Lecture notes by Dr. Arne Hallam, Iowa State University, for an introductory course (Econ 500): Quantitative Methods in Economic Analysis

SINGLE VARIABLE OPTIMIZATION

1. Definition of local maxima and local minima

1.1. Note on open and closed intervals.

1.1.1. *Open interval.* If a and b are two numbers with a < b, then the **open interval** from a to b is the collection of all numbers which are both larger than a and smaller than b. The open interval consists of all numbers between a and b. A compact way of writing this is a < x < b. We denote an open interval with parentheses as (a, b).

1.1.2. *Closed interval.* If a and b are two numbers with a < b, then the **closed interval** from a to b is the collection of all numbers which are both greater than or equal to a and less than or equal to b. The closed interval consists of all points between a and b including a and b. A compact way of writing this is a ≤ x ≤ b. We denote a closed interval with brackets as [a, b].

1.1.3. *Half-open intervals.* Intervals that are closed on one end and open on the other are called half-open intervals. We can denote these as half open on the right or the left and use a mixture of brackets and parentheses as appropriate.

1.2. Max-min theorem for continuous functions.

Theorem 1. *If f is continuous at every point of a closed interval [a, b] ⊂ R, then f assumes both an absolute maximum value* ***M*** *and an absolute minimum value* ***m*** *somewhere in [a, b]. That is, there are numbers x_1 and x_2 in [a, b] with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in [a, b].*

1.3. Absolute (global) extreme values.

1.3.1. *Absolute maximum.* Let f be a function with domain D. Then f has an absolute maximum value on D at a point c if

$$f(x) \leq f(c) \text{ for all } x \in D \tag{1}$$

1.3.2. *Absolute minimum.* Let f be a function with domain D. Then f has an absolute minimum value on D at a point d if

$$f(x) \geq f(d) \text{ for all } x \in D \tag{2}$$

1.4. Local extreme values.

1.4.1. *Local maximum.* Consider a real valued function f defined with domain D. Then f is said to have a local maximum at an interior point $x^* \in D$ if there exists a real number $\delta > 0$ such that

$$f(x) \leq f(x^*) \text{ for all } x \text{ satisfying } \| x - x^* \| < \delta \tag{3}$$

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1.4.2. *Local minimum.* Consider a real valued function f defined with domain D. Then f is said to have a local minimum at an interior point $\tilde{x} \epsilon$ D if there exists a real number $\delta > 0$ such that

$$f(x) \geq f(\tilde{x}) \; \forall \, x \text{ satisfying } \| x - \tilde{x} \| < \delta \tag{4}$$

1.5. **Strict maxima and minima, optimal points, and extreme points.** If the value of f at x* is strictly larger than at any other point in the interval, the x* is a **strict** maximum point. Similarly $\tilde{x}$ is a **strict** minimum point if f (x) $>$ f ($\tilde{x}$) for all x in the interval, x $\neq \tilde{x}$. We often refer to maxima and minima as optimal points or extreme points.

2. Inflection points

2.1. Definition of an inflection point.

Definition 1. The point **c** is called an *inflection point* for the twice-differentiable function f if there exists an interval (a,b) about c such that:

(a) $f''(x) \geq 0$ in (a,c) and $f''(x) \leq 0$ in (c,b) or
(b) $f''(x) \leq 0$ in (a,c) and $f''(x) \geq 0$ in (c,b)

The point x=c is an inflection point if f″(x) changes sign at x=c.

2.2. Test for inflection points.

Theorem 2. *Let f be a function with a continuous second derivative in an interval I, and let c be in interior point of I.*

(a) If c is an inflection point for f, then f″(c)=0.
(b) If f″(c) = 0 and f″ changes sign at c, then c is an inflection point for f.

Proof.

(a) Because $f''(x) \leq 0$ on one side of c and $f''(x) \geq 0$ on the other, and because f″ is continuous, it must be true that $f''(c) = 0$.
(b) If f″ changes sign at c, then c is an inflection point for f, according to definition 1.

This theorem implies that $f''(c) = 0$ is a necessary condition for c to be an inflection point. it is not a sufficient condition, however, because $f''(c) = 0$ does not imply that f″ changes sign at x = c. Typical cases are given in later examples. □

3. Extreme points and derivatives

3.1. **A first derivative test for extreme points.** Let f be a real valued function defined on a domain D. If f has a local maximum or a local minimum at an interior point c ε D, and if f′ is defined at c, then

$$f'(c) = \left. \frac{d\,f(\cdot)}{dx} \right|_c = 0. \tag{5}$$

Proof. To show that f′(c) is zero at a local extremum, we show first that f′(c) cannot be positive and second that it cannot be negative. The only number that is neither positive or negative is zero, so that is what f′(c) must be. Suppose that f has a local maximum value at x = c so that f(x) - f(c) $\leq$ 0 for all values of x sufficiently close to c. Because c is an interior point of f′s domain, f′(c) is defined by the two-sided limit □

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}. \tag{6}$$

This means that the right-hand side and left-hand side limits both exist at c and equal f′(c). When we examine these limits separately, we find that

$$\begin{gathered} f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0 \\ because \\ (x - c) > 0 \text{ and } f(x) \leq f(c) \end{gathered} \tag{7}$$

Similarly,

$$\begin{gathered} f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0 \\ because \\ (x - c) < 0 \text{ and } f(x) \leq f(c) \end{gathered} \tag{8}$$

Together 7 and 8 imply that f′(c) = 0. The proof for minimum values is similar.

