Stochastic Dominance and the Maximization of Expected Utility

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In attempting to construct a general framework for the analysis of choice under uncertainty, researchers have sought to establish reasonable criteria for the selection of one prospect over another. Among current researchers the concept of stochastic dominance[1] has attracted considerable attention. This paper attempts to clarify certain basic relationships between stochastic dominance and the maximization of expected utility.

The theorems proposed by Hanoch and Levy in [1] relate stochastic dominance to a broader class of utility functions than considered by previous researchers working in the area. However, errors appear in the statement and proof of these theorems. Modifications which correct these errors are given in Part 1.

The 1971 article [2] by Josef Hadar and William Russell contains a number of theorems which can be generalized using the modified theorems from Part 1. These results are presented in Part 2.

In what follows, F and G denote arbitrary right continuous distribution functions over the real line R. We let $U^* = \{u\colon R\to R \mid u$ non-decreasing and continuous$\}$, $U^*(F, G) = \{u \in U^* \mid \int udF - \int udG$ well defined$\}$, $U^{**} = \{u\colon R\to R \mid u$ non-decreasing and concave$\}$, and $U^{**}(F, G) = \{u \in U^{**} \mid \int udF - \int udG$ well defined, $\int_{-\infty}^{0} udF > -\infty$, and $\int_{-\infty}^{0} udG > -\infty\}$. We use $[FDG \text{ wrt } U]$ to mean $[\int udF - \int udG \geqq 0 \forall u \in U$, strict inequality holding for some $u \in U]$, where U is any specified class of functions $u\colon R\to R$. All integrals are to be interpreted as Riemann-Stieltjes integrals unless otherwise specified. By " well defined " is meant " not of the form $\infty - \infty$ ".

We also use $[F \leqq G]$ to mean $[F(x) \leqq G(x) \forall x \in R]$, $[F < G]$ to mean

$$[F(x) \leqq G(x) \forall x \in R, \quad F \not\equiv G],$$

$[F \underline{\ll} G]$ to mean

$$\left[\int_{-\infty}^{x} [F(t) - G(t)]dt \leqq 0 \forall x \in R\right],$$

and $[F \ll G]$ to mean

$$\left[\int_{-\infty}^{x} [F(t) - G(t)]dt \leqq 0 \forall x \in R, F \not\equiv G\right].$$

The prefix " H-L " abbreviates " Hanoch-Levy " and the prefix " H-R " abbreviates " Hadar-Russell ".

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(1.1) **H-L Lemma 1** ([1], page 336)

If $u\colon R\to R$ is non-decreasing, then $\int udF - \int udG = \int [G-F]du$.

Remark. H-L state [1, p. 340] that all their integrals are Lebesgue-Stieltjes. Some of their integrals do not exist in this sense, as will be indicated below where appropriate.

The statement and proof of this lemma are in need of modification for the following reasons: (1) there is no guarantee that the integral on the right-hand side exists in the Lebesgue sense; (2) contrary to H-L's argument on pages 336-337, it is incorrect that $\int du[F-G] = 0$ for all u as given in Lemma 1;[2] (3) integration by parts (Lebesgue-Stieltjes integration) may not hold if u and F or u and G have a common point of discontinuity[3] (using Riemann-Stieltjes integration, the integrals in question may not even exist); (4) $\int[G-F]du$ may exist while $\int udF - \int udG$ is not well defined.[4]

Lemma 1* (modification of H-L Lemma 1)

$$\int udF - \int udG = \int [G-F]du \forall u \in U^*(F, G).$$

Proof. The proof of this equality for arbitrary $u \in U^*(F, G)$ will be given in three steps.

Step 1.

$$\left[\int_{0-}^{\infty} udF \text{ finite}\right] \Leftrightarrow \left[\int_{0-}^{\infty} [1-F]du \text{ finite}\right],$$

and if either side holds, then

$$\int_{0-}^{\infty} udF = u(0)[1-F(0^-)] + \int_{0-}^{\infty} [1-F]du.$$

Proof. Suppose $\int_{0-}^{\infty} udF$ is finite. Then

$$\infty > \int_{N+}^{\infty} udF \geqq u(N)[1-F(N^+)] \geqq u(0)[1-F(N^+)] \forall N \geqq 0 \Rightarrow 0 = \lim_{N\to\infty} \int_{N+}^{\infty} udF$$

$$\geqq \lim_{N\to\infty} \sup u(N)[1-F(N^+)] \geqq u(0) \lim_{N\to\infty} [1-F(N^+)] = 0 \Rightarrow \lim_{N\to\infty} u(N)[1-F(N^+)] = 0.$$

Using integration by parts,

$$\int_{0-}^{N+} udF = uF\Big|_{0-}^{N+} - \int_{0-}^{N+} Fdu \equiv -u[1-F]\Big|_{0-}^{N+} + \int_{0-}^{N+} [1-F]du \Rightarrow \int_{0-}^{\infty} udF$$

$$= \lim_{N\to\infty} \left[-u[1-F]\Big|_{0-}^{N+} + \int_{0-}^{N+} [1-F]du \right]$$

$$= 0 - [-u(0)[1-F(0^-)] + \lim_{N\to\infty} \int_{0-}^{N+} [1-F]du \Rightarrow \int_{0-}^{\infty} [1-F]du \text{ is finite,}$$

and

$$\int_{0-}^{\infty} udF = u(0)[1-F(0^-)] + \int_{0-}^{\infty} [1-F]du.$$

Suppose $\int_{0-}^{\infty} [1-F]du$ is finite. Then

$$\int_{M-}^{N+} [1-F]du \geqq [1-F(N^+)][u(N)-u(M)] \geqq 0 \forall N > M \Rightarrow \infty > \int_{M-}^{\infty} [1-F]du$$

$$= \lim_{N\to\infty} \int_{M-}^{N+} [1-F]du \geqq \lim_{N\to\infty} \sup [1-F(N^+)][u(N)-u(M)]$$

$$= \lim_{N\to\infty} \sup [1-F(N^+)]u(N) \geqq u(0) \lim_{N\to\infty} [1-F(N^+)] = 0.$$

Hence

$$0 = \lim_{M\to\infty} \int_{M-}^{\infty} [1-F]du \geqq \lim_{M\to\infty} [\lim_{N\to\infty} \sup [1-F(N^+)]u(N)] \geqq 0$$

$$\Rightarrow \lim_{N\to\infty} [1-F(N^+)]u(N) = 0.$$

Using integration by parts, it follows that

$$\int_{0-}^{\infty} [1-F]du = \lim_{M\to\infty} \int_{0-}^{M+} [1-F]du = \lim_{M\to\infty} \left[[1-F]u \Big|_{0-}^{M+} - \int_{0-}^{M+} u[-dF] \right]$$

$$= -u(0)[1-F(0^-)] + \int_{0-}^{\infty} udF.$$

Step 2.

$$\left[\int_{-\infty}^{0-} udF \text{ finite} \right] \Leftrightarrow \left[\int_{-\infty}^{0-} Fdu \text{ finite} \right],$$

and if either side holds, then

$$\int_{-\infty}^{0-} udF = F(0^-)u(0) - \int_{-\infty}^{0-} Fdu.$$

Proof. The proof of Step 2 is similar to the proof of Step 1, and will be omitted.

