PURE STRATEGY NASH BARGAINING SOLUTIONS

by

Leigh Tesfatsion

Discussion Paper No. 75-61, November 1975

Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

ABSTRACT

A broad class of 2-person threat games for which a unique pure strategy Nash bargaining solution exists is characterized in terms of three, simple, empirically meaningful restrictions on the joint objective function: compact domain, continuity, and "corner concavity." Connectedness [in particular, convexity] of the strategy and payoff sets is not required. In addition, conditions are given for the existence of a pure strategy Nash equilibrium threat solution. Connectedness of the strategy and payoff sets is again not required.

PURE STRATEGY NASH BARGAINING SOLUTIONS*

by

Leigh Tesfatsion

1. INTRODUCTION

Two person bargaining situations have been studied by numerous researchers interested in economic and game theory; e.g., A. Cournot, F. Edgeworth, J. von Neumann-O. Morgenstern, D. Gale, J. Nash, H. Raiffa, and L. Shapley [see Owen (1968) for selected bibliographic references]. Since the appearance of Nash's paper (1950), it has generally been assumed that the bargainers are able to correlate their choice of strategies by resort to random devices. For example, bargainers 1 and 2 may agree to implement joint strategy (a,b) if a flipped coin lands heads and joint strategy (c,d) if it lands tails. Since utility is assumed to be linear with lotteries, games with correlated strategies have convex payoff regions. The various solution concepts devised for such games, e.g., the axiomatic bargaining solution of Nash (1950), do not distinguish between pure and correlated strategy solutions.

On the other hand, in real world economic bargaining contexts such as monopoly versus monopsony, union versus management, and two-nation trade negotiations, the flipping of coins is seldom observed. One simple condition sufficient to explain the absence of coin flipping in any given situation would be that all available correlated joint strategies are dominated by available pure joint strategies. As is shown

---

*Research underlying this paper was supported by National Science Foundation Grant GS-31276X.

below [4.5], this condition can be formalized in terms of a simple restriction ["corner concavity"] placed on the joint objective function.

The principal purpose of this paper is to establish the existence of a unique barter rule, defined over the class of all 2-person threat games with compact strategy sets and continuous, corner concave joint objective functions, which assigns to each such game a pure strategy solution in a manner consistent with Nash's six axioms [see 4.4, 4.10]. Connectedness [in particular, convexity] of the strategy and payoff sets is not required.

In addition, conditions are given for the existence of a pure strategy Nash equilibrium threat solution [see 4.11]. Connectedness of the strategy and payoff sets is again not required.

2. BASIC DEFINITIONS AND NOTATION

The primitives for a 2-person pure strategy threat game are given by

$$(\{1,2\},\ U_1 \times U_2:\ \Theta \to R^2,\ \theta^*),$$

where: R denotes the real line and R^2 denotes the cross-product R X R; {1,2} is the <u>player set</u>; $\Theta = \{\theta,\ldots\}$ is the <u>joint strategy set</u>; $U_i: \Theta \to R$ is the <u>objective function</u> for player $i \in \{1,2\}$; and $\theta^* \in \Theta$ is the <u>threat</u>. The function $U_1 \times U_2: \Theta \to R$ will be referred to as the <u>joint objective function</u>. In the special case where Θ is a cross product $\Theta_1 \times \Theta_2$ of individual strategy sets Θ_1 and Θ_2, the game will be said to be <u>free</u>.

It is assumed that bargaining takes place in three stages: 1) The status quo threat θ^* is announced; 2) Players 1 and 2 attempt to come to an agreement on a joint strategy $\theta \in \Theta$; 3) If an agreement on a joint strategy θ is reached, it is implemented and player i receives $U_i(\theta)$. If an agreement on a joint strategy is not reached, then the threat θ^* is enforced and player i receives $U_i(\theta^*)$. As usual, a time restriction is implicitly assumed to constrain the duration of stage 2).

Each player i is assumed to desire the largest "payoff" in the set $\{U_i(\theta) | \theta \varepsilon \Theta\}$. By refusing to come to an agreement, player i can ensure himself a payoff of at least $U_i(\theta^*)$. Hence the effective range of joint payoffs for players 1 and 2 arising solely from pure strategy choices $\theta \varepsilon \Theta$ is given by the <u>barter set</u>

$$B(u^*,v^*) \equiv B \cap \{(u,v) \varepsilon R^2 | u \geqq u^*, v \geqq v^*\},$$

where $u^* \equiv U_1(\theta^*)$, $v^* \equiv U_2(\theta^*)$, and $B \equiv \{(U_1(\theta), U_2(\theta)) | \theta \varepsilon \Theta\}$.

A cross product A X D of topological spaces A and D will always be assumed to carry the product topology. For all $x,y \varepsilon R^2$ we shall write

$y \geqq x$ iff each component of y is at least as large as the corresponding component of x;

$y \geq x$ iff $y \geqq x$ and $y \neq x$;

$y > x$ iff each component of y is strictly larger than the corresponding component of x.

3. THE NASH BARGAINING SOLUTION

Following the lead of Nash (1950), one can ask whether there exists a set of empirically meaningful axioms guaranteeing the existence of a "barter rule" which assigns to every barter set $B(u^*,v^*)$ a "solution" $(u^\circ,v^\circ) \in B(u^*,v^*)$. As will be shown in 3.2 and 3.3, the following Nash axioms provide an affirmative answer to this question for the collection of all 2-person pure strategy threat games with compact, convex barter sets.

