ensure social optimality, and investigates the proper subset of economies for which NSSW and generation welfare exhibit positive correlation.

1. Introduction

Following Feldstein (1974), the real net social security wealth (*NSSW*) accruing to any given generation is defined to be the real present value of lifetime social security benefits received by that generation minus the real present value of lifetime social security taxes paid by that generation. Numerous studies have implicitly or explicitly interpreted *NSSW* as a direct proxy measure for the impact of a social security system on generation welfare. Zero *NSSW* is considered to be the equitable net benefit position for each generation. Any generation receiving positive *NSSW* is receiving an unearned entitlement, and hence a net welfare gain. See, for example, Boskin et al. (1980), Burkhauser and Turner (1978), Derthick (1979), Kotlikoff (1979), Parsons and Munro (1978), and Pellachio (1979).

Positive correlation between *NSSW* and generation welfare clearly holds for partial equilibrium life-cycle models with price-taking agents constrained only by the present value of their lifetime disposable incomes. However, in models with liquidity constraints and endogenous prices, the relationship between *NSSW* and generation welfare is less obvious. The timing of income receipts and disbursements then becomes potentially critical for welfare, and *NSSW* becomes an endogenous variable along with other real income flows.

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Even if government manages to engineer an increase in *NSSW* for any given generation, the shape of the total real income profile of that generation will be altered beyond its control both directly, through taxes and benefits, and indirectly, through price effects. The end result could be a net decrease in welfare for that generation.

The present paper investigates the relationship between *NSSW* and generation welfare in the context of a stationary pure-exchange¹ overlapping generations model with three-period-lived agents. A balanced budget government imposes an income tax T on the heterogeneous real income endowments of young and middle-aged agents, and also distributes nominal social security benefits NS to old agents, who have no real income endowment.² Liquid assets are represented by a fixed stock of fiat money issued by government, and illiquid assets are represented by the opportunity for young agents to buy or sell two-period bonds (voluntary retirement annuities) in a nominal net aggregate amount \bar{B} determined by government policy, $-\infty < \bar{B} < +\infty$.³ Young agents select feasible consumption, money, and bond profiles to maximize the log-linear utility⁴ of their lifetime consumption, $u(c) = \log(c^1) + \alpha \log(c^2) + \beta \log(c^3)$, for arbitrary positive taste parameters α and β . Prices and interest rates are endogenously determined via market clearing conditions.

It is shown for this class of economies that *NSSW* and generation welfare can indeed be simultaneously negatively correlated for every agent in every generation under economically plausible conditions. For example, suppose middle-aged agents are relatively better endowed (more productive) than young agents, and can provide for their old age with a surplus of income to

¹Previous studies, extensively surveyed by Mitchell and Fields (1982), have established the complexity of the relation between social security and endogenous labor supply. For clarity, the present initial study avoids this issue.

²Alternatively, in place of real income endowments, one can interpret young and middle-aged agents as having heterogeneous labor skills which they supply inelastically in return for real wage incomes. Old agents are then interpreted as retired workers who do not market labor skills.

³According to Derthick (1979, p. 289), early plans for the social security system included a provision that government provide for the sale of voluntary retirement annuities. The provision was later deleted on the grounds that it could be a threat to the private insurance industry. In the present context, $\bar{B} = 0$ corresponds to the case of a completely private annuities market, with government acting solely as a clearing house. The private demands and supplies for annuities by agent types $k = 1, 2$ must then sum to zero in each period in order for equilibrium to hold, but the individual agent demands and supplies for annuities will typically differ from zero. In general, depending on the size and sign of \bar{B} and the liquidity preferences of young and middle-aged agents, the economy as a whole can be either 'classical' (net debtor) or 'Samuelson' (net creditor) in the sense of Gale (1973).

⁴The long-run strategy underlying this paper is to investigate the relationship between social security and generation welfare for various specific classes of utility functions while retaining at least the same level of complexity for the choice environment, e.g. liquidity constraints, endogenous prices, and heterogeneous agents. Studies using more generally specified utility functions typically have to resort to less complex choice environments, e.g. perfect borrowing and lending, to obtain analytical tractability.

spare. Their problem is then to transfer income back from their more productive middle years to their relatively less productive early years. The imposition of a social security system, as a forced forward shifting of income from early to later years, then represents a nonbeneficial distortion of income profiles. In the presence of liquidity constraints (positive bond interest rates), the attempts by agents to counteract this distortion can lead, through price effects, to an absolutely reduced level of utility for all agents. Thus, contrary to common usage, *NSSW* can fail to provide a useful proxy measure for the impact of a social security system on generation welfare.

More generally, the present paper examines the extent to which a social security system is needed in this class of economies to ensure social optimality, and characterizes the various proper subsets of economies for which *NSSW* and generation welfare exhibit positive correlation.

The special case of identical agents is examined first. It is shown that a social security system is not essential for promoting the welfare of identical agents as long as government is able to set net aggregate bonds \bar{B} at arbitrary levels. However, *NSSW* and generation welfare turn out to be positively correlated with respect to both level and direction of change if either: (a) the bond market is perfect (zero equilibrium bond interest rate); or (b) middle-aged agents are poorly enough endowed relative to young agents that they choose not to hold any money balances in equilibrium; or (c) the net aggregate level of indebtedness assumed by young agents in each period is suitably bounded above. More precisely, letting u^* denote the maximum level of lifetime utility for a representative agent over all feasible lifetime consumption allocations, $u(c)$ denote the level of lifetime utility actually achieved by a representative agent in competitive equilibrium, and $\text{sgn}[\cdot]$ denote the sign of a variable, any one of the conditions (a), (b), or (c) implies:

$$\text{sgn}[u(c) - u^*] = \text{sgn}[NSSW], \quad (1a)$$

$$\text{sgn}\left[\frac{\partial u(c)}{\partial NS}\right] = \text{sgn}\left[\frac{\partial NSSW}{\partial NS}\right], \quad (1b)$$

$$\text{sgn}\left[\frac{\partial u(c)}{\partial T}\right] = \text{sgn}\left[\frac{\partial NSSW}{\partial T}\right], \quad (1c)$$

$$\text{sgn}\left[\frac{\partial u(c)}{\partial \bar{B}}\right] = \text{sgn}\left[\frac{\partial NSSW}{\partial \bar{B}}\right]. \quad (1d)$$

As will be clarified below (cf. table 2), the reason why positive correlation (1) holds under conditions (a), (b), or (c) is that perverse price effects are absent.

It is also shown for identical agent economies that the condition $\bar{B} + NS \neq 0$ alone guarantees that *NSSW* provides a simple actuarial

characterization for socially optimal policy configurations in the sense that

$$u(c) = u^* \Leftrightarrow NSSW = 0, \quad (2)$$

i.e. lifetime utility attains its maximum value u^* if and only if net social security wealth $NSSW$ is zero. However, $NSSW$ can still be negatively correlated with generation welfare in this case. That is to say, identical agent economies with $\bar{B} + NS \neq 0$ can easily be constructed for which equalities (1) hold with all right-hand bracketed terms reversed in sign.⁵

The more general case of heterogeneous agents is then considered. Each generation is assumed to consist of two agent types, $k=1, 2$, distinguished by population size, endowment profile, and nominal social security benefits S_k received. The welfare of each generation is measured by a convex combination $W(c) = \theta u(c_1) + [1 - \theta]u(c_2)$ of the utility of lifetime consumptions $u(c_k)$ attained by agent types $k=1, 2$, where again $u(c_k) = \log(c_k^1) + \alpha \log(c_k^2) + \beta \log(c_k^3)$. In contrast to the double implication (2) holding for the identical agent case, it is shown for the heterogeneous agent case that zero net social security wealth $NSSW$ is necessary but *not* sufficient for a social optimum as measured by the generation welfare function $W(c)$. Also, again in contrast to the identical agent case, the social security system (S_1, S_2, T) and the bond instrument \bar{B} are *both* generally needed to ensure that a social optimum is achieved.