Step 3.

$$\int udF - \int udG = \int [G-F]du.$$

Proof. Suppose $\int udF$ and $\int udG$ are finite. Then $\int_{0-}^{\infty} udF$, $\int_{-\infty}^{0-} udF$, $\int_{0-}^{\infty} udG$ and $\int_{-\infty}^{0-} udG$ are finite. Hence, using the identities established in Steps 1 and 2 above, it is easily verified that

$$\int udF - \int udG \equiv \int_{0-}^{\infty} udF + \int_{-\infty}^{0-} udF - \int_{0-}^{\infty} udG - \int_{-\infty}^{0-} udG = \int [G-F]du.$$

Suppose $\int udF - \int udG = +\infty$. Since

$$\int_{0-}^{\infty} udG \geqq u(0)[1-G(0^-)] > -\infty$$

and

$$\int_{-\infty}^{0-} udF \leqq u(0)F(0^-) < +\infty, \left[\int udF - \int udG = +\infty \right] \Rightarrow \left[\int_{0-}^{\infty} udF \leqq +\infty, \right.$$

$$\left. \int_{-\infty}^{0-} udF \text{ finite}, \int_{0-}^{\infty} udG \text{ finite}, \infty > \int_{-\infty}^{0-} udG \geqq -\infty \right],$$

with either $\int_{0-}^{\infty} udF = +\infty$ or $\int_{-\infty}^{0-} udG = -\infty$, say the former. Then $\int_{-\infty}^{0-} Fdu$ and $\int_{0-}^{\infty} [1-G]du$ are finite, $\int_{0-}^{\infty} [1-F]du = +\infty$, and $-\infty < \int_{-\infty}^{0} Gdu \leqq +\infty$. Thus

$$\int_{0-}^{\infty} [1-F]du - \int_{0-}^{\infty} [1-G]du = \int_{0-}^{\infty} [G-F]du = +\infty,$$

and

$$-\infty < \int_{-\infty}^{0-} Gdu - \int_{-\infty}^{0-} Fdu = \int_{-\infty}^{0-} [G-F]du \leqq +\infty$$

$$\Rightarrow \int_{-\infty}^{\infty} [G-F]du = +\infty = \int udF - \int udG.$$

Similarly for $\int_{-\infty}^{0-} udG = -\infty$.

The argument for $\int udF - \int udG = -\infty$ follows immediately from the above by symmetry.

(1.2) **H-L Theorem 1** ([1], p. 337)

Define U to be the class of all non-decreasing, real-valued functions. Then $[FDG \text{ wrt } U] \Leftrightarrow [F < G]$.

This theorem is based on H-L Lemma 1. The class U has therefore been modified as in Lemma 1*.

Theorem 1* (modification of H-L Theorem 1)

$$[FDG \text{ wrt } U^*(F, G)] \Leftrightarrow [F < G].$$

Proof. Suppose $F<G$. Then $G(x)-F(x) \geqq 0 \forall x \rightarrow \int [G-F]du \geqq 0 \forall u \in U^*(F, G) \Rightarrow$ by Lemma 1*, $\int udF - \int udG \geqq 0 \forall u \in U^*(F, G)$. Suppose $G(x')-F(x')>0$. By right continuity of G and F, $\exists$ an interval $[x', r)$ over which $G>F$. Define v: $R \rightarrow R$ by $v(x) = x'$ if $x \leqq x'$, $v(x) = x$ if $x'<x<r$, and $v(x) = r$ if $x \geqq r$. Then $v \in U^*(F, G)$ and by Lemma 1*,

$$\int vdF - \int vdG = \int [G-F]dv = \int_{x'}^{r} [G(x)-F(x)]dx > 0.$$

Hence $[F<G] \Rightarrow [FDG \text{ wrt } U^*(F, G)]$.

Suppose for some x^*, $G(x^*)-F(x^*)<0$. By right continuity of G and F $\exists$ an interval $[x^*, s)$ over which $G<F$. Define w: $R \rightarrow R$ by $w(x) = x^*$ if $x \leqq x^*$, $w(x) = x$ if $x^*<x<s$, and $w(x) = s$ if $x \geqq s$. Then $w \in U^*(F, G)$ and by Lemma 1*,

$$\int wdF - \int wdG = \int [G-F]dw = \int_{x^*}^{s} [G(x)-F(x)]dx < 0.$$

Hence not $[G(x)-F(x) \geqq 0 \forall x]$, which implies not $[FDG \text{ wrt } U^*(F, G)]$. If $G \equiv F$, then $\int udF - \int udG = 0$ for all $u \in U^*(F, G)$; hence not $[FDG \text{ wrt } U^*(F, G)]$. ||

(1.3) **H-L Theorem 2** ([1], p. 338)

Let U_1 be the class of all non-decreasing, concave functions. Then

$$[FDG \text{ wrt } U_1] \Leftrightarrow [F \ll G].$$

The statement and proof of this theorem as given by H-L are in need of modification for the following reasons: (1) the change of variables at the bottom of page 339 is not justified, since T is not necessarily one-to-one over $[t\colon t \leqq x]$ (e.g. T could be identically $-\infty$ over $[t\colon t<x]$); (2) to use their argument for necessity (pp. 338-339), H-L need that $\int_{-\infty}^{x} tdF(t) - \int_{-\infty}^{x} tdG(t)$ is well defined for some (hence every) finite x. Their hypotheses do not guarantee this (see *claim* below); (3) to use their argument for sufficiency, H-L need $\int_{-\infty}^{x} | G-F | dt < \infty$ for some (hence every) finite x. Their hypotheses do not guarantee the existence of the integral in this sense (Lebesgue); (4) in both their necessity and sufficiency arguments H-L rely on Lemma 1; (5) in their argument for sufficiency (p. 339, (4)) H-L decompose $\int_{-\infty}^{x} [G-F](t)du(t)$ into its positive and negative parts for arbitrary $u \in U_1$. This is not justified unless one of the parts is known to be finite. The H-L hypotheses do not guarantee this for $u \in U_1$.

