Soc Choice Welfare (1985) 2:295–309

Social Choice and Welfare

Fair Division with Uncertain Needs and Tastes*

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Received September 18, 1984 / Accepted July 24, 1985

Abstract. Previous studies have shown that egalitarianism maximizes expected social welfare in a contractarian original position with equally likely risk-averse agent tastes. The present paper characterizes agents by subsistence needs as well as by tastes, allows infinitely many possible need and taste profiles, and weakens the requirement that all possibilities be perceived as equally likely. Surplus-egalitarianism (meeting needs, then equally dividing the remainder) is shown to maximize expected social welfare when tastes are uncertain and needs are known, can be met, and are a priori required to be met; but intuitively unfair allocations may result if either of the latter two conditions fails to hold. Conditions under which egalitarianism maximizes expected social welfare with needs and tastes both uncertain are also determined.

Introduction

The classic n-agent cake-cutting problem poses the following question: How can one unit of an infinitely divisible resource be divided among a finite number of agents, $i=1,\ldots,n$, so that each agent i perceives his own share ω_i to be fair? The shares which each agent perceives to be fair for himself are assumed to be exogenously given in the form of an arbitrary collection of resource subsets. Thus, no restrictions are placed on the possibly conflicting perceptions of fairness by individual agents (see Steinhaus 1948; Dubins and Spanier 1961; Kuhn 1967).

Economists studying variants of the n-agent cake-cutting problem have primarily concentrated on the existence of fair allocations $(\omega_1,\ldots,\omega_n)$ characterized in terms of *social* welfare criteria. One line of work, originating with Lerner (1944), has focused on the characterization of the egalitarian allocation

* A previous version of this paper (Tesfatsion 1984) was presented at the Public Economics Workshop, University of Wisconsin, March, 1984. The author is grateful to D. Friedman and H. Quirmbach for helpful comments

$(1/n, \ldots, 1/n)$ as fair in the sense of maximizing expected social welfare in a contractarian original position[1] with equally likely risk-averse agent tastes (see Sen 1969; Lerner 1970; Breit and Culbertson 1970, 1972; McCain 1972; McManus et al. 1972; Sen 1973; Pazner and Schmeidler 1976). A second line of work, originating with Foley (1967), has focused on the ordinal characterization of fair allocations as envy-free allocations, in the sense that no agent envies the share of any other agent (see, e.g., Kolm 1972; Varian 1974; Daniel 1975; Crawford 1977; Thomson 1982; Thomson and Varian 1983).

The present paper generalizes the first line of work in three directions. First and foremost, the n agents are characterized by heterogeneous subsistence needs $\bar{\omega}_1, \ldots, \bar{\omega}_n$ as well as by heterogeneous risk-averse tastes (utility functions) $u_1, \ldots, u_n$, and both needs and tastes may be uncertain in the original position. Second, both the collection of subsistence need profiles $\bar{\omega} = (\bar{\omega}_1, \ldots, \bar{\omega}_n)$ and the collection of taste profiles $u = (u_1, \ldots, u_n)$ deemed possible for society in the original position are allowed to have infinitely many elements. Third, the possible subsistence need profiles $\bar{\omega}$ and taste profiles u need not be perceived as equally likely in the original position in either a Bayesian or an entropy sense.

Specifically, the present paper first addresses the following question: If subsistence needs $\bar{\omega}$ are known and can be met, but tastes u are uncertain, which allocation $\omega = (\omega_1, \ldots, \omega_n)$ could be characterized as fair by contractarians in an original position in the sense of maximizing expected social welfare

$$\mathrm{EW}(u(\omega - \bar{\omega})) \equiv \int W(u_1(\omega_1 - \bar{\omega}_1), \ldots, u_n(\omega_n - \bar{\omega}_n)) \,\mathrm{Prob}(\mathrm{d}u) \tag{1}$$

for any concave increasing social welfare function W? Given simple symmetry restrictions on expectations and the set of possible taste profiles u, and equating below-subsistence shares with death in the sense that $u(x) = \mathbf{0}$ for $x \leqq \mathbf{0}$ for all possible u, it is shown that expected social welfare (1) is uniquely maximized over all feasible allocations ω satisfying $\omega \geqq \bar{\omega}$ by the *surplus-egalitarian allocation*

$$\omega^{\mathrm{SE}} \equiv \left(\bar{\omega}_1 + \left[\frac{1 - \sum_{i=1}^{n} \bar{\omega}_i}{n} \right], \ldots, \bar{\omega}_n + \left[\frac{1 - \sum_{i=1}^{n} \bar{\omega}_i}{n} \right] \right). \tag{2}$$

The allocation (2) directs society first to meet subsistence needs, then to divide the resulting surplus equally among the n agents.

Given certain additional regularity conditions, an analogous result is obtained for a maximin social welfare criterion in place of (1). Briefly, if subsistence needs $\bar{\omega}$ are known and can be met, but tastes u are uncertain, the minimum possible social welfare level

$$\min_u \; W(u(\omega - \bar{\omega})) \tag{3}$$

[1] Contractarianism is an approach to social philosophy which attempts to explain the emergence of social institutions, and to provide norms for changes in those institutions, by conceptually placing persons in an idealized initial state from which mutual agreement might be expected. Following Rawls's terminology (1971, pp. 11–22), an "original position" is an initial state in which no one knows his actual place in society (e.g., class status, wealth, abilities), so that social institutions and norms are to be determined behind a veil of ignorance, free from the vagaries of natural chance and social circumstance

is maximized over all feasible allocations ω satisfying $\omega \geqq \bar{\omega}$ by the surplus-egalitarian allocation (2).