Finally, not surprisingly, $NSSW$ and generation welfare as measured by $W(c)$ generally fail to exhibit any determinate correlation. Nevertheless, in analogy to the identical agent case, it is shown for the heterogeneous agent case that the equilibrium real net social security wealth $NSSW_k$ received by agent type k is weakly positively correlated with the equilibrium level of utility $u(c_k)$ achieved by agent type k for certain interesting special cases in which perverse price effects are mitigated. For example, if the bond market is perfect, then

$$\text{sgn} [u(c_k) - u_k^*] = \text{sgn} [NSSW_k - NSSW_k^*], \quad k = 1, 2, \quad (3a)$$

$$\text{sgn} \left[\frac{\partial u(c_k)}{\partial S_j} \right] = \text{sgn} \left[\frac{\partial NSSW_k}{\partial S_j} \right], \quad k = 1, 2, \quad j = 1, 2, \quad (3b)$$

$$\text{sgn} \left[\frac{\partial u(c_k)}{\partial T} \right] = \text{sgn} \left[\frac{\partial NSSW_k}{\partial T} \right], \quad k = 1, 2, \quad (3c)$$

$$\text{sgn} \left[\frac{\partial u(c_k)}{\partial \bar{B}} \right] = \text{sgn} \left[\frac{\partial NSSW_k}{\partial \bar{B}} \right], \quad k = 1, 2. \quad (3d)$$

⁵Specifically, $\text{sgn} [u(c) - u^*] = \text{sgn} [-NSSW]$, and similarly for equalities (1b)–(1d). An example is given in section 5.

If middle-aged agents choose not to hold any money balances in equilibrium, then (3c) is valid. If the bond market is completely private ($\bar{B}=0$), then (3b) is valid. As in the identical agent case, net social security wealth and generation welfare can otherwise exhibit negative correlation in the sense that equalities (3) hold with all right-hand bracketed terms reversed in sign.

The model is developed in section 2, and the socially optimal solution for the model is characterized and analyzed in section 3. Analytical representations for *NSSW* are developed in section 4. Sections 5 and 6 investigate the relationship between *NSSW* and generation welfare for identical and heterogeneous agents, respectively. Technical notes are included in an appendix.

Using the same basic model as the present paper, Tesfatsion (1982) provides an existence and uniqueness characterization for the stationary competitive equilibria, and investigates the extent to which the macro equilibria corresponding to these micro equilibria are invariant to redistributions of income across agent types. However, the social welfare concerns of the present paper are not discussed.

2. The model

Consider a one-good pure exchange stationary overlapping generations model consisting of a population of heterogeneously endowed three-period-lived agents and a balanced budget government with tax and transfer powers. Specifically, at each time t , $-\infty < t < \infty$, the population consists of N_k newly-born young agents of type k endowed with ω_k^1 units of the commodity good, $k=1, 2$, N_k middle-aged agents of type k born at time $t-1$ and endowed at time t with ω_k^2 units of the commodity good, $k=1, 2$, and N_k old agents of type k born at time $t-2$, $k=1, 2$, who receive no commodity endowment at time t and who will die at the end of the t th period $[t, t+1]$. For each $k=1, 2$, it is assumed that $\omega_k^1 \geq 0$, $\omega_k^2 \geq 0$, and $\omega_k^1 + \omega_k^2 > 0$. Also, the commodity good is assumed to be nonstorable.

At each time t there is a pre-transfer stock of fiat money held in the private sector by middle-aged and old agents of type k in the form of money balances $M_k^1 \geq 0$ and $M_k^2 \geq 0$, respectively, $k=1, 2$. As will be clarified below, the aggregate outstanding stock of fiat money is determined endogenously by market clearing and government budget conditions, and is therefore not a government policy instrument.

Let P denote the fiat money price at each time t of one unit of the consumption good. At each time t the government levies taxes $TP\omega_k^1$ and $TP\omega_k^2$ on type k young and middle-aged agents, respectively, and distributes a nominal social security payment S_k to type k old agents, where $0 \leq T < 1$ and $0 \leq S_k$, $k=1, 2$. In addition, the government stands ready to sell or purchase in net terms a certain nominal quantity $|\bar{B}|$ of two-period bonds

(voluntary retirement annuities) in transactions with young agents, $-\infty < \bar{B} < \infty$, where each bond is a promise to pay one unit of fiat money to the bearer at time $t+2$. Bonds are not privately transferable. The fiat money price of a two-period bond at each time t is denoted by P^B , and the price of a two-period bond at each time t in units of time t consumption is denoted by $p^B \equiv P^B/P$. The two-period bond rate of interest at each time t is thus given by $i \equiv [1 - P^B]/P^B$. For simplicity, the model conditions will henceforth be expressed in terms of i and p^B rather than P and P^B . The structure of the economy is depicted in figs. 1-3.

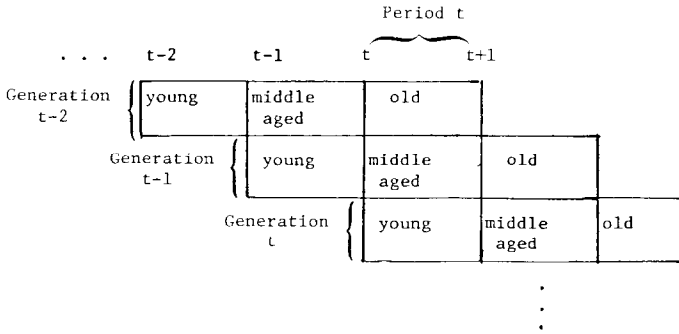


Fig. 1. Overlapping generations model with three-period-lived agents.

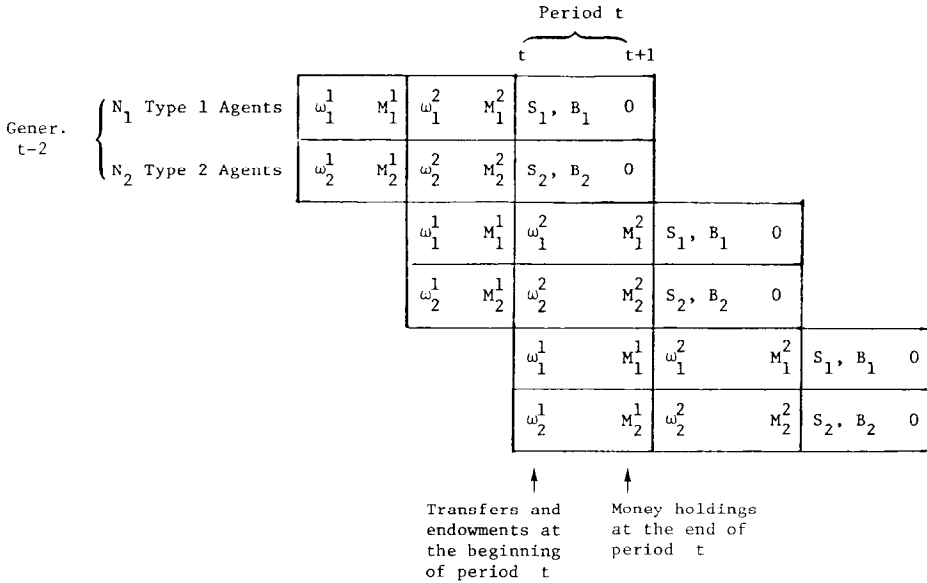


Fig. 2. Transfers, endowments, and money holdings for a typical period t .

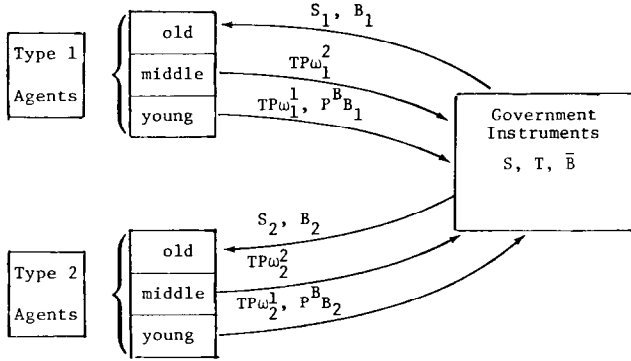


Fig. 3. Nominal transactions for a typical period t .

For each $k = 1, 2$ and each time t , the planning problem of a type k young agent is assumed to be the choice of a consumption profile $c_k \equiv (c_k^1, c_k^2, c_k^3)$, a money holding profile $M_k \equiv (M_k^1, M_k^2)$, and a two-period bond purchase ($B_k \geq 0$) or sale ($B_k < 0$) when young, to maximize utility of life-time consumption:

$$u(c_k^1, c_k^2, c_k^3) \tag{4a}$$

subject to the budget constraints:

$$c_k^1 + (1+i)p^B M_k^1 + p^B B_k = [1-T]\omega_k^1, \tag{4b}$$

$$c_k^2 + (1+i)p^B M_k^2 = [1-T]\omega_k^2 + (1+i)p^B M_k^1, \tag{4c}$$

$$c_k^3 = (1+i)p^B [B_k + S_k + M_k^2], \tag{4d}$$

$$M_k \geq 0, \quad c_k \geq 0, \tag{4e}$$

where $u: R_+^3 \rightarrow R$ is defined by:

$$u(c^1, c^2, c^3) \equiv \log(c^1) + \alpha \log(c^2) + \beta \log(c^3), \tag{4f}$$

for arbitrary positive taste parameters α and β .