In view of the above points, it seems worthwhile to restate and prove this important theorem in detail. We first prove a claim which supports the assertion in (1) above that $\left[\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG \text{ well defined}\right]$ is necessary for the conclusion of H-L Theorem 2.

Claim. If $\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ is not well defined, then $[F \ll G] \not\Rightarrow [FDG \text{ wrt } U_1]$.

Proof. $\left|\int_{-M}^{0} xdF\right|$ and $\left|\int_{-M}^{0} xdG\right|$ are finite for all $M \in R$, $\int_{-\infty}^{0} xdF \leqq 0$, and $\int_{-\infty}^{0} xdG \leqq 0$. For $\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ not to be well defined, it must therefore hold that $\int_{-\infty}^{M} xdF = \int_{-\infty}^{M} xdG = -\infty$ for all $M \in R$. The graph of any non-decreasing, non-constant, real-valued concave function u defined over R is bounded above by some line $\{(x, f(x)) \in R^2 \mid f(x) = bx+c\}$, where $b, c \in R$, $b>0$. Thus for sufficiently negative M, $-\infty = \int_{-\infty}^{M} [bx+c]dF \geqq \int_{-\infty}^{M} u(x)dF(x)$, and similarly for G in place of F. Thus $\int_{-\infty}^{\infty} udF - \int_{-\infty}^{\infty} udG$ is not well defined for any non-constant $u \in U_1$. ||

Theorem 2* (modification of H-L Theorem 2)

Suppose $\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ *is well defined and* $\int_{-\infty}^{0} |G-F|\, dt < \infty$. *Then* $[FDG \text{ wrt } U^{**}(F, G)] \Leftrightarrow [F \ll G]$.

Proof.

A. $[FDG \text{ wrt } U^{**}(F, G)] \Rightarrow [F \ll G]$

Suppose $\exists x'$ s.t. $\int_{-\infty}^{x'} [G-F](t)dt < 0$. Define u^*: $R \rightarrow R$ by $u^*(x) = x'$ if $x \geqq x'$, $= x$ if $x < x'$. $\int_{-\infty}^{0} |G-F|\, dt < \infty$ and $\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ well defined$\Rightarrow$by Lemma 1*, $\int_{-\infty}^{0} xdG$ and $\int_{-\infty}^{0} xdF$ are finite. Hence $u^* \in U^{**}(F, G)$ and by Lemma 1*,

$$\int u^*dF - \int u^*dG = \int_{-\infty}^{x'} [G-F]du^* < 0.$$

Thus not $[FDG \text{ wrt } U^{**}(F, G)]$.

Suppose $F \equiv G$. Then $\int udF - \int udG = 0 \forall u \in U^{**}(F, G) \Rightarrow$not $[FDG \text{ wrt } U^{**}(F, G)]$.

B. $[F \ll G] \Rightarrow [FDG \text{ wrt } U^{**}(F, G)]$

Assume $\int_{-\infty}^{x} [G-F]dt \geqq 0 \forall x$. Then

$$\int_{-\infty}^{x} [G-F]I_A(t)dt + \int_{-\infty}^{x} [G-F]I_B(t)dt \geqq 0 \forall x \Rightarrow \int_{-\infty}^{x} [G-F]I_A(t)dt \geqq \int_{-\infty}^{x} [F-G]I_B(t)dt \forall x,$$

where $I_A(\cdot)$ is the indicator function for $A = \{x\colon G(x) \geqq F(x)\}$, $I_B(\cdot)$ is the indicator function for $B = \{x\colon G(x) < F(x)\}$, and the decomposition is justified by the hypothesis $\int_{-\infty}^{0} |G-F|\, dt < \infty$. Since $0 \leqq \int_{-\infty}^{0} [G-F]I_A(t)dt < \infty$, $\lim_{x\to-\infty} \int_{-\infty}^{x} [G-F]I_A(t)dt = 0$.

The proof that $[FDG \text{ wrt } U^{**}(F, G)]$ will now proceed in four steps.

Step 1. One may define a function $T\colon R \to [-\infty, +\infty)$ s.t. the following hold:

1. For each x, $\int_{-\infty}^{T(x)} [G-F](t)I_A(t)dt = \int_{-\infty}^{x} [F-G](t)I_B(t)dt$; ...(1)
2. $T(x) \leqq x \forall x$;
3. T is a monotone non-decreasing function, continuous and differentiable a.e. over its essential domain $D = \{x\colon T(x) > -\infty\}$.

Proof. By the above, for a value x' (possibly $-\infty$) to exist for each x s.t.

$$\int_{-\infty}^{x'} [G-F]I_A(t)dt = \int_{-\infty}^{x} [F-G]I_B(t)dt,$$

it is sufficient that $\int_{-\infty}^{x} [G-F]I_A(t)dt$ be a continuous function of x. But this is immediate, since for any x and y

$$\left|\int_{-\infty}^{x} [G-F]I_A(t)dt - \int_{-\infty}^{y} [G-F]I_A(t)dt\right| = \left|\int_{y}^{x} [G-F]I_A(t)dt\right| \leqq 2\,|x-y|.$$

Define

$$T(x) = \inf\left\{x' \,\middle|\, \int_{-\infty}^{x'} [G-F]I_A(t)dt = \int_{-\infty}^{x} [F-G]I_B(t)dt\right\}.$$

Then T is a well-defined function of x, and, since $[G-F](t)I_A(t) \geqq 0$ for all t, it is clear that $T(x) \leqq x$ for all x. By definition of T and positivity of $[F-G](t)I_B(t)$ for all t,

$$[x \geqq x'] \Rightarrow \left[\int_{-\infty}^{x} [F-G](t)I_B(t)dt \geqq \int_{-\infty}^{x'} [F-G](t)I_B(t)dt\right] \Rightarrow [T(x) \geqq T(x')].$$

Hence T is a monotone non-decreasing function of x, continuous and differentiable a.e. over its essential domain $D = \{x\colon T(x) > -\infty\}$.

Step 2.

$$I_A(T(t))[G(T(t)) - F(T(t))]\tilde{T}(t) = I_B(t)[F(t) - G(t)] \quad \text{...(2)}$$

a.e. on R, where $\tilde{T}(t) = T'(t)$ if $T'(t)$ exists, $\tilde{T}(t) = 1$ if $T'(t)$ does not exist.