3.2. **Implication of theorem 2.** The only places where a function f can possibly have an extreme value (local or global) are

- **a:** interior points where f′(·) = 0,
- **b:** interior points where f′(·) is undefined,
- **c:** endpoints of the domain of f.

3.3. **Critical (stationary) points.** An interior point of the domain of a function f where f′ is zero or undefined is a **critical (stationary) point** of f. The only domain points where a function f can assume extreme values are critical points and endpoints.

3.4. **Graphical illustration of extreme points.**

3.4.1. *A relative maximum.* Figure 1 shows the relative maximum of a function.

3.4.2. *A relative minimum.* Figure 2 shows the relative minimum of a function.

3.5. **Examples of finding relative maximums and minimums.**

3.5.1. *Example 1.* Find the absolute maximum and minimum values of $f(x) = x^2,\ x \in [-2, 1]$.

The only critical value of f(x) is at x = 0 because f′(·) = 2x = 0 ⇒ x = 0. At x = 0, f(x) = 0. The endpoint values are f(-2) = 4 and f(1) = 1. So the function has an absolute maximum value of 4 at x = -2 and an absolute minimum value of 0 at x = 0. Figure 3 shows the graph of example 1.

FIGURE 1. Relative maximum of function $y = 20 + 10x - 0.5x^2$

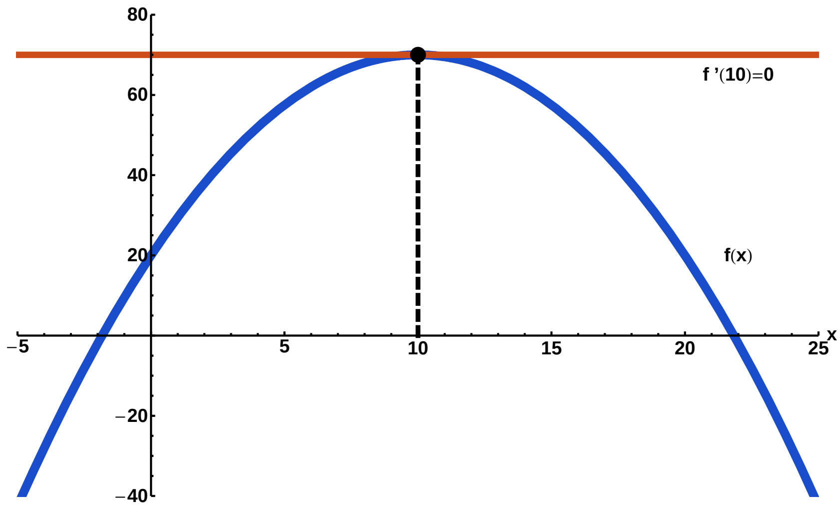
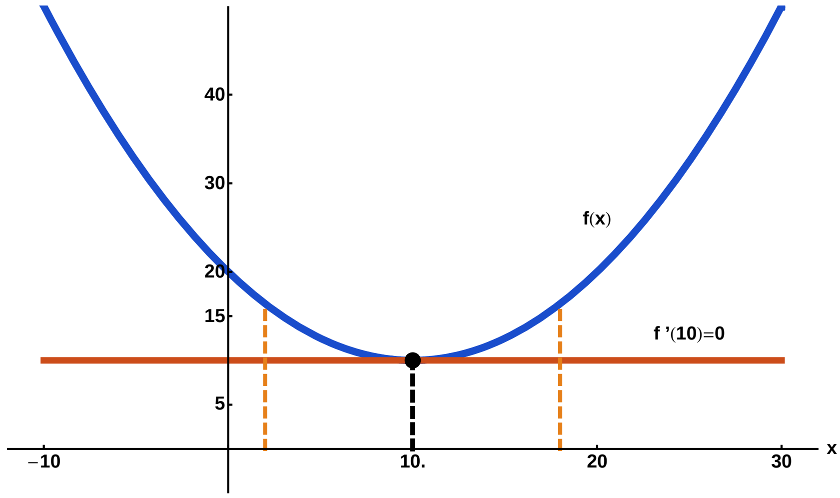


