Optimization Basics for Electric Power Markets

Leigh Tesfatsion

Professor of Economics
Courtesy Professor of Mathematics
and Electrical & Computer Engineering
Iowa State University
Ames, Iowa 50011-1070
https://www2.econ.iastate.edu/tesfatsi/
tesfatsi@iastate.edu

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Topic Outline

- ISO Market Optimization on a Typical Operating Day D
- Alternative modeling formulations
- Optimization illustration: Real-time economic dispatch
- Classic Nonlinear Programming Problem (NPP): Minimization subject to equality constraints
- NPP via the Lagrange multiplier approach
- NPP Lagrange multipliers as shadow prices
- Real-time economic dispatch: Numerical example
- General Nonlinear Programming Problem (GNPP): Minimization subject to equality and inequality constraints
- GNPP via the Lagrange multiplier approach
- GNPP Lagrange multipliers as shadow prices
- Necessary versus sufficient conditions for optimization
- Technical references

Key Objective of EE/Econ 458

- Understand the optimization processes undertaken by participants in restructured wholesale power markets

- For Independent System Operators (ISOs), these processes include:
  - **Security-Constrained Unit Commitment (SCUC)** to determine which *Generating Companies (GenCos)* must be available to produce energy (MWh) during designated future operating periods in response to received ISO dispatch instructions.
  - **Security-Constrained Economic Dispatch (SCED)** to determine GenCo dispatch schedules and *Locational Marginal Prices (LMP)* ($/MWh) for grid-delivered energy (MWh) in short-run – i.e., day-ahead and intra-day – markets.

ISO Market Optimization on a Typical Operating Day D:
Timing from Midwest ISO (MISO)

Real-time spot market (balancing mechanism) for day D

Real-time settlement

00:00	**Day-ahead market for day D+1** — ISO collects demand bids from LSEs and supply offers from GenCos
11:00	ISO evaluates LSE demand bids and GenCo supply offers
16:00	ISO solves D+1 Security Constrained Unit Commitment (SCUC) & Security Constrained Economic Dispatch (SCED) & posts D+1 commitment, dispatch, and LMP schedule
23:00	Day-ahead settlement

Types of Model Representations
(from Bradley et al. [2])

Main focus of 458 (LT)

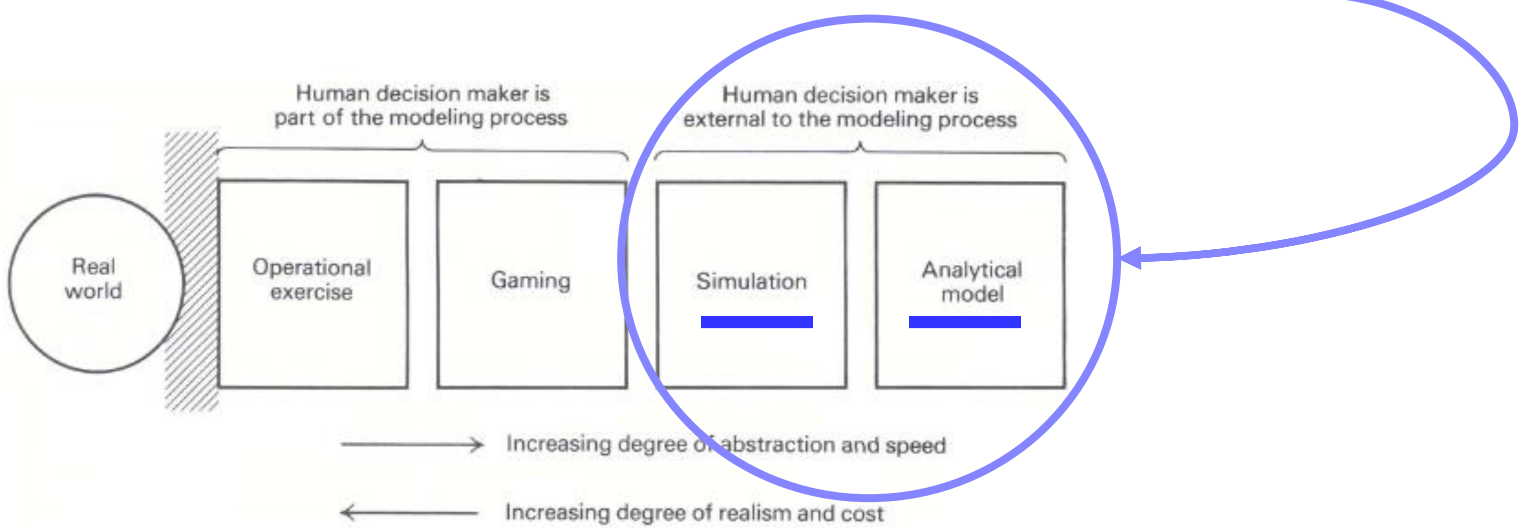


Fig. 1.1 Types of model representation.

Classification of Modeling Tools
(from Bradley et al. [2])

Main focus of 458 (LT)

Table 1.1 Classification of Analytical and Simulation Models

	Strategy evaluation	*Strategy generation*
Certainty	Deterministic simulation Econometric models Systems of simultaneous equations Input-output models	Linear programming Network models Integer and mixed-integer programming Nonlinear programming Control theory
Uncertainty	Monte Carlo simulation Econometric models Stochastic processes Queueing theory Reliability theory	Decision theory Dynamic programming Inventory theory Stochastic programming Stochastic control theory

Statistics and subjective assessment are used in all models to determine values for parameters of the models and limits on the alternatives.