It can be shown [Tesfatsion (1982)] that the solution (M_k, B_k, c_k) for problem (4) is a well-defined continuous nondifferentiable function of p^B and i over R_+^2 . Infinitely many solutions exist for problem (4) when $i=0$, since agents are then indifferent between money and bonds. As simple arbitrage arguments demonstrate, problem (4) has no (finite) solution if either $p^B=0$ or $i < 0$, and no solution satisfying both $M_k^1 > 0$ and $M_k^2 > 0$ if $i > 0$.

Before presenting the government budget constraint and market clearing conditions for this model, certain compact notations will be introduced for describing aggregates.

2.1. Notational conventions

Let

$$N \equiv (N_1, N_2), \quad \omega \equiv (\omega_1^1, \omega_1^2, \omega_2^1, \omega_2^2) \equiv (\omega_1, \omega_2), \quad S \equiv (S_1, S_2),$$

denote the vector of population sizes, endowments, and social security benefits, respectively, for agent types 1 and 2, and let

$$M \equiv (M_1^1, M_1^2, M_2^1, M_2^2) \equiv (M_1, M_2),$$

$$B \equiv (B_1, B_2),$$

$$c \equiv (c_1^1, c_1^2, c_1^3, c_2^1, c_2^2, c_2^3) \equiv (c_1, c_2),$$

denote the vector of money holdings, bond holdings, and consumption levels, respectively, for agent types 1 and 2. Note, by stationarity, that aggregating endowments over time for a generation also yields aggregate endowments at each time t , and similarly for consumption, money and bond holdings, and social security benefits (see fig. 2). The following notational conventions will be used to denote the indicated endowment aggregates at each time t :

$$N\omega \equiv N_1[\omega_1^1 + \omega_1^2] + N_2[\omega_2^1 + \omega_2^2] \quad (\text{aggregate endowment}),$$

$$N\omega^1 \equiv N_1\omega_1^1 + N_2\omega_2^1 \quad (\text{aggregate youth endowment}),$$

$$N\omega^2 \equiv N_1\omega_1^2 + N_2\omega_2^2 \quad (\text{aggregate middle-age endowment}),$$

$$N_k\omega_k \equiv N_k[\omega_k^1 + \omega_k^2] \quad (\text{aggregate endowment of agent type } k).$$

Similarly, the following notational conventions will be used to denote the indicated consumption, money, bond, and social security benefit aggregates at each time t :

$$Nc \equiv N_1[c_1^1 + c_1^2 + c_1^3] + N_2[c_2^1 + c_2^2 + c_2^3] \quad (\text{aggregate consumption});$$

$$Nc^j \equiv N_1c_1^j + N_2c_2^j, \quad j = 1, 2, 3 \quad (\text{aggregate age } j \text{ consumption});$$

$$NM \equiv N_1[M_1^1 + M_1^2] + N_2[M_2^1 + M_2^2] \quad (\text{aggregate money holdings});$$

$NM^j \equiv N_1M_1^j + N_2M_2^j$, $j=1, 2$ (aggregate age j money holdings);

$NB \equiv N_1B_1 + N_2B_2$ (aggregate bond holdings);

$NS \equiv N_1S_1 + N_2S_2$ (aggregate social security benefits).

Using these notational conventions, the government budget constraint at each time t is given by:

$$p^B NB + TN\omega = (1+i)p^B[\bar{B} + NS]; \quad (5)$$

the goods market clearing condition at time t is given by:

$$Nc = N\omega; \quad (6)$$

and the bond market clearing condition at each time t is given by:

$$NB = \bar{B}. \quad (7)$$

Finally, the money market clearing condition at each time t is given by:

$$(1+i)p^B NM + p^B NB + TN\omega = (1+i)p^B[NB + NS + NM]. \quad (8)$$

Clearly (8) holds if (5) and (7) hold. Moreover, the cross-sectional budget constraints for young, middle-aged, and old agents at each time t together with (6) and (7) can be shown to imply (5); hence, both (5) and (8) are superfluous conditions in the present stationary context.

Definitions will now be given for an *economy* and for a *stationary competitive equilibrium*, or *equilibrium* for short.

2.2. Definitions

A parameter vector $e = (N, \omega, \alpha, \beta, S, T, \bar{B})$ in R^{12} will be called an *economy* if the parameter values are admissible in the sense that $N \in R_+^2$ with $N_1 + N_2 > 0$, $\omega \in R_+^4$ with $\omega_k^1 + \omega_k^2 > 0$, $k=1, 2$, $\alpha \in R_{++}$, $\beta \in R_{++}$, $S \in R_+^2$, $T \in [0, 1)$, and $\bar{B} \in R$. Given any economy e , a vector $v^e = (p^B, i, M, B, c)$ in $R_{++} \times R_+ \times R_+^4 \times R^2 \times R_{++}^6$ will be called an *equilibrium for e* if (M_k, B_k, c_k) solves the young agent planning problem (4) for the given real bond price p^B , bond interest rate i , and parameter values defining e , $k=1, 2$, and the market clearing conditions (6) and (7) are also satisfied.

Let \mathcal{E} denote the set of all economies e which support at least one equilibrium; and, for each e in \mathcal{E} , let $\varphi(e)$ denote the set of all equilibria v^e

corresponding to e . As discussed in more detail in the appendix to this paper, the set \mathcal{E} partitions into ten liquidity preference regions according to the liquidity preference behavior exhibited by agent types $k=1,2$ in the corresponding equilibria. For example, region I (money–bond indifference) is given by:

$$I \equiv \{e \in \mathcal{E} \mid i=0 \text{ for all } v^e \in \varphi(e)\}, \quad (9)$$

and region II (youth money holding only) is given by:

$$II \equiv \{e \in \mathcal{E} \mid i > 0, M_k^1 > 0, M_k^2 = 0, k = 1, 2, \text{ for all } v^e \in \varphi(e)\}. \quad (10)$$

Throughout the rest of this paper it will often be necessary to examine \mathcal{E} region by region in order to reveal uniformities in the corresponding equilibria.

An economy e in \mathcal{E} will be called *regular* if the components of any equilibrium v^e corresponding to e are right and/or left differentiable functions of the policy instruments (S, T, \bar{B}) at the point e . Conditions guaranteeing regularity are detailed in the appendix.

3. Social optimality

Following Samuelson (1958), the welfare of each generation will be measured by a convex combination,

$$W(c) = \theta u(c_1) + [1 - \theta] u(c_2), \quad (11)$$

of the utility of lifetime consumptions $u(c_k)$ achieved by agent types $k=1,2$ in competitive equilibrium, where θ is any fixed arbitrarily selected number between zero and one. The welfare function (11) reduces to the classical utilitarian welfare function advocated by Lerner (1959), i.e. the cross-sectional total utility of all agents alive at any time t , if and only if $\theta = N_1 / [N_1 + N_2]$.

The socially optimal consumption allocation c^* corresponding to any given values for N , ω , α , and β is then characterized as the solution to:

$$\max_{c \in \mathcal{R}_+^6} W(c) \quad (12a)$$

subject to the feasibility condition:

$$Nc \leq N\omega. \quad (12b)$$

It is easily established that c^* takes the form:

$$c_1^{1*} = \frac{\theta N \omega}{N_1(1 + \alpha + \beta)}, \quad c_1^{2*} = \alpha c_1^{1*}, \quad c_1^{3*} = \beta c_1^{1*}, \quad (13a)$$

$$c_2^{1*} = \frac{[1 - \theta] N \omega}{N_2(1 + \alpha + \beta)}, \quad c_2^{2*} = \alpha c_2^{1*}, \quad c_2^{3*} = \beta c_2^{1*}. \quad (13b)$$

Thus, c^* depends *only* on $(N, \alpha, \beta, \omega, \theta)$. In particular, c^* is independent of the government policy instruments (S, T, \bar{B}) .