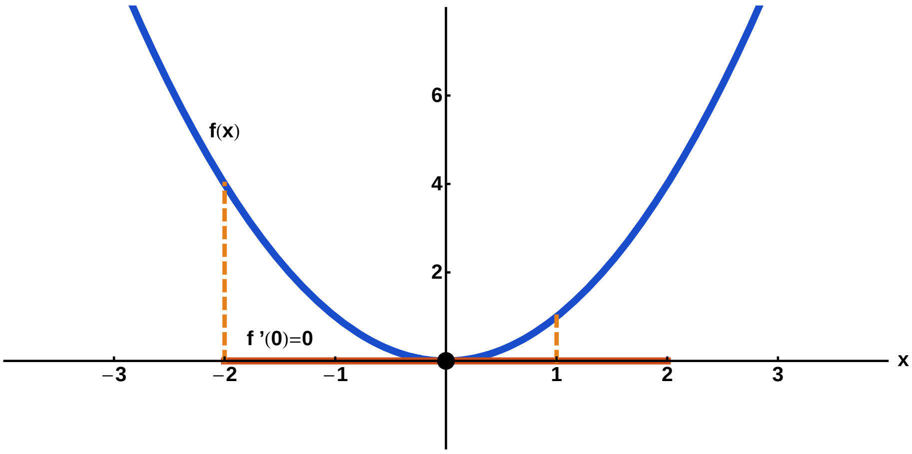
FIGURE 2. Relative minimum of function $y = 20 - 2x - 0.1x^2$



3.5.2. *Example 2.* Find the absolute maximum and minimum values of $f(x) = 8x - x^4,\ x \in [-2, 1]$.

Setting the derivative equal to zero we obtain

FIGURE 3. Maximum and minimum of function $y = x^2,\ x \in [-2, 1]$



$$\begin{aligned} f(x) &= 8x - x^4 \\ \Rightarrow f'[x] = 8\ -\ 4x^3 &= 0 \\ \Rightarrow\ 4x^3 &= 8 \\ \Rightarrow\ x^3 &= 2 \\ \Rightarrow\ x &= 2^{\frac{1}{3}} \\ &= \sqrt[3]{2} \end{aligned}$$

The point $\sqrt[3]{2}$ is not in the interval [-2, 1]. So we check the endpoints of the interval. This yields $f(-2) = -32$ and $f(1) = 7$. So the absolute maximum of the function $f(x) = 8x - x^4$ in the interval [-2, 1] is 7 at x = 1 and the absolute minimum in the interval is -32 at x = -2. Figure 4 shows the graph of example 2.

3.6. **Increasing and decreasing functions.**

3.6.1. *Definitions of increasing and decreasing functions.* Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

a: f **increases** on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
b: f **decreases** on I if $x_1 < x_2 \Rightarrow f(x_2) < f(x_1)$.

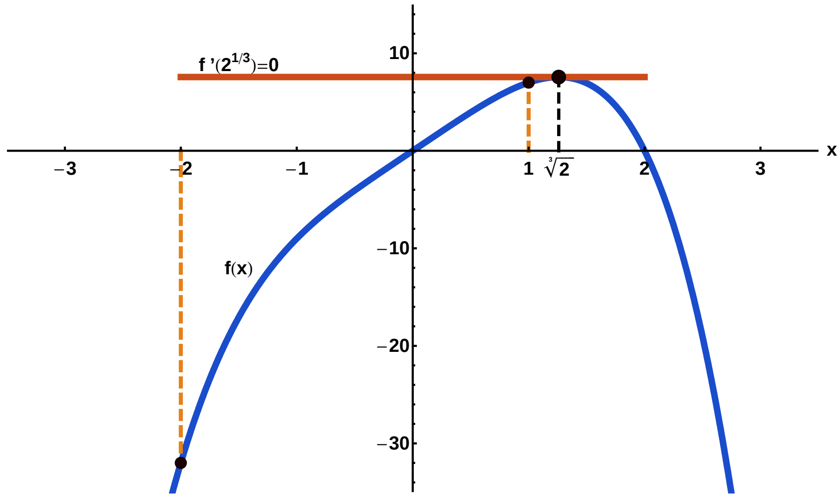
3.6.2. *First derivative test for increasing and decreasing functions.* Suppose that f is continuous on [a, b] and differentiable on (a, b).

a: If $f'(\ ) > 0$ at each point of (a, b), then f increases on [a, b].
b: If $f'\ < 0$ at each point of (a, b), then f decreases on [a, b].

4. Tests for local extreme values

4.1. **Necessary versus sufficient conditions.**

FIGURE 4. Maximum and minimum of function $y = 8x - x^4$, $x \, \varepsilon \, [-2, 1]$

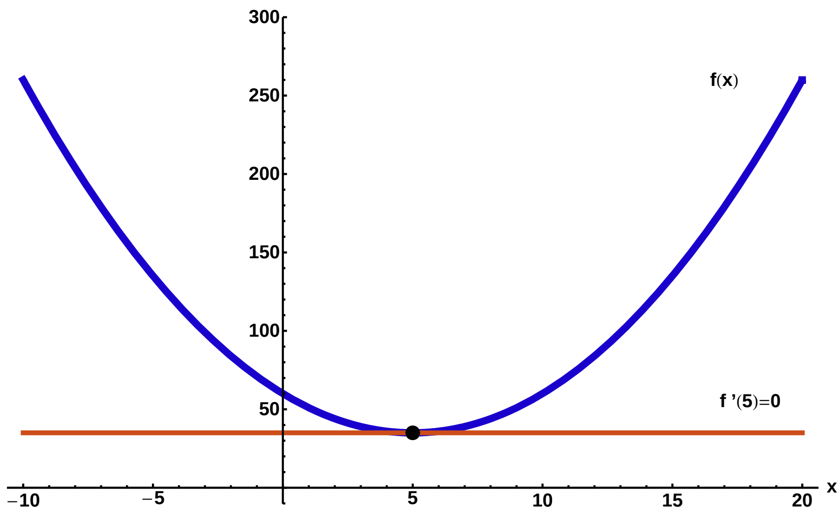


While the zero derivative condition (critical value) is necessary for a maximum or minimum, it is clear that it is not sufficient. In other words, while all local extreme points are critical values, not all critical values are extreme points. Consider the three different cases below.