Proof. Let $D^C \equiv \{t\colon T(t) = -\infty\}$. If $D^C = \varnothing$, then (2) holds vacuously over D^C. Suppose $D^C \neq \varnothing$. Let $t' = \sup\{t \mid t \in D^C\}$. Then $T(t) = -\infty$ for all $t < t' \Rightarrow$

$$\int_{-\infty}^{s} [F(t) - G(t)]I_B(t)dt = 0$$

for all $s < t' \Rightarrow [F(t) - G(t)]I_B(t) = 0$ a.e. over $\{t \mid t < t'\}$. Since

$$G(T(t)) - F(T(t)) = G(-\infty) - F(-\infty) = 0 \forall t < t',$$

(2) holds a.e. over $\{t \mid t < t'\}$, and hence a.e. over D^C. Differentiating (1) (see Step 1) at any point t where $T'(t)$ exists, one gets

$$I_A(T(t))[G(T(t)) - F(T(t))]T'(t) = I_B(t)[F(t) - G(t)].$$

Hence (2) holds a.e. over $D = \{t\colon T(x) > -\infty\}$. Since $D^C \cup D = R$, the proof of Step 2 is complete.

Step 3. For any $u \in U^{**}(F, G)$, $\int_{-\infty}^{x} [G(t)-F(t)]du(t) \geqq 0 \forall x$; hence $\int udF - \int udG \geqq 0$.

Proof. Let $u \in U^{**}(F, G)$. Then for any x, using Step 2 in the proof of Lemma 1*,

$$\int_{-\infty}^{x} |G-F|\, du \leqq \int_{-\infty}^{x} Gdu + \int_{-\infty}^{x} Fdu = -\left[\int_{-\infty}^{0} udG + \int_{-\infty}^{0} udF\right] + \text{constant} < \infty.$$

Hence the following decomposition is justified:

$$\begin{aligned}\int_{-\infty}^{x} [G-F]du &= \int_{-\infty}^{x} [G-F](t)I_A(t)du(t) + \int_{-\infty}^{x} [G-F](t)I_B(t)du(t)\\ &= \int_{-\infty}^{x} [G-F](t)I_A(t)du(t) - \int_{-\infty}^{x} [F-G](t)I_B(t)du(t)\\ &= \int_{-\infty}^{x} [G-F](t)I_A(t)du(t) - \int_{-\infty}^{x} I_A(T(t))[G(T(t))-F(T(t))]\tilde{T}(t)du(t), \quad \ldots(3)\end{aligned}$$

where the last substitution follows from Step 2. T monotone non-decreasing and u non-decreasing, concave and finite over R imply that $M_1 \equiv \{t \mid T'(t)$ does not exist and $T(t) > -\infty\}$ and $M_2 \equiv \{t \mid u'(t)$ or $u'(T(t))$ does not exist$\} \cap \{t \mid T'(t) > 0\}$ have Lebesgue measure zero, u is absolutely continuous, and $u'(t_2) \leqq u'(t_1)$ whenever both derivatives exist and $t_1 \leqq t_2$. Let $M_3 \equiv \{t \mid T(x) = -\infty$ or $T'(t) = 0\}$. Then

$$I_A(T(t))[G(T(t))-F(T(t))]\tilde{T}(t) = 0$$

whenever $t \in M_3$. Defining $S = \{t \mid t \leqq x\} \cap [M_1 \cup M_2 \cup M_3]^C$, the following holds:

$$\begin{aligned}&\int_{-\infty}^{x} I_A(T(t))[G(T(t)-F(T(t))]\tilde{T}(t)du(t)\\ &\qquad = \int_S I_A(T(t))[G(T(t))-F(T(t))]T'(t)u'(t)dt\\ &\qquad \leqq \int_S I_A(T(t))[G(T(t))-F(T(t))]T'(t)u'(T(t))dt\\ &\qquad = \int_S I_A(T(t))[G(T(t))-F(T(t))]du(T(t))\\ &\qquad = \int_{T(S)} I_A(z)[G(z)-F(z)]du(z),\end{aligned}$$

where the last inequality follows by concavity of u and monotonicity of T and the last equality follows since T is one-to-one over S. Since the integrand and the measure du are positive, integrating over a larger set can only increase the value of the integral. Thus

$$\int_{T(S)} I_A(z)[G(z)-F(z)]du(z) \leqq \int_{[z \leqq T(x)]} I_A(z)[G(z)-F(z)]du(z).$$

Substituting this result into (3),

$$\begin{aligned}\int_{-\infty}^{x} [G-F]du &\geqq \int_{-\infty}^{x} [G-F](t)I_A(t)du(t) - \int_{[t \leqq T(x)]} I_A(t)[G(t)-F(t)]du(t)\\ &= \int_{[T(x) \leqq t \leqq x]} [G-F](t)I_A(t)du(t) \geqq 0\end{aligned}$$

since $T(x) \leqq x \forall x$. Hence by Lemma 1*,

$$\int udF - \int udG = \int_{-\infty}^{\infty} [G-F]du = \lim_{x\to\infty} \int_{-\infty}^{x} [G-F]du \geqq 0.$$

Step 4. If $F \not\equiv G$ in addition to $\int_{-\infty}^{x} [G-F]dt \geqq 0 \forall x$, *then* $\exists u \in U^{**}(F, G)$ s.t.

$$\int udF - \int udG > 0.$$

Proof. $F \not\equiv G$ and $\int_{-\infty}^{x} [G-F]dt \geqq 0 \forall x \Rightarrow \exists x' \in R$ s.t. $G(x') > F(x') \Rightarrow$(by right continuity of F and G) there exists an interval $[x', r)$ over which $G(x) > F(x)$. Define $u(x) = r$ if $x \geqq r$, $u(x) = x$ if $x < r$. $\int_{-\infty}^{0} | G-F | dt < \infty$ and $\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ well defined implies by Lemma 1* that $\int_{-\infty}^{0} xdG$ and $\int_{-\infty}^{0} xdF$ are finite. Hence $u \in U^{**}(F, G)$ and by Lemma 1*,

$$\int udF - \int udG = \int [G-F]du = \int_{-\infty}^{r} [G-F]dt = \int_{x'}^{r} [G-F]dt + \int_{-\infty}^{x'} [G-F]dt > 0.$$

This completes Step 4.

Combining Steps 3 and 4,

$$\left[\int_{-\infty}^{x} [G-F]dt \geqq 0 \forall x, G \not\equiv F\right] \equiv [F \ll G] \Rightarrow [FDG \text{ wrt } U^{**}(F, G)]. \quad \|$$

Corollary 2*. *Under the hypotheses of Theorem* 2*,

$$\left[\int udF \geqq \int udG \forall u \in U^{**}(F, G)\right] \Leftrightarrow [F \leqslant G].$$

Proof. Immediate from the proof of Theorem 2*.