It might be surmised that surplus-egalitarianism maximizes expected and minimum social welfare without the additional constraint $\omega \geqq \bar{\omega}$ guaranteeing subsistence shares for all agents. However, this is not the case. The difficulty is that expected and minimum social welfare can be increased if high subsistence needs are not met, as a matter of general policy. More precisely, recall that agents receive zero utility from below-subsistence shares. Thus, if agents with high subsistence needs are allowed to die with zero shares, the substantial resources which would have gone to these agents simply to keep them alive, with no gain in utility, can instead be allocated to agents with low subsistence needs and positive marginal utilities.[2]

As Rawls (1971, Chap. 1) stresses, utilitarian social welfare functions $W(u_1, \ldots, u_n)$ in principle permit some agents to be sacrificed for the greater total benefit of other agents. The concavity assumptions commonly imposed on the utility functions u_j and welfare function W obscure this fact. In the present context the individual utility functions $u_1, \ldots, u_n$ exhibit a fundamental non-concavity at subsistence share levels, where death occurs, and expected social welfare maximization can result in the intuitively unfair sacrifice of some agents for others.

In general,[3] then, surplus-egalitarianism is not derivable from utilitarian principles alone. The right to receive subsistence shares must be imposed as a lexicographically prior principle of fairness.

Conversely, for high-scarcity economies in which the needs of all agents *cannot* be met, i.e., $\sum_{i=1}^{n} \bar{\omega}_i > 1$, surplus-egalitarianism (2) becomes deficit-egalitarianism dictating that all agents must die even if a proper subset could be saved. Clearly the latter "fair" allocation is not a reasonable allocation from the viewpoint of social survival and perpetuation. Thus, for high-scarcity economies, the unconstrained maximization of either expected or minimum social welfare is evidently superior to surplus-egalitarianism, although the particular way these utilitarian criteria resolve the lifeboat ethics problem of who shall live and who shall die is certainly controversial (see Lucas and Ogletree 1976).

The paper next addresses the following question: If subsistence needs and tastes are *both* uncertain in the original position, which feasible allocation ω could be characterized as fair in the sense of maximizing expected social welfare? Given the previous identification of below-subsistence shares with death, no general answer is

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[2] Consider a two-agent society with a known subsistence need profile $\bar{\omega} = (0, \beta)$ for agents 1 and 2, two possible equally likely utility function profiles $u = (f, g)$ and $u^* = (g, f)$ for agents 1 and 2, and a social welfare function W defined by $W(z_1, z_2) \equiv z_1 + z_2$. Suppose $f(x) = g(x) \equiv [1 - e(-x)]$ for all $x \geqq 0$, and $f(x) = g(x) = 0$ for $x \leqq 0$. Then, for all values of β greater than 0.24, expected social welfare $\mathrm{EW}(u(\omega - \bar{\omega}))$ is uniquely maximized over all allocations $\omega = (\omega_1, \omega_2) \geqq \mathbf{0}$ satisfying $\omega_1 + \omega_2 \leqq 1$ by giving the entire unit of resource to agent 1 and letting agent 2 die with a zero share. Small perturbations in f and g, so that $f \neq g$, only result in small perturbations in the cut-off level 0.24 for β without changing the qualitative nature of the solution

[3] As is clear from the proofs of Theorems 1 and 2 in Sect. II, below, surplus-egalitarianism is derivable from the utilitarian criteria (1) and (3) alone if needs can be met and all agents *infinitely* prefer life to death in the sense that $u_i(0) = -\infty$, $i = 1, \ldots, n$

possible. Indeed, examples can easily be constructed for which the allocations maximizing expected social welfare seem patently unfair.[4] The difficulty once again stems from the nonconcavity exhibited by agents' utility functions at subsistence share levels. However, if the meaning of subsistence needs is suitably weakened to a poverty line definition, then the *egalitarian allocation*

$$\omega^E \equiv (1/n, \ldots, 1/n) \tag{4}$$

emerges as the unique maximizer of expected social welfare. In addition, the egalitarian allocation (4) then also maximizes the minimum possible social welfare level averaged over needs.

As indicated in the above summary remarks, even the simple modelling of subsistence needs attempted here raises subtle and difficult issues for the contractarian characterization of fair allocations. Nevertheless, methods devised for the decentralized attainment of perceived fair allocations also encounter difficulties when subsistence needs are explicitly recognized; and these difficulties high-light the importance of characterizing and institutionalizing fair allocation rules in a social contract sense. For example, consider the well-known Divide-and-Choose method for solving the classic two-agent cake-cutting problem: the Divider, chosen randomly by the toss of an unbiased coin, divides the cake into two portions; the Chooser then selects one of the portions for his own share, the remaining portion going to the Divider. Suppose the Divider or Chooser needs more than half the cake to survive, and perceives as unfair any allocation which awards him less than this amount. In such a case, even if both agents know each other's needs and the cake is large enough to meet these needs, a perceived fair allocation will only result under the Divide-and-Choose method if the agents happen to exhibit altruism towards each other.

The basic allocation problem analyzed in the present paper is detailed in the following Sect. I. Contractarian fair allocations with known needs and uncertain tastes are characterized in Sect. II. Contractarian fair allocations with both needs and tastes uncertain are characterized in Sect. III. Proofs of theorems are provided in Sect. IV.