4.2. **Types of critical points.**

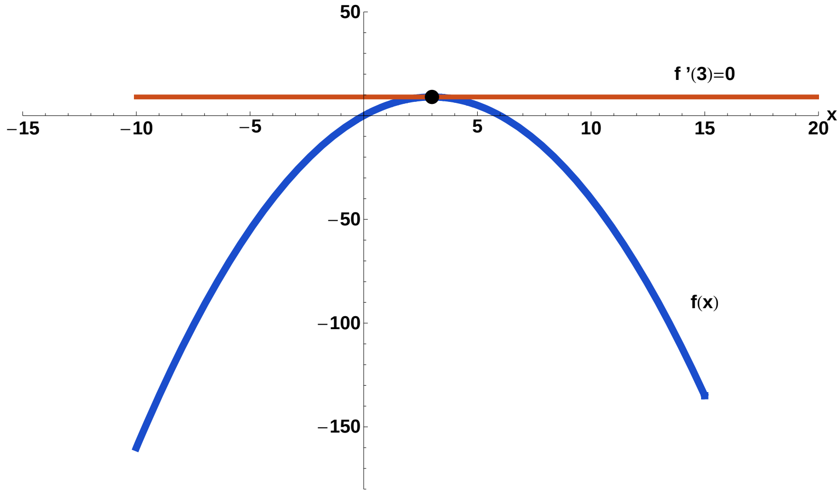
4.2.1. *Unique global minimum.* First consider a function where there is a unique global minimum, so that a relative minimum is also the global minimum. The function $y = x^2 - 10x + 60$ is a parabola that has a global minimum at x = 5. The graph of this function is given in figure 5.

FIGURE 5. Global minimum of function $y = x^2 - 10x + 60$



4.2.2. *Unique global maximum.* Now consider a case where the relative maximum is a global maximum. The function $y = 6x - x^2$ is a parabola that has a global maximum at x = 3. The graph of this function is given in figure 6.

FIGURE 6. Global maximum of function $y = 6x - x^2$



4.2.3. *Local maxima and minima but no global extreme values over the entire domain (in this case the real numbers).* Now consider a case where the function has local maxima and minima and there are no global maximums or minimums. Specifically consider the function $y = 50 - 6x + 1/18x^3$. The graph of this function is given in figure 7. The relative maximum is at x = -6 and the relative minimum is at x = 6.

4.2.4. *No relative maxima or minima at critical points.* Consider the case where the first derivative is equal to zero but there is no relative maximum or minimum. Such a point is called an inflection point. The second derivative is zero at this point. A graph of a function with an inflection point at a critical value is given in figure 8.

4.3. **The first derivative test for local extreme values.**

4.3.1. *Local extreme values at critical points (points where f′(c) = 0).*

- **a:** If f′( ) changes from positive to negative at c (f′(c) $>$ 0 for x $<$ c and f′(c) $<$ 0 for x $>$ c), then f has a local maximum value at c. This case in shown in figure 9.
- **b:** If f′( ) changes from negative to positive at c, (f′( ) $<$ 0 for x $<$ c and f′( ) $>$ 0 for x $>$ c), then f has a local minimum value at c. This case in shown in figure 10.
- **c:** If f′( ) does not change sign at c (f′ has the same sign on both sides of c), then f has no local extreme value at c. This case in shown in figure 11.

4.3.2. *Local extreme values at endpoints.*

- **a:** At the left endpoint a, if f′ $<$ 0 (f′ $>$ 0) for x $>$ a, then f has a local maximum (minimum) at a. The case with the minimum or maximum at the left end point is shown in figure 12
- **b:** At the right endpoint b, if f′ $<$ 0 (f′ $>$ 0) for x $<$ b, then f has a local minimum (maximum) at b. This case is shown in figure 13.