Game Theory

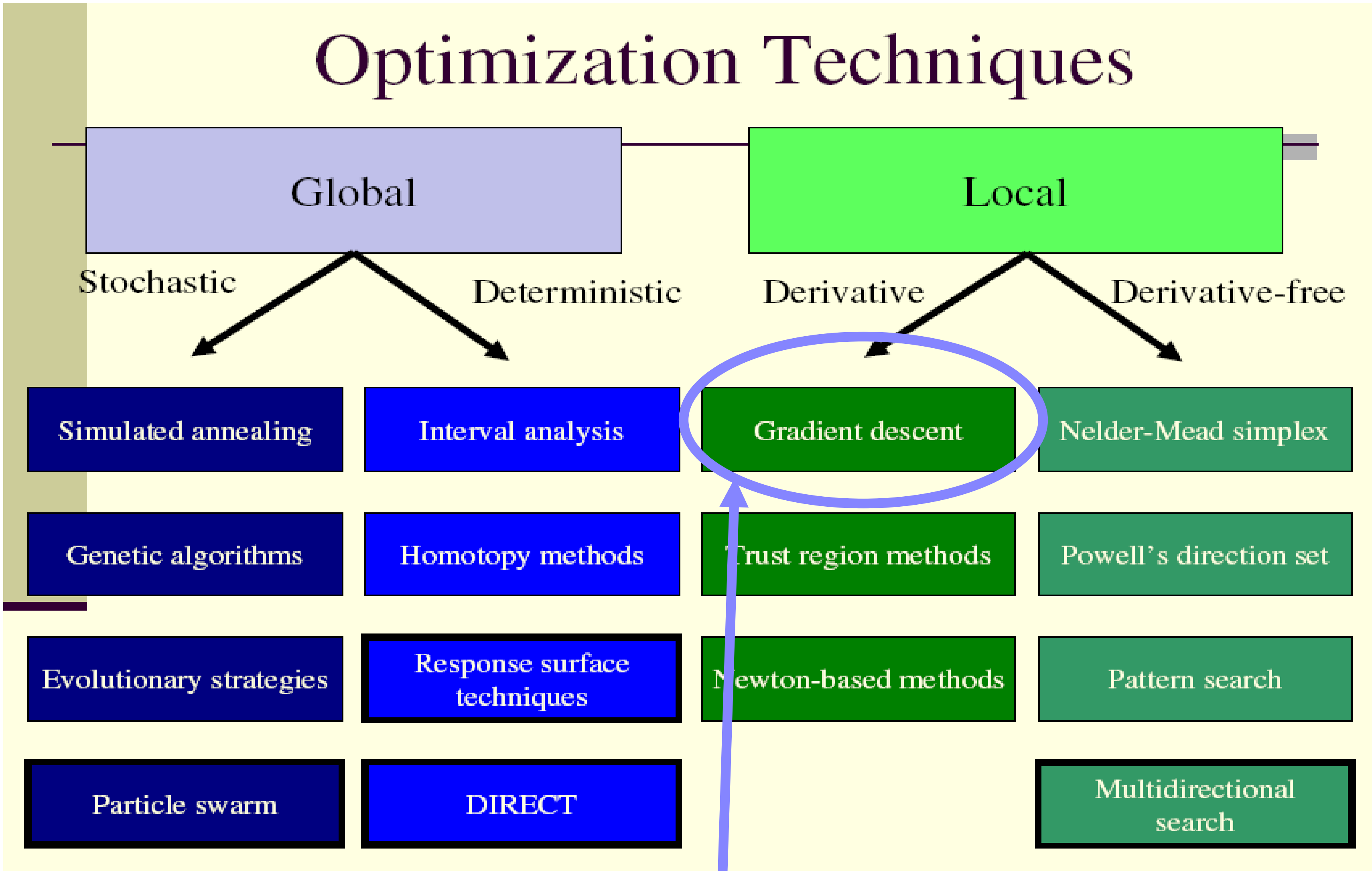
Optimization in Practice
(from Bradley et al. [2])

Main focus of 458 (LT)

Table 5.1 Distinct Characteristics of Strategic, Tactical, and Operational Decisions

Characteristics	*Strategic planning*	*Tactical planning*	*Operations control*
Objective	Resource acquisition	Resource utilization	Execution
Time horizon	Long	Middle	Short
Level of management involvement	Top	Medium	Low
Scope	Broad	Medium	Narrow
Source of information	(External	& Internal)	Internal
Level of detail of information	Highly aggregate	Moderately aggregate	Low
Degree of uncertainty	High	Moderate	Low
Degree of risk	High	Moderate	Low

Source: M. Wachowiak, Optim Lecture Notes (On-Line)



Main focus of 458 (LT)

Optimization Illustration:

Economic Dispatch for a Designated Future Hour H

- **Initial Problem Simplification**

Ignore Generation Company (GenCo) capacity limits, transmission constraints, line limit losses, and all costs except variable costs.

- **Problem Formulation for Future Hour H**

Determine scheduled real-power dispatch levels P_{Gi} (MW) to be maintained by GenCos i = 1, 2, … , I during hour H that minimize total variable cost TVC, subject to the constraint that total grid-delivered energy (MWh) during H equals total energy demand P_D x 1h (MWh) during hour H, where P_D (MW) is the anticipated average demand for real power (MW) during H.