3.1 NASH AXIOMS. Let $\mathcal{D}^*$ denote any collection of subsets of the form

$$D(u^*,v^*) \equiv D \cap \{(u,v) \in R^2 \mid (u,v) \geqq (u^*,v^*)\}$$

with $(u^*,v^*) \in D \quad R^2$. A function $\phi: \mathcal{D}^* \to R^2$ will be said to satisfy the Nash axioms with respect to $\mathcal{D}^*$ if for every $D(u^*,v^*) \in \mathcal{D}^*$:

Axiom 1 (Feasibility). $\phi(D(u^*,v^*)) \in D$;

Axiom 2 (Individual Rationality). $\phi(D(u^*,v^*)) \geqq (u^*,v^*)$;

Axiom 3 (Pareto Optimality). If $(u,v) \in D$ and $(u,v) \geqq \phi(D(u^*,v^*))$, then $(u,v) = \phi(D(u^*,v^*))$;

Axiom 4 (Independence of Irrelevant Alternatives). If $A(u^*,v^*) \in \mathcal{D}^*$ with $(u',v') \in A(u^*,v^*) \subseteq D(u^*,v^*)$ and $(u',v') = \phi(D(u^*,v^*))$, then $(u',v') = \phi(A(u^*,v^*))$.

Axiom 5 (Independence of Linear Transformations). Suppose for some vectors $r,s \in R^2$

$E \equiv D(u^*,v^*) \cdot r^T + s^T \in \mathcal{D}^*$,

where T denotes transpose and $\cdot$ denotes vector multiplication. Then if $\phi(D(u^*,v^*)) = (u',v')$, it must hold that $\phi(E) = (u',v') \cdot r^T + s^T$;

Axiom 6 (Symmetry). Suppose $(u,v) \in D(u^*,v^*)$ iff $(v,u) \in D(u^*,v^*)$, and suppose $u^* = v^*$ and $\phi(D(u^*,v^*)) = (u',v')$. Then $u' = v'$.

<u>Remark</u>. See Luce and Raiffa (1957) for a critical appraisal of the Nash axioms; also Nydegger and Owen (1975) for an interesting experimental test of these axioms.

Let M^* denote the collection of all subsets of R^2 of the form

$$M(u^*,v^*) \equiv M \cap \{(u,v) \in R^2 \,|\, (u,v) \geqq (u^*,v^*)\}$$

with $M \subseteq R^2$ compact and convex, and $(u^*,v^*) \in M$.

3.2 THEOREM [Nash (1950)]. There exists a unique function $\phi: M^* \to R^2$ which satisfies the Nash axioms with respect to M^*:

<u>Remark</u>. Specifically, for any $M(u^*,v^*) \in M^*$:

a. If $(u',v') > (u^*,v^*)$ for some $(u',v') \in M$, then $\phi(M(u^*,v^*)) = (u^\circ,v^\circ)$, where (u°,v°) is the unique point which maximizes $(u-u^*)(v-v^*)$ over $M(u^*,v^*)$;

b. If $v \leqq v^*$ for every $(u,v) \in M$ with $u > u^*$, then $\phi(M(u^*,v^*)) = (u',v^*)$, where $u' = \max\{u \in R \,|\, (u,v^*) \in M\}$;

c. If $u \leqq u^*$ for every $(u,v) \in M$ with $v > v^*$, then $\phi(M(u^*,v^*)) = (u^*,v')$, where $v' = \max\{v \in R \,|\, (u^*,v) \in M$.

The function ϕ in 3.2 can only be interpreted as a "barter rule" when its domain is restricted to those elements of M^* which can be interpreted as payoff sets, preferably for a well-defined class of 2-person games. Uniqueness of ϕ with respect to the Nash axioms does not necessarily imply uniqueness of this associated barter rule ϕ^* with respect to these axioms unless $\phi^* = \phi : M^* \to R^2$. The following theorem demonstrates that M^* precisely characterizes the collection of compact, convex barter sets corresponding to 2-person pure strategy threat games, as defined in section 2; hence ϕ is a barter rule for a well-defined class of 2-person games.

3.3 THEOREM. The collection M^* in 3.2 coincides with the collection $M°$ of all compact convex barter sets corresponding to 2-person pure strategy threat games as defined in section 2.

Proof. Since each $B(u^*,v^*) \in M°$ can be written in the form

$$B(u^*,v^*) \cap \{(u,v) \in R^2 \mid (u,v) \geqq (u^*,v^*)\}$$

with $B(u^*,v^*)$ compact and convex and $(u^*,v^*) \in B(u^*,v^*)$, it is clear that $M° \subseteq M^*$.

Let $M(u^*,v^*) \in M^*$. To prove the converse, it suffices to show that

$$M = \{(u_1(\theta), u_2(\theta)) \in R^2 \mid \theta \in \Theta\} \qquad (2)$$

for some "joint strategy set" Θ and "joint objective function" $U_1 \times U_2 : \Theta \to R^2$.