For later purposes, it is useful to note that c^* is alternatively characterized by three distinct types of conditions:

Individual optimality conditions

$$c_k^2 = \alpha c_k^1, \quad c_k^3 = \beta c_k^1, \quad k = 1, 2; \quad (14a)$$

Social welfare condition

$$0 = [\theta N_2 c_2^1 - [1 - \theta] N_1 c_1^1]; \quad (14b)$$

Efficiency condition

$$Nc = N\omega. \quad (14c)$$

The individual optimality conditions (14a) restrict the shape of the individual consumption profiles. The social welfare condition (14b) imposes a condition across agent types, making use of the welfare function parameter θ . The efficiency condition (14c) guarantees that no resources are wasted.

Let $e = (N, \omega, \alpha, \beta, S, T, \bar{B})$ be any economy in \mathcal{E} , and let $v^e = (p^e, i, M, B, c)$ be any element of $\varphi(e)$, the set of equilibria corresponding to e . A detailed characterization of the equilibrium correspondence $e \mapsto \varphi(e)$ is provided in Tesfatsion (1982). Using this characterization, it is possible to determine, qualitatively, how the equilibrium micro consumption profile c and the equilibrium macro consumption profile (Nc^1, Nc^2, Nc^3) corresponding to c differ from their socially optimal values c^* and $(Nc^{1*}, Nc^{2*}, Nc^{3*})$, depending on which particular liquidity preference region of \mathcal{E} contains e .⁶

Table 1 summarizes these results for regions I–VII,⁷ making use of the

⁶The micro consumption profile c^* is defined as in (13) for the fixed value of θ and the particular values $(N, \omega, \alpha, \beta)$ appearing in e . The macro consumption profile $(Nc^{1*}, Nc^{2*}, Nc^{3*})$ corresponds to c^* in the usual way. (See section 2.)

⁷See the appendix to this paper for a detailed description of the seven basic liquidity preference regions I–VII and their three symmetrical counterparts V*, VI* and VII*. Given any result for region V, VI or VII, the corresponding result for region V*, VI* or VII* is obtained by *interchanging* agent subscripts. The appendix also contains a discussion of the technical derivation for table 1.

Table 1
Deviation of macro and micro consumption profiles from their socially optimal values.

Liquidity preference region	Macro deviations			Micro deviations, $k = 1, 2$		
	ΔNc^1	ΔNc^2	ΔNc^3	Δc_k^1	Δc_k^2	Δc_k^3
I $i=0$	0	0	0	X_k^1	X_k^1	X_k^3
II $i>0, M^2>0, M^2=0$	—	—	+	X_k^1	X_k^1	X_k^3
III $i>0, M^1=0, M^2=0$	—	?	+	X_k^1	?	X_k^3
IV $i>0, M^1=0, M^2\geq 0$	—	+	+	X_k^1	X_k^3	X_k^3
V $i>0, M^1\geq 0, M^2=0$	—	?	+	X_k^1	X_k^2	X_k^3
VI $i>0, M^1=0, M^2>0$	—	?	+	X_k^1	X_k^1	X_k^3
VII $i>0, M^1\geq 0, M^2\geq 0$	—	?	+	X_k^1	X_k^2	X_k^3

Note: For regions V–VII, the top sign for Δc_k^2 is for $k=1$ and the bottom sign for Δc_k^2 is for $k=2$.

following abbreviations:

$$\Delta Nc^j \equiv \text{sgn} [Nc^j - Nc^{j*}], \quad j = 1, 2, 3, \quad (15a)$$

$$\Delta c_k^j \equiv \text{sgn} [c_k^j - c_k^{j*}], \quad j = 1, 2, 3, \quad k = 1, 2, \quad (15b)$$

$$\theta_1 \equiv \theta, \quad \theta_2 \equiv [1 - \theta], \quad (15c)$$

$$Z \equiv \theta_1 N_2 c_2^1 - \theta_2 N_1 c_1^1, \quad (15d)$$

$$X_k^1 \equiv \text{sgn} ([-1]^k Z + \theta_k [Nc^1 - Nc^{1*}]), \quad k = 1, 2, \quad (15e)$$

$$X_k^3 \equiv \text{sgn} ([-1]^k (1+i)\beta Z + \theta_k [Nc^3 - Nc^{3*}]), \quad k = 1, 2, \quad (15f)$$

$$(y \geq 0) \equiv (y \geq \mathbf{0} \quad \text{and} \quad y \neq \mathbf{0}), \quad \text{for any vector } y. \quad (15g)$$

It is seen from table 1 that $c=c^*$ if and only if e lies in region I ($i=0$) and c satisfies the social welfare condition (14b), so that $Z = X_k^1 = X_k^3 = 0$, $k=1, 2$. If e lies in region I but c does not satisfy (14b), then the macro consumption profile (Nc^1, Nc^2, Nc^3) is socially optimal but the micro consumption profile c

is not. If e lies outside of region I, then neither the macro nor the micro consumption profile is socially optimal.⁸

The macro consumption profile deviations depicted in table 1 have an intuitively plausible explanation. Liquidity constraints are in effect outside of region I because the two-period bond interest rate i is positive. In consequence, agents wishing to carry income back from old age to youth are discouraged, and agents wishing to carry income forward from youth to old age are encouraged. The social security system, as a forced forward shifting of income from early to later years, reinforces this distortion. Thus, compared to the socially optimal consumption profile $(Nc^{1*}, Nc^{2*}, Nc^{3*})$, macro young age consumption Nc^1 is too low and macro old age consumption Nc^3 is too high outside of region I. Macro middle age consumption Nc^2 may go either way.

The deviations in the micro consumption profiles mimic the deviations in the macro consumption profiles to a certain extent, but exact signs are ambiguous. For example, if e lies in region II, the sign pattern for $(\Delta Nc^1, \Delta Nc^2, \Delta Nc^3)$ is $(-, -, +)$; but the sign pattern for $(\Delta c_k^1, \Delta c_k^2, \Delta c_k^3, k=1, 2)$ is $(X_k^1, X_k^1, X_k^3, k=1, 2)$, which may take on any one of nine possible configurations in conformity with the macro sign pattern $(-, -, +)$.

4. Analytical representations for NSSW

At each time t a young type k agent faces a nominal social security benefit stream $(0, 0, S_k)$ and a nominal tax assessment stream $(TP\omega_k^1, TP\omega_k^2, 0)$, $k=1, 2$. Since the only interest rate is the two-period bond interest rate i , the present value of benefits received minus taxes paid by each generation is:

$$\frac{1}{1+i}(N_1S_1 + N_2S_2) - TP(N_1[\omega_1^1 + \omega_1^2] + N_2[\omega_1^2 + \omega_2^2]) \equiv P^B NS - TPN\omega. \quad (16)$$

Hence, by definition, the real net social security wealth *NSSW* accruing to each generation is $p^B NS - TN\omega$, where $p^B \equiv P^B/P$. From the government budget constraint (5), it follows immediately that:

$$NSSW = p^B NS - TN\omega = -ip^B[\bar{B} + NS]. \quad (17)$$

From the cross-sectional budget constraints for young, middle-aged, and old agents at each time t , together with the bond market clearing condition

⁸Specifically, the individual optimality conditions (14a) are only satisfied in region I. The social welfare condition (14b) holds only for a proper subset of economies in each region. The efficiency condition (14c) is satisfied in all regions.