I. The Basic Allocation Problem

Consider the following variant of the classic n-agent cake-cutting problem. A single unit of an infinitely divisible resource is to be divided among n agents in a society. Letting ω_i denote the resource share allocated to agent i, $i=1,\ldots,n$, the set of feasible resource allocations $\omega=(\omega_1,\ldots,\omega_n)$ is given by[5]

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[4] Consider the example described in footnote 2 with the following changes: the subsistence profile $\bar{\omega}$ is $(\frac{1}{2},\frac{1}{2})$ with probability 1, and the utility functions $f(x)$ and $g(x)$ are arbitrary, strictly increasing functions over $x \geqq 0$ satisfying $f(x)=g(x)=0$ for $x \leqq 0$. Then either one of the feasible allocations $\omega'=(1,0)$ or $\omega''=(0,1)$ yields strictly higher expected social welfare than any other feasible allocation $\omega=(\omega_1,\omega_2)$, including in particular the egalitarian allocation $\omega^E=(\frac{1}{2},\frac{1}{2})$

[5] As usual, R^n_+ is defined to be the set of all real n-tuples $(z_1,\ldots,z_n)$ satisfying $z_i \geqq 0$ for $i=1,\ldots,n$, and R^n_{++} is defined to be the set of all real n-tuples $(z_1,\ldots,z_n)$ satisfying $z_i > 0$ for $i=1,\ldots,n$

$$\Omega^F \equiv \left\{ \omega \in R_+^n \mid \sum_{i=1}^{n} \omega_i \leqq 1 \right\}. \tag{5}$$

The n agents have heterogeneous subsistence needs and tastes. Let $\bar{\omega}_i$ denote the resource subsistence need of agent i, and let $u_i(\omega_i - \bar{\omega}_i)$ denote the utility level achieved by agent i when he is allocated ω_i, $i=1,\ldots,n$. It will be supposed that each utility function u_i is well-defined over all of R, and has been normalized to satisfy $u_i(x)=0$ for $x \leqq 0$. Thus, anything less than or equal to a subsistence share $\omega_i = \bar{\omega}_i$ yields zero utility to agent i, $i=1,\ldots,n$. In effect, then, agents die unless they receive at least their subsistence need.

Finally, suppose the n agents inhabit a moderate scarcity society in the sense that needs can be met, i.e., the subsistence need profile $\bar{\omega}=(\bar{\omega}_1,\ldots,\bar{\omega}_n)$ lies in the feasible allocation set Ω^F. The set of feasible allocations permitting survival for all, denoted by

$$\Omega^F(\bar{\omega}) \equiv \{\omega \in \Omega^F \mid \omega \geqq \bar{\omega}\}, \tag{6}$$

is then nonempty.

II. Contractarian Fair Allocations With Known Needs and Uncertain Tastes

A group of contractarians wishes to select a fair allocation ω for the society described in Sect. I from the vantage point of an original position. Social welfare is to be measured in terms of a real-valued function $W(u)$ of society's utility function profile $u=(u_1,\ldots,u_n)$.[6]

Suppose, first, that the subsistence need profile $\bar{\omega}$ in Ω^F is known but the utility function profile u is uncertain. Let

$$\mathscr{U} = \{u_\alpha : R^n \rightarrow R^n \mid \alpha \in A\} \tag{7}$$

denote the set of possible utility function profiles $u_\alpha = (u_{\alpha 1},\ldots,u_{\alpha n})$ indexed by a set A with topology $\mathscr{A}$,[7] where $u_\alpha(x_1,\ldots,x_n) \equiv (u_{\alpha 1}(x_1),\ldots,u_{\alpha n}(x_n))$; and, for each α in A, let

$$W_\alpha(\cdot) \equiv W(u_\alpha(\cdot)) \tag{8}$$

denote the composition of W with u_α. The contractarians' uncertainty concerning the true utility function profile u_α in $\mathscr{U}$ is then equivalently and more simply expressable as an uncertainty concerning the true social welfare function W_α in

$$\mathscr{W} \equiv \{W_\alpha : R^n \rightarrow R \mid \alpha \in A\}. \tag{9}$$

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6 For example, in Harsanyi (1955) the social welfare level $W(u)$ corresponding to any given profile $u=(u_1,\ldots,u_n)$ is the own expected utility of each contractarian in the original position, assuming an equal probability of $1/n$ that each contractarian will become agent i, $i=1,\ldots,n$; i.e., $W(u) \equiv \left[\sum_{i=1}^{n} u_i\right] \Big/ n$

7 A topology describes the subsets of a set to be considered "open". If A is a countable set, $\mathscr{A}$ can be taken to consist of all subsets of A

In particular, suppose the contractarians' uncertainty concerning W_α is representable in terms of a probability measure P on $(A, \mathscr{A})$.[8] Thus, $P(\alpha)$ denotes the perceived probability that W_α is the true social welfare function in $\mathscr{W}$. Assuming W_α is a continuous function of α over $(A, \mathscr{A})$, the expected social welfare associated with any feasible allocation ω in Ω^F is then given by

$$\mathrm{EW}(\omega; \bar{\omega}, P) \equiv \int_A W_\alpha(\omega - \bar{\omega}) P(\mathrm{d}\alpha). \tag{10}$$

It will now be shown that the surplus-egalitarian allocation ω^{SE} defined as in (2) uniquely maximizes expected social welfare (10) over all feasible allocations ω allowing survival, i.e., over all ω in $\Omega^F(\bar{\omega})$, if the welfare function probability measure P and the set $\mathscr{W}$ of possible social welfare functions W_α satisfy certain simple symmetry and concavity conditions.