FIGURE 7. Relative minimum and maximum of function $y = 50 - 6x + 1/18x^3$

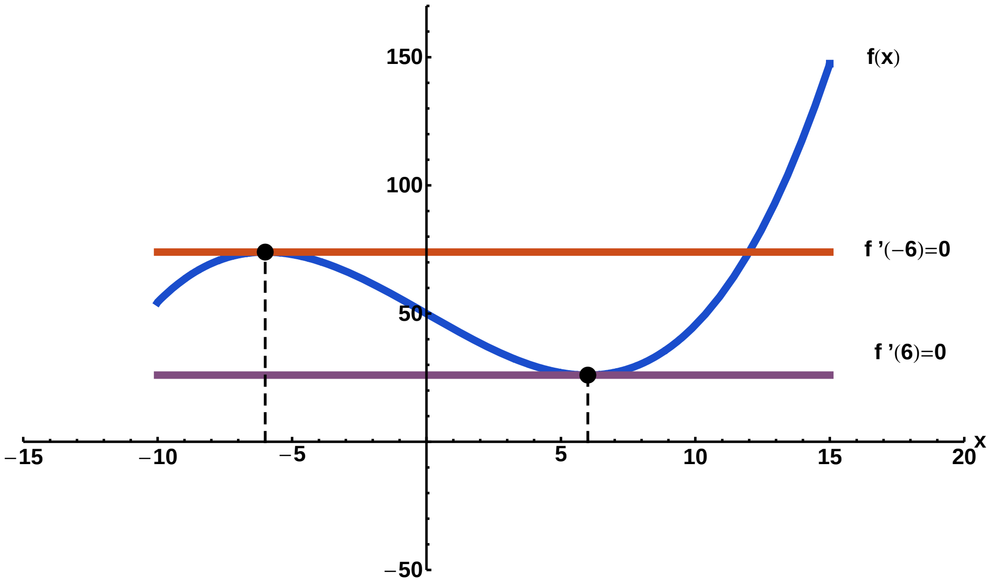
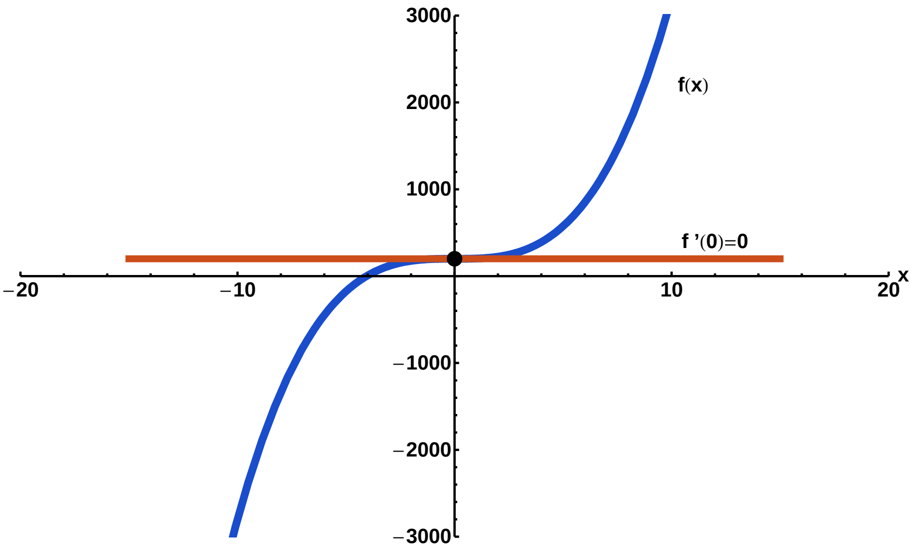


FIGURE 8. Inflection Point of function $y = 3x^3 + 200$



4.4. **The second derivative test for local extreme values.**

4.4.1. *The second derivative test.*

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at x = c.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at x = c.