To demonstrate that (2) holds for arbitrary, compact, convex, nonempty $M \subseteq R^2$, it suffices to show that (2) is satisfied by the one-element set $\{e°\} \subseteq R^2$, the closed line segment $[e°,e'] \subseteq R^2$, and the closed triangle

$\Delta \subseteq R^2$ determined by $\{e^\circ, e', e''\}$, where $e^\circ \equiv (0,0), e' \equiv (0,1)$, and $e'' \equiv (1,0)$. For, if M contains no interior point, then by convexity and compactness M is either a point or a closed line segment, hence homeomorphic to $\{e^\circ\}$ or $[e^\circ, e']$, respectively. And if M contains an interior point, then by convexity and compactness M is homeomorphic to the standard 2-simplex [Dold (1972, 1.3 and 1.4, p. 55)], hence also to Δ. Thus in every case it would hold that

$$M = \{(\Phi_1 \circ W_1(\theta), \Phi_2 \circ W_2(\theta)) \mid \theta \varepsilon \Theta\}$$

where $W_1 \times W_2$ is a suitable function mapping a "joint strategy set" Θ onto either $\{e^\circ\}$, $[e^\circ, e']$, or Δ, and $\Phi_1 \times \Phi_2$ is the appropriate homeomorphism mapping the image set $W_1 \times W_2\ \Theta$ onto M.

Let $\Theta = I \times I$, where I denotes the unit interval [0,1]. Then the set $\{e^\circ\}$ satisfies (2) with $W_1 \times W_2 \colon I \times I \to R^2$ given by the continuous coordinate functions

$$W_1(\theta) = 0, \quad W_2(\theta) = 0, \quad \theta \varepsilon I \times I.$$

The line segment $[e^\circ, e']$ satisfies (2) with $W_1 \times W_2 \colon I \times I \to R^2$ given by the continuous coordinate functions

$$W_1(\theta_1, \theta_2) = 0, \quad W_2(\theta_1, \theta_2) = \theta_2, \quad (\theta_1, \theta_2) \varepsilon I \times I.$$

Finally, the triangle Δ satisfies (2) with $W = W_1 \times W_2 \colon I \times I \to R^2$ given by the continuous function

$$W(ae' + be'') = \begin{cases} ae' + be'' & \text{if } 0 \leqq a + b \leqq 1; \\ [1-b]e' + [1-a]e'' & \text{if } 1 \leqq a + b \leqq 2, \end{cases}$$

where $a,b \varepsilon [0,1]$. [For $1 \leqq a + b \leqq 2$, W reflects ae' + be" across the line $\{(x,y) \varepsilon R^2 | x + y = 1\}$].

Q.E.D.

3.4 COROLLARY. Every element $M(u^*,v^*) \varepsilon M^*$ can be interpreted as a compact, convex barter set corresponding to a free 2-person pure strategy threat game with compact pure strategy sets $\Theta_1 = \Theta_2 = [0,1]$, continuous joint objective function, threat (0,0), and threat payoff (u^*,v^*).

4. CHARACTERIZATION FOR GAMES WITH PURE STRATEGY SOLUTIONS

As seen in section 3, there exists a unique barter rule ϕ satisfying the six Nash axioms (3.1) which assigns to every 2-person pure strategy threat game with compact convex barter set $B(u^*,v^*)$ a (necessarily pure strategy) solution $\phi(B(u^*,v^*))$. However, Nash axioms 1-4 imply that the existence of this barter rule depends only on the existence of suitably shaped subsets of "pareto optimal" points in the barter sets and not on global properties of the barter sets such as convexity. The purpose of this section is to precisely characterize "suitably shaped pareto optimal sets" in terms of a property of the joint objective function. This in turn will permit the extension of the barter rule ϕ to a broader class of pure strategy games characterized in terms of three simple restrictions on the joint objective function.

4.1 DEFINITIONS AND NOTATION. Let $D \subseteq R$. A point $(u',v') \varepsilon D$ will be said to be pareto optimal if $(u,v) \leqq (u',v')$ implies $(u,v) = (u',v')$, for all $(u,v) \varepsilon D$. Let D^p denote the set of all pareto optimal points in D, and let D^- denote the closed convex hull of D. Then D will be said to be <u>corner concave</u> if and only if D^P is a compact nonempty set which coincides with $(D^-)^P$. Finally, D will be said to be <u>upper (lower) bounded</u> if there exists $(u',v') \in R^2$ such that $(u,v) \leqq (\geqq) (u',v')$ for all $(u,v) \in D$. Clearly D is bounded if and only if it is both upper and lower bounded.

4.2 LEMMA. Let $K \subseteq R^2$ be compact and nonempty. Then K^P is compact and nonempty.

<u>Proof</u>. For each $(u',v') \in K$, define

$$K_{u'v'} \equiv K \cap \{(u,v) \in R^2 \mid (u,v) \geqq (u',v')\},$$

a compact set containing (u',v'). Let $g:K \to K$ be the multivalued map given by $g(u,v) = K_{uv}$. As is easily verified, g is a closed mapping. Since K is compact, it follows that g is upper semi-continuous [Berge (1968, Corollary, p. 112)]. It is also straightforward to establish that for each [relatively] open ball $V \subseteq K$ the set $\{x \in K \mid g(x) \cap B \neq \phi\}$ is [relatively] open in K. Hence [Berge (1968, Theorem 1, p. 109)] $g:K \to K$ is also lower semi-continuous.

Define $F:K \to R$ by

$$F(u',v') \equiv \max \{(u-u') + (v-v') \mid (u,v) \in g(u',v')\}.$$

Combining Theorem 1 and Theorem 2 [Berge (1968, pp. 115-116)], F is continuous. Thus

$$K^P = \{(u,v) \in K \mid F(u,v) = 0\}$$

is a closed, hence compact subset of K.

