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# DYNAMIC INVESTMENT, RISK AVERSION, AND FORESIGHT SENSITIVITY*

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Since optimal investment strategies generally cannot be obtained in closed form when utility functions exhibit non-constant risk aversion, most dynamic investment studies have focused on the constant risk aversion case. The present paper investigates a general class of dynamic investment models with final-period expected wealth objective for which the final-period utility of wealth function is not restricted to be constantly risk averse. Existence, monotonicity, concavity, differentiability, and absolute risk aversion properties are established for the optimal feedback investment strategies and dynamic programming indirect utility functions. The loss in final-period expected utility resulting from the use of limited foresight investments is shown to be bounded above by terms dependent both on the variance of myopically achievable utility and on the relative size of myopic and global absolute risk aversion. Finally, simulation results are presented which indicate the optimality of a rolling 2-period foresight horizon for a class of exponential utility functions exhibiting decreasing absolute risk aversion.

## 1. Introduction

In a previous paper, Hildreth (1974a) introduces an investment model of the form

$$\max_{v \in V} E\varphi(\bar{x}+\omega^1+\omega^2 v), \tag{1}$$

where $x = Ex + [x - Ex] \equiv \bar{x} + \omega^1$ is a random variable representing the decision maker's current wealth prospects, $\omega^2$ is a random vector representing a possible new venture, $\varphi(\cdot)$ is the decision maker's utility of wealth function, and $v$ is the amount invested by the decision maker in the new venture. The Hildreth investment model generalizes a classic investment model due to Arrow (1971) in which current wealth prospects are assumed given. In subsequent papers, Hildreth and Tesfatsion (1974, 1977) and Hildreth (1974b, 1979) establish various comparative static properties of the Hildreth investment model (1), and several economic applications are developed.

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In the present paper a dynamic version of the Hildreth investment model is considered, having the general form

$$\max_{v_0,\ldots,v_N} E\varphi(f_N(\omega_N, v_N, x_N)), \tag{2a}$$

subject to dynamic wealth constraints

$$x_0 = \bar{x}_0, \quad x_{n+1} = f_n(\omega_n, v_n, x_n) \in X \subseteq R, \qquad 0 \leqq n \leqq N, \tag{2b}$$

and investment vector constraints

$$v_n \in V(n, x_n), \qquad 0 \leqq n \leqq N, \tag{2c}$$

where the random vectors $\omega_n$ are serially independent drawings from time-varying distributions. In addition to providing a dynamic generalization for many of the static investment problems studied by Hildreth–Tesfatsion *op. cit.*, the general format of model (2) has also been used by a variety of previous researchers to investigate linear-quadratic macro policy problems, dynamic portfolio problems, and problems in discrete-time optimal growth.

The purpose of the present paper is two-fold. First, it is well-known that optimal feedback investment strategies generally cannot be obtained in closed form for investment models with utility of wealth functions exhibiting non-constant risk aversion. In consequence, such utility function specifications have generally been avoided in dynamic investment contexts in favor of more analytically tractable, if less empirically plausible, constant risk aversion forms.[1] Following a more careful development of the basic investment model (2) in section 2, our first objective will therefore be to establish, in section 3, various qualitative characteristics of the optimal feedback investment strategy $\boldsymbol{v}^\circ = (v_0^\circ(x), \ldots, v_N^\circ(x))$ and associated dynamic programming indirect (return-to-go) utility functions $F_0(x), \ldots, F_N(x)$ for the basic model which do not depend on the utility function $\varphi(\cdot)$ exhibiting constant absolute or relative risk aversion. Specifically, various sets of sufficient conditions are given for guaranteeing the monotonicity, concavity, differentiability, and decreasing absolute risk aversion of the indirect utility functions $F_n(x)$, for determining the sign of the optimal investments $v_n^\circ(x)$ and their first derivatives $\partial v_n^\circ(x)/\partial x$, and for ensuring the existence of an optimal feedback investment strategy $\boldsymbol{v}^\circ$.

Secondly, the simulation results of Rausser and Freebairn (1974) and Johnson and Tse (1978) suggest that in some cases the myopic selection of controls based on a rolling $M$-period foresight horizon ('$M$-measurement

[1] Decreasing absolute risk aversion and non-constant relative risk aversion are generally considered to be the norm. See, e.g., the discussion of this point in Hirshleifer and Riley (1979, sec. 1.1.5).

feedback controls') yields satisfactory global performance for $N$-period stochastic control problems, $N>M$. This is a potentially important finding, since limited foresight controls are often resorted to in practice. Nevertheless, little in the way of sensitivity analysis has yet been undertaken concerning the selection of $M$.

For a certain class of dynamic stochastic control models that includes model (2) as a special case, it is known [Tesfatsion (1980a)] that a rolling 1-period foresight horizon yields optimal global return if period-by-period myopic returns are perfectly positively correlated. Perfect positive correlation holds for various versions of model (2) with myopically specified intermediate-period utility functions if each state function $f_n(\cdot)$ is strictly increasing in $x$, and the final-period utility function $\varphi(\cdot)$ exhibits constant absolute or relative risk aversion. What can be said for model (2) given the empirically more plausible assumption that $\varphi(\cdot)$ exhibits decreasing absolute risk aversion and non-constant relative risk aversion? Specifically, is the final-period expected utility resulting from the sequential use of $M$-period foresight investments a monotonically increasing function of $M$?

In partial answer to this question, various foresight sensitivity results are presented in section 4 for model (2) which do not depend on $\varphi(\cdot)$ exhibiting constant risk aversion. The loss in final-period expected utility resulting from the use of myopically selected investments $v_n^*(x)$ in place of the optimal investments $v_n^\circ(x)$ is first shown to be bounded above by terms dependent both on the variance of myopically achievable utility of wealth and on the relative size of myopic and global absolute risk aversion, where the latter is defined in terms of the indirect utility functions $F_n(x)$. Foresight sensitivity simulation results are then presented for a special case of the basic investment model (2) with the final-period utility function $\varphi(\cdot)$ specified to lie in the class of concave, monotone increasing, decreasingly risk averse functions $\varphi: R \to R$ having the general form

$$\varphi(x)=a+bx-\sum_{k=1}^{K} c_k e^{-d_k x}, \qquad b, c_k d_k \geqq 0. \tag{3}$$

The class of utility functions (3) was apparently first introduced by Pratt (1964), and was used by Hildreth (1979) in a combined empirical and theoretical investigation of grain storage and hedging by farmers. Surprisingly, the simulation results indicate that a rolling 2-period foresight horizon yields optimal final-period expected utility for the class of utility functions (3) whenever $b=0$. A rolling 1-period foresight horizon is generally suboptimal unless $c_k=d_k=0$, $k\geqq 2$. In all cases, final-period expected utility is a monotone increasing function of the foresight range $M$.

Concluding comments are given in section 5. Proofs of theorems are outlined in an appendix.

## 2. The basic investment model

Consider a finite-horizon investment model described by equations of the form

$$x_0 = \bar{x}_0 \quad \text{(given initial wealth state)}, \tag{4a}$$

$$x_{n+1} = f_n(\omega_n, v_n, x_n), \qquad 0 \leqq n \leqq N, \tag{4b}$$

where, for each $n \in \{0,\ldots,N\}$, the $n$th period initial wealth state $x_n$ is an element of an open set $X \subseteq R$, the $n$th period investment $v_n$ is constrained to lie in an admissible investment set $V(n, x_n) \subseteq V$ for some open set $V \subseteq R^r$, the $n$th period random 'disturbance' $\omega_n$ is an element of a set $\Omega \subseteq R^s$, and $f_n : \Omega \times V \times X \to X$ is a continuous[2] state function, strictly increasing in $x$. Letting $\mathscr{F}$ denote the $\sigma$-algebra generated by the open subsets of $\Omega$, it will be assumed that each $\omega_n$ is independently governed by a probability distribution function $p_n : \mathscr{F} \to R$ conditioned on the current time $n$. In addition, it will be assumed that the return associated with each possible disturbance, investment, and state configuration $(\omega, v, x)$ for the final period $N$ is measured by $\varphi(f_N(\omega, v, x)) = \varphi(x_{N+1})$ for some continuous strictly increasing utility of wealth function $\varphi : X \to R$.