FIGURE 9. A local maximum with f′(c) = 0 and with f′(c) undefined

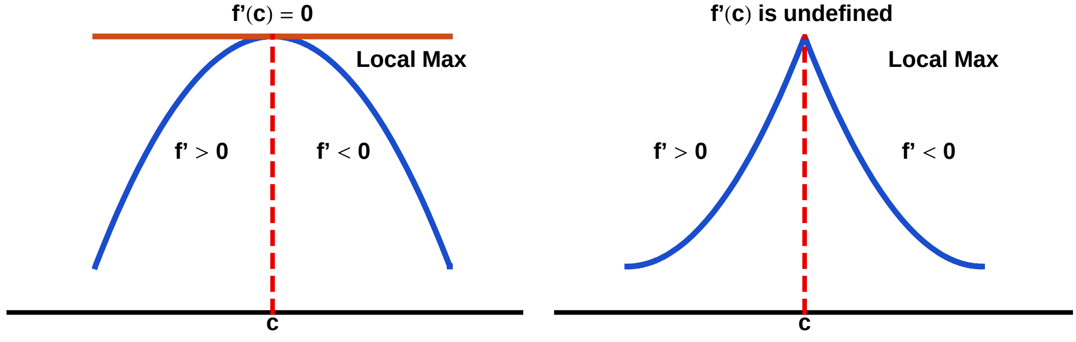


FIGURE 10. A local minimum with f′(c) = 0 and with f′(c) undefined

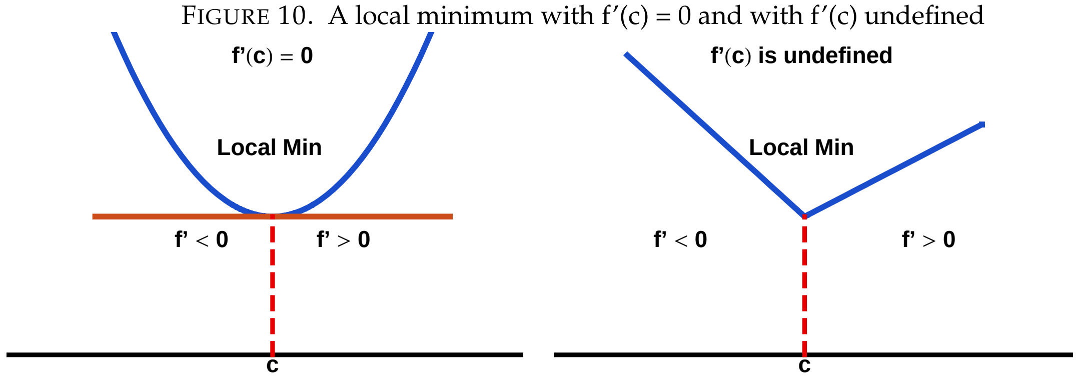
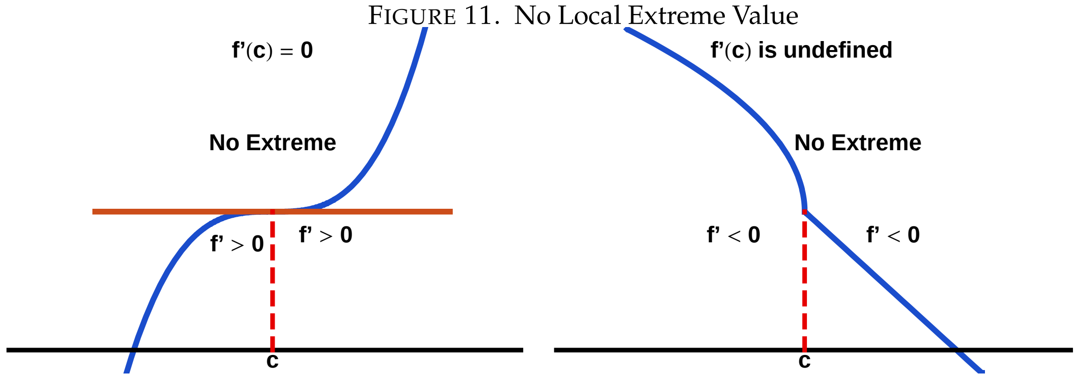


FIGURE 11. No Local Extreme Value



4.4.2. *Sufficient condition for a maximum or minimum.* For some integer $n \geq 1$, let f have a continuous n^{th} derivative in the open interval (a,b). Suppose also that for some interior point c ∈ (a,b) we have

$$f'(c) = f''(c) = \cdots = f^{n-1} = 0, \text{but} f^{n}(c) \neq 0 \tag{9}$$

Then for n even, f has a local minimum at c if $f^{n}(c) > 0$, and a local maximum if $f^{n}(c) < 0$. If n is odd, there is neither a local maximum nor a local minimum at c.

FIGURE 12. A local minimum and maximum at left endpoints

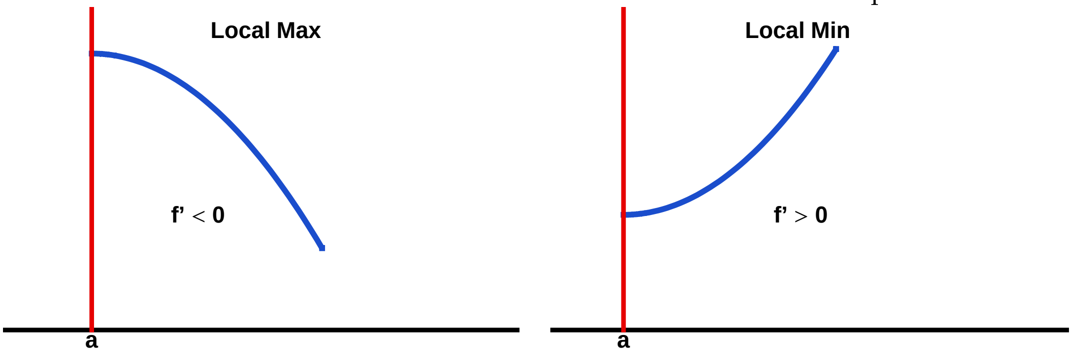
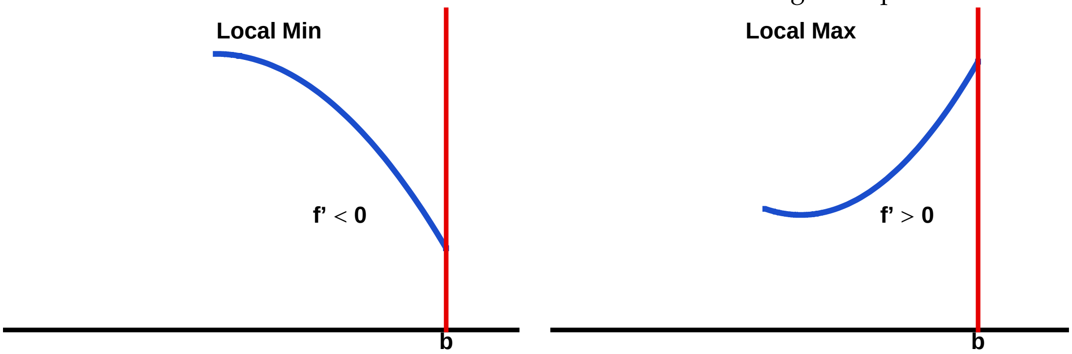