(7), one obtains:

$$Nc^3 = (1+i)p^B[\bar{B} + NS + NM^2]. \quad (18)$$

Using the detailed solution characterization provided in Tesfatsion (1982), it can be shown that:

$$Nc^3 = (1+i)\beta Nc^1 \quad (19)$$

in all liquidity preference regions of \mathcal{E} . Combining (17) through (19) yields:

$$NSSW = p^B NS - TN\omega = -ip^B[\bar{B} + NS] = i[p^B NM^2 - \beta Nc^1]. \quad (20)$$

Certain interesting facts are immediately obtainable from (20). If $(\bar{B} + NS) \neq 0$, then

$$NSSW = 0 \Leftrightarrow i = 0. \quad (21)$$

If $NM^2 = 0$, then

$$NSSW \begin{pmatrix} < \\ = \end{pmatrix} 0 \Leftrightarrow i \begin{pmatrix} > \\ = \end{pmatrix} 0. \quad (22)$$

In general, however, it is easy to construct economies with reasonable parameter specifications for which $NSSW > 0$.⁹

Can $NSSW$ be used in any meaningful sense to compare the welfare of individuals existing in two arbitrarily selected economies e and e' in \mathcal{E} ? The answer is no. Consider the following partition of \mathcal{E} :

$$\{e \in \mathcal{E} \mid i = 0 \text{ for all } v^e \in \varphi(e)\}, \quad (23a)$$

$$\{e \in \mathcal{E} \mid i > 0 \text{ and } NSSW > 0 \text{ for all } v^e \in \varphi(e)\}, \quad (23b)$$

$$\{e \in \mathcal{E} \mid i > 0 \text{ and } NSSW = 0 \text{ for all } v^e \in \varphi(e)\}, \quad (23c)$$

$$\{e \in \mathcal{E} \mid i > 0 \text{ and } NSSW < 0 \text{ for all } v^e \in \varphi(e)\}, \quad (23d)$$

It is easily shown that the equilibrium utility levels $u(c_k)$ attained by agent

⁹For example, consider the economy $e = (N, \omega, \alpha, \beta, S, T, \bar{B})$ satisfying $N_1 = N_2 = 1$, $N\omega^1 = 0$, $\omega_1^2 = \omega_2^2 > 0$, $\alpha = \beta = 1$, $S_1 = S_2 = \frac{1}{2}$, $0 \leq T < \frac{1}{6}$, $\bar{B} = -2$. Using Tesfatsion (1982, Theorem 3.5), it can be shown that a unique positive interest rate equilibrium $v^e = (p^B, i, M, B, c)$ corresponds to e . Since $(\bar{B} + NS) = (-2 + 1) < 0$, it follows from (20) that $NSSW > 0$ in this equilibrium.

types $k=1, 2$ take on all values between plus and minus infinity over *each* of the four subsets (23a)–(23d).

The following sections focus more narrowly on the ability of *NSSW* to indicate *local* welfare distortions and directions of improvement.

5. NSSW and generation welfare: Identical agents

In this section attention is focused on economies in \mathcal{E} for which $N_2=0$, i.e. for which all agents are of type 1. By construction, any such identical agent economy must lie in one of the liquidity preference regions I–IV, where agents exhibit qualitatively identical liquidity preference behavior. Also, the social welfare function parameter θ is set equal to 1, so that the social welfare function (11) places no weight on the utility of consumption $u(c_2)$ for agent type 2.

Let $e^1=(N_1, 0, \omega_1, 0, \alpha, \beta, S_1, 0, T, \bar{B})$ be any identical agent economy in \mathcal{E} , and let $v^{e^1}=(p^B, i, M_1, 0, B_1, 0, c_1, 0)$ be any equilibrium for e^1 .¹⁰ From section 3, the socially optimal consumption profile c_1^* corresponding to e^1 is:

$$c_1^{1*} = \frac{\omega_1^1 + \omega_1^2}{(1 + \alpha + \beta)}, \quad c_1^{2*} = \alpha c_1^{1*}, \quad c_1^{3*} = \beta c_1^{1*}. \quad (24)$$

As always, c_1^* is independent of the government policy instruments (S_1, T, \bar{B}) ; and, as table 1 indicates, the equilibrium consumption profile c_1 will generally differ from c_1^* .

The first question posed in this section is as follows. Starting from the given parameter values defining e^1 , is it possible to move c_1 to c_1^* (in a comparative static sense) by appropriate manipulation of the policy instruments (S_1, T, \bar{B}) ?

To answer this question, first note that the social welfare condition (14b) holds trivially for v^{e^1} since $(N_2, \theta)=(0, 1)$. It follows from table 1 that:

$$c_1 = c_1^* \Leftrightarrow i = 0. \quad (25)$$

However, it can be shown [Tesfatsion (1982)] that government *non*-interference ($S_1 = T = \bar{B} = 0$) implies:

$$i = 0 \Leftrightarrow 0 \leq (\alpha + \beta)\omega_1^1 - \omega_1^2; \quad (26)$$

hence, *some* government interference is necessary for ensuring $i=0$ unless

¹⁰Given $N_2=0$, the remaining variables $(\omega_2, S_2, M_2, B_2, c_2)$ for type 2 agents could be assigned arbitrary values without affecting the solution (p^B, i, M_1, B_1, c_1) for type 1 agents [Tesfatsion (1982)]. Here they are set equal to zero for clarity.

youth endowments ω_1^1 are sufficiently larger than middle-age endowments ω_1^2 in the sense of (26).

Not surprisingly, the social security system (S_1, T) turns out to be neither necessary nor sufficient for ensuring $i=0$. The key policy instrument is the aggregate net level \bar{B} for bonds (voluntary retirement annuities). Given any fixed admissible values for S_1 and T satisfying either $TS_1 > 0$ or $S_1 = T = 0$, it is *always* possible to ensure $i=0$ by a suitable manipulation in \bar{B} , $-\infty < \bar{B} < \infty$.¹¹

The second question posed in this section concerns the degree to which $NSSW$ is positively correlated with generation welfare for the identical agent economy $e1$. From (20) and (25), the level of real net social security wealth $NSSW^*$ corresponding to the socially optimal consumption profile c_1^* is:

$$NSSW^* = 0. \quad (27)$$

Consider the deviations:

$$\Delta u(c_1) \equiv u(c_1) - u(c_1^*), \quad (28a)$$

$$\Delta NSSW \equiv NSSW - NSSW^*, \quad (28b)$$

between equilibrium and socially optimal lifetime utility and real net social security wealth for $(e1, v^{e1})$. Assuming $e1$ is regular (section 2 and appendix), under what conditions is $\Delta u(c_1)$ positively correlated with $\Delta NSSW$ in the sense of (1)?

Table 2 qualitatively characterizes the comparative static sensitivity of the deviations $(\Delta u(c_1), \Delta NSSW)$, the real bond price p^B , and the bond interest rate i to changes in the policy variables (S_1, T, \bar{B}) according to which liquidity preference region contains $e1$. It is seen that $\Delta u(c_1)$ and $\Delta NSSW$ exhibit positive correlation in the sense of (1) if $e1$ lies in regions I–III. However, in region IV, where all middle-aged agents choose to hold positive money balances in equilibrium, perverse price effects can result in *negative* correlation between $\Delta u(c_1)$ and $\Delta NSSW$.

For example, consider the particular identical agent economy $e1' = (N_1, 0, \omega_1, 0, \alpha, \beta, S_1, T, \bar{B})$ given by:

$$N_1 = 1, \quad \omega_1^1 = 0, \quad \omega_1^2 = 1, \quad \alpha = \beta = 1, \quad S_1 = 1, \quad T = 0, \quad \bar{B} = -2. \quad (29)$$

¹¹As established in Tesfatsion (1982), $i=0$ for $e1$ in \mathcal{E} only if either $S_1 = T = 0$ or $TS_1 > 0$. Given $S_1 = T = 0$:

$$[i=0] \Leftrightarrow [q(1 + \alpha + \beta)\bar{B} \leq \min \{(\alpha + \beta)N_1\omega_1^1 - N_1\omega_1^2, \beta N_1\omega_1\} \text{ for some } q > 0].$$

Given $TS_1 > 0$, then $i=0$ if and only if condition (A.1) holds in the appendix to this paper. Clearly, by choosing \bar{B} to be suitably negative, either of these conditions for guaranteeing $i=0$ can always be met. The sign restrictions $S_1 \geq 0$ and $1 > T \geq 0$ on S_1 and T naturally make these social security instruments more rigid than the bond instrument \bar{B} .

Table 2
Sensitivities for regular economies composed entirely of type 1 agents.