An admissible feedback investment strategy for model (4) is defined to be any vector $\boldsymbol{v} = (v_0(\cdot), \ldots, v_N(\cdot))$ of measurable functions $v_n : X \to V$ satisfying $v_n(x) \in V(n, x)$ for each $x \in X$. The symbol $\mathscr{L}$ will be used to denote the set of all admissible feedback investment strategies $\boldsymbol{v}$. The objective assumed for the investor will be the maximization of final-period expected utility,

$$E\varphi(f_N(\omega_N, v_N(x_N), x_N)), \tag{4c}$$

via selection of a feedback investment strategy $\boldsymbol{v} \in \mathscr{L}$.[3]

[2]It is assumed throughout the paper that $X$, $V$, and $\Omega$ have the usual relative topology, and that products of $X$, $V$, and $\Omega$ have the corresponding product topology. Each of the spaces $X$, $V$, and $\Omega$ will also be regarded as a measurable space, with $\sigma$-algebra generated by its open sets.

[3]The expectation operator $E[\cdot]$ is more precisely defined as follows. Let $\Omega^N$ denote the set of all disturbance sequences $\omega^N = (\omega_0, \ldots, \omega_N)$ satisfying $\omega_n \in \Omega$, $0 \leqq n \leqq N$, and let $\mathscr{F}^N$ denote the product $\sigma$-algebra generated by all cylinder sets of the form

$$\prod_{n=0}^{N} A_n = \{\omega^N \in \Omega^N \mid \omega_0 \in A_0, \ldots, \omega_N \in A_N\},$$

where $A_n \in \mathscr{F}, 0 \leqq n \leqq N$. Finally, let $p^N(\cdot)$ denote the unique probability measure on $(\Omega^N, \mathscr{F}^N)$ satisfying

$$p^N\left(\prod_{n=0}^{N} A_n\right) = \int_{A_0} \int_{A_1} \cdots \int_{A_N} p_N(\mathrm{d}\omega_N) \ldots p_1(\mathrm{d}\omega_1) p_0(\mathrm{d}\omega_0),$$

for each cylinder set $\prod_{n=0}^{N} A_n \in \mathscr{F}^N$ [see Hinderer (1971, thm. A.5, p. 148)]. Expectation with respect to $\langle \Omega^N, \mathscr{F}^N, p^N \rangle$ is then denoted by $E[\cdot]$.

For brevity, any investment problem satisfying specifications (4) will be referred to as a *basic investment model*.[4] The following additional restrictions will be imposed as needed:

(A.1) $\varphi(x)$ is twice continuously differentiable, and $f_n(\omega, v, x)$ is twice continuously differentiable in $v$ and $x$ for each $\omega \in \Omega$ and $n \in \{0, \ldots, N\}$.
(A.2) $\Omega$ is compact.[5]
(A.3) An optimal admissible feedback investment strategy $\mathbf{v}^{\circ}$ exists.
(A.4) A regular[6] optimal admissible feedback investment strategy $\mathbf{v}^{\circ}$ exists.
(A.5) $\varphi(x)$ is thrice continuously differentiable, and $f_n(\omega, v, x)$ is thrice continuously differentiable in $v$ and $x$ for each $\omega \in \Omega$ and $n \in \{0, \ldots, N\}$.
(A.6) The admissible investment sets are increasing in $x$; i.e., $V(n, x^1) \subseteq V(n, x^2)$ if $x^1 \leqq x^2$, for all $x^1$, $x^2 \in X$ and $n \in \{0, \ldots, N\}$.
(A.7) The admissible investment sets are convex in $x$; i.e., $tV(n, x^1) + [1-t]V(n, x^2) \subseteq V(n, tx^1 + [1-t]x^2)$ for all $t \in [0,1]$, $x^1$ and $x^2 \in X$, and $n \in \{0, \ldots, N\}$.
(A.8) $V$ and $X$ are convex sets, $\varphi(x)$ is strictly concave, and $f_n(\omega, v, x)$ is jointly concave in $v$ and $x$ for each $\omega \in \Omega$ and $n \in \{0, \ldots, N\}$.
(A.9) The disturbance terms $\omega_0, \ldots, \omega_N$ are independent and identically distributed drawings from a common distribution $(\Omega, \mathscr{F}, p)$, the state functions have the stationary form $f: \Omega \times V \times X \to X$ for each $n \in \{0, \ldots, N\}$, and the admissible investment sets have the stationary form $V(x)$, $x \in X$, for each $n \in \{0, \ldots, N\}$.

*Example 2.1. Portfolio Model* [cf. Arrow (1971), Mossin (1968), Hakansson (1971), Bellman and Kalaba (1957), and Kalaba and Tesfatsion (1978)]. In each period $n \in \{0, \ldots, N\}$ an investor must decide how to allocate his current wealth $x_n \in R_{++} \equiv X$ between two investment opportunities $A$ and $B$, the first yielding a positive or negative net return rate $\omega_n$, $0 \leqq |\omega_n| \leqq 1$, governed by a probability distribution function $p_n(\cdot)$, and the second yielding a known net return rate $r_n$, $0 \leqq r_n \leqq 1$. The investor's objective is to maximize the expected utility of his wealth $x_{N+1}$ at the end of period $N$ via feedback control.

[4]An axiomatization for a one-period version of the basic investment model with discrete probability distributions is provided in Tesfatsion (1980b). The symmetrical treatment of utility and probability in the basic investment model has proved to be useful in the development of a new approach to adaptive control, direct criterion function updating. See Tesfatsion (1978, 1979).

[5]The sole purpose of the compact $\Omega$ restriction is to allow the interchange of expectation and differentiation operations. Various alternative restrictions would serve equally well for this purpose.

[6]An optimal admissible feedback investment strategy $\mathbf{v}^{\circ} = (v_0^{\circ}(\cdot), \ldots, v_N^{\circ}(\cdot))$ will be called *regular* if for each $x \in X$ the solution $v_n^{\circ}(x)$ to $\max E_n[F_{n+1} \circ f_n(\omega, v, x)] \equiv Q(v, x)$ over $v \in V(n, x)$ lies in the interior of $V(n, x)$, with $Q_{vv}(v_n^{\circ}(x), x)$ negative definite, where the indirect utility function $F_n(\cdot)$ is as defined in section 3. Alternative sets of sufficient conditions guaranteeing the existence of optimal admissible feedback investment strategies for the basic investment model can be derived using the results of Hinderer (1971), Leland (1972), and Hildreth (1974a). See, for example, Theorem 3.6 in section 3.

Assuming the investor's initial wealth $x_0$ for period 0 is positive, his initial wealth $x_{n+1}$ for period $n+1$ is a simple function of his initial wealth $x_n$ for period $n$, the net return rate $\omega_n \in [-1,1] \equiv \Omega$ observed for investment opportunity $A$ in period $n$, and the amount $v_n \in [0, x_n] \equiv V(n, x_n)$ of wealth he allocated to $A$ in period $n$; namely, $x_{n+1} = x_n + \omega_n v_n + r_n[x_n - v_n] \equiv f_n(\omega_n, v_n, x_n)$. Assuming utility of wealth at the end of period $N$ is measured by $\varphi(x_N + \omega_N v_N + r_N[x_N - v_N]) = \varphi(x_{N+1})$ for some strictly concave, strictly increasing, thrice continuously differentiable function $\varphi: X \to R$, this portfolio problem has the basic model format, and satisfies (A.1)–(A.3) and (A.5)–(A.8).[7]

*Example 2.2. Insurance Model* [cf. Hildreth–Tesfatsion (1977)]. Suppose in each period $n$, $0 \leqq n \leqq N$, a decision maker in an initial capital state $x_n$ stands to lose an amount $d_n > 0$ if a particular random event $A(n)$ occurs, where $A(n)$ is independent of $A(j)$, $j \neq n$. In exchange for a premium $c_n > 0$ he is offered an insurance policy that will cover this contingent loss. Suppose he can also elect partial coverage at a proportionally reduced premium, i.e., he can elect to pay a premium $v_n c_n$, $0 \leqq v_n \leqq 1$, and be reimbursed $v_n d_n$ if the loss occurs. Letting $\omega_n \equiv (\omega_n^1, \omega_n^2)$ be defined by

$$\omega_n^1 \equiv b_n - d_n I_{A(n)}, \qquad 0 \leqq n \leqq N,$$

$$\omega_n^2 \equiv d_n I_{A(n)} - c_n, \qquad 0 \leqq n \leqq N,$$

where $I_{A(n)}$ denotes the indicator function for $A(n)$, and $b_n$ denotes assets or debts accumulated over period $n$ from additional sources, the decision maker's objective is to maximize the expected utility $E\varphi(x_N + \omega_N^1 + \omega_N^2 v_N)$ of his final capital state $x_{N+1} = x_N + \omega_N^1 + \omega_N^2 v_N$ subject to

$$x_0 = \bar{x}_0 \quad \text{(initial conditions)},$$

$$x_{n+1} = x_n + \omega_n^1 + \omega_n^2 v_n, \qquad 0 \leqq n \leqq N.$$

It follows that this insurance model has the basic model format, and satisfies (A.1)–(A.3) and (A.5)–(A.8), if $\varphi(\cdot)$ is strictly increasing, strictly concave, and thrice continuously differentiable.[8]

## 3. Properties of the basic investment model

Consider any basic investment model (4). For each $n \in \{0, \ldots, N\}$ and $x \in X$, let $F_n(x)$ denote the maximum attainable final-period expected utility

[7]For existence (A.3) of an optimal admissible feedback investment strategy $v^\circ$, see Theorem 3.6 in section 3.