Let $u' \equiv \max\{u \mid (u,v) \in K \text{ for some } v\}$ and $v' = \max\{v \mid (u',v) \in K\}$. As is easily verified, (u',v') is a pareto optimal point in K; hence K^p is nonempty.

Q.E.D.

4.3 LEMMA. Let $D \subseteq R^2$ be a lower bounded nonempty set. Then

$$(D^-)^p \text{ is compact and nonempty} \Leftrightarrow D \text{ is bounded.}$$

In particular, every lower bounded corner concave subset of R^2 is bounded.

<u>Proof</u>. Suppose $(D^-)^p$ is compact and nonempty. Then there exists $(u',v') \in R^2$ satisfying $u' \equiv \max\{u \mid (u,v) \in (D^-)^p \text{ for some } v\}$ and $v' = \max\{v \mid (u,v) \in (D^-)^p \text{ for some } u\}$. Clearly $(D^-)^p \subseteq K \equiv \{(u,v) \mid (u,v) \leqq (u',v')\}$. As is easily verified, this implied $D^- \subseteq K$, hence also $D \subseteq K$. It follows that D, lower bounded by assumption, is bounded.

Conversely, suppose D is bounded. Then D^- is a compact set, nonempty by assumption. Hence, by 4.2, $(D^-)^p$ is compact and nonempty.

The last statement of the lemma is immediate from the definition of corner concavity [4.1] and the first part of the proof.

Q.E.D.

Let C^* denote the collection of all closed, corner concave subsets having the form

$$C(u^*,v^*) \equiv C \cap \{(u,v) \in R^2 \mid (u,v) \geqq (u^*,v^*)\}$$

with $(u^*,v^*) \in C \subseteq R^2$.

4.4 THEOREM. There exists a unique function $\phi°: C^* \to R^2$ which satisfies the Nash axioms with respect to C^* [see 3.1].

Proof. Let $C(u^*,v^*) \in C^*$. By 4.3, $C(u^*,v^*)$ is bounded. It follows that $C(u^*,v^*)^-$ has the form

$$[C(u^*,v^*)^-]\cap\{(u,v)\,|\,(u,v) \geqq (u^*,v^*)\}$$

with $C(u^*,v^*)^-$ compact and convex and $(u^*,v^*) \in C(u^*,v^*)^-$; i.e., $C(u^*,v^*)^- \in M^*$ [see 3.2].

Define a function $\phi°: C^* \to R^2$ by

$$\phi°(D) = \phi(D^-),\ D \in C^* ,$$

where $\phi: M^* \to R^2$ is the function appearing in theorem 3.2. Then for every $C(u^*,v^*) \in C^*$,

$$\phi°(C(u^*,v^*)) = \phi(C(u^*,v^*)^-) \in (C(u^*,v^*)^-)^P = C(u^*,v^*)^P ;$$

hence $\phi°$ satisfies Nash axioms 1 - 3 with respect to C^*.

Suppose $A(u^*,v^*), C(u^*,v^*) \in C^*$ with $(u',v') \in A(u^*,v^*) \subseteq C(u^*,c^*)$ and $(u',v') = \phi°(C(u^*v^*)) \equiv \phi(C(u^*,v^*)^-)$. Then $(u',v') \in A(u^*,v^*)^- \subseteq C(u^*,v^*)^-$, hence by 3.2 $(u',v') = \phi(A(u^*,v^*)^-) \equiv \phi°(A(u^*,v^*))$. Thus $\phi°$ satisfies Nash axiom 4 with respect to C^*.

Suppose for some $C(u^*,v^*) \in C^*$ and vectors $r,\ s \in R^2$

$$E \equiv C(u^*,v^*) \cdot r^T + s^T \in C^* ,$$

where T denotes transpose and $\cdot$ denotes vector multiplication; and suppose $(u',v') = \phi^\circ(C(u^*,v^*)) \equiv \phi(C(u^*,v^*)^-)$. Since $E^- = [C(u^*,v^*)^-] \circ r^T + s^T$, it follows by 3.2 that $\phi^\circ(E) \equiv \phi(E^-) = (u',v') \circ r^T + s^T$. Thus ϕ° satisfies Nash axiom 5 with respect to C^*.

Let $C(u^*,v^*) \in C^*$, and suppose $(u,v) \in C(u^*,v^*)$ if and only if $(v,u) \in C(u^*,v^*)$, $u^* = v^*$, and $(u',v') = \phi^\circ(C(u^*,v^*))$. Since $C(u^*,v^*)^- \in M^*$ retains the symmetry of $C(u^*,v^*)$, it follows by 3.2 that $u' = v'$. Hence ϕ° satisfies Nash axiom 6 with respect to C^*.

Combining the above, ϕ satisfies all six Nash axioms with respect to C^*. It remains to show that ϕ° is the unique function $C^* \to R^2$ having this property.

Suppose $\phi':C^* \to R$ is another function satisfying the Nash axioms with respect to C^*. By 4.3 and the definition of corner concavity [4.1], every nonempty compact convex set is corner concave. In particular, $C^- \subseteq M^* \subseteq C^*$, where $C^- \equiv \{B \subseteq R^2 \mid B = D^- \text{ for some } D \in C^*\}$. It follows that both ϕ' and ϕ°, restricted to M^*, satisfy the Nash axioms with respect to M^*. By 3.2, ϕ' and ϕ° must therefore agree on M^*, and hence also on $C^- \subseteq M^*$.