Economic Dispatch for the Future Hour H: Mathematical Formulation

Variable cost of GenCo i

minimize $\mathbf{TVC}(\mathbf{P_G}) = \sum_{i=1}^{I} VC_i(P_{Gi})$ **($/h)**

with respect to $\mathbf{P_G} = (P_{G1},\ldots,P_{GI})^{\mathbf{T}}$ **(MW)**

subject to *the balance constraint*

$$\sum_{i=1}^{I} P_{Gi} = P_D$$ **(MW)**

Fixed (non-price-sensitive) demand as a constraint constant for power dispatch

5-Bus Transmission Grid Example

5 GenCos (I=5), Total Load P_D = 500MW

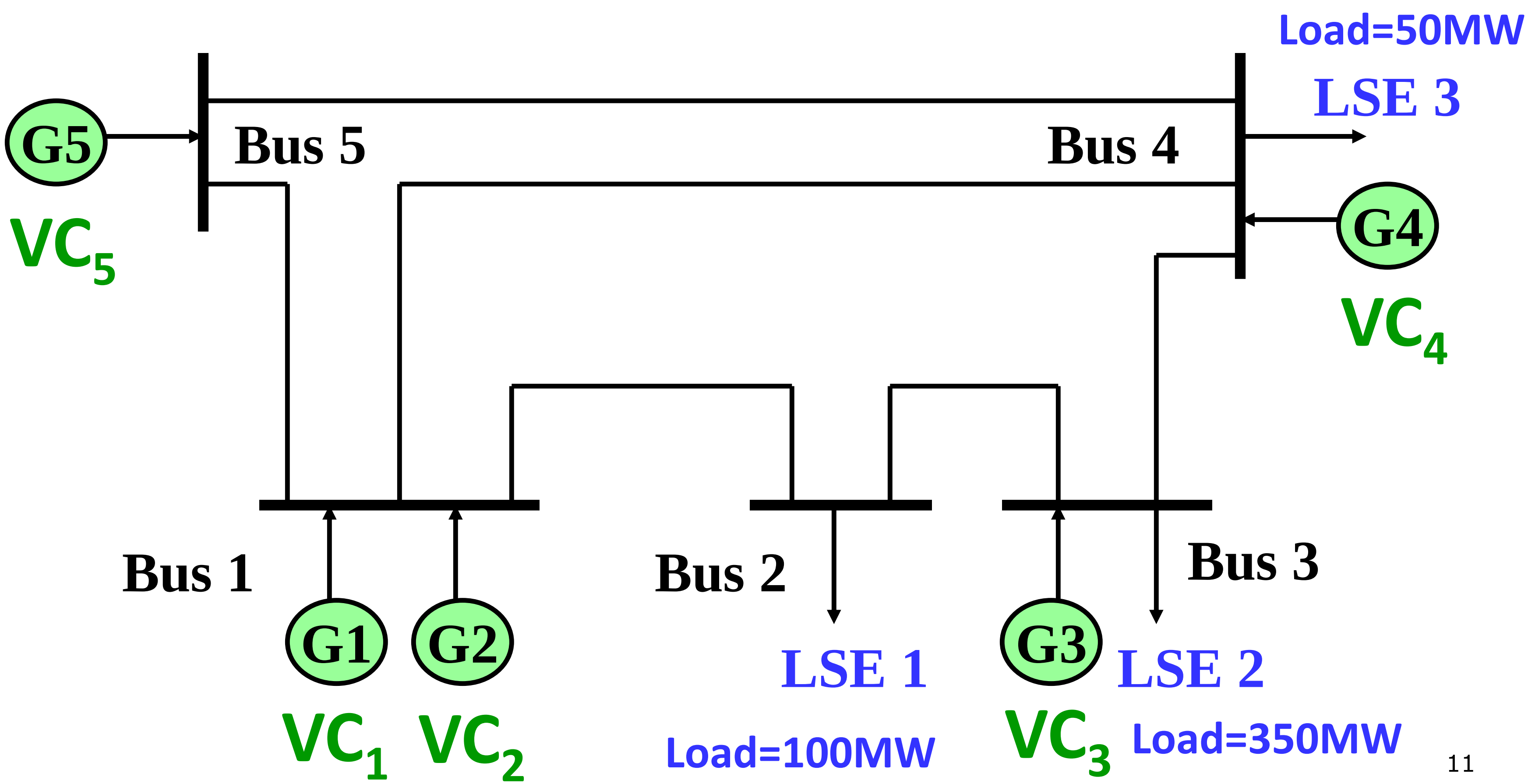
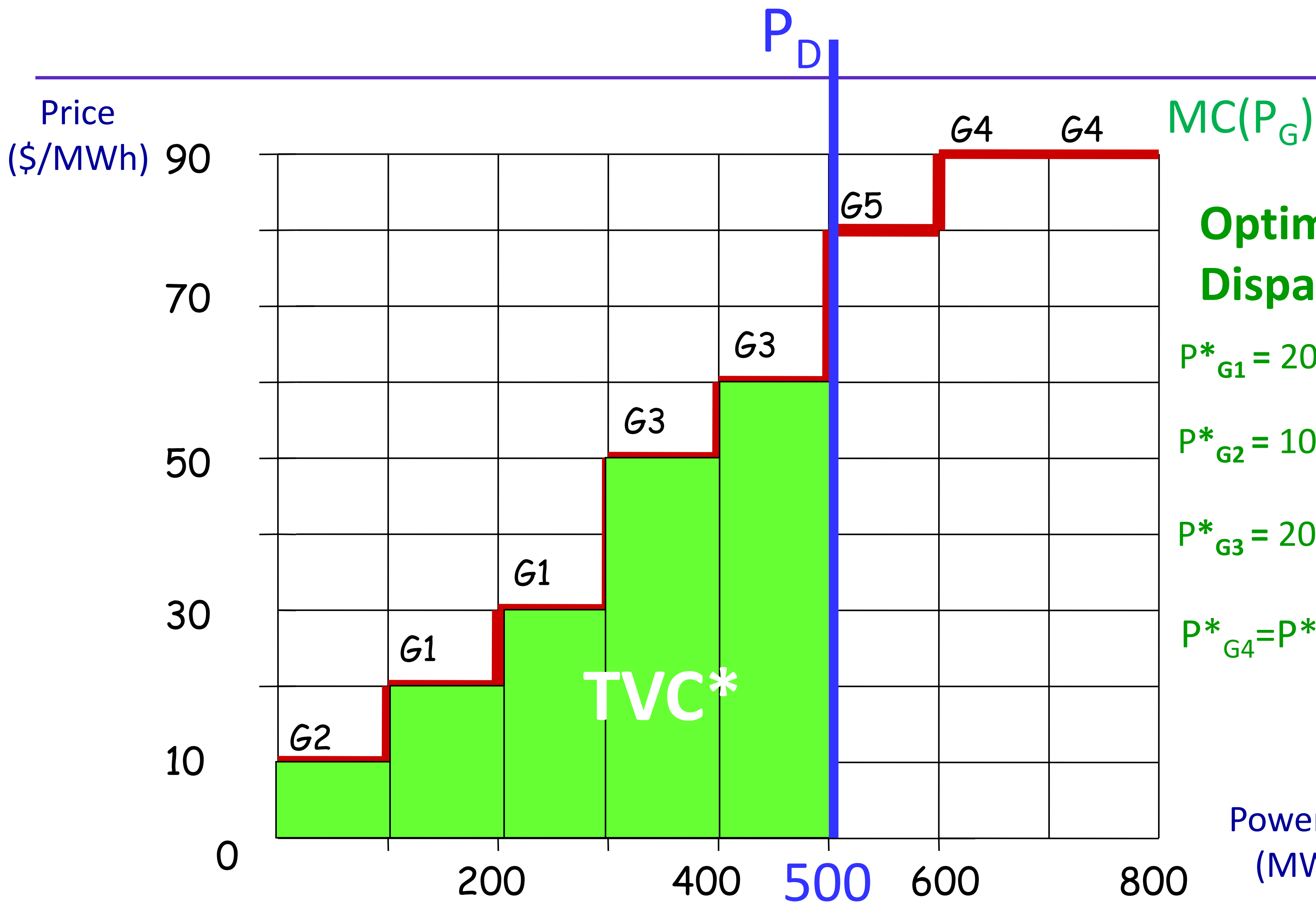