Let the set consisting of all $n!$ permutations π of $\{1, \ldots, n\}$ be denoted by

$$\Pi \equiv \{\pi 1, \ldots, \pi n!\}, \tag{11}$$

where $\pi 1$ is the identity permutation. For example, if $n=2$, Π consists of the two permutations

$$\pi 1 \equiv \begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 1 & 2 \end{pmatrix}, \quad \pi 2 \equiv \begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 2 & 1 \end{pmatrix}. \tag{12}$$

For any α in A and π in Π, define the *π-symmetrical counterpart* W_α^π of the social welfare function W_α by

$$W_\alpha^\pi(x_1, \ldots, x_n) \equiv W_\alpha(x_{\pi(1)}, \ldots, x_{\pi(n)}). \tag{13}$$

The set $\mathscr{W}$ of possible social welfare functions will be called *symmetrically closed* if, given any W_α in $\mathscr{W}$, the π-symmetrical counterpart W_α^π of W_α also belongs to $\mathscr{W}$ for each π in Π.

If the basic underlying social welfare function W is itself symmetric, i.e., if

$$W(z_1, \ldots, z_n) = W(z_{\pi(1)}, \ldots, z_{\pi(n)}) \tag{14}$$

for every $(z_1, \ldots, z_n)$ in R^n and π in Π, so that agents are not discriminated among by the contractarians on the basis of subscript identification alone, then symmetric closure of $\mathscr{W}$ is equivalent to symmetric closure of the set $\mathscr{U}$ of possible utility function profiles in the following sense: every permutation $u_\pi = (u_{\pi(1)}, \ldots, u_{\pi(n)})$ of a utility function profile $u = (u_1, \ldots, u_n)$ in $\mathscr{U}$ is also a utility function profile in $\mathscr{U}$.

The remaining restrictions to be placed on the social welfare functions W_α in $\mathscr{W}$ are standard in the literature. The complete list of restrictions is given below for later reference.

For each α in A, $W_\alpha : R^n \to R$ is differentiable, monotone increasing, and strictly concave over R^n_{++}, and continuous over R^n_+, with $W_\alpha(\mathbf{0}) = 0$; (15a)

[8] Here A is being regarded as a measurable space with σ-algebra generated by its open sets $\mathscr{A}$ [cf. footnote 7]

For each x in R^n, $W_\alpha(x)$ is a continuous function of α over $(A, \mathscr{A})$;[9] (15b)

$\mathscr{W}$ is symmetrically closed. (15c)

Finally, let $\mathscr{P}$ denote the set of all probability measures P on $(A, \mathscr{A})$ satisfying the following two conditions:

For each $\bar{\omega}$ in Ω^F and ω in $\Omega^F(\bar{\omega})$,

$$\int_A |W_\alpha(\omega - \bar{\omega})| P(\mathrm{d}\alpha) < \infty; \quad (16a)$$

If $\mathscr{W}$ is symmetrically closed, then, for each α in A, P assigns equal (possibly zero) probability to the $n!$ symmetrical counterparts $\{W_\alpha^\pi | \pi \in \Pi\}$ of W_α. (16b)

Condition (16a) guarantees that expected social welfare (10) is finite for each subsistence need profile $\bar{\omega}$ in Ω^F and each feasible allocation ω in $\Omega^F(\bar{\omega})$. Condition (16b) has an especially simple interpretation when the basic underlying social welfare function W is symmetric: If a collection $\{u_1, \ldots, u_n\}$ of utility functions is considered by the contractarians to be at all possible for the n-agent society, then any one of the $n!$ possible utility function profiles $\{u_\pi | \pi \in \Pi\}$ matching these n utility functions with the n agents is perceived to be equally likely.[10]

The following theorem can now be stated.

Utilitarian Theorem 1 *[Known Needs and Uncertain Tastes]: Suppose restrictions* (15) *hold. Then, given any subsistence need profile $\bar{\omega}$ in Ω^F and any welfare function probability measure P in $\mathscr{P}$, expected social welfare* $\mathrm{EW}(\omega; \bar{\omega}, P)$ *is uniquely maximized over all allocations ω in $\Omega^F(\bar{\omega})$ by the surplus-egalitarian allocation ω^{SE} defined as in* (2).

Suppose, instead, that the contractarians in the original position desire to place a safety net under society by hedging against the worst possible social outcome. More precisely, suppose in place of maximizing expected social welfare (10) they consider the selection of a feasible allocation ω in $\Omega^F(\bar{\omega})$ to maximize the minimum possible level of social welfare,

$$\min_{\alpha \in A} [W_\alpha(\omega - \bar{\omega})]. \quad (17)$$

As the following theorem indicates, the surplus-egalitarian allocation once again emerges as the contractarians' choice if certain additional regularity conditions hold.

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9 If A is countable and $\mathscr{A}$ consists of all subsets of A, then every function mapping A into R is continuous on $(A, \mathscr{A})$ and restriction (15b) holds trivially [cf. footnote 7]

10 Thus, when W is symmetric, condition (16b) reduces to a direct generalization of Assumption 4 in Sen (1973, p. 1023), which requires that the probability of agent i having utility function u_j be the same as the probability of agent m having utility function u_j, $i, m \in \{1, \ldots, n\}$, for each of n given possible utility functions u_j. In the present paper the contractarians perceive particular utility function *profiles* to be equally likely for *society* rather than particular utility *functions* to be equally likely for *individuals*

Maximin Theorem 2 *[Known Needs and Uncertain Tastes]: Suppose restrictions* (15) *hold. In addition, suppose A is a compact convex subset of some vector space, and, for each x in R^n_+, $W_\alpha(x)$ is a convex function of α over A. Then, given any subsistence need profile $\bar{\omega}$ in Ω^F, minimum social welfare* (17) *is uniquely maximized over all allocations ω in $\Omega^F(\bar{\omega})$ by the surplus-egalitarian allocation ω^{SE} defined as in* (2).