[8]See footnote 7.

beginning in period $n$ with initial state $x$, and using feedback investment; and let $E_n[\cdot]$ denote expectation with respect to $\langle \Omega, \mathscr{F}, p_n \rangle$. Then [Hinderer (1971, thm. 14.4, p. 101; pp. 104–105; thm. 17.6, p. 111)]

$$F_N(x) = \sup_{v \in V(N,x)} E_N[\varphi \circ f_N(\omega, v, x)], \tag{5a}$$

$$F_n(x) = \sup_{v \in V(n,x)} E_n[F_{n+1} \circ f_n(\omega, v, x)], \qquad 0 \leqq n \leqq N-1, \tag{5b}$$

and a feedback investment strategy $\mathbf{v}^\circ \in \mathscr{L}$ satisfies the final-period expected utility objective

$$\max_{\mathbf{v} \in \mathscr{L}} E[\varphi \circ f_N(\omega_N, v_N(x_N), x_N)], \tag{6}$$

if and only if it satisfies the dynamic programming optimality equations

$$F_N(x_N) = E_N[\varphi \circ f_N(\omega_N, v_N^\circ(x_N), x_N)], \tag{7a}$$

$$F_n(x_n) = E_n[F_{n+1} \circ f_n(\omega_n, v_n^\circ(x_n), x_n)], \qquad 0 \leqq n \leqq N-1, \tag{7b}$$

for almost every disturbance sequence $(\omega_0, \ldots, \omega_N)$.[9] The functions $F_n : X \to R$ are generally referred to as return-to-go functions or indirect utility functions, the former term prevailing among system scientists and control engineers, and variations of the latter term prevailing among economists.

For any twice differentiable function $Q : X \to R$ with $Q' > 0$, let $R_Q : X \to R$ denote the Pratt–Arrow measure of absolute risk aversion for $Q(\cdot)$, defined by $R_Q(x) \equiv -Q''(x)/Q'(x)$. Interpreting $Q(\cdot)$ as a utility of wealth function for an expected utility maximizing decision maker, it can be shown [Pratt (1964, p. 125)] that $R_Q(x)$ is approximately equal to $2r(x, z^*)/\sigma_z^{2*}$ for any zero-mean random variable $z^*$ with small variance $\sigma_z^{2*}$, where the risk premium $r(x, z^*) \in R$ is the maximum amount the decision maker would be willing to pay to avoid a gamble on $z^*$; i.e., $r(x, z^*)$ satisfies $Q(x - r(x, z^*)) = E_z Q(x + z)$. Thus, in principle, $R_Q(x)$ can be directly elicited by suitable gamble experiments even if $Q(\cdot)$ is unknown to the experimenter. The corresponding function $xR_Q(x)$ is generally referred to as the Pratt–Arrow measure of relative risk aversion for $Q(\cdot)$.

Since optimal feedback investment strategies generally cannot be obtained in closed form for investment models with utility of wealth functions exhibiting non-constant absolute and relative risk aversion, dynamic investment studies have generally focused on the constant risk aversion case.

[9]More precisely, using the definitions presented in footnote 3, the optimality equations must hold for $p^N$-almost every disturbance sequence $\omega^N \in \Omega^N$. The symbol $\circ$ denotes function composition, e.g., $h \circ s(x) \equiv h(s(x))$.

For example, see Mossin (1968), Merton (1969), Samuelson (1969), and Hakansson (1970, 1971). Non-constant risk averse utility function specifications have for the most part been studied in the context of static single-period decision problems, e.g., by Arrow (1971), Sandmo (1969), Cass–Stiglitz (1972), Hildreth (1974a, b), and Hildreth–Tesfatsion (1974, 1977).[10] The following theorems establish characteristics of the basic model optimal investment strategy $v^{\circ}$ and indirect utility functions $F_n(\cdot)$ which do not depend on the final-period utility function $\varphi(\cdot)$ exhibiting any form of constant risk aversion. Besides being of interest in themselves, these results also proved to be a tremendous aid in the design of a computer program for the numerical generation of $v^{\circ}$, which was subsequently used for the foresight sensitivity study summarized in section 4.[11]

The first result, Theorem 3.1, establishes analytical representations for the derivatives of the optimal investment functions $v_n^{\circ}(\cdot)$ and indirect utility functions $F_n(\cdot)$ in the interior solution case.

*Theorem 3.1. For any basic investment model satisfying assumptions* (A.1)–(A.4), *the optimal investment function $v_n^{\circ}(x)$ is a continuously differentiable function of the wealth state $x$, and the indirect utility function $F_n(x)$ is a strictly increasing and twice continuously differentiable function of $x$, for each $n \in \{0,\ldots,N\}$. In particular, letting $F_{N+1}(\cdot) \equiv \varphi(\cdot)$, superscript* T *denote transpose, and $d$ denote the point $(\omega, v_n^{\circ}(x), x)$, the $r \times 1$ gradient vector of $v_n^{\circ}(x)$ is*

$$\frac{\partial v_n^{\circ}}{\partial x}(x) = -D_n(x)^{-1} S_n(x), \tag{8a}$$

*where*

$$S_n(x) \equiv E_n\left[F''_{n+1}(f_n(d))\frac{\partial f_n}{\partial x}(d)\frac{\partial f_n^{\mathrm{T}}}{\partial v}(d) + F'_{n+1}(f_n(d))\frac{\partial^2 f_n^{\mathrm{T}}}{\partial x \partial v}(d)\right]_{r\times 1}, \tag{8b}$$

*and $D_n(x)$ is a negative definite $r \times r$ matrix given by*

$$D_n(x) \equiv E_n\left[F''_{n+1}(f_n(d))\frac{\partial f_n^{\mathrm{T}}}{\partial v}(d)\frac{\partial f_n}{\partial v}(d) + F'_{n+1}(f_n(d))\frac{\partial^2 f}{\partial v^2}(d)\right]. \tag{8c}$$

[10] An exception is Neave (1974), who investigates a scalar investment model of the form $\max E[\sum_{n=1}^{N} \beta^n u_n(x_n - v_n) + \varphi(k + \omega_N v_N)]$ with respect to $v_n \in [0, x_n]$ subject to $x_{n+1} = k + \omega_n v_n$, $n \in \{1,\ldots,N\}$, with utility functions $u_n(\cdot)$ and $\varphi(\cdot)$ assumed to be decreasingly absolute risk averse and increasingly relative risk averse.

[11] For example, concavity avoids the problem of local maxima, and theoretically determined monotonicity properties aid the selection of appropriate step sizes for the grid storage of the indirect utility functions. The complete program, with extensive comments, is available upon request.

*Moreover,*

$$E_n\left[F'_{n+1}(f_n(d))\frac{\partial f_n^{\mathrm{T}}}{\partial v}(d)\right]=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}_{r\times 1}, \tag{9a}$$

$$F'_n(x)=E_n\left[F'_{n+1}(f_n(d))\frac{\partial f_n}{\partial x}(d)\right]>0, \tag{9b}$$

$$F''_n(x)=E_n\left[F''_{n+1}(f_n(d))\left(\frac{\partial f_n}{\partial x}(d)\right)^2+F'_{n+1}(f_n(d))\frac{\partial^2 f_n}{\partial x^2}(d)\right]$$

$$+S_n(x)^{\mathrm{T}}\frac{\partial v_n^{\mathrm{o}}}{\partial x}(x). \tag{9c}$$

*If in addition assumption* (A.5) *holds, then the indirect utility functions* $F_n:X\to R$ *are thrice continuously differentiable,* $0\leqq n\leqq N$.