Illustration of TVC Determination for hour H



Optimal Dispatch

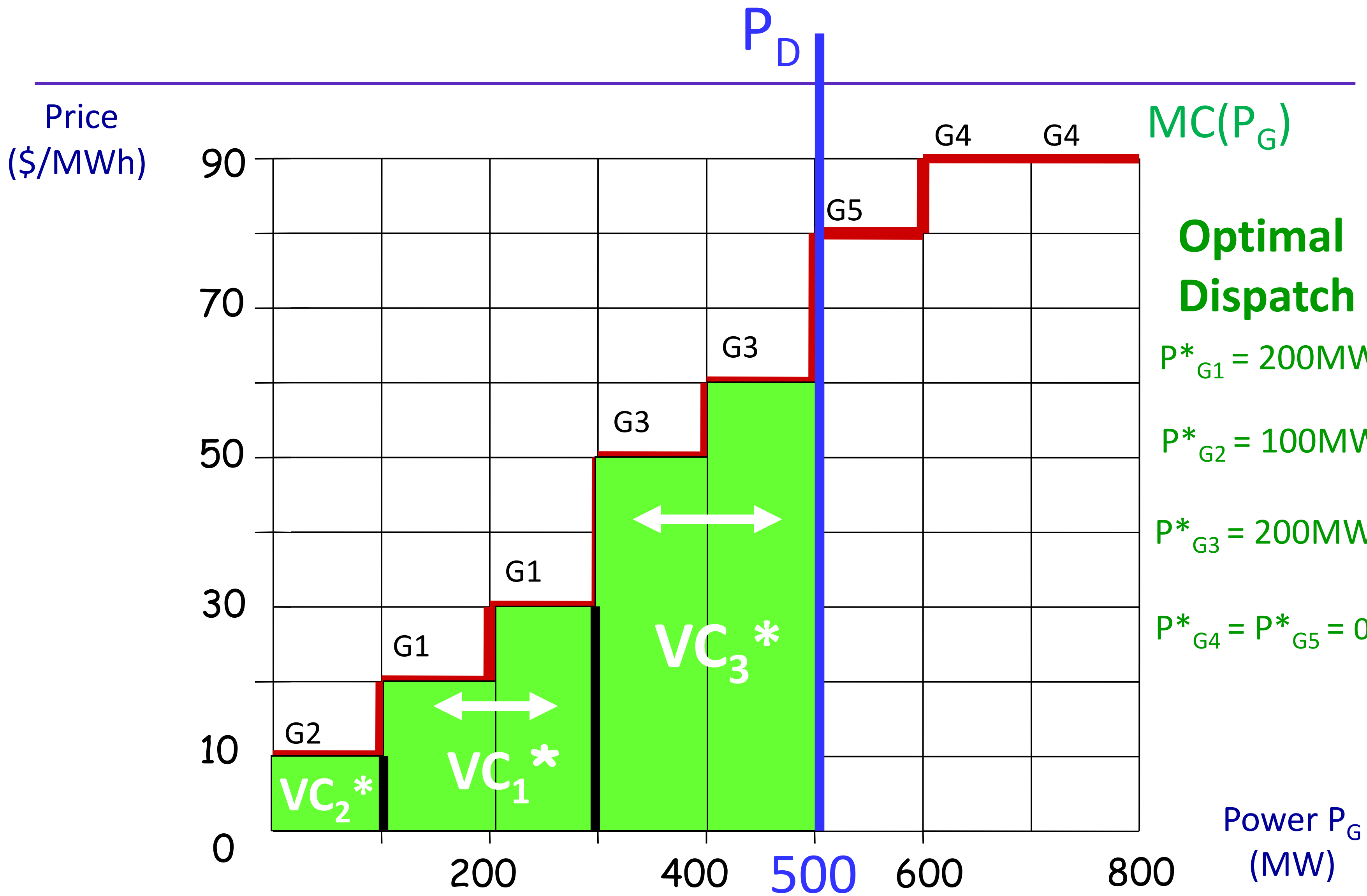
$P^*_{G1} = 200MW$

$P^*_{G2} = 100MW$

$P^*_{G3} = 200MW$

$P^*_{G4} = P^*_{G5} = 0MW$

Illustration of TVC Determination for hour H



Optimal Dispatch

$P^*_{G1} = 200MW$

$P^*_{G2} = 100MW$

$P^*_{G3} = 200MW$

$P^*_{G4} = P^*_{G5} = 0MW$

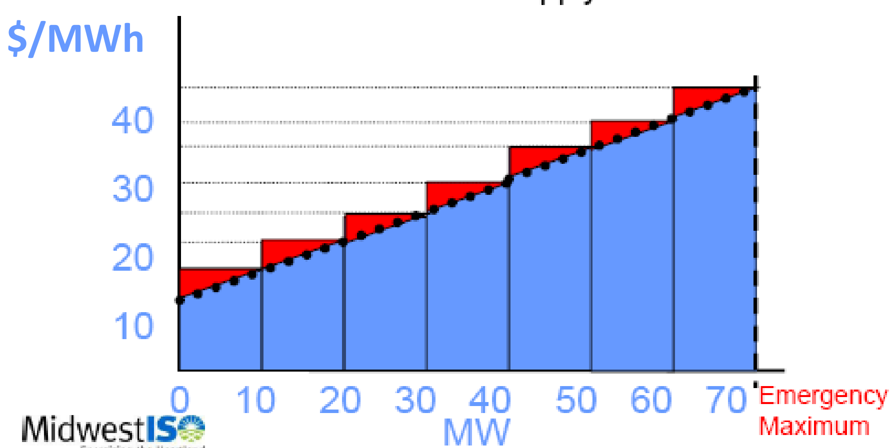
Day-Ahead Supply Offers in MISO

Minimum acceptable price (“sale reservation price”) for each ΔMW

Resource Offers

Energy Offer Curves

- An Offer Curve is an offer to sell generation by a Resource
  - Slope (“true”) vs. block (“false”) offer
  - Monotonically increasing in price and non-decreasing in MW
  - Can vary hourly by location (CPNode)
  - Can submit up to 10 MW/price pairs
  - Previous DA offer carries over to DA and the previous day's RT offer carries over to RT if no supply offer is submitted for the next day



$/MW	MW	ΔMW
10	0	10
15	10	10
20	20	10
25	30	10
30	40	10
35	50	10
40	60	10
45	70	10

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Supply Offers in the MISO … Cont'd

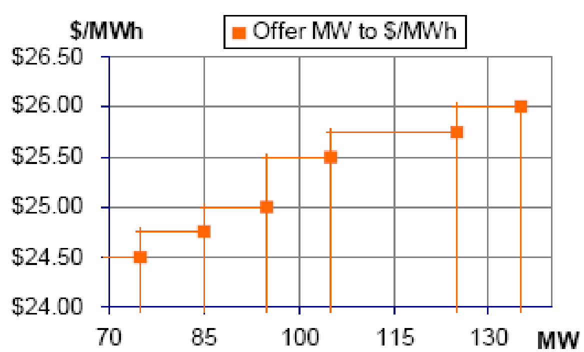
Resource Offers

Offer Curves Exercise Key

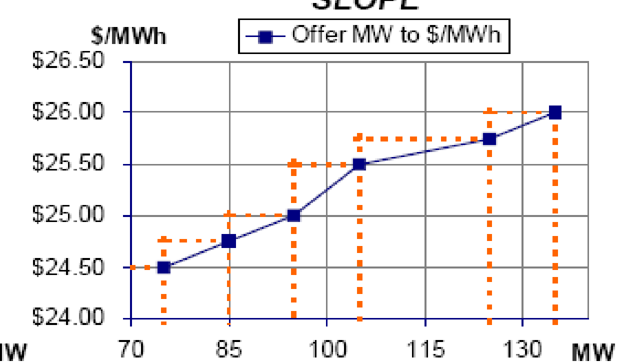
- Diagram this data set as a Block Offer and Slope Offer

	Offer MW to $/MWh	
Segment	MW	$/MWh
1	75	$24.50
2	85	$24.75
3	95	$25.00
4	105	$25.50
5	125	$25.75
6	135	$26.00

BLOCK



SLOPE



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Important Remark on the Form of GenCo Supply Offers

- ***Linear programming (LP)*** is used to handle economic dispatch when supply offers have *block (step-function) form*.

- ***Nonlinear Programming (NP)*** techniques are used to handle economic dispatch when supply offers take a *slope (piecewise differentiable or differentiable) form*. **(Kirschen/Strbac assumption)**

- The remainder of these notes use NP techniques for economic dispatch, assuming supply offers take a differentiable form.

- When we later treat GenCo capacity constraints, we will need to relax this to permit supply offers to take a *piecewise* differentiable form.

Back to Economic Dispatch Problem for hour H

Variable cost of GenCo i

minimize $\quad TVC(\mathbf{P_G}) = \sum_{i=1}^{I} VC_i(P_{Gi}) \qquad (\$/h)$

with respect to $\mathbf{P_G} = (P_{G1},\ldots,P_{GI})^{\mathbf{T}} \qquad (MW)$

subject to the *power balance constraint*

$$\sum_{i=1}^{I} P_{Gi} = P_D \qquad (MW)$$

constraint constant (MW)

Minimization with Equality Constraints: General Solution Method?

- For a differentiable function f(**x**) of an n-dimensional vector $\mathbf{x} = (x_1,\ldots,x_n)^T$, a necessary (but not sufficient) condition for $\mathbf{x}^*$ to minimize f(**x**) is that $\nabla_{\mathbf{x}} f(\mathbf{x}^*) = \mathbf{0}$.