III. Contractarian Fair Allocations with Uncertain Needs and Tastes

Suppose now that both needs and tastes are uncertain in the original position. Assuming the perceived distribution of needs is symmetric, one might surmise that egalitarianism would maximize expected social welfare. However, as discussed in the introduction and illustrated in footnote 4, this is not necessarily the case. The nonconcavity of the agents' utility functions at subsistence share levels results in the nonconcavity of the expected social welfare function, leading to the possibility of boundary solutions.

It will now be shown that egalitarianism does maximize expected social welfare with symmetrically distributed needs if the interpretation of subsistence needs is suitably modified.

Previously, each possible utility function $u_i(\omega_i - \bar{\omega}_i)$ for agent i was assumed to take on the value zero for all shares ω_i less than the subsistence need $\bar{\omega}_i$, $i = 1, \ldots, n$; hence, below-subsistence shares were equated with death. Suppose instead that each possible utility function for each agent i is normalized to take the value zero *at* $\omega_i = \bar{\omega}_i$, but is strictly increasing in a *neighborhood* of $\omega_i = \bar{\omega}_i$. In this case, subsistence needs represent a socially determined norm rather than a physical survival requirement. For example, $\bar{\omega}_i$ may represent the amount of resource required to bring agent i up to some socially determined poverty line defined independently of preferences, $i = 1, \ldots, n$.

The following restriction on the derived set $\mathscr{W}$ of possible social welfare functions W_α embodies this changed interpretation of subsistence needs, and will be used in place of condition (15a) in subsequent theorems.

For each α in A, $W_\alpha: R^n \to R$ is differentiable, monotone increasing, and strictly concave over

$$X \equiv \{x \in R^n | x = (\omega - \bar{\omega}) \text{ for some } \omega, \bar{\omega} \in \Omega^F\},$$

with $W_\alpha(\mathbf{0}) = 0$. (18)

Finally, let $\mathscr{F}$ denote the topology induced on Ω^F by the usual topology on R^n,[11] and let $\bar{\mathscr{P}}$ denote the set of all subsistence need probability measures $\bar{P}$ on $(\Omega^F, \mathscr{F})$ satisfying the following two conditions:[12]

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[11] That is, a subset B of Ω^F is open in Ω^F, hence an element of the collection of sets $\mathscr{F}$, if and only if it can be represented as the intersection of Ω^F with some open subset V of R^n, where openess in R^n is defined in the usual way using Euclidean distance (see Takayama 1974, pp. 19–20)

[12] Here Ω^F is being regarded as a measurable space with σ-algebra generated by its open sets $\mathscr{F}$. By choosing the support of $\bar{P}$ appropriately, the possible subsistence need profiles $\bar{\omega}$ can in effect be restricted to any symmetric subset of Ω^F

For each P in $\mathscr{P}$ and ω in Ω^F,

$$\int_A \int_{\Omega^F} |W_\alpha(\omega - \bar{\omega})| \bar{P}(\mathrm{d}\bar{\omega}) P(\mathrm{d}\alpha) < \infty; \tag{19a}$$

For any subset B in $\mathscr{F}$ and any permutation π in Π,

$$\bar{P}(B) = \bar{P}(B_\pi),$$

where

$$B_\pi \equiv \{(\bar{\omega}_{\pi(1)}, \ldots, \bar{\omega}_{\pi(n)}) | (\bar{\omega}_1, \ldots, \bar{\omega}_n) \in B\}. \tag{19b}$$

Condition (19a) guarantees the finiteness of the expected social welfare level

$$\mathrm{EW}(\omega; \bar{P}, P) \equiv \int_A \int_{\Omega^F} W_\alpha(\omega - \bar{\omega}) \bar{P}(\mathrm{d}\bar{\omega}) P(\mathrm{d}\alpha) \tag{20}$$

associated with any feasible allocation ω in Ω^F and any subsistence need and welfare function probability measures $\bar{P}$ in $\bar{\mathscr{P}}$ and P in $\mathscr{P}$. Condition (19b) essentially requires that the permutations $\{\bar{\omega}_\pi | \pi \in \Pi\}$ of any given subsistence need profile $\bar{\omega}$ in Ω^F be perceived as having equal (possibly zero) probability by the contractarians in the original position. For example, the uniform distribution over $(\Omega^F, \mathscr{F})$ defined by $\bar{P}(\mathrm{d}\bar{\omega}) \equiv n!\mathrm{d}\bar{\omega}$ satisfies (19b).

The following theorem can now be stated

Utilitarian Theorem 3 *[Uncertain Needs and Tastes]: Suppose restrictions* (15b), (15c), *and* (18) *hold. Then, given any subsistence need probability measure $\bar{P}$ in $\bar{\mathscr{P}}$ and any welfare function probability measure P in $\mathscr{P}$, expected social welfare* $\mathrm{EW}(\omega; \bar{P}, P)$ *is uniquely maximized over all allocations ω in Ω^F by the egalitarian allocation ω^E defined as in* (4).

Suppose, instead, that the contractarians in the original position wish to place a safety net under society by maximizing the minimum possible level of social welfare averaged over needs, i.e.,

$$\min_{\alpha \in A} \left[\int_{\Omega^F} W_\alpha(\omega - \bar{\omega}) \bar{P}(\mathrm{d}\bar{\omega}) \right]. \tag{21}$$

As the following theorem indicates, the egalitarian allocation will still be the selected allocation if certain additional regularity conditions are met.