FIGURE 13. A local minimum and maximum at right endpoints



4.4.3. *Examples.*

a: Consider a case where the function has a global minimum.

$$\begin{aligned} y &= f(x) = x^2 \\ f'(x) &= 2x = 0 \\ &\Rightarrow x = 0 \text{ is an extreme point} \\ f''(x) &= 2 \text{ which is not zero (n is even) and positive} \\ &\Rightarrow x = 0 \text{ is a local minimum} \end{aligned}$$

Figure 14 shows the graph for the example in part a.

b: Consider a case where the function has a global maximum.

$$\begin{aligned} y &= f(x) = 6x - x^2 \\ f'(x) &= 6 - 2x = 0 \\ &\Rightarrow x = 3 \text{ is an extreme point} \\ f''(x) &= -2 \text{ which is not zero (n is even) and negative} \\ &\Rightarrow x = 3 \text{ is a local maximum} \end{aligned}$$

Figure 15 shows the graph for the example in part b.

FIGURE 14. Local minimum of function $y = x^2$

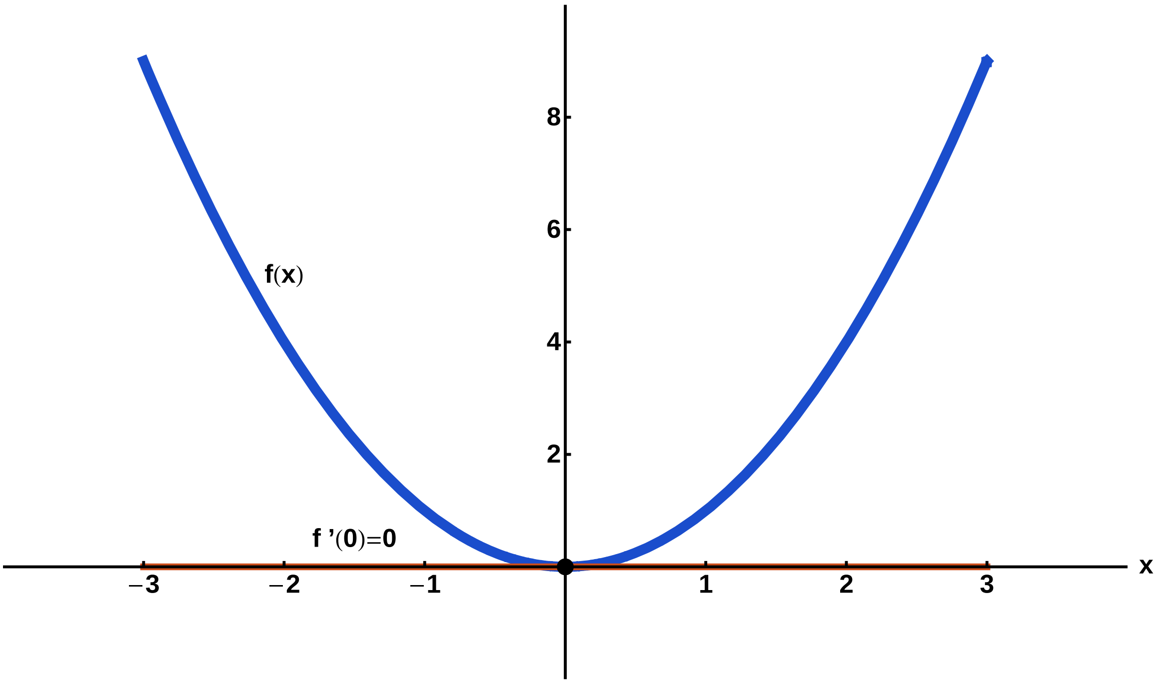
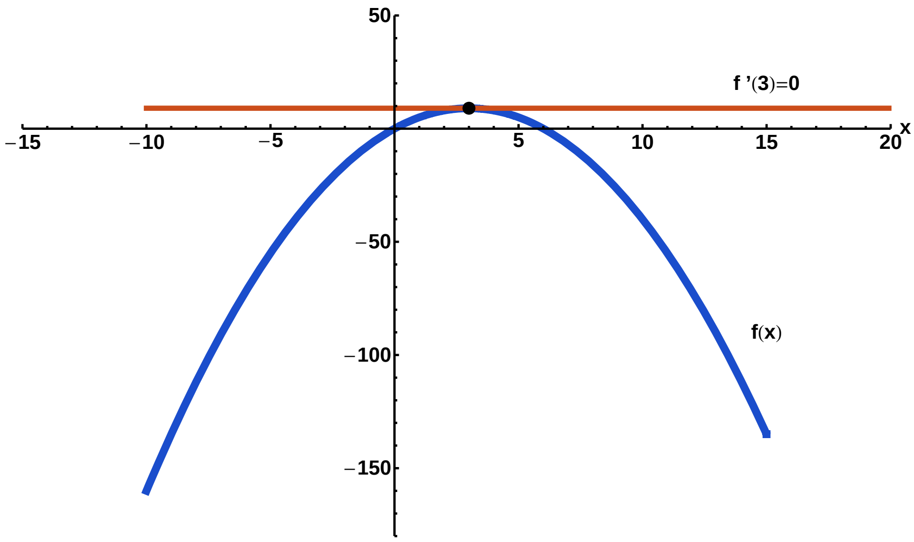