Let $D \in C^*$. Then $D^- \in C^-$, with $D^P = (D^-)^P$ by definition of C^*. By Nash axioms 1-3, $\phi'(D^-) \in D^P = (D^-)^P \subseteq D^-$; i.e. $\phi'(D^-) \subseteq D \subseteq D^-$. Hence by Nash axiom 4, $\phi'(D) = \phi'(D^-)$. Similarly, $\phi^\circ(D) = \phi^\circ(D^-)$. Since ϕ' and ϕ° agree on C^-, it follows that

$$\phi'(D) = \phi'(D^-) = \phi^\circ(D^-) = \phi^\circ(D).$$

Thus ϕ° is unique.

Q.E.D.

4.5 DEFINITIONS AND REMARK. For any $a,b \in R$ and $r \in [0,1]$, let $arb \equiv ra + [1-r]b$. A function $J:D \to R^2$, D an arbitrary set, will be said to be <u>corner concave</u> if for every pair d, d' $\in$ D and every $r \in [0,1]$ there exists $d^* \in D$ such that

$$J(d)rJ(d') \leqq J(d^*) \ . \tag{3}$$

If D is interpreted as a collection of pure joint strategies and J is interpreted as a joint objective function, then (3) has an obvious interpretation: Each available correlated joint strategy [lottery among pure joint strategies] involving at most two pure joint strategies is "dominated" by at least one available pure joint strategy in the sense that, for each player i, the expected utility of the correlated joint strategy is no greater than the utility of the pure joint strategy.

A function $J:D \to R^2$ will be said to be <u>corner concave with respect to $(u^*,v^*) \in R^2$</u> if the restriction of J to $J^{-1}(\{(u,v) \mid (u,v) \geqq (u^*,v^*)\})$ is corner concave.

In 4.10 below it will be shown that the collection C^* in 4.4 coincides with the collection of all barter sets corresponding to 2-person pure strategy threat games with compact joint strategy sets and continuous joint objective functions, corner concave with respect to the threat payoff. Certain needed intermediary results will first be established.

4.6 Let $D \subseteq R^2$. The following conditions will be referred to in 4.7.

a. There exists a point $(u^\circ,v^\circ) \in D$ such that $(u,v) \in D \Rightarrow v < v^\circ$ or $v = v^\circ$ and $u \leqq u^\circ$;

b. For some $u' \geqq u°$, the [possibly degenerate] interval $[u°, u'] = \{u \mid u \geqq u°, (u,v) \in D \text{ for some } v\}$;

c. For every $u^* \in [u°,u']$ there exists $(u^*,f(u^*)) \in D$ such that $f(u^*) = \max \{v \mid (u^*,v) \in D\}$.

d. The function $f: [u°,u'] \to R$ is concave.

4.7 LEMMA. Let D be a nonempty lower bounded subset of R^2. Then

D corner concave <=> D satisfies a.,b.,c.,d. in 4.6.

<u>Proof</u>. The proof for the forward implication proceeds by straightforward verification [using 4.3] that $(D^-)^P = \{(u,f(u)) \mid u \in [u°,u']\}$, with $v° = f(u°)$, where $v° \equiv \max\{v \mid (u,v) \in D^- \text{ for some } u\}$, $u° \equiv \max\{u \mid (u,v°) \in D^-\}$, $u' \equiv \max\{u \mid (u,v) \in D^- \text{ for some } v\}$, and $f(u^*) \equiv \max\{v \mid (u^*,v) \in D^-\}, u^* \in [u°,u']$, are well-defined and have all the properties required in a.,b.,c.,d.

Conversely, it is easily verified that $D^P = (D^-)^P = \{(u,f(u)) \mid u \in [u°,u']\}$ for the given f, u°, and u'.

Q.E.D.

4.8 THEOREM. Let $J_1 \times J_2: A \to R^2$ be given, where

1) A is a nonempty compact space;

2) $J_1 \times J_2$ is continuous.

Then

the set $J_1 \times J_2$ (A) is corner concave	<=>	the function $J_1 \times J_2$ is corner concave.

<u>Remark</u>. Corner concavity for sets is defined in 4.1; corner concavity for functions is defined in 4.5. The reason for the similar terminology now becomes evident.

<u>Proof</u>. Assume $D \equiv J_1 \times J_2(A)$ is corner concave. By 1) and 2), D is a nonempty bounded subset of R^2; hence, by 4.7, D satisfies a.,b.,c.,d. in

4.6. Let $(u^\circ,v^\circ) \varepsilon D$ be as in a., $[u^\circ,u']$ be as in b., and $f: [u^\circ,u'] \to R$ be as in c. and d. By concavity of f and definition of (u°,v°), it must hold that $(u,f(u)) \varepsilon (D^-)^P$ for all $u \varepsilon [u^\circ,u']$. Hence for all $r \varepsilon [0,1]$, for all (u,v), $(u^\#,v^\#) \varepsilon D$, $u^\# ru \leqq u'$ and

$$u^\# r\, u \leqq u^\circ \Rightarrow (u^\# ru,\ v^\# rv) \leqq (u^\circ,v^\circ); \qquad (4)$$
$$u^\# ru > u^\circ \Rightarrow (u^\# ru,\ v^\# rv) \leqq (u^\# ru,\ f(u^\# ru)),$$

where the last inequality follows by pareto optimality of $(u^\# ru, f(u^\# ru))$ in D^-. In terms of $J_1 \times J_2: A \to R^2$, (4) guarantees that $J_1 \times J_2$ is corner concave.