The next result, Theorem 3.2, establishes monotonicity and concavity properties of the indirect utility functions $F_n(\cdot)$ that do not depend on any differentiability or interiority assumptions:

*Theorem 3.2. If a basic investment model satisfies assumptions* (A.3) *and* (A.6)–(A.8), *then each indirect utility function* $F_n:X\to R$ *is continuous, strictly increasing, and strictly concave,* $0\leqq n\leqq N$.

It is natural to conjecture that the absolute risk aversion properties assumed for the utility function $\varphi(\cdot)$ are inherited by the indirect utility functions $F_n(\cdot)$. However, as indicated by Theorem 3.3 and Corollaries 3.1 and 3.2, below, inheritance of absolute risk aversion properties depends strongly on the structure of the state functions $f_n(\cdot)$ and the admissible investment sets $V(n,x)$. In addition, to rigorously establish absolute risk aversion inheritance, it seems necessary to impose differentiability and interiority assumptions such as (A.1)–(A.4), which guarantee the differentiability of the indirect utility functions $F_n(\cdot)$.

The preliminary two Lemmas 3.1 and 3.2 appear to be of independent interest. The first establishes preservation of decreasing absolute risk aversion under integration, and the second establishes preservation of decreasing absolute risk aversion under maximization:

*Lemma 3.1. Let* $X$ *be an open subset of* $R$, $\Omega$ *be a subset of* $R^s$, $s\geqq 1$, *and* $\mathscr{F}$ *be a* $\sigma$*-algebra of subsets of* $\Omega$. *Let* $\mu$ *be a finite positive measure on* $(\Omega,\mathscr{F})$

*with $\mu(\Omega)>0$, and let $g:\Omega\times X\to R$ be integrable over $(\Omega,\mathscr{F},\mu)$ for each $x\in X$ and thrice continuously differentiable and strictly increasing over $X$ for each $\omega\in\Omega$. Define $h:X\to R$ by*

$$h(x)\equiv\int_{\Omega} g(\omega,x)\mu(\mathrm{d}\omega). \tag{10}$$

*If*

$$R_g(\omega,x)\equiv -g_{xx}(\omega,x)/g_x(\omega,x) \tag{11}$$

*is a non-increasing function of $x$ for $\mu-$a.e.$\omega$, then for any $x\in X$, $R_h'(x)\leqq 0$, and $R_h'(x)<0$ unless there exist non-negative real numbers $A_x$ and $B_x$, not both zero, such that*

$$A_x g_x(\omega,x)=B_x g_{xxx}(\omega,x) \quad \mu-\text{a.e.}\omega. \tag{12}$$

*Remark.* Pratt (1964, thm. 5, p. 132) proves that a finite positive sum of decreasingly risk averse utility functions on the real line is itself a decreasingly risk averse utility function. Lemma 3.1 generalizes this result to integrals. Neave (1974, p. 43) states and uses a special case of Lemma 3.1 for functions $g:\Omega\times X\to R$ of the form $g(\omega,x)=f(k+\omega x)$, where $\Omega\subset R$ and $k$ is any constant.

*Lemma 3.2. Let $X$ be any open subset of $R$ and $V$ be any open subset of $R^r$, $r\geqq 1$, and let $\{V(x)\mid x\in X\}$ be a collection of open subsets of $V$. Let $h:V\times X\to R$ be any thrice continuously differentiable function, strictly increasing in $x$, such that*

$$R_h(v,x)\equiv -h_{xx}(v,x)/h_x(v,x) \tag{13}$$

*is a non-increasing (strictly decreasing) function of $x$ for each $v\in V$. Suppose there exists a continuous piecewise differentiable function $v:X\to V$ satisfying $v(x)\in V(x)$ and*

$$\sup_{v\in V(x)} h(v,x)=h(v(x),x)\equiv s(x), \qquad x\in X. \tag{14}$$

*Then $R_s(x)\equiv -s''(x)/s'(x)$ is a well-defined non-increasing (strictly decreasing) function of $x$ over $X$.*

*Theorem 3.3. Consider a basic investment model which satisfies* (A.1)–(A.8). *Suppose $R_\varphi'(x)\leqq 0$, $x\in X$, and*

$$\frac{\partial f_n}{\partial x}\,\frac{\partial^3 f_n}{\partial x^3}\geqq\left[\frac{\partial^2 f_n}{\partial x^2}\right]^2, \tag{15}$$

*for all* $(\omega, v, x) \in \Omega \times V \times X$ *and* $n \in \{0, \ldots, N\}$. *Then* $R_{F_n}(x) \geqq 0$ *and* $R'_{F_n}(x) \leqq 0$ *for all* $x \in X$ *and* $n \in \{0, \ldots, N\}$; *i.e., the indirect utility functions* $F_n: X \rightarrow R$ *exhibit non-negative decreasing absolute risk aversion.*

*Corollary 3.1. If a basic investment model satisfies* (A.1)–(A.5), *and* $R'_{\varphi}(x) \leqq 0$ *and* $\partial^2 f_n(\omega, v, x)/\partial x^2 = 0$ *for all* $(\omega, v, x) \in \Omega \times V \times X$ *and* $n \in \{0, \ldots, N\}$, *then* $R'_{F_n}(x) \leqq 0$ *for all* $x \in X$ *and* $n \in \{0, \ldots, N\}$.

*Corollary 3.2. Consider a basic investment model which satisfies* (A.1)–(A.4). *Suppose* $R_{\varphi}(x) = c$ *for all* $x \in X$, *for some constant* $c \in R$, *and suppose each state function* $f_n(\cdot)$ *has the form* $f_n(\omega, v, x) = G_n(\omega, v) + b_n x$ *for some function* $G_n: \Omega \times V \rightarrow R$ *and constant* $b_n \in R_{++}$. *Then*

$$R_{F_n}(x) = c_n, \quad x \in X, \qquad n \in \{0, \ldots, N\}, \tag{16}$$

*where*

$$c_{N+1} = c, \tag{17a}$$

$$c_n = c_{n+1} b_n, \qquad 0 \leqq n \leqq N. \tag{17b}$$

The next result, Theorem 3.4, demonstrates how Theorems 3.1, 3.2, and 3.3 can be used to establish sign and monotonicity properties for the optimal investment strategy $v^{\circ}$, given certain state function specifications $f_n(\cdot)$ (see, e.g., Example 2.1 in section 2):

*Theorem 3.4. Consider a basic investment model satisfying* (A.1)–(A.4) *and* (A.6)–(A.8). *Suppose* $0 \in V \subseteq R$, *and each state function* $f_n(\cdot)$ *has the form*

$$f_n(\omega, v, x) = G_n(\omega, v, x)v + H_n(v, x). \tag{18}$$

*Then*

$$v_n^{\circ}(x) \gtreqless 0 \Leftrightarrow \left[ E_n G_n(\omega, 0, x) + \frac{\partial H_n}{\partial v}(0, x) \right] \gtreqless 0. \tag{19}$$

*Suppose in addition* (A.5) *holds,* $R'_{\varphi}(x) \leqq 0$ *for all* $x \in X$, *and each state function* $f_n(\cdot)$ *has the form*

$$f_n(\omega, v, x) = G_n(\omega)v + H_n(x), \tag{20}$$

*for some functions* $G_n(\cdot)$ *and* $H_n(\cdot)$, *satisfying*

$$H'_n(x) H'''_n(x) \geqq [H''_n(x)]^2, \qquad x \in X. \tag{21}$$

*Then*

$$E_n G_n(\omega) \gtreqless 0 \Rightarrow \frac{\partial v_n^o(x)}{\partial x} \gtreqless 0, \qquad x \in X. \tag{22}$$

*Remark.* It is interesting to note that (22) is false if $R'_\varphi \leqq 0$ is weakened to $\varphi''' \geqq 0$.