Liquidity preference region	Variable	sgn [\cdot]	sgn $\left[\frac{\partial}{\partial S_1} \right]$	sgn $\left[\frac{\partial}{\partial T} \right]$	sgn $\left[\frac{\partial}{\partial \bar{B}} \right]$
I $i=0$	$\Delta u(c_1)$	0	0	0	0
	$\Delta NSSW$	0	0	0	0
	p^B	+	-	+	0
	i	0	0	0	0
II-III $i>0, M_1^1 \geq 0, M_1^2 = 0$	$\Delta u(c_1)$	-	sgn $[\bar{B}]$	-	sgn $[-S_1]$
	$\Delta NSSW$	-	sgn $[\bar{B}]$	-	sgn $[-S_1]$
	p^B	+	-	-	-
	i	+	sgn $[\bar{B}]$	+	sgn $[S_1]$
IV $i \geq 0, M_1^1 = 0, M_1^2 > 0$	$\Delta u(c_1)$	-	sgn $[\bar{B}]$	-	sgn $[-S_1]$
	$\Delta NSSW$?	?	?	?
	p^B	+	-	?	?
	i	+	sgn $[-\bar{B}]$	+	sgn $[S_1]$

Using (24), the socially optimal consumption profile c_1^* corresponding to $e1'$ is:

$$c_1^{1*} = c_1^{2*} = c_1^{3*} = \frac{1}{3}; \tag{30}$$

and, as always for identical agent economies, $NSSW^* = 0$. It can be shown [Tesfatsion (1982, theorem 3.5)] that $e1'$ is a regular economy in region IV, and has a unique equilibrium $v^{e1'} = (p^B, i, M_1, 0, B_1, 0, c_1, 0)$. Moreover, contrary to (1), generation welfare and $NSSW$ are negatively correlated for $v^{e1'}$ in the sense that

$$\text{sgn} [\Delta u(c_1)] = \text{sgn} [-\Delta NSSW] = \text{sgn} [-1], \tag{31a}$$

$$\text{sgn} \left[\frac{\partial u(c_1)}{\partial S_1} \right] = \text{sgn} \left[\frac{-\partial NSSW}{\partial S_1} \right] = \text{sgn} [\bar{B}], \tag{31b}$$

$$\text{sgn} \left[\frac{\partial u(c_1)}{\partial \bar{B}} \right] = \text{sgn} \left[\frac{-\partial NSSW}{\partial \bar{B}} \right] = \text{sgn} [-S_1]. \tag{31c}$$

In particular, it follows from (31a) that the level $NSSW$ of real net social security wealth attained by each generation is positive in $v^{e1'}$, despite the fact that the equilibrium level of utility $u(c_1)$ attained by each agent in $v^{e1'}$ is strictly less than the socially optimal level $u(c_1^*)$.

A sufficient condition for $\Delta u(c_1)$ and $\Delta NSSW$ to be positively correlated in

the sense of (1) for an identical agent economy e_1 in region IV is:

$$(1 + \alpha + \beta)\bar{B} + N_1 S_1 > 0, \quad (32)$$

which imposes an upper bound on $-\bar{B}$, the net aggregate level of indebtedness that young agents can assume in each period. In particular, positive correlation holds if $\bar{B} > 0$, so that each young agent in equilibrium plans to privately support his old age by means of principal plus interest from young-age lending in addition to money carryover from middle age. In this case the social security system's forced shifting of income from early to later years clearly complements rather than impedes the private attempts by agents to smooth their consumption profiles.¹²

6. NSSW and generation welfare: Heterogeneous agents

The two questions posed in the previous section for identical agent economies will now be investigated more generally for heterogeneous agent economies. Specifically, attention will be focussed on an arbitrary economy $e = (N, \omega, \alpha, \beta, S, T, \bar{B})$ in \mathcal{E} with $N \equiv (N_1, N_2) > 0$, so that some agents of each type $k=1, 2$ are present. In addition, it will be assumed that the welfare function parameter θ lies in the open interval $(0, 1)$, so that the social welfare function $W(\cdot)$ defined by (11) gives positive weight to the utility of consumption for each agent type.

From section 3, the socially optimal consumption profile c^* corresponding to e is given by (13). Let $v^e = (p^B, i, M, B, c)$ be any equilibrium for e . As indicated in table 1:

$$[c = c^*] \Leftrightarrow [i = 0 \text{ and } Z = 0]. \quad (33)$$

In particular, in contrast to the identical agent case, (33) asserts that it is necessary but *not* sufficient for social optimality that e lie in region I ($i=0$). Starting from the given parameter values defining e , is it possible to move c to c^* (in a comparative static sense) by appropriate manipulation of the government policy instruments (S, T, \bar{B}) , where $S \equiv (S_1, S_2)$?

The answer to this question is affirmative. As in the identical agent case, given either $TNS > 0$ or $NS = T = 0$, manipulation of the bond instrument \bar{B} is necessary and sufficient for ensuring that $i=0$. However, in contrast to the identical agent case, manipulation of the social security instruments (S_1, S_2, T) is now both necessary and sufficient for ensuring that $Z=0$.

Specifically, using the solution characterization provided in Tesfatsion (1982), it can be shown that e lies in region I ($i=0$) only if Z has the reduced

¹²Condition (32) guarantees that $\partial p^B / \partial \bar{B} < 0$ in region IV, as in regions II and III; but, surprisingly, even $\bar{B} > 0$ does not resolve the sign ambiguity of $\partial p^B / \partial T$ in region IV.

form:

$$Z = \frac{N\omega}{1 + \alpha + \beta} \left[\theta - [1 - T] \frac{N_1\omega_1}{N\omega} - T \frac{N_1S_1}{NS} \right]. \quad (34)$$

Since θ lies in the open interval $(0, 1)$, one can always find positive admissible values (S^*, T^*) for which the bracketed term in (34) is zero.¹³ (These values are not unique.) It then follows from the appendix conditions (A.1) that the modified economy,

$$e^* = (N, \omega, \alpha, \beta, S^*, T^*, \bar{B}^*), \quad (35)$$

will lie in region I ($i=0$) if \bar{B} is chosen to be suitably negative; and, by construction, Z will be zero for this economy. By (33), any equilibrium v^{e^*} corresponding to e^* will therefore satisfy $c = c^*$.

Is social welfare $W(c)$ positively correlated with $NSSW$ for heterogeneous agent economies such as e^* ? Not surprisingly, the answer is negative. Social welfare $W(c)$ depends upon which agent type gets what; $NSSW$ ignores this distributional consideration. In general, $NSSW$ and $W(c)$ exhibit no determinate correlation.

A similar question, posed on a more micro level, turns out to have a more interesting answer. As indicated by (20) and (33), the level of aggregate real net social security wealth corresponding to c^* is $NSSW^* = 0$, just as in the identical agent case. However, the aggregate variable $NSSW$ now decomposes nontrivially into two subaggregates, i.e.

$$\begin{aligned} NSSW &\equiv p^B NS - TN\omega \\ &\equiv [p^B N_1 S_1 - TN_1 \omega_1] + [p^B N_2 S_2 - TN_2 \omega_2] \\ &\equiv NSSW_1 + NSSW_2, \end{aligned} \quad (36)$$

where $NSSW_k$ denotes the aggregate real net social security wealth attained by type k agents in any given generation, $k=1, 2$. The particular subaggregates $NSSW_k^*$ corresponding to c^* take the form:¹⁴

¹³Define $\pi_1 \equiv N_1\omega_1/N\omega$ and $\delta_1 \equiv N_1S_1/NS$, and note that both π_1 and δ_1 lie in $[0, 1]$ by construction. For any given T satisfying $\pi_1/[1 + \pi_1] \leq T < 1$, the expression $[1 - T]\pi_1 + T\delta_1$ in (34) covers the open interval $([1 - T]\pi_1, [1 - T]\pi_1 + T)$ as δ_1 varies over $(0, 1)$. Thus, by letting $T \rightarrow 1$, one can always find a pair of values δ_1^* and T^* in $(0, 1)$ for which $\theta = [1 - T^*]\pi_1 + T^*\delta_1^*$. Now define $S_1^* \equiv \delta_1^* NS/N_1 > 0$ and $S_2^* \equiv [1 - \delta_1^*] NS/N_2 > 0$ for any arbitrary scale factor $NS > 0$.

¹⁴In region I ($i=0$), the consumption profile c has the form $c_k^1 = [N_k\omega_k + NSSW_k]/N_k(1 + \alpha + \beta)$, $c_k^2 = \alpha c_k^1$, $c_k^3 = \beta c_k^1$, $k=1, 2$. It follows from (13) that $c = c^*$ only if $NSSW_k$ satisfies (37), $k=1, 2$. Note that the socially optimal net social security wealth allocation is 'equitable', i.e. $NSSW_k^* = 0$, $k=1, 2$, if and only if the social welfare function (11) weights agent types according to their proportion of total endowments (real wages) rather than, for example, according to their proportion of total population as suggested by Lerner (1959).

$$NSSW_1^* = N\omega \left[\theta - \frac{N_1\omega_1}{N\omega} \right], \quad (37a)$$

$$NSSW_2^* = N\omega \left[[1 - \theta] - \frac{N_2\omega_2}{N\omega} \right] = -NSSW_1^*. \quad (37b)$$

Assuming e is regular, under what conditions are the *micro* deviations,

$$\Delta u(c_k) \equiv u(c_k) - u(c_k^*), \quad (38a)$$

$$\Delta NSSW_k \equiv NSSW_k - NSSW_k^*, \quad (38b)$$

for (e, v^e) positively correlated in the sense of (3)?