(1.4) **H-L Theorem 3** ([1], p. 341)

Let F and G have mean values μ_1, μ_2 respectively. Suppose for some $x' < \infty$, $F(x) \leqq G(x)$ for all $x \leqq x'$, $F(x'') < G(x'')$ for some $x'' < x'$, and $F(x) \geqq G(x)$ for all $x \geqq x'$. Then FDG wrt *concave utility functions if and only if $\mu_1 \geqq \mu_2$.*

Since this theorem is based on Lemma 1 and Theorem 2, it has been modified in accordance with the changes made there.

Theorem 3* (modification of H-L Theorem 3)

Suppose $\int xdF - \int xdG$ *is well defined and* $\int_{-\infty}^{0} | G-F | dt < \infty$. *If*

$$\exists x' \in R \text{ s.t. } F(x) \leqq G(x) \forall x \leqq x', F(x) \geqq G(x) \forall x \geqq x',$$

and $F \not\equiv G$, *then*

$$[FDG \text{ wrt } U^{**}(F, G)] \Leftrightarrow \left[\int xdF - \int xdG \geqq 0\right].$$

Proof. Suppose $\int xdF - \int xdG \geqq 0$. Then by Lemma 1*

$$\int xdF - \int xdG = \int [G-F]dx = \int_{-\infty}^{x'} [G-F]dt - \int_{x'}^{\infty} |G-F|\, dt \geqq 0,$$

which implies $\int_{-\infty}^{x} [G-F]dt \geqq 0$ for all x. By hypothesis $\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ is well defined, $\int_{-\infty}^{0} |G-F|\, dt < \infty$ and $F \not\equiv G$; hence by Theorem 2* [FDG wrt $U^{**}(F, G)$].

Suppose not [$\int xdF - \int xdG \geqq 0$]. Since $\int xdF - \int xdG$ is well defined by hypothesis,

$$\text{not}\left[\int xdF - \int xdG \geqq 0\right] \Rightarrow \int xdF - \int xdG = \int_{-\infty}^{x'} [G-F]dt - \int_{x'}^{\infty} |G-F|\, dt < 0.$$

Hence for some finite $x'' > x'$,

$$\int_{-\infty}^{x''} [G-F]dt = \int_{-\infty}^{x'} [G-F]dt - \int_{x'}^{x''} |G-F|\, dt < 0,$$

whereas $\int_{-\infty}^{x'} [G-F]dt > 0$. By Theorem 2*, not [$FDG$ wrt $U^{**}(F, G)$]. ||

(1.5) **H-L Theorem 4** ([1], p. 343)

Let F and G be two distinct distributions with means μ_F and μ_G and variances σ_F^2 and σ_G^2, respectively, such that $F(x) = G(y)$ for all x and y which satisfy $[x-\mu_F]/\sigma_F = [y-\mu_G]/\sigma_G$. Let $\mu_F \geqq \mu_G$ and $F(x_1) > G(x_1)$ for some x_1 (i.e. F and G intersect), and let U_1 be the class of all non-decreasing concave functions. Then [FDG wrt U_1]$\Leftrightarrow$[$\sigma_F^2 \leqq \sigma_G^2$].

Since H-L Theorem 4 is based on Theorem 1, Theorem 2 and Theorem 3, it has been modified in accordance with the changes made in those theorems. The proof of the modified theorem will not be given, since the only non-trivial change to be made is the replacement of U_1 with $U^{**}(F, G)$.

Theorem 4* (modification of H-L Theorem 4)

Let F and G have finite[5] *means and variances (μ_F, σ_F^2), (μ_G, σ_G^2) respectively. Suppose $\mu_F \geqq \mu_G$, $F(x) = G(y) \forall x$ and y satisfying $[x-\mu_F]/\sigma_F = [y-\mu_G]/\sigma_G$ and $F(x') > G(x')$ for some x'. Then* [FDG wrt $U^{**}(F, G)$]$\Leftrightarrow$[$\sigma_F^2 \leqq \sigma_G^2$].

(1.6) **H-R Theorem 1**[6] ([2], p. 290)

Define U_1 to be the class of all bounded, strictly increasing, real-valued functions on R s.t. the first derivative exists and is continuous everywhere. Then

$$[\int udG - \int udF > 0 \forall u \in U_1] \Leftrightarrow [G < F].$$

Corollary 1* strengthens the sufficiency direction of H-R Theorem 1, since the class of functions appearing in the corollary is considerably larger. For this same reason, the necessity direction of H-R Theorem 1 is a strengthening of Theorem 1*.

The proof of necessity in H-R Theorem 1 is incomplete. A complete necessity proof is given below.

Theorem 5* (modification of H-R Theorem 1)

$$[GDF \text{ wrt } U_1] \Rightarrow [G < F].$$

Proof. Suppose $G(x^*)>F(x^*)$ for some $x^*\in R$. By right continuity of F and $G\exists$ an interval $[x^*, r]$ over which $G(x)>F(x)$. Let U^0 be the class of all real-valued, continuous, positive functions u^0 over R possessing a bell-shaped graph with peak at

$$([x^*+r]/2,\ u^0([x^*+r]/2)).$$

By choosing $u^0\in U^0$ so that $u^0([x^*+r]/2)$ is sufficiently large, $u^0(x^*)$ and $u^0(r)$ are sufficiently near zero, and $|u^0(x)|$ approaches zero sufficiently rapidly as $|x|\to+\infty$, it is clear that the sign of

$$\int [F-G](x)u^0(x)dx = \int_r^{\infty} [F-G](x)u^0(x)dx+\int_{x^*}^{r} [F-G](x)u^0(x)dx+\int_{-\infty}^{x^*} [F-G](x)u^0(x)dx$$

can be made negative.

It is also clear that one may require $|u^0(x)|$ to approach zero so rapidly for $|x|\to+\infty$ that $u(x)\equiv\int_{-\infty}^{x} u^0(t)dt$ is a bounded function of x. In this case $u\in U_1$, $\int udF$ and $\int udG$ both exist and are finite, and by Lemma 1* $\int udG-\int udF=\int[F-G](x)u^0(x)dx<0$; hence not $[GDF$ wrt $U_1]$. If $F\equiv G$, then no $u\in U_1$ exists with $\int udG-\int udF>0$; hence again not $[GDF$ wrt $U_1]$. ||

(1.7) **H-R Theorem 2** ([2], p. 292)

Define $U_2=[u\in U_1 \mid u''$ *continuous and non-positive*]. *Let F and G be the distribution functions for two non-negative random variables with finite means. Then*

$$[GDF \text{ wrt } U_2]\Leftrightarrow[G\ll F].$$

The hypotheses made in H-R Theorem 2 are stronger and the class of functions U_2 smaller than the corresponding hypotheses and functions appearing in Theorem 2*. Hence the sufficiency direction in Theorem 2* strengthens that of H-R Theorem 2, whereas the necessity direction in H-R Theorem 2 is a strengthening of Theorem 2* for distribution functions corresponding to non-negative random variables.