The next result, Theorem 3.5, demonstrates how the restrictions (18) and (20) on $f_n(\cdot)$ in Theorem 3.4 can be weakened if each random vector $\omega_n$ is composed of two independently distributed subvectors $\omega_n^1$ and $\omega_n^2$:

*Theorem 3.5. Consider a basic investment model satisfying* (A.1)–(A.4) *and* (A.6)–(A.8). *Suppose* $0 \in V \subseteq R$, *each random bector* $\omega_n$ *has the form* $\omega_n = (\omega_n^1, \omega_n^2)$, *where* $\omega_n^1$ *is distributed independently of* $\omega_n^2$, *and each state function* $f_n(\cdot)$ *has the form*

$$f_n(\omega, v, x) = G_n(\omega^2, v, x)v + H_n(v, x) + I_n(\omega^1, x). \tag{23}$$

*Then*

$$v_n^o(x) \gtreqless 0 \Leftrightarrow \left[ E_n G_n(\omega_n^2, 0, x) + \frac{\partial H_n}{\partial v}(0, x) \right] \gtreqless 0. \tag{24}$$

*Suppose in addition* (A.5) *holds,* $R'_\varphi(x) \leqq 0$ *for all* $x \in X$, *and each state function* $f_n(\cdot)$ *has the form*

$$f_n(\omega, v, x) = G_n(\omega^2)v + H_n(x) + I_n(\omega^1), \tag{25}$$

*for some functions* $G_n(\cdot)$, $H_n(\cdot)$, *and* $I_n(\cdot)$, *satisfying*

$$H'_n(x)H'''_n(x) \geqq [H''_n(x)]^2, \qquad x \in X. \tag{26}$$

*Then*

$$E_n G_n(\omega_n^2) \gtreqless 0 \Rightarrow \frac{\partial v_n^o(x)}{\partial x} \gtreqless 0. \tag{27}$$

Although the state functions in Example 2.2 of section 2 satisfy hypotheses (23) and (25) of Theorem 3.5, the subvectors $\omega_n^1$ and $\omega_n^2$ for that insurance model are not independent. Some sign properties for $v^o$ are established in Hildreth–Tesfatsion (1977) for the case of correlated $\omega^1$ and $\omega^2$ in a static single-period context, but the dynamic generalization of these results will not be treated in the present paper.

The final result of this section, Theorem 3.6, provides sufficient conditions for the existence of an optimal admissible feedback investment strategy $\mathbf{v}^\circ \in \mathscr{L}$, i.e., for assumption (A.3) to be satisfied. Define

$$H_0 \equiv \{\bar{x}_0\} \times \Omega, \tag{28a}$$

$$H_{n+1} = \{(h, v, \omega) \mid h \in H_n, v \in D_n(h), \omega \in \Omega\}, \qquad 0 \leqq n \leqq N, \tag{28b}$$

where, for any element $h_n = (\bar{x}_0, \omega_0, v_0, \ldots, \omega_{n-1}, v_{n-1}, \omega_n)$ in $H_n$, $D_n: H_n \to 2^V$ is defined by

$$D_n(h_n) \equiv V_n(t_n(h_n)), \tag{29}$$

and $t_n: H_n \to X$ is defined by

$$t_n(h_n) \equiv f_{n-1}(\omega_{n-1}, v_{n-1}, f_{n-2}(\omega_{n-2}, v_{n-2}, f_{n-3}(\ldots)))$$

$$= f_{n-1}(\omega_{n-1}, v_{n-1}, x_{n-1}) = x_n. \tag{30}$$

Finally, define

$$K_n \equiv \{(h, v) \mid h \in H_n, v \in D_n(h)\}, \qquad 0 \leqq n \leqq N. \tag{31}$$

*Theorem 3.6. Existence Theorem* [cf. Hinderer (1970, thm. 17.12, p. 116)]. *Consider any basic investment model* (4). *Assume the following conditions hold:*

(i) *There exists a compact set $B \subset V$ such that, for each $n \in \{0, \ldots, N\}$ and $x \in X$, the admissible investment set $V(n, x)$ is contained in $B$.*
(ii) *Each set $K_n$ is a closed subset of $H_n \times B$, and contains the graph of a measurable map.*
(iii) $\sup_{(h,v) \in K_N} |E_N \varphi \circ f_N(\omega, v, t_N(h))| < \infty.$

*Then there exists an optimal admissible feedback investment strategy $\mathbf{v}^\circ \in \mathscr{L}$; i.e., assumption* (A.3) *is satisfied.*

## 4. Foresight sensitivity results

The simulation results of Rausser and Freebairn (1974) and Johnson and Tse (1978) indicate that in certain cases the myopic selection of controls based on a rolling $M$-period horizon yields satisfactory global return. A sensitivity analysis focusing on the magnitude of $M$ would therefore seem to be of interest. In the present basic investment model framework, this sensitivity analysis reduces to asking for the relative expected final-period utility associated with myopic indirect utility function specifications $U_n(x)$, $0 \leqq n \leqq N-1$, based on alternative foresight ranges $M$, where each function $U_n(x)$ can be interpreted as a proxy representation for the optimal indirect utility function $F_n(x)$ based on complete foresight.

Specifically, consider an investor in period $n$ of an $(N+1)$-period basic investment problem for which the current wealth state is $x$. As in section 3, let $F_{n+1}(x')$ denote the maximum attainable final-period expected utility beginning in period $n+1$ with initial wealth state $x'$, $x' \in X$. Then, for any currently admissible investment selection $v \in V(n,x)$, maximum attainable final-period expected utility beginning in period $n$ is given by

$$G(v) \equiv E_n[F_{n+1} \circ f_n(\omega, v, x)]. \tag{32}$$

Assuming (A.3) holds, there exists an investment selection $v_n^{\circ}(x) \in V(n,x)$ which maximizes $G(\cdot)$ over $V(n,x)$. The following theorem provides an upper bound for the loss $G(v_n^{\circ}(x)) - G(v_n^*(x))$ in final-period expected utility resulting from the use of a myopically selected investment $v_n^*(x)$ in place of $v_n^{\circ}(x)$:

*Theorem 4.1. Consider a basic investment model satisfying assumptions* (A.1)–(A.4). *If an investment $v_n^*(x)$ is selected in period n which satisfies*

$$\max_{v \in V(n,x)} E_n[U \circ f_n(\omega, v, x)], \tag{33}$$

*for some strictly increasing twice differentiable function $U: X \to R$, then*

$$0 \leqq G(v_n^{\circ}(x)) - G(v_n^*(x)) \leqq \sigma^2 \Psi, \tag{34}$$

*where*

$$\sigma^2 \equiv \sup_{v \in V(n,x)} E_n[U \circ f_n(\omega, v, x) - E_n[U \circ f_n(\omega, v, x)]]^2, \tag{35}$$

$$\Psi \equiv \sup_{y \in Y} \left(\frac{F'_{n+1}(y)}{U'(y)}\right) \left|\frac{R_U(y) - R_{F_{n+1}}(y)}{U'(y)}\right|, \tag{36}$$

$$Y \equiv \{ y \in R \mid y = f_n(\omega, v, x) \text{ for some } (\omega, v) \in \Omega \times V(n,x)\}. \tag{37}$$

As indicated by inequality (34), myopic optimization in the form of (33) is equivalent to global optimization for the basic investment model if either all of the probability distributions $p_n(\cdot)$ are degenerate, implying $\sigma^2 = 0$, or $U(\cdot)$ and $F_{n+1}(\cdot)$ have the same absolute risk aversion characteristics, implying $\Psi$

$=0$. For example, using Corollary 3.2, it is easily shown that $\Psi=0$ if $R'_\varphi=0$, $f_n(\omega,v,x)=G_n(\omega,v)+b_n x$ for some function $G_n(\cdot)$ and some constant $b_n\in R_{++}$, $U:X\to R$ is defined by $U(x)\equiv\varphi(x\prod_{j=1}^{N-n}b_{n+j})$, and (A.1)–(A.4) hold.

As demonstrated more generally in Tesfatsion (1980a), $\Psi$ can be interpreted as an inverse measure for positive correlation in returns. Specifically, the correlation coefficient $\rho^v(U\circ f_n, F_{n+1}\circ f_n)$ for the two random variables $U\circ f_n(\cdot,v,x)$ and $F_{n+1}\circ f_n(\cdot,v,x)$ increases as $\Psi$ decreases, equalling 1.0 when $\Psi=0$. The case $\Psi=0$ is of course highly special.

Now consider a basic investment model satisfying (A.3) and (A.9). By the stationarity assumption (A.9), the optimal investment function $v_n^o(\cdot)$ for any period $n$ of this $(N+1)$-period investment problem is also the optimal investment function for period 0 of a similar $(N+1-n)$-period investment problem. Stated somewhat differently, for any $M\in\{1,\dots,N+1\}$, $v_{N+1-M}^o(\cdot)$ is the optimal investment function for period 0 of a foreshortened $M$-period basic investment model.