Maximin Theorem 4 *[Uncertain Needs and Tastes]: Suppose restrictions* (15b), (15c), *and* (18) *hold. In addition, suppose A is a compact convex subset of some vector space, and, for each x in X, $W_\alpha(x)$ is a convex function of α over A. Then, given any subsistence need probability measure $\bar{P}$ in $\bar{\mathscr{P}}$, minimum average social welfare* (21) *is uniquely maximized over all allocations ω in Ω^F by the egalitarian allocation ω^E defined as in* (4).

IV. Proofs

Proof of Utilitarian Theorem 1

Let α in A, $\bar{\omega}$ in Ω^F, and P in $\mathscr{P}$ be given, and let $W_\alpha^{\pi j}, j=1,\ldots,n!$, denote the $n!$ symmetrical counterparts of W_α. Consider the problem of maximizing expected social welfare over $\Omega^F(\bar{\omega})$ with respect to the restricted universe of socal welfare functions $\{W_\alpha^{\pi j} | j=1,\ldots,n!\}$, where each function $W_\alpha^{\pi j}$ is assigned equal probability $n!$; i.e., consider the problem:

$$\max_{\omega\in\Omega^F(\bar{\omega})} \left[\frac{\sum_{j=1}^{n!} W_\alpha^{\pi j}(\omega-\bar{\omega})}{n!}\right]. \tag{22}$$

Restriction (15a) guarantees that problem (22) is a concave programming problem. Define the Lagrangian function $L: R^n \times R^n \times R \to R$ for problem (22) by

$$L(\omega,\lambda,\theta) \equiv \left[\frac{\sum_{j=1}^{n!} W_\alpha^{\pi j}(\omega-\bar{\omega})}{n!}\right] + \lambda[\omega-\bar{\omega}] + \theta\left[1-\sum_{i=1}^{n}\omega_i\right]. \tag{23}$$

Then (cf. Takayama 1974, Theorem 1.D.5, 94–95), in order for ω^* in R^n to solve (22), it is necessary and sufficient that there exists a vector (λ^*, θ^*) in R_+^{n+1} such that, evaluated at $(\omega^*, \lambda^*, \theta^*)$, the following conditions hold:

$$\mathbf{0} \geqq \frac{\partial L}{\partial \omega}, \quad 0=\left[\frac{\partial L}{\partial \omega}\right]\cdot \omega; \tag{24a}$$

$$\mathbf{0} \leqq \frac{\partial L}{\partial \lambda}, \quad 0=\left[\frac{\partial L}{\partial \lambda}\right]\cdot \lambda; \tag{24b}$$

$$0 \leqq \frac{\partial L}{\partial \theta}, \quad 0=\left[\frac{\partial L}{\partial \theta}\right]\cdot \theta. \tag{24c}$$

For any given i, $i=1,\ldots,n$, it can be shown that

$$\frac{\partial\left[\sum_{j=1}^{n!} W_\alpha^{\pi j}(\omega-\bar{\omega})\right]}{\partial \omega_i} = \sum_{k=1}^{n} \sum_{\pi\in\Pi_{ik}} \left[\frac{\partial W_\alpha}{\partial \omega_k}(\omega_\pi - \bar{\omega}_\pi)\right], \tag{25}$$

where

$$(\omega_\pi - \bar{\omega}_\pi) \equiv (\omega_{\pi(1)} - \bar{\omega}_{\pi(1)}, \ldots, \omega_{\pi(n)} - \bar{\omega}_{\pi(n)}), \tag{26}$$

and Π_{ik} consists of the particular $(n-1)!$ permutations in Π which map i into k. Clearly the derivative (25) takes on the same value for each i, $i=1,\ldots,n$, if $(\omega_\pi - \bar{\omega}_\pi)$ is independent of the permutation π, i.e., if the n components of $(\omega_\pi - \bar{\omega}_\pi)$ are set equal to each other. This yields $\omega = \omega^{SE}$, the surplus-egalitarian allocation.

It follows from these observations that one solution to (24) is given by:

$$\omega^* = \omega^{\mathrm{SE}}; \tag{27a}$$

$$\lambda^* = (0, \ldots, 0); \tag{27b}$$

$$\theta^* = \frac{\left[\sum_{j=1}^{n!} \frac{\partial W_\alpha^{\pi j}}{\partial \omega_i} (\omega^* - \bar{\omega})\right]}{n!}, \qquad i = 1, \ldots, n, \tag{27c}$$

hence ω^{SE} solves (22). By strict concavity of the maximand in (22), ω^{SE} is thus the unique solution for (22).

By construction, $\mathscr{W}$ contains a function W_α if and only if it contains each of the $n!$ symmetrical counterparts $W_\alpha^{\pi j}$ of W_α, and the given welfare function probability measure P in $\mathscr{P}$ assigns equal probability to symmetrical counterparts. It follows that $\mathscr{W}$ can be represented as

$$\mathscr{W} = \{W_\alpha^{\pi j} | \alpha \in A\} \tag{28}$$

for any πj in Π; and, for any ω in $\Omega^F(\bar{\omega})$, expected social welfare can be represented as

$$\begin{aligned} \mathrm{EW}(\omega; \bar{\omega}, P) &= \frac{1}{n!} \sum_{j=1}^{n!} [\mathrm{EW}(\omega; \bar{\omega}, P)] \\ &= \frac{1}{n!} \sum_{j=1}^{n!} \left[\int_A W_\alpha^{\pi j}(\omega - \bar{\omega}) P(\mathrm{d}\alpha)\right] \\ &= \int_A \left[\frac{\sum_{j=1}^{n!} W_\alpha^{\pi j}(\omega - \bar{\omega})}{n!}\right] P(\mathrm{d}\alpha). \end{aligned} \tag{29}$$

Since the surplus-egalitarian allocation ω^{SE} uniquely maximizes the final bracketed sum in (29) over ω in $\Omega^F(\bar{\omega})$ for each α in A, ω^{SE} must also uniquely maximize expected social welfare $\mathrm{EW}(\omega; \bar{\omega}, P)$ over ω in $\Omega^F(\bar{\omega})$.