The proof of necessity in H-R Theorem 2 is incomplete. A complete necessity proof is given below.

Theorem 6* (modification of H-R Theorem 3)

$$\int_0^{x'} G(t)dt>\int_0^{x'} F(t)dt \quad \textit{for some} \quad x'\in R\Rightarrow \textit{not}\ [GDF \text{ wrt } U_2].$$

Proof. Suppose $\int_0^{x'} G(t)dt>\int_0^{x'} F(t)dt$. Let $\varepsilon, \Delta\in(0, 1)$, and let $u'_{\Delta,\varepsilon}(x)$ equal 1 for $x\in[0, x']$, have range in $(0, \Delta x'^2/x^2)$ for $x\in[x'+\varepsilon, +\infty)$ and be defined over $(x', x'+\varepsilon)$ in such a way that $u'_{\Delta,\varepsilon}$ is positive, differentiable, and non-increasing over $(0, \infty)$. Then $\lim_{\Delta,\varepsilon\to 0} u'_{\Delta,\varepsilon}(x)=1$ if $0\leqq x\leqq x'$, $=0$ if $x>x'$, and for all $\varepsilon, \Delta\in(0, 1)$,

$$|[F(x)-G(x)]u'_{\Delta,\varepsilon}(x)|\leqq g(x),$$

where $g(x)=2x'^2/x^2$ if $x\geqq x'+1$, $g(x)=2$ if $0\leqq x<x'+1$, and $g(x)=0$ if $x<0$.

It is clear that for each $\Delta, \varepsilon\in(0, 1)$, $u_{\Delta,\varepsilon}(x)\equiv\int_0^{x} u'_{\Delta,\varepsilon}(t)dt$ is in U_2. In particular,

$u_{\Delta,\varepsilon}$ is bounded and hence $\int u_{\Delta,\varepsilon}dG - \int u_{\Delta,\varepsilon}dF$ is well defined. Using Lemma 1* and Lebesgue's Dominated Convergence Theorem,

$$\lim_{\Delta,\varepsilon\to 0}\left[\int_0^\infty u_{\Delta,}1_\varepsilon dG - \int_0^\infty u_{\Delta,\varepsilon}dF\right] = \lim_{\Delta,\varepsilon\to 0}\left[\int_0^\infty [F-G](t)u'_{\Delta,\varepsilon}(t)dt\right]$$

$$= \int_0^\infty \lim_{\Delta,\varepsilon\to 0}\left[[F-G](t)u'_{\Delta,\varepsilon}(t)\right]dt = \int_0^{x'} [F-G](t)dt < 0.$$

One can therefore choose Δ^*, $\varepsilon^* \in (0, 1)$ s.t. $\int_0^\infty u_{\Delta^*,\varepsilon^*}dG - \int_0^\infty u_{\Delta^*,\varepsilon^*}dF < 0$, and hence not $[GDF \text{ wrt } U_2]$. ||

2

The conclusions of certain theorems in [2] can be strengthened by using the modified Hanoch-Levy Theorems 1* and 2* in place of the Hadar-Russell Theorems 1 and 2. Also, non-negativity, independence, and density assumptions can be relaxed as well as certain restrictions on coefficients. These generalized theorems are presented below.

In Theorem 1′, a generalization of Theorem 4 ([2], p. 295), the decision maker is confronted with the choice of transforming his current portfolio containing a random prospect into a diversified portfolio containing a sure prospect and a specified amount of the original random prospect. Part (ii), in particular, gives a necessary and sufficient condition for the second degree stochastic dominance of one portfolio over the other, assuming the diversified portfolio contains a positive " percentage " of the random prospect.

Theorem 1′. *Let X be a random variable with finite mean $\bar{x}$, and define $Y = a+bX$, $a, b \in R$. Let F and G be the distribution functions of X and Y respectively. Then:*

(i) *If $F(x) = 0$ for $x<0$, $a \geqq 0$ and $b \geqq 1$, then $[G \leqq F]$.*

(ii) *If $0 \leqq b \leqq 1$, then $[a+b\bar{x} \geqq \bar{x}] \Leftrightarrow [G \underline{\ll} F]$.*

Proof of (i). By definition of Y and right continuity of F, $G(y) = F([y-a]/b)$ if $y \geqq a$, and $G(y) = 0$ if $y<a$. Since

$$[y-a]/b \geqq y \forall y \geqq a,\ G(y) = F([y-a]/b) \leqq F(y)\forall y \geqq a.$$

Hence $G(y) \leqq F(y)\forall y$.

Proof of (ii) *Sufficiency.*

Case 1: $b = 1$

$G(y) = F(y-a)\forall y \in R$. But $a+\bar{x} \geqq \bar{x} \Rightarrow a \geqq 0$. Hence $F(y) \geqq G(y)\forall y$, and $[G \underline{\ll} F]$.

Case 2: $0<b<1$

$F(y) \gtreqless F([y-a]/b) = G(y)$ for $y \gtreqless [y-a]/b$, i.e., for $y \lesseqgtr y^* = a/[1-b]$. By hypothesis $\bar{y} = a+b\bar{x} \geqq \bar{x}$, $\bar{x}$ finite; hence by Lemma 1*

$$\bar{y}-\bar{x} = \int xdG - \int xdF = \int [F-G]dx \geqq 0.$$

For any $x \geqq y^*$,

$$\int_{-\infty}^{x} [F-G]dt = \int_{-\infty}^{\infty} [F-G]dt - \int_{x}^{\infty} [F-G]dt = \int_{-\infty}^{\infty} [F-G]dt + \int_{x}^{\infty} [G-F]dt \geqq 0$$

by the above. And similarly, for $x<y^*$, $\int_{-\infty}^{x} [F-G]dt \geqq 0$. Hence $[G \underline{\ll} F]$.

Case 3: $b = 0$

In this case $Y \equiv a$. If $F \equiv G$ the corollary follows trivially. Assume $F \not\equiv G$. The hypotheses guarantee that $\int_{-\infty}^{0} | G(t)-F(t)| \, dt<\infty$ and $\int xdF - \int xdG$ is well defined. Thus $EY = a \geqq \bar{x}$, $G(x) \geqq F(x)$ for $x \geqq a$, and $F(x) \geqq G(x)$ for $x<a \Rightarrow$ by Theorem 3*, $[GDF \text{ wrt } U^{**}(F, G)]$. It then follows from Theorem 2* that $[G \underline{\ll} F]$.