Consider, first, the two important special cases of a perfect bond market ($i=0$) and a completely private bond market ($\bar{B}=0$). Table 3 exhibits the macro and micro deviation sensitivities for both of these cases, making use of the following additional abbreviations:

$$\Delta W(c) \equiv W(c) - W(c^*), \quad (39a)$$

$$H_k \equiv \frac{N_k S_k}{NS} - \frac{N_k \omega_k}{N\omega}, \quad k=1, 2, \quad (39b)$$

$$\Pi^B \equiv \text{region II} \cap \{e \in \mathcal{E} \mid \bar{B}=0\}, \text{ etc.} \quad (39c)$$

As table 3 indicates, there is a steady diminution in the degree of positive correlation exhibited by both the macro deviations ($\Delta W(c)$, $\Delta NSSW$) and the micro deviations ($\Delta u(c_k)$, $\Delta NSSW_k$) as one moves from region I to region VII^B. Positive correlation holds for the macro deviations in regions I–IV^B with respect to both level and first-order differentiation if and only if the social welfare condition (14b) is satisfied; moving into region V^B and beyond, eventually only the levels $\Delta W(c)$ and $\Delta NSSW$ necessarily retain the same sign. A similar phenomenon is true for the micro deviations. Only condition (3b), positive correlation with respect to changes in S_k , $k=1, 2$, necessarily holds throughout all regions.

The diminution in positive correlation from region I to region VII^B in table 3 is paralleled by a diminution in the degree to which agent types exhibit qualitatively similar liquidity preference behavior. Comparing table 3 to table 2, the bond market condition $\bar{B}=0$ prevents middle-aged money holding *by itself* from being much of a factor in the breakdown of positive correlation, just as condition (32) prevented any breakdown in the identical agent case.

Table 3
Sensitivities for regular economies with either perfect ($i=0$) or completely private ($\bar{B}=0$) bond markets.

Liquidity preference region	Variable	sgn [·]	$\text{sgn}\left(\frac{\partial}{\partial S_1}\right)$	$\text{sgn}\left(\frac{\partial}{\partial S_2}\right)$	$\text{sgn}\left(\frac{\partial}{\partial T}\right)$
I $i=0$	$\Delta W(c)$	$\text{sgn}(-Z^2)$	$\text{sgn}(S_2 Z)$	$\text{sgn}(-S_1 Z)$	$\text{sgn}(H_1 Z)$
	$\Delta NSSW$	0	0	0	0
	$\Delta u(c_k)$ $\Delta NSSW_k$	$(-1)^k \text{sgn}(Z)$ $(-1)^k \text{sgn}(Z)$	$(-1)^k \text{sgn}(-S_2)$ $(-1)^k \text{sgn}(-S_2)$	$(-1)^k \text{sgn}(S_1)$ $(-1)^k \text{sgn}(S_1)$	$\text{sgn}(H_k)$ $\text{sgn}(H_k)$
II^B-IV^B					
Qualitatively identical liquidity preferences with $i>0$ and $\bar{B}=0$	$\Delta W(c)$	—	$\text{sgn}(S_2 Z)$	$\text{sgn}(-S_1 Z)$	—
	$\Delta NSSW$	—	0	0	—
	$\Delta u(c_k)$?	$(-1)^k \text{sgn}(-S_2)$	$(-1)^k \text{sgn}(S_1)$	—
	$\Delta NSSW_k$?	$(-1)^k \text{sgn}(-S_2)$	$(-1)^k \text{sgn}(S_1)$	—
V^B					
Qualitatively dissimilar liquidity preferences with $i>0$, $NM^2=0$, and $\bar{B}=0$	$\Delta W(c)$	—	\pm	\mp	—
	$\Delta NSSW$	—	$\text{sgn}(S_2)$	$\text{sgn}(-S_1)$	—
	$\Delta u(c_k)$?	$(-1)^k \text{sgn}(-S_2)$	$(-1)^k \text{sgn}(S_1)$	—
	$\Delta NSSW_k$?	$(-1)^k \text{sgn}(-S_2)$	$(-1)^k \text{sgn}(S_1)$	—
VI^B-VII^B					
Qualitatively dissimilar liquidity preferences with $i>0$, $NM^2>0$, and $\bar{B}=0$	$\Delta W(c)$	—	\pm	\mp	?
	$\Delta NSSW$	—	$\text{sgn}(S_2)$	$\text{sgn}(-S_1)$?
	$\Delta u(c_k)$?	$(-1)^k \text{sgn}(-S_2)$	$(-1)^k \text{sgn}(S_1)$?
	$\Delta NSSW_k$?	$(-1)^k \text{sgn}(-S_2)$	$(-1)^k \text{sgn}(S_1)$?

The retention of positive correlation with respect to (S_1, S_2) throughout regions II^B-VII^B for the micro deviations ($\Delta u(c_k), \Delta NSSW_k$) is directly attributable to the force of the symmetry condition $\bar{B}=0$. For example, given $\bar{B}=0$, the following relations hold in all liquidity preference regions of \mathcal{E} :¹⁵

$$S_1 \frac{\partial i}{\partial S_1} + S_2 \frac{\partial i}{\partial S_2} = 0, \tag{40a}$$

$$S_1 \frac{\partial W(c)}{\partial S_1} + S_2 \frac{\partial W(c)}{\partial S_2} = 0, \tag{40b}$$

$$S_1 \frac{\partial NSSW}{\partial S_1} + S_2 \frac{\partial NSSW}{\partial S_2} = 0, \tag{40c}$$

¹⁵Conditions (40) hold in region I whether or not $\bar{B}=0$.

$$S_1 \frac{\partial u(c_k)}{\partial S_1} + S_2 \frac{\partial u(c_k)}{\partial S_2} = 0, \quad k = 1, 2, \quad (40d)$$

$$S_1 \frac{\partial NSSW_k}{\partial S_1} + S_2 \frac{\partial NSSW_k}{\partial S_2} = 0, \quad k = 1, 2. \quad (40e)$$

Conditions (40) guarantee that any positive correlation between welfare and net social security wealth with respect to changes in S_1 will be matched by an analogous positive correlation with respect to changes in S_2 . Also, in the analytically difficult regions, V, VI, and VII, where agents exhibit qualitatively dissimilar liquidity preference behavior, $\bar{B} = 0$ implies:

$$\text{sgn} \left[\frac{\partial p^B}{\partial S_k} \right] = \text{sgn} [-1], \quad k = 1, 2, \quad (41a)$$

$$\text{sgn} \left[\frac{\partial i}{\partial S_1} \right] = \text{sgn} [-S_2], \quad (41b)$$

$$\text{sgn} \left[\frac{\partial i}{\partial S_2} \right] = \text{sgn} [S_1]. \quad (41c)$$

When $\bar{B} \neq 0$, the signs for these sensitivities are ambiguous.

What can be said regarding positive correlation when the heterogeneous agent economy e lies in one of the regions II–VII, and no additional restrictions are placed on e ? If e lies in region II, III, or V, then:

$$\text{sgn} [\Delta W(c)] = \text{sgn} [\Delta NSSW] = \text{sgn} [-1], \quad (42a)$$

$$\text{sgn} \left[\frac{\partial W(c)}{\partial T} \right] = \text{sgn} \left[\frac{\partial NSSW}{\partial T} \right] = \text{sgn} [-1], \quad (42b)$$

$$\text{sgn} \left[\frac{\partial u(c_k)}{\partial T} \right] = \text{sgn} \left[\frac{\partial NSSW_k}{\partial T} \right] = \text{sgn} [-1], \quad k = 1, 2. \quad (42c)$$

These are precisely the regions where $i > 0$ and *no* agents choose to hold positive money balances in middle age. Elsewhere, indeterminate signs prevail, and examples are easily constructed of economies for which net social security wealth and generation welfare are negatively correlated on both the macro and micro level.

Appendix: Technical notes

Section 2

As in section 2, let \mathcal{E} denote the set of all economies e which have at least one equilibrium v^e ; and, for each e in \mathcal{E} , let $\varphi(e)$ denote the set of all equilibria v^e corresponding to e . Finally, let $V \equiv \varphi(\mathcal{E})$ denote the set of all equilibria v^e corresponding to some e in \mathcal{E} .