Conversely, assume that $J_1 \times J_2$ is corner concave. It will be shown that the bounded nonempty set $D \equiv J_1 \times J_2 (A)$ must then satisfy a., b., c., d. in 4.6. By 4.7, this will imply that D is corner concave.

By assumptions 1) and 2), J_2 attains a maximum v° over the compact set A. By 1) and 2) again, J_1 attains a maximum over the closed hence compact set $J_2^{-1}(v^\circ)$ at some point $a' \varepsilon J_2^{-1}(v^\circ)$. As is easily verified, $(u^\circ,v^\circ) = (J_1(a'), J_2(a'))$ satisfies condition a. in 4.6.

By 1) and 2), $J_1^{-1}(u'')$ is compact and nonempty for every $u'' \varepsilon S^\circ \equiv J_1(A) \cap \{u \varepsilon R \mid u \geqq u^\circ\}$; hence by 2) $f(u'') \equiv \max \{J_2(a) \mid a \varepsilon J_1^{-1}(u'')\}$ exists for every $u'' \varepsilon S^\circ$. In particular, $f(u^\circ) = v^\circ$. It will now be shown that f is strictly decreasing over S°.

If $S^\circ = \{u^\circ\}$, then trivially f is strictly decreasing over S°; and if S° consists of two distinct points u°, $u\#$, then $f(u^\circ) = v^\circ > f(u\#)$ by definition of (u°,v°).

Suppose S° contains at least three points. Let u^-, $u^\sim \varepsilon S^\circ$ satisfy $u^\circ < u^- < u^\sim$, and suppose $f(u^-) < f(u^\sim)$. Define

$v^{\#} \equiv \max \{ v \mid (u,v) \varepsilon D \text{ for some } u \geqq u^{-} \}.$

$u^{\#} \equiv \max \{ u \mid (u,v^{\#}) \varepsilon D \}.$

[By 1) and 2), $D \equiv J_1 X J_2 (A)$ is compact; hence $v^{\#}$ and $u^{\#}$ exist. Note that $u^{\#} \geqq u^{-}$]. Then

$$f(u^{-}) < f(u^{\sim}) \leqq v^{\#} ;$$

hence by definition of f and $u^{\#}$, $u^{-} < u^{\#}$. Choose $r \varepsilon (0,1)$ such that $u^{-} = u^{\circ}\, r\, u^{\#}$. Since $v^{\circ} = f(u^{\circ}) > v^{\#}$ by definition of (u°,v°) and $v^{\#}$, it holds that $f(u^{\circ}) r\, v^{\#} > v^{\#}$. By corner concavity of $J_1 X J_2$, there exists $a' \varepsilon A$ such that

$$\begin{aligned}(J_1(a'), J_2(a')) &\geqq (u^{\circ} r u^{\#}, f(u^{\circ}) r v^{\#}) \\ &= (u^{-}, f(u^{\circ}) r v^{\#}) \\ &\geqq (u^{-}, v^{\#}).\end{aligned} \tag{5}$$

Since (5) contradicts the definition of $v^{\#}$, the original supposition that $f(u'') < f(u^{\sim})$ for $u^{-} < u^{\sim}$ must be false.

Combining the above two paragraphs, f is non-increasing over S°. Suppose there exist u^{-}, $u^{\sim} \varepsilon S^{\circ}$ with $u^{-} < u^{\sim}$ and $f(u^{-}) = f(u^{\sim})$. By definition of (u°,v°) and f it must hold that $u^{\circ} < u^{-}$. Choose $r \varepsilon (0,1)$ such that $u^{-} = u^{\circ}\, r\, u^{\sim}$. By corner concavity of $J_1 X J_2$ there exists $a' \varepsilon A$ such that

$$\begin{aligned}(J_1(a'), J_2(a')) &\geqq (u^{\circ} r u^{\sim}, f(u^{\circ})\, r f(u^{\sim})) \\ &= (u^{-}, v^{\circ}\, r\, f(u^{\sim})).\end{aligned}$$

Thus $J_2(a') > f(u^{\sim}) = f(u^{-})$ and $J_1(a') \geqq u^{-} > u^{\circ}$, which implies $J_1(a') \varepsilon S^{\circ}$. It follows that $f(J_1(a')) \geqq J_2(a') > f(u^{-})$ with $J_1(a')) \geqq u^{-}$, contradicting the just proved result that f is non-increasing over S°. The supposition that $f(u^{-}) = f(u^{\sim})$ for distinct $u^{-}, u^{\sim} \varepsilon S^{\circ}$ must therefore be false. Combined with the previous result, this implies f is strictly decreasing over S°.

By 1) and 2), $S^{\circ} \equiv J_1(A) \cap \{u \varepsilon R | u \geqq u^{\circ}\}$ is compact. Define $u' \equiv \max\{u | u \varepsilon S^{\circ}\}$. Then S° is contained in the [possibly degenerate] interval $[u^{\circ}, u']$. To prove that b. and c. in 4.6 hold for $D \equiv J_1 \times J_2(A)$, it suffices to show that S° is dense in $[u^{\circ}, u']$; for, since S° is closed, this would imply immediately that $S^{\circ} = [u^{\circ}, u']$.