Proof of (ii) *Necessity*. Suppose $[G \underline{\ll} F]$. If $F \equiv G$, then $\int xdF = \int xdG$; i.e., $a+b\bar{x} = \bar{x}$. Suppose $F \not\equiv G$. The condition that $\bar{x}$ be finite guarantees that $\int_{-\infty}^{0} | G(t)-F(t)| \, dt<\infty$ and $\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ is well defined; thus by Theorem 2*, $[GDF \text{ wrt } U^{**}(F, G)]$. The function u: $x \to x$ is in $U^{**}(F, G)$. Hence $\int xdG - \int xdF \geqq 0$; i.e., $a+b\bar{x} \geqq \bar{x}$. ||

Theorem 2′ below demonstrates that if X is " preferred " to Y, and W is independent of both X and Y, then $X+W$ will be " preferred " to $Y+W$. Theorem 2′ generalizes Theorem 5 ([2], p. 296) in that the coefficient b is unrestricted, W is not assumed to have a density, and the conclusion (in combination with Theorem 1*) applies to a wider class of utility functions.

Theorem 2′. *Let X, Y, and W denote random variables with distribution functions F, G, and H respectively. Assume that W is independent of X and Y. Let the distribution functions of the random variables $aX+bW$ and $aY+bW$ be denoted by $\hat{F}$ and $\hat{G}$ respectively, where $a \geqq 0$. Then:*

(i) $[G<F] \Rightarrow [\hat{G}<\hat{F}]$;

(ii) $\int | F(w)-G(w)| \, dw<\infty$ *and* $[G \ll F] \Rightarrow [\hat{G} \ll \hat{F}]$.

Proof of (i). Suppose $b = 0$. Then $\hat{F}(z)-\hat{G}(z) = F(z/a)-G(z/a) \geqq 0 \forall z$, strict inequality holding for some z, so $[\hat{G}<\hat{F}]$.

Suppose $b>0$. Then

$$\hat{F}(z)-\hat{G}(z) = F^{*}_{aX}H_{bW}(z)-G^{*}_{aY}H_{bW}(z)$$
$$= \int F\left(\frac{z-t}{a}\right) dH\left(\frac{t}{b}\right) - \int G\left(\frac{z-t}{a}\right) dH\left(\frac{t}{b}\right) = \int \left[F\left(\frac{z-t}{a}\right) - G\left(\frac{z-t}{a}\right)\right] dH\left(\frac{t}{b}\right).$$

Since $[G<F]$ by assumption, the integrand is everywhere non-negative, and positive over some interval. Hence $[\hat{G}<\hat{F}]$.

Suppose $b<0$. Define a new random variable $\tilde{W} = -W$. Then $aX+bW = aX+\tilde{b}\tilde{W}$ and $aY+bW = aY+\tilde{b}\tilde{W}$, where $\tilde{b} = -b>0$ and $\tilde{W}$ is independent of X and Y. Hence the proof goes through as above.

Proof of (ii). Suppose $b = 0$. Then

$$\int_{-\infty}^{x} [\hat{F}(z)-\hat{G}(z)]dz = \int_{-\infty}^{x} [F(z/a) - G(z/a)] \, dz \geqq 0 \forall x,$$

strict inequality holding for some x. Hence $[\hat{F} \ll \hat{G}]$.

Suppose $b>0$. Then

$$\int_{-\infty}^{x} [\hat{F}(z)-\hat{G}(z)]dz = \int_{-\infty}^{x} \left[\int \left[F\left(\frac{z-t}{a}\right) - G\left(\frac{z-t}{a}\right)\right] dH\left(\frac{t}{b}\right)\right] dz$$
$$= \int dH\left(\frac{t}{b}\right) \left[\int_{-\infty}^{x} \left[F\left(\frac{z-t}{a}\right) - G\left(\frac{z-t}{a}\right)\right] dz\right],$$

where the interchange of integration is justified since $\int | F(w)-G(w)| \, dw<\infty$ by hypothesis. Since $[F \ll G]$ by hypothesis, $[\hat{F} \ll \hat{G}]$.

The case $b<0$ follows immediately upon making the transformation introduced in the proof of (i). ||

The next theorem, a simple consequence of Theorem 2′, gives conditions under which a decision maker faced with two random, independent prospects will choose a diversified portfolio.

Theorem 3′. *Let X_1 and X_2 be independent random variables with F_1 and F_2 their respective distribution functions. Let H_1 be the distribution function of aX_1+bX_2, H_2 the distribution function of aX_2+bY_2, where X_2 and Y_2 are independent and identically distributed, a and $b \in R$ with $a \geqq 0$. Assume $[F_1 \ll F_2]$ and $[H_2 \ll F_1]$, and*

$$\int | F_1(w)-F_2(w)| \, dw<\infty.$$

Then $[H_1 \ll F_1]$ and $[H_1 \ll F_2]$.

Proof. By Theorem 2′, $[F_1 \ll F_2]$ and $\int | F_1(w)-F_2(z)| \, dw<\infty \Rightarrow [H_1 \ll H_2]$. Hence by the easily checked transitivity of $\ll$, $[H_1 \ll H_2 \ll F_1 \ll F_2] \Rightarrow [H_1 \ll F_1]$ and $[H_1 \ll F_2]$. ||

If a risk averter (i.e. a decision maker with a concave utility function) is to choose between two portfolios, each containing a mixture of a sure prospect and a random prospect, and the value of the sure prospect exceeds the mean of the random prospect, it seems reasonable that he should prefer the portfolio containing the larger proportion of the sure prospect. This is demonstrated by Hadar-Russell ([2], Theorem 10, p. 303, Theorem 2, p. 292) for non-negative prospects when the utility function of the decision maker is bounded, strictly increasing, twice continuously differentiable, and concave. Theorem 4′ below, in combination with Theorem 2*, generalizes this result.