One natural way to define myopic optimization in period 0 with foresight range $M$ for the original $(N+1)$-period basic investment model is thus to postulate the use of $v_{N+1-M}^o(\cdot)$ in period 0 in place of the investment function $v_0^o(\cdot)$ that is optimal for period 0. Equivalently, this can be interpreted as the use of the myopic indirect utility function $U(\cdot)\equiv F_{N+2-M}(\cdot)$ in period 0 in place of the indirect utility function $F_1(\cdot)$ that is optimal for period 0.

Letting $G_0(v)\equiv E_0[F_1\circ f(\omega,v,x)]$ denote the maximum final-period expected utility starting in period 0 with initial wealth state $x$ and arbitrary investment $v\in V(x)$, a plausible conjecture would then be

$$G_0(v_{N+1-M}^o(x))\leqq G_0(v_{N+1-(M+1)}^o(x)),\qquad M\in\{1,\dots,N\},\qquad x\in X; \tag{38}$$

i.e., final-period expected utility is a monotone increasing function of the foresight range $M$ for every $x\in X$. An equivalent formulation of conjecture (38) is

$$G_0(v_{n+1}^o(x))\leqq G_0(v_n^o(x)),\qquad n\in\{0,\dots,N-1\},\qquad x\in X. \tag{39}$$

To test the monotonicity conjecture (39), consider a special basic investment model, abbreviated Model (S), given by

$$\text{(S)}\qquad \max_{v\in\mathcal{L}} E_N[\varphi(x_N+\omega_N v_N(x_N))],$$

subject to

$$x_0=\bar{x}_0, \quad x_{n+1}=x_n+\omega_n v_n, \qquad 0 \leqq n \leqq N,$$

where

$$X \equiv V \equiv R,$$

$$\Omega \equiv \{s_1, s_2\}, \qquad s_1<0<s_2,$$

$$V(n,x) \equiv [0,x], \qquad 0 \leqq n \leqq N,$$

$$p_n(s_2)=1-p_n(s_1) \equiv p, \qquad 0 \leqq n \leqq N,$$

and $\varphi(\cdot)$ lies in the $\varphi^*$ class of functions, defined to be the set of all functions $\varphi: R \to R$ having the general form

$$(\varphi^*) \qquad \varphi(x)=a+bx-\sum_{k=1}^{K} c_k \mathrm{e}^{-d_k x}, \qquad b, c_k, d_k \geqq 0.$$

Aside from the $\varphi^*$ specification for $\varphi(\cdot)$, Model (S) is the portfolio model used by Mossin (1968), which in turn is a dynamic generalization of the static portfolio model used by Arrow (1971). (See Example 2.1 in section 2.) Mossin investigates his portfolio model using constant risk aversion utility functions. In contrast, using Lemma 3.1, it is readily checked that each element $\varphi: R \to R$ belonging to the $\varphi^*$ class of utility functions is a concave monotone increasing function satisfying $R'_\varphi \leqq 0$, with $R'_\varphi < 0$ if and only if there exist integers $i$ and $j$ such that $d_i \neq d_j$ and $c_i d_i c_j d_j > 0$. In fact, the $\varphi^*$ functions form a proper subset of the set of concave monotone increasing infinitely differentiable functions $\Psi: R \to R$ with completely monotonic first derivative (i.e., $[-1]^k \Psi^{(k+1)} \geqq 0$, $k=0,1,\ldots$), all of which satisfy $R'_\Psi \leqq 0$. See Widder (1941, thm. 16, p. 167).

To test the monotonicity conjecture (39) for Model (S), comparative simulation studies were carried out on an IBM 370/Model 158. To obtain a general foresight sensitivity overview, tests were first undertaken for the following range of parameter values:

*Initial Wealth State* $\bar{x}_0 \in [0,9]$.

*Time Horizon* $N=2,3$.

*Probability* $p=0.6667$, $0.7500$.

| | | |
|---|---|---|
| *Disturbance Terms* | $s_1 = -0.5, \ -1.0,$ | |
| | $s_2 = 1.0.$ | |
| *Utility Coefficients* | $a = 1.0,$ | |
| | $b = 0.0, \ 0.5,$ | |
| | $c_k = 0.5, \ 1.0, \ 2.0,$ | $k = 1, 2, 3,$ |
| | $d_k = 0.5, \ 1.0, \ 2.0,$ | $k = 1, 2, 3,$ |
| | $c_k = d_k = 0.0,$ | $k \geqq 4.$ |

The indirect utility functions $F_n(x)$ were linearly interpolated using a grid step size of 0.50 for $x$. The optimal investments $v_n^\circ(x)$ were found by direct search over $[0, x]$ using a search step size of 0.05.

Surprisingly, in all cases it was found that the monotonicity conjecture (39) was satisfied for 'almost every' initial wealth state $x \equiv \bar{x}_0$ in the following strengthened form:

$$G_0(v_{n+1}^\circ(x)) = G_0(v_n^\circ(x)), \qquad 0 \leqq n \leqq N-2, \tag{40a}$$

$$G_0(v_N^\circ(x)) \leqq G_0(v_{N-1}^\circ(x)); \tag{40b}$$

i.e., given an initial wealth state $x$ in period 0, the maximum final-period expected utility achieved by using $M$-period foresight in period 0, $2 \leqq M \leqq N$, precisely coincided with the maximum final-period expected utility achieved by using complete $(N+1)$-period foresight in period 0. The completely myopic case of 1-period foresight [use of $v_N^\circ(x)$] in period 0 was generally suboptimal, although even here optimality held for some parameter configurations. The exceptional initial wealth states $x$ where (40) failed to hold were isolated and scattered, suggested that round-off error may have been the cause.

A more intensive investigation of both (39) and (40) was then undertaken. Since a cross-comparison of the previous results revealed no apparent sensitivity of either myopic-global equivalence or exceptional point behavior to the time horizon $N$, probability $p$, disturbance values $s_1$ and $s_2$, and utility coefficients $c_3$ and $d_3$, these parameters were held fixed at $N=2$, $p=0.7500$, $s_1 = -1.0$, $s_2 = 1.0$, and $c_3 = d_3 = 0.0$. The grid step size for $x$ was decreased to 0.25 to reduce round-off error.

Finally, to allow more meaningful comparison of numbers, a base $\varphi^*$ utility function was selected,

$$\varphi(x) \equiv 1 + 0.5x - e^{-0.5x}, \tag{41a}$$

$$\varphi(0) = 0, \tag{41b}$$

$$\varphi'(0) = 1, \tag{41c}$$

for which it was determined, numerically, that $M$-period foresight was globally optimal, $1 \leqq M \leqq N$, with $v_0(x) \equiv v_1(x) \equiv v_2(x) \equiv x$.[12] Small variations were then made in the parameters $(b, c_1, d_1, c_2, d_2)$, away from their values in (41), while retaining the normalizations $\varphi(0)=0$ and $\varphi'(0)=1$; i.e., a set of $\varphi^*$ utility functions was tested of the form

$$\varphi(x) = 1 + bx - c_1 e^{-d_1 x} - c_2 e^{-d_2 x}, \tag{42a}$$

$$\varphi(0) = 1 - c_1 - c_2 = 0, \tag{42b}$$

$$\varphi'(0) = b + c_1 d_1 + c_2 d_2 = 1.0, \tag{42c}$$

$$(b, c_1, d_1, c_2, d_2) \cong (0.5, 1.0, 0.5, 0.0, 0.0). \tag{42d}$$

In all cases the monotonicity conjecture (39) was upheld to at least four decimal places. For the 2-period foresight conjecture (40), the pivotal parameter turned out to be the linear coefficient $b$. For each $b$ value, as $b$ was varied by 0.1 from 0.5 to 0.0, a number of runs were made for various $c_k$ and $d_k$ values consistent with the normalizations (42b) and (42c). Investment functions, indirect utility functions, and final-period expected utilities were evaluated to four decimal places. Representative runs are depicted in figs. 1 through 4 for $b$ taking on the values 0.5, 0.3, 0.1, and 0.0.

For $b=0.5$, both 1-period and 2-period foresight yielded optimal final-period expected utility; i.e. strict equalities held in conjecture (40), regardless of the $c_k$ and $d_k$ values. The normalization conditions (42b) and (42c) guaranteed that the linear part $a+bx$ of $\varphi(x)$ would be dominant. A representative run is depicted in fig. 1.