Q.E.D.

Proof of Maximin Theorem 2

Let ω in Ω^F be given, and define

$$\tilde{\omega}^{\mathrm{SE}} \equiv (\omega^{\mathrm{SE}} - \bar{\omega}) = \left(\frac{1 - \sum_{i=1}^{n} \bar{\omega}_i}{n}, \ldots, \frac{1 - \sum_{i=1}^{n} \bar{\omega}_i}{n}\right). \tag{30}$$

For any α in A and ω in $\Omega^F(\bar{\omega})$, it follows by the monotonicity and concavity of W_α over $\Omega^F(\bar{\omega})$ that

$$\frac{1}{n!}\sum_{j=1}^{n!} W_\alpha^{\pi j}(\omega-\bar{\omega}) \equiv \frac{1}{n!}\sum_{j=1}^{n!} W_\alpha(\omega_{\pi j}-\bar{\omega}_{\pi j})$$
$$\leqq W_\alpha\left(\frac{1}{n!}\sum_{j=1}^{n!}[\omega_{\pi j}-\bar{\omega}_{\pi j}]\right)$$
$$\leqq W_\alpha(\bar{\omega}^{\mathrm{SE}}), \tag{31}$$

hence

$$\min_{\pi\in\Pi}[W_\alpha^\pi(\omega-\bar{\omega})] \leqq W_\alpha(\bar{\omega}^{\mathrm{SE}}). \tag{32}$$

Setting $\omega=\omega^{\mathrm{SE}}$, one obtains

$$\min_{\pi\in\Pi}[W_\alpha^\pi(\omega^{\mathrm{SE}}-\bar{\omega})] = \min_{\pi\in\Pi}[W_\alpha^\pi(\bar{\omega}^{\mathrm{SE}})] = W_\alpha(\bar{\omega}^{\mathrm{SE}}). \tag{33}$$

Thus

$$\max_{\omega\in\Omega^F(\bar{\omega})}\left[\min_{\pi\in\Pi} W_\alpha^\pi(\omega-\bar{\omega})\right] = W_\alpha(\bar{\omega}^{\mathrm{SE}}) = W_\alpha(\omega^{\mathrm{SE}}-\bar{\omega}). \tag{34}$$

Returning to the original maximin criterion function (17), it follows by (34), together with the general min-max theorem (cf. Karlin 1959, Theorem 1.5.1, p. 28), that

$$\max_{\omega\in\Omega^F(\bar{\omega})}\left[\min_{\alpha\in A} W_\alpha(\omega-\bar{\omega})\right] = \max_{\omega\in\Omega^F(\bar{\omega})}\left[\min_{\alpha\in A}\left(\min_{\pi\in\Pi} W_\alpha^\pi(\omega-\bar{\omega})\right)\right]$$
$$= \min_{\alpha\in A}\left[\max_{\omega\in\Omega^F(\bar{\omega})}\left(\operatorname{Min}_{\pi\in\Pi} W_\alpha^\pi(\omega-\bar{\omega})\right)\right]$$
$$= \min_{\alpha\in A}[W_\alpha(\omega^{\mathrm{SE}}-\bar{\omega})]. \tag{35}$$

Q.E.D.

Proof of Utilitarian Theorem 3

Let α in A, $\bar{P}$ in $\bar{\mathscr{P}}$, and P in $\mathscr{P}$ be given. Consider the problem

$$\max_{\omega\in\Omega^F}\int_{\Omega^F}\left[\sum_{j=1}^{n!} W_\alpha^{\pi j}(\omega-\bar{\omega})\right]\bar{P}(\mathrm{d}\bar{\omega}). \tag{36}$$

Restriction (18) guarantees that (36) is a concave programming problem. Define the Lagrangian $L: R^n\times R^n\times R\rightarrow R$ for problem (36) by

$$L(\omega,\lambda,\theta) \equiv \int_{\Omega^F}\left[\sum_{j=1}^{n!} W_\alpha^{\pi j}(\omega-\bar{\omega})\right]P(\mathrm{d}\bar{\omega})+\lambda\omega+\theta\left[1-\sum_{i=1}^{n}\omega_i\right]. \tag{37}$$

Necessary and sufficient conditions for an allocation ω^* in R^n to solve (36) again take the form (24). As in (25), it can be shown that

$$\frac{\partial\left[\int\limits_{\Omega^F}\left[\sum\limits_{j=1}^{n!} W_\alpha^{\pi j}(\omega-\bar\omega)\right]\bar P(\mathrm{d}\bar\omega)\right]}{\partial\omega_i}=\sum_{k=1}^{n}\sum_{\pi\in\Pi_{ik}}\left[\int\limits_{\Omega^F}\frac{\partial W_\alpha}{\partial\omega_k}(\omega_\pi-\bar\omega_\pi)\bar P(\mathrm{d}\bar\omega)\right] \tag{38}$$

for each $i=1,\dots,n$. If ω in (38) is set equal to the egalitarian allocation ω^E, so that $\omega_\pi\equiv(\omega_{\pi(1)},\dots,\omega_{\pi(n)})$ is independent of π, then the bracketed term on the right-hand side of (38) is independent of π.