- This multi-variable gradient condition generalizes the first derivative condition for 1-variable min problems:

n-dimensional row vector !

$$\nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}) = \left[\frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_1}, \frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_2}, \ldots, \frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_n} \right]$$

Minimization with Equality Constraints…Cont'd

- When a minimization problem involves equality constraints, we can solve the problem using the ***method of Lagrange Multipliers***
- The key idea is to transform the constrained minimization problem into an unconstrained problem

Minimization with Equality Constraints…Cont'd

- n-dimensional *choice vector* $\mathbf{x} = (x_1,\ldots,x_n)^T$ ➔ n choice variables
- m-dimensional *constraint vector* $\mathbf{c} = (c_1,\ldots,c_m)^T$ ➔ m constraint constants
- R^n = all real n-dimensional vectors, R^m = all real m-dimensional vectors
- $f{:}R^n \rightarrow R$ ➔ *objective function* mapping $\mathbf{x} \rightarrow f(\mathbf{x})$ on real line
- $\mathbf{h}{:}R^n \rightarrow R^m$ ➔ *constraint function* mapping $\mathbf{x} \rightarrow \mathbf{h}(\mathbf{x}) = (h_1(\mathbf{x}),\ldots,h_m(\mathbf{x}))^T$

Nonlinear Programming Problem:

(NPP) minimize $f(\mathbf{x})$ with respect to the choice vector $\mathbf{x}$
subject to the constraint $\mathbf{h}(\mathbf{x}) = \mathbf{c}$

Minimization with Equality Constraints…Cont'd

- The *Lagrange Function* for this problem can be expressed in parameterized form as

$$L(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{c}) = f(\mathbf{x}) - \boldsymbol{\lambda}^T \cdot [\mathbf{h}(\mathbf{x}) - \mathbf{c}]$$

Making explicit the dependence of the Lagrange Function L on the constraint vector **c**

or equivalently,

$$L(\mathbf{x}, \boldsymbol{\lambda}, c) = f(\mathbf{x}) + \boldsymbol{\lambda}^T \cdot [\mathbf{c} - \mathbf{h}(\mathbf{x})]$$

where $\boldsymbol{\lambda}^T = (\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m)$ = *vector of m Lagrange multipliers*

Minimization with Equality Constraints…Cont'd (cf. Fletcher [3])

Math Regularity Conditions: Suppose f, **h** are differentiable, and either

rank($\nabla_x \mathbf{h}(\mathbf{x}^*)$) = m or **h**(**x**) has form $\mathbf{A}_{mxn}\mathbf{x}_{nx1} + \mathbf{b}_{mx1}$.

m by n matrix

$$\nabla_x \mathbf{h}(\mathbf{x}) = \begin{bmatrix} \frac{\partial h_1(x)}{\partial x_1} & \frac{\partial h_1(x)}{\partial x_2} & \cdots & \frac{\partial h_1(x)}{\partial x_n} \\ \frac{\partial h_2(x)}{\partial x_1} & \frac{\partial h_2(x)}{\partial x_2} & \cdots & \frac{\partial h_2(x)}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_m(x)}{\partial x_1} & \frac{\partial h_m(x)}{\partial x_2} & \cdots & \frac{\partial h_m(x)}{\partial x_n} \end{bmatrix}_{(m \times n)}$$

Minimization with Equality Constraints…Continued (cf. Fletcher [3])

First-Order Necessary Conditions (FONC) for **x*** to solve Problem (NPP), given math regularity conditions:

There exists a value **λ*** for the vector **λ** of Lagrange multipliers such that (**x***, **λ***) satisfies:

(1) $\mathbf{0} \;=\; \nabla_x L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \mathbf{c})_{1xn} \;=\; \nabla_x f(\mathbf{x}^*) - \boldsymbol{\lambda}^T \bullet \nabla_x \mathbf{h}(\mathbf{x}^*)$

(2) $\mathbf{0} \;=\; \nabla_{\lambda} L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \mathbf{c})_{1xm} \;=\; [\mathbf{c} - \mathbf{h}(\mathbf{x}^*)]^T$

Lagrange Multipliers as "Shadow Prices"?

- By construction, the solution $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ is a function of the exogenously given constraint vector $\mathbf{c}$:

$$(\mathbf{x}^*, \boldsymbol{\lambda}^*) = (\mathbf{x}(\mathbf{c}), \boldsymbol{\lambda}(\mathbf{c}))$$

- From FONC condition (2) on previous page:

$$f(\mathbf{x}(\mathbf{c})) = L(\mathbf{x}(\mathbf{c}), \boldsymbol{\lambda}(\mathbf{c}), \mathbf{c})$$

Lagrange Multipliers as "Shadow Prices"…

Total Differential

- Then from implicit function theorem, the chain rule, and FONC (1), (2), for each constraint k,

Partial Differentials

$$\frac{df}{dc_k}(x(c)) = \frac{dL}{dc_k}(x(c), \lambda(c), c)$$

$$= \nabla_x L \cdot \frac{\partial \mathbf{x}}{\partial c_k} + \nabla_\lambda L \cdot \frac{\partial \lambda}{\partial c_k} + \frac{\partial L}{\partial c_k}$$

$$= 0 + 0 + \lambda_k(c)$$

This "+" sign follows from our assumed form of the Lagrange function L

Lagrange Multipliers as "Shadow Prices"…

In summary, for each constraint k:

$$\frac{df}{dc_k}(x(c)) \quad = \quad \lambda_k(c)$$

Thus, the Lagrange multiplier solution $\lambda_k(\mathbf{c})$ for the kth constraint gives the change in the ***optimized*** objective function f(**x***) = f(**x**(**c**)) with respect to a change in the constraint constant c_k for the kth constraint appearing in the constraint **h**(**x**) = **c** .

Example: Economic Dispatch Again

For the economic dispatch we have a minimization constrained with a single equality constraint

Constraint Constant

$$L(\mathbf{P}_G,\lambda) = \sum_{i=1}^{I} VC_i(P_{Gi}) + \lambda(P_D - \sum_{i=1}^{I} P_{Gi}) \quad \text{(no losses)}$$

The necessary conditions for a minimum are

$$\frac{\partial L(\mathbf{P}_G,\lambda)}{\partial P_{Gi}} = \frac{dVC_i(P_{Gi})}{dP_{Gi}} - \lambda = 0 \quad (\text{for } i = 1 \text{ to } I)$$

$$P_D - \sum_{i=1}^{I} P_{Gi} = 0$$

Total Dispatch

Economic Dispatch for a Future Hour H:
Numerical Example (Baldick/Overbye)