As $b$ was decreased from 0.5 to 0.3, 1-period foresight became suboptimal for sufficiently large initial wealth states $x$, but 2-period foresight remained optimal; i.e., conjecture (40) held, but not with strict equality in (40b). A representative run is depicted in fig. 2. Note the increased concavity of the

[12]Although myopic-global equivalence is to be expected for the base $\varphi$ (41) on the basis of Corollary 3.2 and Theorem 4.1, the interiority regularity condition (A.4) used in these two propositions prevents a claim of analytical proof.

indirect utility functions $F_n(x)$ and the departure of the 1-period foresight investment function $v_2^o(x)$ from linearity.

As $b$ was decreased still further, from 0.3 to 0.1, final-period expected utility became a strictly monotone increasing function of the foresight range $M$; i.e., strict inequalities held in conjecture (39), and conjecture (40) failed to hold. A representative run is depicted in fig. 3. Note that all three investment functions are now nonlinear, and have a cross-point.

Finally, for $b=0.0$, 2-period foresight once again yielded optimal final-period expected utility; i.e., conjecture (40) held, but not with strict equality in (40b). A representative run is depicted in fig. 4. Comparing fig. 2 and fig. 4, note that the investment functions take on a reverse order although conjecture (40) holds in both cases. Also, the qualitative behavior of the indirect utility functions is markedly different.

For any initial wealth state $x \in X$, the final-period expected utility function $G_0(v)$ defined by $G_0(v) \equiv E_0[F_1 \circ f(\omega, v, x)]$ is concave in $v$, and takes on a maximum at $v_0^o(x)$. An analytical proof of the monotonicity conjecture (39) would therefore follow immediately if it could be shown that the investment functions $v_n(x)$ are monotone in $n$ in the sense that, for each $n \in \{1, \ldots, N-1\}$ and $x \in X$, either

$$v_{n+1}^o(x) \leqq v_n^o(x) \leqq v_0^o(x), \tag{43a}$$

or

$$v_0^o(x) \leqq v_n^o(x) \leqq v_{n+1}^o(x). \tag{43b}$$

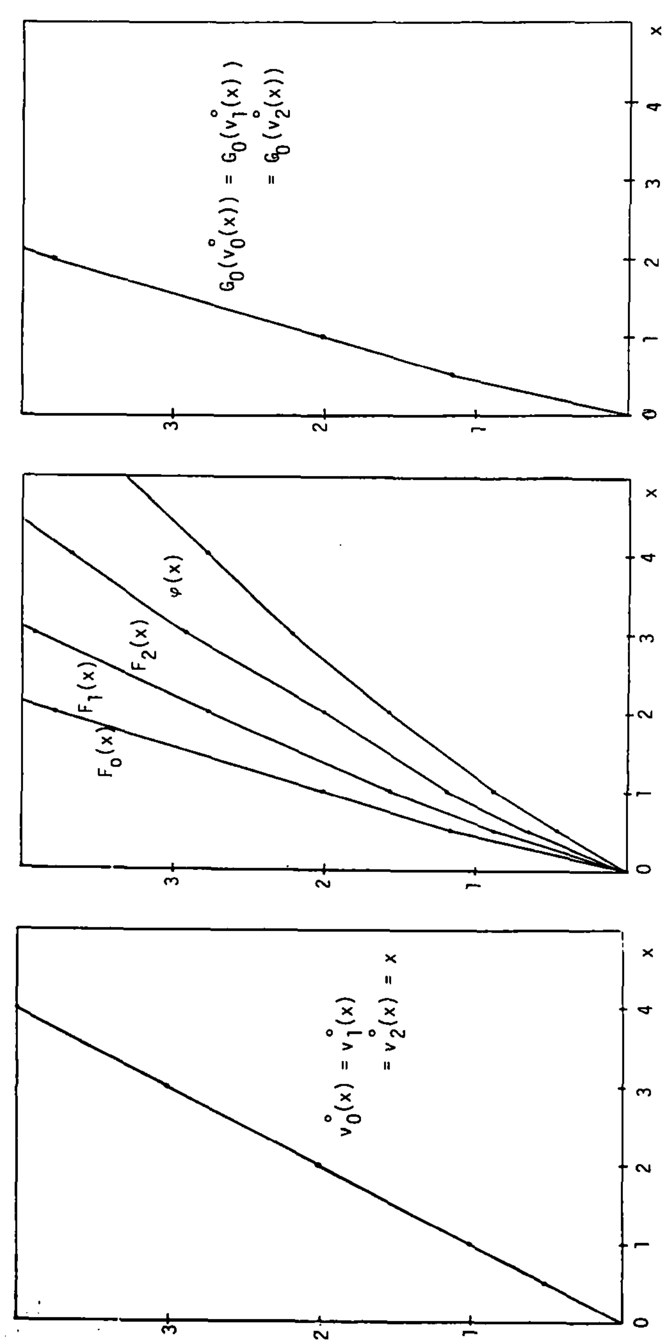
Although (43) held in all simulation runs, as indicated in figs. 1 through 4, the possible presence of cross-points for the investment functions seemingly makes an analytical proof of (43) difficult.

## 5. Discussion

The dynamic characteristics of a generalized Arrow–Hildreth investment model have been investigated, with special attention paid to risk aversion and foresight sensitivity. Three basic questions are posed:

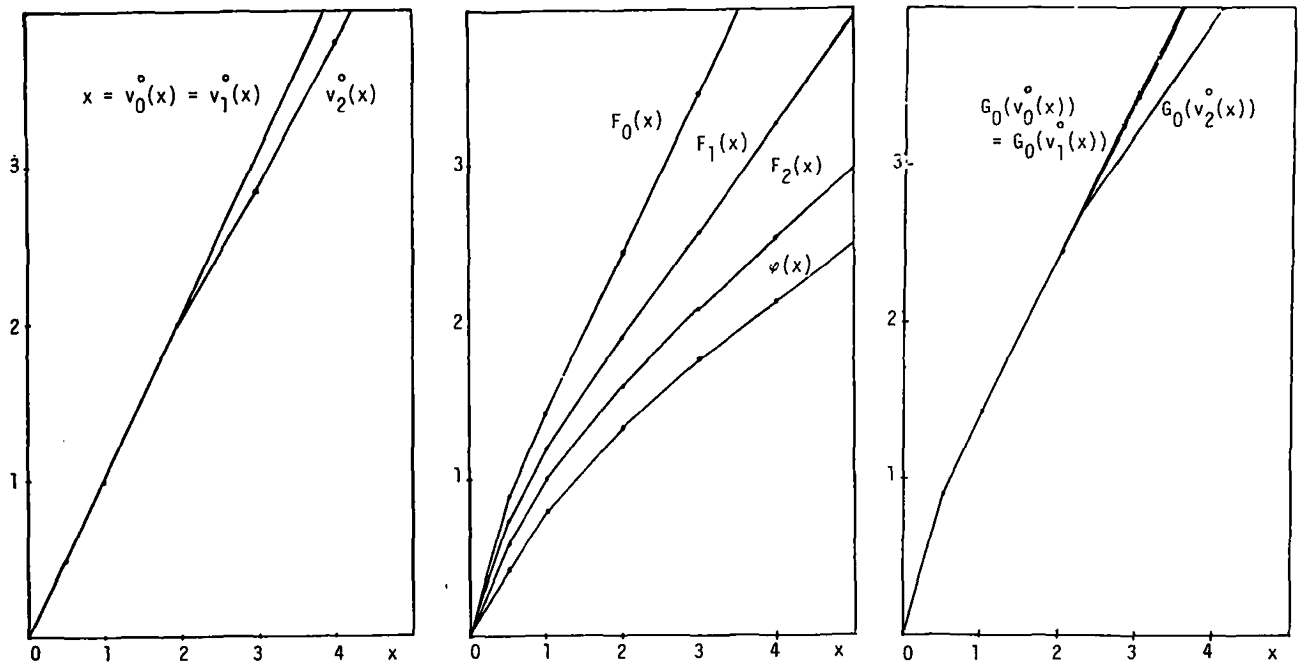
(i) Under what conditions are the monotonicity, concavity, differentiability, and absolute risk aversion properties of the final-period utility of wealth function $\varphi(x)$ inherited by the dynamic programming indirect utility functions $F_n(x)$?

(ii) Under what conditions does an optimal investment strategy $\mathbf{v}^o = (v_0^o(x), \ldots, v_N^o(x))$ exist, and when is it possible to sign the component functions $v_n^o(x)$ and their derivatives $\partial v_n^o(x)/\partial x$?