Each element $v^e = (p^B, i, M, B, c)$ of V can be uniquely categorized according to the particular sign configuration of its components:

$$(i, M^1, M^2) \equiv (i, M_1^1, M_2^1, M_1^2, M_2^2).$$

This categorization results in a partition of V into ten subsets of the form:

$$V_I \equiv \{v \in V \mid i = 0\}, \quad V_{II} \equiv \{v \in V \mid i > 0, M^1 > \mathbf{0}, M^2 = \mathbf{0}\}, \dots$$

This partition of V in turn induces a partition on \mathcal{E} in one-to-one fashion. For example, V_I corresponds to:

$$I \equiv \{e \in \mathcal{E} \mid v^e \in V_I \text{ for all } v^e \in \varphi(e)\},$$

and V_{II} corresponds to

$$II \equiv \{e \in \mathcal{E} \mid v^e \in V_{II} \text{ for all } v^e \in \varphi(e)\}.$$

These ten subsets I, II, ... of \mathcal{E} are referred to as *liquidity preference regions*.

The ten liquidity preference regions of \mathcal{E} will now be generally characterized for reference purposes. Below the general characterization for each region are listed certain restrictions which must hold in that region for each *regular* economy, i.e. each economy $e = (N, \omega, \alpha, \beta, S, T, \bar{B})$ for which all equilibria v^e in $\varphi(e)$ are right and/or left differentiable with respect to (S, T, \bar{B}) at the point e . These restrictions are an incomplete characterization for the regular economies, since the real bond price p^B and interest rate i are not given in terms of the basic economy parameters. Nevertheless, they suggest how the partition of \mathcal{E} into liquidity preference regions depends on the shape of the individual endowment profiles after modification by government tax-transfer policies.

A complete basic parameter characterization of the partition of \mathcal{E} into liquidity preference regions is provided for both regular and nonregular economies in Tesfatsion (1982). The latter paper also graphically illustrates the complex crystalline aspects of the partition.

I. Money-bond indifference [$i=0$]

$TNS \neq 0$ (regularity condition),

$$\left[\frac{(1+\alpha+\beta)N_k\bar{B}_k + N_k S_k}{NS} \right] \leq \left[\frac{1-T}{T} \right] \left[\frac{(\alpha+\beta)N_k\omega_k^1 - N_k\omega_k^2}{N\omega} \right], \quad k=1, 2, \quad (\text{A.1})$$

$$\left[\frac{(1+\alpha+\beta)N_k\bar{B}_k + (1+\alpha)N_k S_k}{NS} \right] \leq \beta \left[\frac{1-T}{T} \right] \left[\frac{N_k\omega_k}{N\omega} \right], \quad k=1, 2,$$

for some values \bar{B}_1 and \bar{B}_2 in R satisfying $N_1\bar{B}_1 + N_2\bar{B}_2 = \bar{B}$.

II. Youth money holding only [$i > 0, M^1 > 0, M^2 = 0$]

$$N_k\omega_k^2 < \left(\frac{\alpha}{1+\beta} \right) [N_k\omega_k^1 + p^B[1-T]^{-1}N_k S_k], \quad \text{if } N_k > 0, k=1, 2.$$

III. No money holding [$i > 0, M^1 = 0, M^2 = 0$]

$D \equiv (1+\beta)\bar{B} + NS \neq 0$ (regularity condition),

$$\left(\frac{\alpha}{1+\beta} \right) [N_k\omega_k^1 + p^B[1-T]^{-1}N_k S_k] \leq N_k\omega_k^2$$

$$\leq (1+i) \left(\frac{\alpha}{1+\beta} \right) [N_k\omega_k^1 + p^B[1-T]^{-1}N_k S_k], \quad k=1, 2.$$

IV. Middle-age money holding only [$i > 0, M^1 = 0, M^2 > 0$]

$Q \equiv (1+\alpha+\beta)\bar{B} + NS \neq 0$ (regularity condition),

$$(1+i) \left(\frac{\alpha}{1+\beta} \right) [N_k\omega_k^1 + p^B[1-T]^{-1}N_k S_k] < N_k\omega_k^2 \quad \text{if } N_k > 0, k=1, 2.$$

V. Weakly dissimilar youth money holding [$N > 0, i > 0, M_i^1 = 0, M_i^2 = 0, M_n^1 > 0, M_n^2 = 0, (l, n) = (1, 2)$]

$$\left(\frac{\alpha}{1+\beta} \right) [\omega_l^1 + p^B[1-T]^{-1}S_l] \leq \omega_l^2$$

$$\leq (1+i) \left(\frac{\alpha}{1+\beta} \right) [\omega_l + p^B[1-T]^{-1}S_l],$$

$$\omega_n^2 < \left(\frac{\alpha}{1+\beta} \right) [\omega_n^1 + p^B [1-T]^{-1} S_n].$$

V*. Same as V except $(l, n) = (2, 1)$.

VI. *Weakly dissimilar middle-age money holding* [$N > 0, i > 0, M_l^1 = 0, M_l^2 > 0, M_n^1 = 0, M_n^2 = 0, (l, n) = (1, 2)$]

$$W_l \equiv (1 + \alpha + \beta)[(1 + \beta)\bar{B} + NS] - \alpha N_l S_l \neq 0 \quad (\text{regularity condition}),$$

$$(1 + i) \left(\frac{\alpha}{1 + \beta} \right) [\omega_l^1 + p^B [1 - T]^{-1} S_l] < \omega_l^2,$$

$$\left(\frac{\alpha}{1 + \beta} \right) [\omega_n^1 + p^B [1 - T]^{-1} S_n] \leq \omega_n^2$$

$$\leq (1 + i) \left(\frac{\alpha}{1 + \beta} \right) [\omega_n^1 + p^B [1 - T]^{-1} S_n].$$

VI*. Same as VI except $(l, n) = (2, 1)$.

VII. *Strongly dissimilar money holding* [$N > 0, i > 0, M_l^1 = 0, M_l^2 > 0, M_n^1 > 0, M_n^2 = 0, (l, n) = (1, 2)$]

$$R_{l,n} \equiv (1 + \alpha + \beta)\bar{B} + N_l S_l + (1 + \alpha)N_n S_n \neq 0 \quad (\text{regularity condition}),$$

$$(1 + i) \left(\frac{\alpha}{1 + \beta} \right) [\omega_l^1 + p^B [1 - T]^{-1} S_l] < \omega_l^2,$$

$$\omega_n^2 < \left(\frac{\alpha}{1 + \beta} \right) [\omega_n^1 + p^B [1 - T]^{-1} S_n].$$

VII*. Same as VII except $(l, n) = (2, 1)$.

Section 3

The deviation of the equilibrium macro consumption profiles from their socially optimal values depicted in table 1 can be determined from (13), together with the general shape of the equilibrium micro consumption profiles in each liquidity preference region of \mathcal{E} . For example, in region II any equilibrium micro consumption profile c must satisfy:

$$c_k^2 = \alpha c_k^1, c_k^3 = (1 + i)\beta c_k^1, \quad k = 1, 2.$$

Thus, $Nc^1 - Nc^{1*} \geq 0$ would imply both

$$Nc^2 - Nc^{2*} \geq 0$$

and

$$\begin{aligned} Nc^3 - Nc^{3*} &= \beta[(1+i)Nc^1 - Nc^{1*}] \\ &> \beta[Nc^1 - Nc^{1*}] \geq 0. \end{aligned}$$

However, this in turn would imply that:

$$Nc = Nc^1 + Nc^2 + Nc^3 > Nc^{1*} + Nc^{2*} + Nc^{3*} = N\omega,$$

a contradiction of the market clearing condition (6) which holds by definition for each equilibrium c . It follows (proof by contradiction) that $Nc^1 < Nc^{1*}$ in region II.

The deviations for the micro consumption profiles are determined even more directly from (13) and general profile shape. For example,

$$\begin{aligned} N_1c_1^1 - N_1c_1^{1*} &= N_1c_1^1 - \theta Nc^{1*} \\ &= N_1c_1^1 - \theta N_1c_1^1 + \theta N_1c_1^1 \\ &\quad - \theta N_2c_2^1 + \theta N_2c_2^1 - \theta Nc^{1*} \\ &= [(1-\theta)N_1c_1^1 - \theta N_2c_2^1] + \theta[Nc^1 - Nc^{1*}] \\ &\equiv -Z + \theta[Nc^1 - Nc^{1*}] \\ &\equiv X_1^1. \end{aligned}$$

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