FIGURE 15. Global maximum of function $y = 6x - x^2$

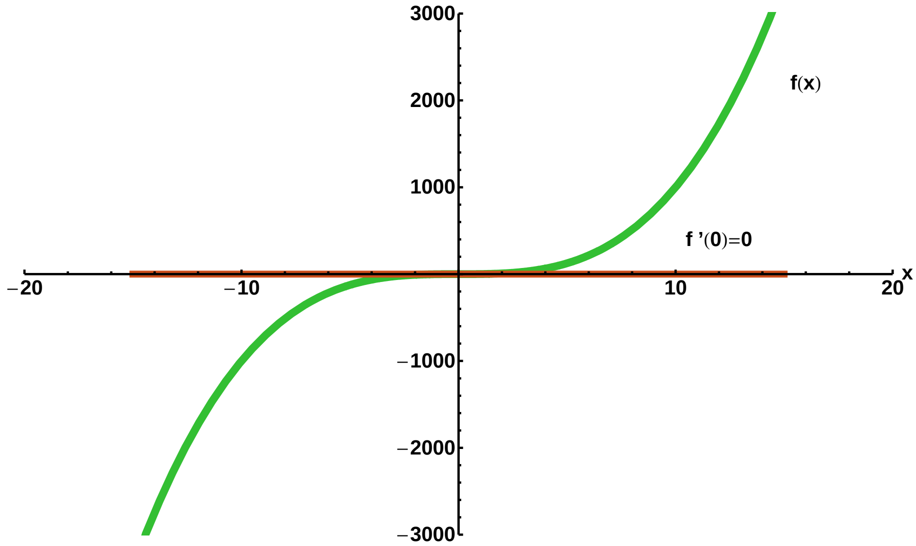


c: Consider a case where the function has no global maximum nor global minimum.

$$
\begin{aligned}
y &= f(x) = x^3 \\
f'(x) &= 3x^2 = 0 \\
&\Rightarrow x = 0 \text{ is an extreme point} \\
f''(x) &= 6x \text{ which is zero at zero (n is even)} \\
f'''(x) &= 6 \text{ which is not zero but positive (n is odd)} \\
&\Rightarrow x \ = 0 \text{ is neither a local maximum nor a local minimum}
\end{aligned}
$$

Figure 16 shows the graph for example c.

FIGURE 16. Function which has no global maximum nor global minimum y = x^3



4.4.4. *Partial converse of second derivative test.* Suppose f″(a) exists. If f has a local minimum at a, then f″(a) $\geq$ 0 if f has a local maximum at a, then f″(a) $\leq$ 0.

4.5. **A cookbook for finding and characterizing critical and inflection points.** The following steps provide a cookbook that is useful for finding and characterizing most critical points and identifying points of inflection for functions that are twice differentiable.

1. If the function is defined over an interval, find the value of the function at the endpoints of the interval.
2. Find the first derivative of the function.
3. Set the first derivative of the function equal to zero and find all the real roots of this equation.
4. Find the second derivative of the function.
5. For each real root, determine whether it is a maximum or a minimum point using the second derivative test where c is a critical point.
   a. If f′(c) = 0 and f″(c) < 0, then f has a local maximum at x = c.
   b. If f′(c) = 0 and f″(c) > 0, then f has a local minimum at x = c.
   c. If f′(c) = 0 and f″(c) $= 0$, then the second derivative test fails.
   d. If the second derivative test fails, consider a higher order test. If f has continuous n^{th} derivatives in the open interval (a,b) and f′(c) = f″(c) = 0 at some point c, then we consider higher order derivatives of f at c. Specifically, if at an interior point of the interval (a,b),

   $$f'''(c) = f^4(c) = \cdots = f^{n-1} = 0, \text{ but } f^n(c) \neq 0$$

   then for n even, f has a local minimum at c if $f^n(c) > 0$, and a local maximum if $f^n(c) < 0$. If n is odd, there is neither a local maximum nor a local minimum at c.
   e. Compare the local maxima or minima from previous steps. An endpoint may be the maximum or minimum value of the function over the interval.
   f. Identify potential inflection points by setting the second derivative of the function equal to zero and finding all the real roots of this equation.
   g. Determine which of the points in 5f are actual inflection points by seeing if f″ changes sign at c.