What is economic dispatch for a two generator system $P_D = P_{G1} + P_{G2} = 500$ MW and

$$VC_1(P_{G1}) = 1000 + 20 P_{G1} + 0.01 P_{G1}^2 \ \$/\text{h}$$

$$VC_2(P_{G2}) = 400 + 15 P_{G2} + 0.03 P_{G2}^2 \ \$/\text{h}$$

This requires cost coefficients to have units, omitted here for simplicity

Using the Lagrange multiplier method we know

$$\frac{dVC_1(P_{G1})}{dP_{G1}} - \lambda = 20 + 0.02 P_{G1} - \lambda = 0$$

$/MWh

$$\frac{dVC_2(P_{G2})}{dP_{G2}} - \lambda = 15 + 0.06 P_{G2} - \lambda = 0$$

$$500 - P_{G1} - P_{G2} = 0$$

Economic Dispatch Example for Hour H …Continued

We therefore need to solve three linear equations

$$20 + 0.02 P_{G1} - \lambda = 0$$

$$15 + 0.06 P_{G2} - \lambda = 0$$

$$500 - P_{G1} - P_{G2} = 0$$

Important Note: **The power levels P_{Gi} (MW) in these expressions denote power levels to be** ***maintained*** **during the future operating hour H.** ***Thus, these equations in power levels P_{Gi} (MW) are in fact equations for determination of grid-delivered energy amounts P_{Gi} x 1h (MWh) for hour H.***

$$\begin{bmatrix} 0.02 & 0 & -1 \\ 0 & 0.06 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} P_{G1} \\ P_{G2} \\ \lambda \end{bmatrix} = \begin{bmatrix} -20 \\ -15 \\ -500 \end{bmatrix}$$

$$\begin{bmatrix} P_{G1}^{*} \\ P_{G2}^{*} \\ \lambda^{*} \end{bmatrix} = \begin{bmatrix} 312.5 \text{ MW} \\ 187.5 \text{ MW} \\ 26.2 \text{ \$/MWh} \end{bmatrix}$$

Solution Values for Hour H

Economic Dispatch Example...Continued

- The solution values for this Economic Dispatch problem for hour H are:

Optimal Dispatch Vector

$$\boldsymbol{P_G}^* = (P_{G1}^*, P_{G2}^*)^T = (312.5\text{MW},\ 187.5\text{MW})^T$$

$$\lambda^* = \$26.2/\text{MWh}$$ (common across all grid buses for hour H)

- By construction, the solution values $\boldsymbol{P_G}^*$ and λ^* for the dispatch vector $\boldsymbol{P_G}$ and Lagrange multiplier λ are functions of the constraint constant $P_D = 500$.

- That is, a change in P_D would result in a change in these solution values.

Economic Dispatch Example … Continued

Applying previous developments for Lagrange multipliers as shadow prices,

$$\lambda^* = \frac{\partial TVC(\boldsymbol{P}_G^*)}{\partial P_D} =: LMP(H) \quad \text{(measured in \$/MWh)}$$

= Change in minimized total variable cost TVC($\boldsymbol{P}_G$*) for hour H (measured in \$/h) with respect to a change in the total power demand P_D (MW) to be maintained during hour H, evaluated at the optimal dispatch vector $\boldsymbol{P}_G$*.

Recall that P_D is the constraint constant for the *power balance constraint* for hour H, which takes the form *Total Genco Dispatch = P_D*

Economic Dispatch Example … Continued

Roughly stated, the ***Locational Marginal Price*** LMP**(**k**,**H**)** at a grid bus k for an operating hour H is the least cost of servicing a "next unit" (MWh) of load (energy demand) at bus k during hour H.

Assume for simplicity that there are no losses of power from grid transmission lines and no binding transmission constraints, hence no LMP separation across grid buses. Then, given suitable math regularity conditions:

$$\boldsymbol{\lambda^*} = \frac{\partial TVC(\mathbf{P}_G\mathbf{*})}{\partial P_D} \qquad \text{(in \$/MWh)}$$

= ***Locational marginal price (LMP)*** for grid-delivered energy (MWh) <u>at each grid bus k</u> during hour H.

Extension to Inequality Constraints

(cf. Fletcher [3])

General Nonlinear Programming Problem (GNPP):

- **x** = n x 1 choice vector;
- **c =** m x 1 vector & **d** = s x 1 vector (constraint constants)
- f(**x**) maps **x** into R (all real numbers)
- **h**(**x**) maps **x** into R^m (all m-dimensional vectors)
- **z**(**x**) maps **x** into R^s (all s-dimensional vectors)

GNPP: Minimize f(x) with respect to x subject to

$$\mathbf{h}(\mathbf{x}) = \mathbf{c}$$

$$\mathbf{z}(\mathbf{x}) \geq \mathbf{d}$$

Important Remark on the Representation of Inequality Constraints

Note: The inequality constraint for (GNPP) can ***equivalently*** be expressed in a variety of ways, as indicated below:

(1) $\mathbf{z}(\mathbf{x}) \geq \mathbf{d}$;

(2) $\mathbf{z}(\mathbf{x}) - \mathbf{d} \geq \mathbf{0}$;

(3) $- \mathbf{z}(\mathbf{x}) - [-\mathbf{d}] \leq \mathbf{0}$;

(4) $\mathbf{r}(\mathbf{x}) - \mathbf{e} \leq \mathbf{0}$ $\quad (\mathbf{r}(\mathbf{x}) =: -[\mathbf{z}(\mathbf{x})],\ \mathbf{e} =: -[\mathbf{d}])$

(5) $\mathbf{r}(\mathbf{x}) \leq \mathbf{e}$

Why Our Form of Inequality?

Our GNPP Form:

Minimize f(**x**) with respect to **x** subject to

$$\mathbf{h}(\mathbf{x}) = \mathbf{c}$$

$$(*) \quad \mathbf{z}(\mathbf{x}) \geq \mathbf{d}$$

Given suitable regularity conditions

- Given this form, we know that an ***INCREASE*** in **d** has to result in a new value for the minimized objective function f(**x***) that is ***AT LEAST AS GREAT*** as before.
- Why? When **d** increases the feasible choice set for **x** ***SHRINKS***, hence [min f] either ↑ or stays same.
- ➔ $0 \leq \partial f(\mathbf{x}^*)/\partial \mathbf{d} = \boldsymbol{\mu}^{*T}$ = Shadow price vector for (*)

Extension to Inequality Constraints…Continued

- Define the ***Lagrangean Function*** as

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{c}, \mathbf{d}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T[\mathbf{c} - \mathbf{h}(\mathbf{x})] + \boldsymbol{\mu}^T[\mathbf{d} - \mathbf{z}(\mathbf{x})]$$

- Assume ***Kuhn-Tucker Constraint Qualification (KTCQ)*** holds at **x***, roughly stated as follows:

The true set of feasible directions away from **x***

= Set of feasible directions away from **x*** assuming a linearized set of constraints in place of the original set of constraints.

Extension to Inequality Constraints…Continued

- Given KTCQ, the ***First-Order Necessary Conditions (FONC)*** for $\mathbf{x}^*$ to solve the (GNPP) are as follows: There exist $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ such that $(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ satisfy:

$$0 = \nabla_x L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \mathbf{c}, \mathbf{d})$$

$$= [\, \nabla_x f(\mathbf{x}^*) - \boldsymbol{\lambda}^{*T} \bullet \nabla_x \mathbf{h}(\mathbf{x}^*) - \boldsymbol{\mu}^{*T} \bullet \nabla_x \mathbf{z}(\mathbf{x}^*) \,] \,;$$

$$\mathbf{h}(\mathbf{x}^*) = \mathbf{c} \,;$$

$$\mathbf{z}(\mathbf{x}^*) \geq \mathbf{d}; \quad \boldsymbol{\mu}^{*T} \cdot [\mathbf{d} - \mathbf{z}(\mathbf{x}^*)] = 0; \quad \boldsymbol{\mu}^* \geq \mathbf{0}$$

- These FONC are known as the ***Karush-Kuhn-Tucker (KKT) Conditions***.