Suppose to the contrary that S° is not dense in $[u^{\circ}, u']$, hence there exists some maximal open interval $W \quad \{u | u \varepsilon [u^{\circ}, u'],\ u \notin S^{\circ}\}$ whose closure had endpoints $u^{\sim}$, $u^{\approx}$ [in S°] with $u^{\sim} < u^{\approx}$. Since f is strictly decreasing over S°, $f(u^{\sim}) > f(u^{\approx})$. By corner concavity of $J_1 \times J_2$ there exists $a^* \varepsilon A$ such that

$$(J_1(a^*), J_2(a^*)) \geqq (u^{\sim}\ {}^{1}/_{2}\ u^{\approx},\ f(u^{\sim})\ {}^{1}/_{2}\ f(u^{\approx})) > (u^{\sim},\ f(u^{\approx})).$$

Clearly, $J_1(a^*) \varepsilon S^{\circ}$. It cannot hold that $J_1(a^*) \geqq u^{\approx}$, since then $f(J_1(a^*)) \geqq J_2(a^*) > f(u^{\approx})$ contradicts f strictly decreasing over S°. On the other hand, $u^{\sim} < J_1(a^*) < u^{\approx}$ would be a contradiction since by definition the open interval W with endpoints $u^{\sim}, u^{\approx}$ satisfies $W \cap S^{\circ} = \phi$.
Hence in every case a contradiction results. It follows that no such interval W exists; i.e., S° is dense in $[u^{\circ}, u']$ as was to be shown.

It remains to prove that d. in 4.6 holds; i.e., $f:[u^{\circ}, u'] \to R$ is concave. We first note that for all $u \varepsilon S^{\circ} = [u^{\circ}, u']$ and $r \varepsilon [0,1]$,

$$f(u) \geqq \max\{J_2(a')rJ_2(a'') | a',a''\varepsilon A, J_1(a')rJ_1(a'') = u\}. \qquad (6)$$

For suppose there exist a', a'' ε A, r ε [0,1], and u ε S° such that

$$J_2(a')rJ_2(a'') > f(u); \qquad (7)$$
$$J_1(a')rJ_1(a'') = u.$$

Since f is strictly decreasing over S°, there exists no a* ε A such that $(J_1(a^*),J_2(a^*)) \geq (u,f(u))$. Together with (7), this implies there exists no a* ε A such that $(J_1(a^*),J_2(a^*)) \geqq (J_1(a')rJ_1(a''), J_2(a')rJ_2(a''))$. But this contradicts the corner concavity of $J_1 \times J_2$. Hence (6) must hold.

Now let $u^{\sim}$, $u^{\approx}$ ε [u°,u'] = S° and r ε [0,1] be given. Then $u^{\sim}$ r $u^{\approx}$ ε S°. Let $a^{\sim}$ ε $J_1^{-1}(u^{\sim})$ and $a^{\approx}$ ε $J_1^{-1}(u^{\approx})$ satisfy

$$J_2(a^{\sim}) = f(u^{\sim}) \equiv \max\{J_2(a) | a \varepsilon J_1^{-1}(u^{\sim})\};$$
$$J_2(a^{\approx}) = f(u^{\approx}) \equiv \max\{J_2(a) | a \varepsilon J_1^{-1}(u^{\approx})\}.$$

Using (6),

$$f(u^{\sim}ru^{\approx}) \geqq \max\{J_2(a')rJ_2(a'') | a',a''\varepsilon A, J_1(a')rJ_1(a'') = u^{\sim}ru^{\approx}\}$$
$$\geqq J_2(a^{\sim})rJ_2(a^{\approx})$$
$$= f(u^{\sim})rf(u^{\approx}).$$

Hence f:[u°,u'] $\rightarrow$ R is concave, as was to be shown.

Q.E.D.

4.9 COROLLARY. Let a 2-person pure strategy threat game

$$(\{1,2\}, U_1 \times U_2 : \Theta \to R^2, \theta^*)$$

be given, with barter set B(u*,v*) [see section 2]. Assume:
1) Θ is a nonempty compact space; and 2) $U_1 \times U_2$ is continuous. Then

B(u*,v*) is corner concave	<=>	$U_1 \times U_2$ is corner concave with respect to (u*,v*).

Proof. Let $U \equiv U_1 \times U_2$ and $K \equiv \{(u,v) \mid (u,v) \geqq (u^*,v^*)\}$. By definition, $B(u^*,v^*) = [U\,\Theta\,] \cap K = U(U^{-1}(K))$. The assumption "U corner concave with respect to (u*,v*)" is equivalent to the assumption "$U:U^{-1}(K) \to R$ corner concave." Letting $J_i \equiv U_i$ and $A \equiv U^{-1}(K)$, compact and nonempty by 1) and 2), 4.9 now follows immediately from 4.8.

Q.E.D.

4.10 THEOREM. The collection C^* in 4.4 coincides with the collection $C°$ of all barter sets corresponding to 2-person pure strategy threat games with compact joint strategy sets and continuous joint objective functions, corner concave with respect to the threat payoff.

Proof. By 4.9, $C° \subseteq C^*$. It remains to show that $C^* \subseteq C°$.

Let $C(u^*,v^*) \in C^*$. By definition, C(u*,v*) is a closed, corner concave subset of R^2, bounded below by $(u^*,v^*) \in C(u^*,v^*)$. Using 4.3, the [closed] convex hull $C(u^*,v^*)^-$ of C(u*,v*) is therefore compact, convex, and bounded below by $(u^*,v^*) \in C(u^*,v^*)^-$; i.e., $C(u^*,v^*)^- \in M^*$ [see 3.2]. Thus, by 3.4,

$C(u^*,v^*)^-$ can be interpreted as the barter set for a free 2-person pure strategy threat game with compact pure strategy sets $\Theta_1 = \Theta_2 = [0,1] \equiv I$, continuous joint objective function $U_1 \times U_2 : I \times I \to R^2$, threat $(0,0)$, and threat payoff (u^*,v^*).