Theorem 4′. *Let X be a random variable with finite $\bar{x}$ and v the value of a sure prospect satisfying $v \geqq \bar{x}$. Let H_k be the distribution function for $kv+[1-k]X$. Then $0 \leqq k<k' \leqq 1 \Rightarrow [H_{k'} \ll H_k]$.*

Proof. Let $k<k'$, $k, k' \in [0, 1]$, and define $k'' = [k'-k]/[1-k]$, i.e., $k' = k+[1-k]k''$. Then

$$\begin{aligned} k'v+[1-k']X &= [k+[1-k]k'']v+[1-k][1-k'']X \\ &= k''v+[1-k''][kv+[1-k]X] = a+bY \end{aligned}$$

where $a = k''v$, $b = 1-k''$, and $Y = kv+[1-k]X$. The mean of Y satisfies

$$\bar{y} = kv+[1-k]\bar{x} \leqq v = a/[1-b].$$

Hence by Theorem 1′, part (ii), $[F_{a+bY} \ll F_Y]$. But $F_{a+bY} \equiv H_{k'}$ and $F_Y \equiv H_k$. ||

Theorem 5′ below demonstrates that diversification will be optimal for any risk averter faced with a choice between two identically distributed random variables with finite mean. In particular, the portfolio containing equal proportions of the two prospects will be most preferred. Hadar-Russell obtain this conclusion for two independent, identically distributed, non-negative random variables with densities ([2], Theorem 8, p. 300, and Theorem 9, p. 301).

Theorem 5′. *Let X and Y be identically distributed random variables with finite mean and distribution function F, and let H_k be the distribution function for $kX+[1-k]Y$. Then $[H_{\frac{1}{2}} \ll H_k \ll F] \forall k \in [0, 1]$.*

Proof. Let $k \in [0, 1]$. $EX = E[kX+[1-k]Y]$ finite by hypothesis implies $\int_{-\infty}^{0} xdF(x)$ and $\int_{-\infty}^{0} xdH_k(x)$ are finite. Using Step 2 in the proof of Lemma 1*, it follows that $\int_{-\infty}^{0} | H_k(x)-F(x)| \, dx < \infty$.

Let $u \in U^{**}(H_k, F)$. By concavity of u,

$$\int udH_k = Eu(kX+[1-k]Y) \geqq E[ku(X)+[1-k]u(Y)] = \int udF,$$

where the last equality follows since X and Y are identically distributed.

Similarly, for any $u \in U^{**}(H_{\frac{1}{2}}, H_k)$,

$$\int udH_{\frac{1}{2}} = Eu([X+Y]/2) = Eu([kX+[1-k]Y]/2+[[1-k]X+kY]/2)$$
$$\geqq E[u(kX+[1-k]Y)+u([1-k]X+kY)]/2 = Eu(kX+[1-k]Y) = \int udH_k,$$

where the next to last equality follows since X and Y are identically distributed.

Combining the above three paragraphs, it follows from Corollary 2* that $[H_{\frac{1}{2}} \underline{\ll} H_k \underline{\ll} F]$. ||

5′ Corollary A. *Let X and Y be identically distributed non-negative random variables with distribution function F, and let H_k be the distribution function for $kX+[1-k]Y$. Then $[H_{\frac{1}{2}} \underline{\ll} H_k \underline{\ll} F] \forall k \in [0, 1]$.*

Proof. By non-negativity, the regularity conditions in Theorem 2* are automatically satisfied. The proof then proceeds as in Theorem 5′.

5′ Corollary B. *Let X and Y be random variables with distribution functions F and G, respectively. Assume either X has a finite mean or the support of F is bounded below; similarly for Y and G. Let H_k denote the distribution function of $kX+[1-k]Y$. Then $\forall k \in [0, 1]$,*

$$\left[\int udG \geqq \int udF \forall u \in U^{**}(H_k, F)\right] \Rightarrow \left[H_k \underline{\ll} F\right].$$

Proof. Let $k \in [0, 1]$. The hypotheses guarantee that the regularity conditions in Theorem 2* are satisfied.

Let $u \in U^{**}(H_k, F)$. By concavity of u,

$$\int udH_k = Eu(kX+[1-k]Y) \geqq E[ku(X)+[1-k]u(Y)] \geqq E[ku(X)+[1-k]u(X)] = \int udF,$$

where the last inequality follows since $\int udG \geqq \int udF$ by hypothesis.

Combining the above two paragraphs, the claim follows from Corollary 2*.

First version received February 1974; *final version accepted May* 1975 (*Eds.*).

Research underlying this paper was supported by National Science Foundation Grant GS-3317 to the University of Minnesota.

NOTES

1. First degree stochastic dominance is said to hold between two distribution functions F and G when $F(x) \leqq G(x) \forall x \in R$; and second degree stochastic dominance when $\int_{-\infty}^{x} [F(t)-G(t)]dt \leqq 0 \forall x \in R$.

2. *Example.* Let $u(x) = x \forall x$; $F(x) = 1-1/x$ if $x \geqq 1$, $= 0$ if $x<1$; $G(x) = 1-2/x$ if $x \geqq 2$, $= 0$ if $x<2$. Then for $N>2$ and $M>0$,

$$u(x)[F(x)-G(x)]\Big|_{-M}^{N} = N[2/N-1/N] = 1 \Rightarrow \lim_{\substack{N\to+\infty \\ N\to+\infty}} U[F-G]\Big|_{-M}^{N} = \int du[F-G] = 1.$$

3. *Example*. Let $u(x) = 1$ if $x \geqq 1$, $= 0$ if $x<1$; $F(x) = 1$ if $x \geqq 1$, $= 0$ if $x<1$; and $G(x) = 1$ if $x \geqq 2$, $= x/2$ if $0 \leqq x<2$, and $= 0$ if $x<0$. Suppose integration by parts were valid. Then for any $N>2$ and $M>0$,

$$\left[\int_{-M}^{N} u(x)dF(x) - \int_{-M}^{N} u(x)dG(x)[F(x)-G(x)]\Big|_{-M}^{N} - \int_{-M}^{N} [F(x)-G(x)]du(x)\right]$$

$$\Rightarrow \left[u(1).1 - \int_{1}^{2} 1d[x/2] = G(1)-F(1)\right] \Rightarrow 1/2 = -1/2, \quad \text{a contradiction.}$$

4. *Example*. Let $u(x) = x$ for all x; $F(x) = G(x) = 1-1/x$ if $x \geqq 1$, $= 0$ if $x<1$. Then $\int [G-F]du = 0$, but $\int udF - \int udG = \int_{1}^{\infty} x(1/x^2)dx = +\infty$. Hence $\int udF - \int udG$ is not well defined.

5. Finite means implies $\int_{-\infty}^{0} xdF$ and $\int_{-\infty}^{0} xdG$ are finite, and hence by Lemma 1*,

$$\int_{-\infty}^{0} |G-F|\, dt \leqq \int_{-\infty}^{0} Gdt + \int_{-\infty}^{0} Fdt = -\left[\int_{-\infty}^{0} xdG + \int_{-\infty}^{0} xdF\right] + \text{constant} < +\infty.$$

Theorem 2* can therefore be used.

6. The original H-R Theorems 1 and 2 are stated in terms of densities. This restriction has been eliminated here.

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