Solution Vector as a Function of (c,d)

By construction, the components of the solution vector ($\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*$) are *functions* of the constraint constant vectors **c** and **d:**

- $\mathbf{x}^* = \mathbf{x}(\mathbf{c},\mathbf{d})$
- $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}(\mathbf{c},\mathbf{d})$
- $\boldsymbol{\mu}^* = \boldsymbol{\mu}(\mathbf{c},\mathbf{d})$

GNPP Lagrange Multipliers as Shadow Prices

Given additional math regularity conditions…

- The solution $\boldsymbol{\lambda}^*$ for the m x 1 Lagrange multiplier vector $\boldsymbol{\lambda}$ is the derivative of the solution value $f(\mathbf{x}^*)$ of the objective function $f(\mathbf{x})$ with respect to the constraint vector $\mathbf{c}$, all other problem aspects remaining the same.

$$\partial f(\mathbf{x}^*)/\partial \mathbf{c} = \partial f(\mathbf{x}(\mathbf{c},\mathbf{d}))/\partial \mathbf{c} = \boldsymbol{\lambda}^{*T}$$

GNPP Lagrange Multipliers as Shadow Prices…

Given additional math regularity conditions…

- The solution $\boldsymbol{\mu}^*$ for the s x 1 multiplier vector $\boldsymbol{\mu}$ is the derivative of the solution value f($\mathbf{x}^*$) of the objective function f($\mathbf{x}$) with respect to the constraint vector $\mathbf{d}$, all other problem aspects remaining the same.

$$0 \leq \partial f(\mathbf{x}^*)/\partial \mathbf{d} = \partial f(\mathbf{x}(\mathbf{c},\mathbf{d}))/\partial \mathbf{d} = \boldsymbol{\mu}^{*T}$$

GNPP Lagrange Multipliers as Shadow Prices…

In this case…

- The solution $\boldsymbol{\lambda}^*$ for the multiplier vector $\boldsymbol{\lambda}$ thus essentially gives the *prices (values)* associated with unit changes in the components of the constraint vector $\mathbf{c}$.
- The solution $\boldsymbol{\mu}^*$ for the multiplier vector $\boldsymbol{\mu}$ thus essentially gives the *prices (values)* associated with unit changes in the components of the constraint vector $\mathbf{d}$.
- Each component of $\boldsymbol{\lambda}^*$ can take on ***any sign***
- Each component of $\boldsymbol{\mu}^*$ must be ***nonnegative***

Sufficient Conditions for Minimization?

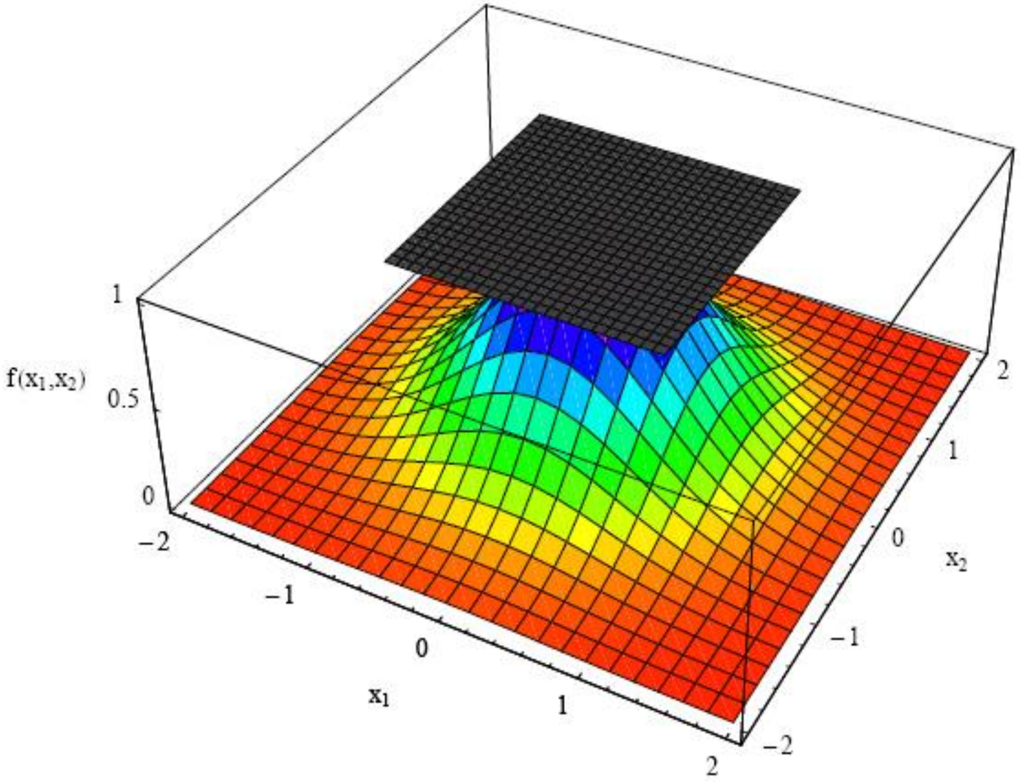
First-order ***necessary*** conditions for **x*** to solve NPP/GNPP are ***not sufficient in general*** to ensure **x*** solves NPP/GNPP, or to ensure **x*** solves NPP/GNPP uniquely.

- **What can go wrong?**
  - (1) Local maximum rather than local minimization
  - (2) Inflection point rather than minimum point
  - (3) Local minimum rather than global minimum
  - (4) Multiple minimizing solution points

- Conditions on second derivatives are needed to rule out 1 & 2, and "global" methods/conditions are needed to rule out 3 & 4 .

Example: Local Max Rather Than Local Min

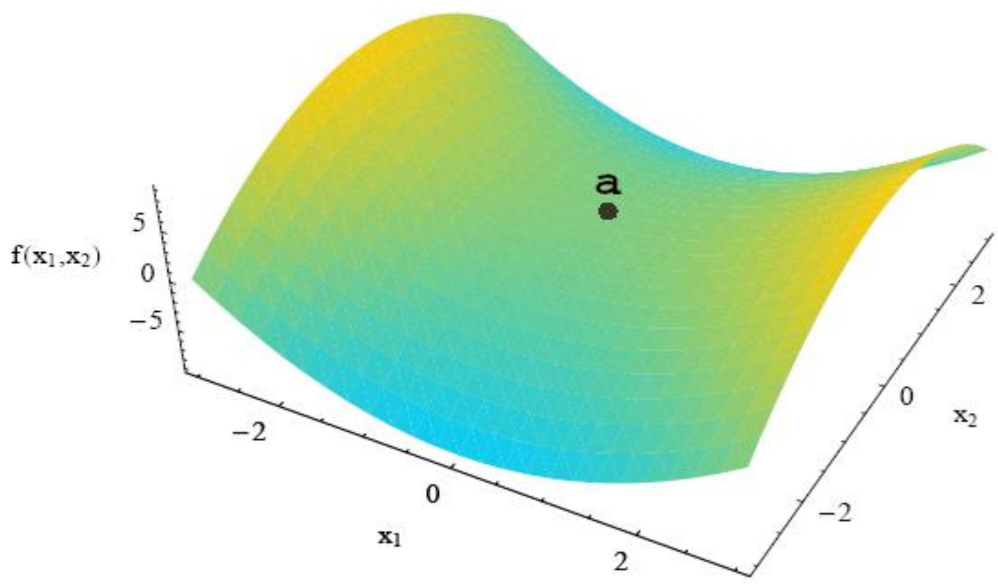
FIGURE 1. Local maximum of function $f(x_1, x_2) = e^{-(x_1^2+x_2^2)}$