Define $B \equiv (U_1 \times U_2)^{-1} (C(u^*,v^*)) \subseteq I \times I$. Since $C(u^*,v^*)$ is closed by assumption and bounded by 4.3, and $U_1 \times U_2$ is continuous, B is compact. Moreoever, B contains the threat $(0,0)$ since $(u^*,v^*) = U_1 \times U_2(0,0) \in C(u^*,v^*)$. Thus $C(u^*,v^*)$ is the [corner concave] barter set for the 2-person pure strategy threat game

$$(\{1,2\}\ ,\ U_1 \times U_2 : B \to R^2,\ (0,0))$$

with compact joint strategy set B and continuous joint objective function $U_1 \times U_2 : B \to R^2$. By 4.9, $U_1 \times U_2 : B \to R^2$ is corner concave with respect to the threat payoff (u^*,v^*).

Q.E.D.

Suppose Γ is a free game with compact individual pure strategy sets Θ_1 and Θ_2 and continuous corner concave objective function $U \equiv U_1 \times U_2 : \Theta_1 \times \Theta_2 \to R^2$. The question arises where there exists a Nash equilibrium choice of joint threat for Γ; i.e., letting $W_1 \times W_2(\cdot) \equiv W(\cdot) \equiv \phi(B(U(\cdot)))$, where ϕ is the barter rule in 4.4, a joint strategy $(\theta_1^*,\theta_2^*) \in \Theta_1 \times \Theta_2$ such that

$$W_1(\theta_1^*,\theta_2^*) \geqq W_1(\theta_1,\theta_2^*); \tag{7}$$

$$W_2(\theta_1^*,\theta_2^*) \geqq W_2(\theta_1^*,\theta_2),$$

for all $(\theta_1,\theta_2) \varepsilon\ \Theta_1 \times \Theta_2$. The following result answers this question in the affirmative, given certain additional restrictions.

4.11 THEOREM [Tesfatsion (1975, 2.8, p.8)]. Let $W_1 \times W_2 : \Theta_1 \times \Theta_2 \rightarrow R^2$ be given, where:

1) $\Theta_1 \times \Theta_2$ is a compact, metrizable, absolute neighborhood retract;
2) $W_1 \times W_2$ is continuous;
3) $T(\theta)$ is C_F- acyclic [i.e., acyclic with respect to Čech homology over a field F] for each $\theta \varepsilon\ \Theta_1 \times \Theta_2$, where $T \equiv T_1 \times T_2 : \Theta_1 \times \Theta_2 \rightarrow \Theta_1 \times \Theta_2$ is defined by

$$T_1(\theta_1',\theta_2') \equiv \{\theta_1{}^\circ \varepsilon\ \Theta_1 \mid W_1(\theta_1{}^\circ,\theta_2') = \max_{\theta_1 \varepsilon\ \Theta_1} W_1(\theta_1,\theta_2')\};$$

$$T_2(\theta_1',\theta_2') \equiv \{\theta_2{}^\circ \varepsilon\ \Theta_2 \mid W_2(\theta_1',\theta_2^\circ) = \max_{\theta_2 \varepsilon\ \Theta_2} W_2(\theta_1',\theta_2)\};$$

4) The Lefschetz number of T [with respect to Čech homology over F] is not zero.

Then there exists at least one point $(\theta_1{}^*,\theta_2{}^*) \varepsilon\ \Theta_1 \times \Theta_2$ satisfying (7) with respect to $W_1 \times W_2$.

Remark. As discussed in Tesfatsion (1975, section 3), most of the spaces commonly used in economic and game theory are compact, metrizable, absolute neighborhood retracts: for example, compact convex subsets of Banach spaces; finite dimensional, locally contractible, compact metrizable spaces [e.g., finite discrete spaces]; and locally euclidean compact metrizable spaces [e.g., compact n-manifolds]. Contractable subsets of compact Hausdorff spaces are C_F-acyclic. If $\Theta_1 \times \Theta_2$ is a C_F-acyclic compact Hausdorff space, then the Lefschetz number of T is equal to 1. In general, however, the hypotheses of 4.11 do not require any kind of global connectedness for $\Theta_1 \times \Theta_2$.

REFERENCES

Berge, C., 1963, Topological Spaces (The Macmillan Company, N.Y.).

Dold, A., 1972, Lectures on Algebraic Topology (Springer-Verlag, Berlin).

Luce, R. D., and H. Raiffa, 1957, Games and Decisions (John Wiley & Sons, Inc., N.Y.).

Nash, J., 1950, "The Bargaining Problem," Econometrica 18, 155-162.

Nydegger, R. V., and G. Owen, 1974, "Two Person Bargaining: An Experimental Test of the Nash Axioms," International Journal of Game Theory 3, Issue 4, 239-249.

Owen, G., 1968, Game Theory (W. B. Saunders Company, Philadelphia).

Tesfatsion, L., 1975, "Pure Strategy Nash Equilibrium Points and the Lefschetz Fixed Point Theorem," Discussion Paper 75-60, Center for Economic Research, University of Minnesota.