More precisely, introducing the change of variable $z=\bar\omega_\pi$, and noting that z varies over Ω^F as $\bar\omega$ varies over Ω^F, with

$$\mathrm{Prob}\,(z\in B)=\bar P(B_\pi)=\bar P(B) \tag{39}$$

for any B in $\mathscr{F}$ by definition of z and the assumed symmetry of $\bar P$ in $\bar{\mathscr{P}}$, it follows that the bracketed term on the right-hand side of (38) is equal to

$$\left[\int\limits_{\Omega^F}\frac{\partial W_\alpha}{\partial\omega_k}(\omega_\pi-z)\bar P(\mathrm{d}z)\right]. \tag{40}$$

Thus, when ω is set equal to ω^E, each of the derivatives (38) takes on the same value, $i=1,\dots,n$; namely,

$$\sum_{k=1}^{n}(n-1)!\left[\int\limits_{\Omega^F}\frac{\partial W_\alpha}{\partial\omega_k}(\omega^E-z)\bar P(\mathrm{d}z)\right]. \tag{41}$$

It follows from these observations that one solution to (36) is $\omega=\omega^E$. By strict concavity of the maximand in (36), $\omega=\omega^E$ is the unique solution to (36).

As in the utilitarian theorem 1, the proof is now completed by noting that

$$\begin{aligned}\mathrm{EW}(\omega;\bar P,P)&=\frac{1}{n!}\sum_{j=1}^{n!}[\mathrm{EW}(\omega;\bar P,P)]\\&=\frac{1}{n!}\sum_{j=1}^{n!}\left[\int\limits_{A}\int\limits_{\Omega^F}W_\alpha^{\pi j}(\omega-\bar\omega)\bar P(\mathrm{d}\bar\omega)P(\mathrm{d}\alpha)\right]\\&=\int\limits_{A}\left[\int\limits_{\Omega^F}\left(\frac{\sum\limits_{j=1}^{n!}W_\alpha^{\pi j}(\omega-\bar\omega)}{n!}\right)\bar P(\mathrm{d}\bar\omega)\right]P(\mathrm{d}\alpha).\end{aligned} \tag{42}$$

Since ω^E uniquely maximizes the final bracketed term in (42) over all ω in Ω^F for each α in A, it must also uniquely maximize $E(\omega;\bar P,P)$ over all ω in Ω^F.

Q.E.D.

Proof of Maximin Theorem 4

Let α in A and $\bar P$ in $\bar{\mathscr{P}}$ be given, and consider the following problem:

$$\max_{\omega\in\Omega^F}\left(\min_{\pi\in\Pi}\left[\int\limits_{\Omega^F}W_\alpha^{\pi}(\omega-\bar\omega)\bar P(\mathrm{d}\bar\omega)\right]\right). \tag{43}$$

Clearly (43) is bounded below by

$$\min_{\pi\in\Pi}\left[\int_{\Omega^F} W_\alpha^\pi(\omega^E-\bar{\omega})\bar{P}(\mathrm{d}\bar{\omega})\right]. \tag{44}$$

However, as established in the proof of utilitarian theorem 2, the bracketed term in (44) is independent of π; hence, in particular, (43) is bounded below by

$$\int_{\Omega^F} W_\alpha(\omega^E-\bar{\omega})\bar{P}(\mathrm{d}\bar{\omega}). \tag{45}$$

On the other hand, (43) is bounded above by

$$\max_{\omega\in\Omega^F}\left[\frac{1}{n!}\sum_{j=1}^{n!}\int_{\Omega^F} W_\alpha^{\pi_j}(\omega-\bar{\omega})\bar{P}(\mathrm{d}\bar{\omega})\right]. \tag{46}$$

As established in the proof of utilitarian theorem 2, the egalitarian allocation ω^E uniquely solves (46); and, when $\omega=\omega^E$, the bracketed term in (46) reduces again to (45). Thus it has been shown that ω^E uniquely solves (43).

Returning to the original maximin criterion function (21), it follows by use of the general min-max theorem (cf. Karlin 1959) that

$$\begin{aligned}
&\max_{\omega\in\Omega^F}\left[\min_{\alpha\in A}\left(\int_{\Omega^F} W_\alpha(\omega-\bar{\omega})\bar{P}(\mathrm{d}\bar{\omega})\right)\right]\\
&\quad=\max_{\omega\in\Omega^F}\left[\min_{\alpha\in A}\left(\min_{\pi\in\Pi}\left[\int_{\Omega^F} W_\alpha^\pi(\omega-\bar{\omega})\bar{P}(\mathrm{d}\bar{\omega})\right]\right)\right]\\
&\quad=\min_{\alpha\in A}\left[\max_{\omega\in\Omega^F}\left(\min_{\pi\in\Pi}\left[\int_{\Omega^F} W_\alpha^\pi(\omega-\bar{\omega})\bar{P}(\mathrm{d}\bar{\omega})\right]\right)\right]\\
&\quad=\min_{\alpha\in A}\left[\int_{\Omega^F} W_\alpha(\omega^E-\bar{\omega})\bar{P}(\mathrm{d}\bar{\omega})\right].
\end{aligned} \tag{47}$$

Q.E.D.

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