From: A. Hallam, "Simple Multivariate Optimization" (on-line)

Example: Inflection Rather than Minimum Point

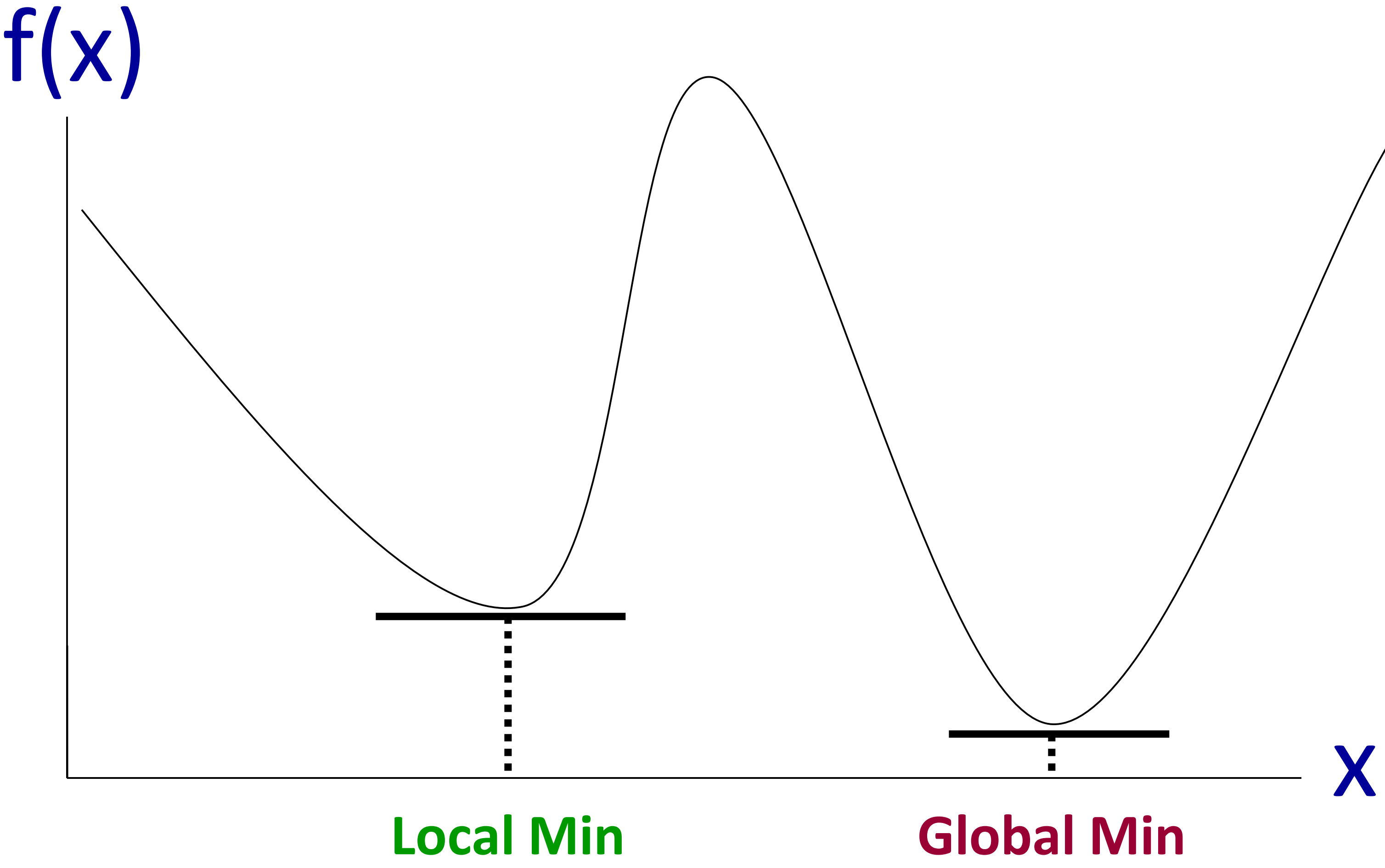
FIGURE 2. Saddle point of the function $f(x_1, x_2) = x_1^2 - x_2^2$



From: A. Hallam, "Simple Multivariate Optimization" (on-line)

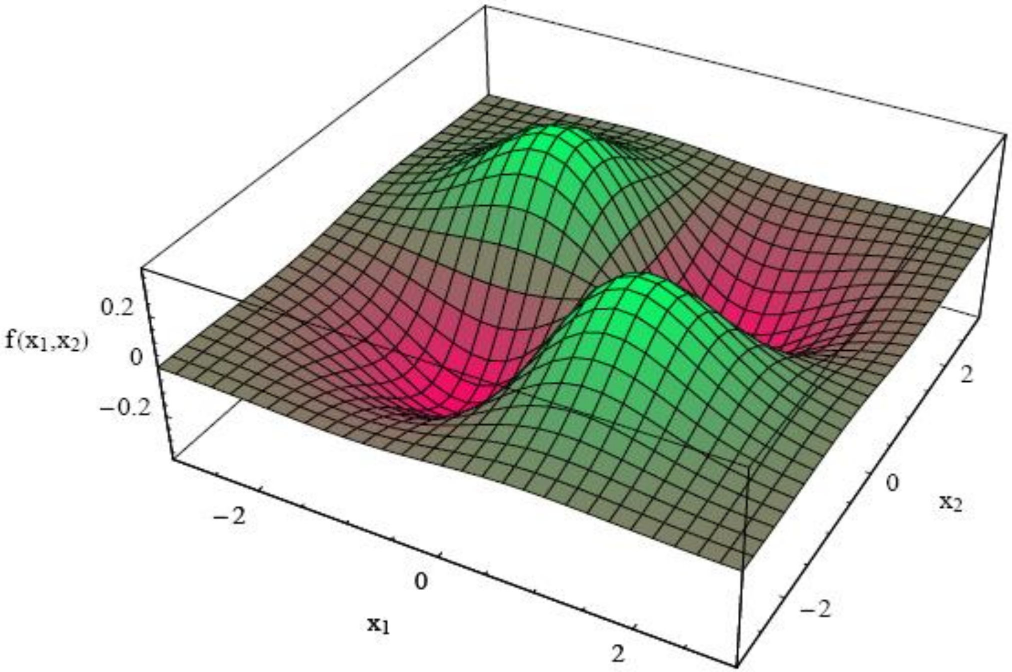
Example: Local Min Rather than Global Min

Note that <u>both</u> points satisfy the FONC given by df(x)/dx = 0 .



Example: Multiple Minimization Points

FIGURE 18. Graph of the function $f(x_1, x_2) = -x_1 x_2 e^{\frac{-(x_1^2 + x_2^2)}{2}}$



From: A. Hallam, "Simple Multivariate Optimization" (on-line)

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