(a) Investment Functions $v_n^o(x)$ (b) Indirect Utility Functions $F_n(x)$ (c) Final-Period Expected Utility $G_0(v_n^o(x))$

Fig. 1. $\varphi(x)=1.0+0.5x-0.5e^{-0.75x}-0.5e^{-0.25x}$; both 1-period and 2-period foresight yield optimal final-period expected utility $G_0(v_0^o(x))$.




(a) Investment Functions $v_n^\circ(x)$ (b) Indirect Utility Functions $F_n(x)$ (c) Final-Period Expected Utility $G_0(v_n^\circ(x))$

Fig. 2. $\varphi(x) = 1.0 + 0.3x - 0.5\,e^{-0.8x} - 0.5\,e^{-0.6x}$; only 2-period foresight yields optimal final-period expected utility $G_0(v_0^\circ(x))$.

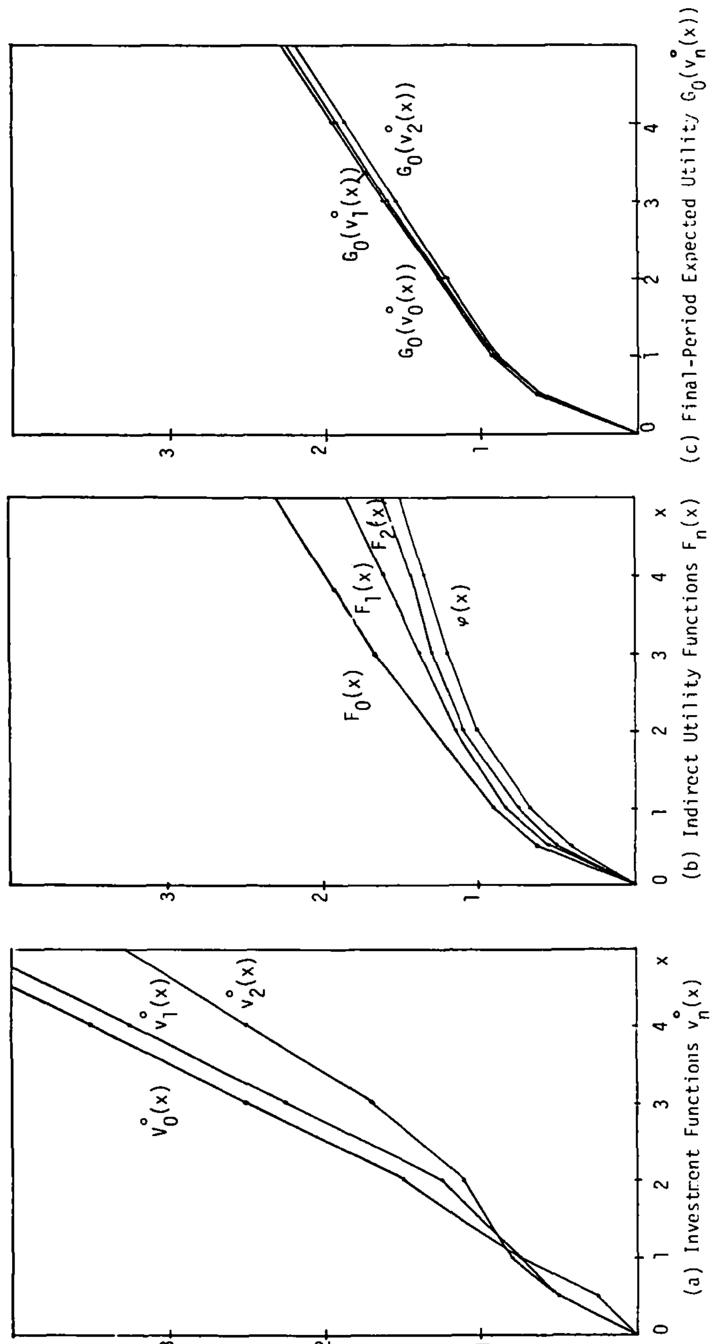
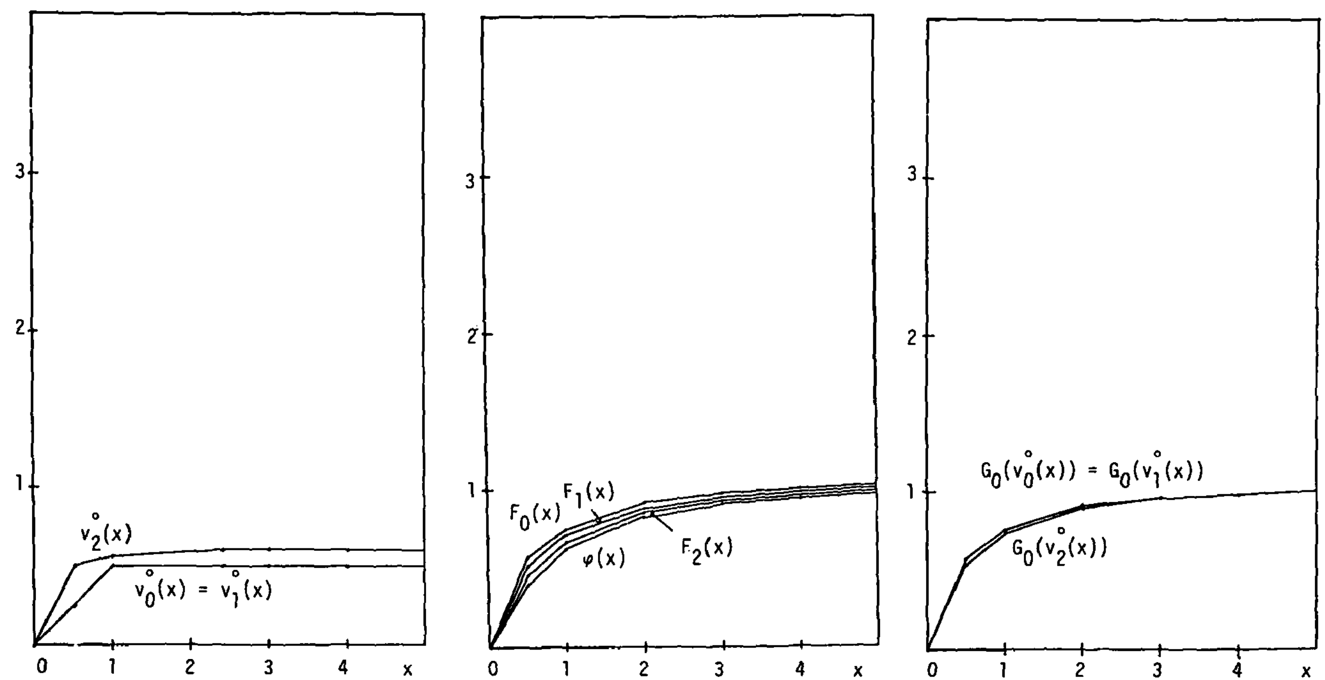



Fig. 3. $\varphi(x)=1.0+0.1x-0.5e^{-1.2x}-0.5e^{-0.6x}$; final-period expected utility is a strictly monotone increasing function of the foresight range $M$.




(a) Investment Functions $v_n^\circ(x)$ (b) Indirect Utility Functions $F_n(x)$ (c) Final-Period Expected Utility $G_0(v_n^\circ(x))$

Fig. 4. $\varphi(x)=1.0+0.0x-0.5e^{-1.2x}-0.5e^{-0.8x}$; only 2-period foresight yields optimal final-period expected utility $G_0(v_0^\circ(x))$.

(iii) What are the roles played by absolute risk aversion and uncertainty in determining the sensitivity of final-period expected utility to the foresight range $M$ incorporated into intermediate-period investments?

Questions (i) and (ii) are addressed in section 3. Relatively weak conditions are found sufficient to guarantee the existence of an optimal investment strategy $\mathbf{v}^{\circ}$ and the inherited monotonicity and concavity of the indirect utility functions $F_n(x)$. However, an interiority assumption is resorted to in order to rigorously establish inherited differentiability and decreasing absolute risk aversion properties for the functions $F_n(x)$, and various linearity restrictions are imposed on the dynamic wealth constraint state functions $f_n(\cdot)$ in order to sign the investment functions $v_n^{\circ}(x)$ and their derivatives.

Question (iii) is addressed in section 4. An upper bound provided for the loss in final-period expected utility resulting from myopic intermediate-period investments is shown to vary directly with the variance of myopically achievable utility, the ratio of global to myopic marginal utility, and the absolute difference between global and myopic absolute risk aversion. The interdependent roles played by the curvature of the utility function $\varphi(x)$ and the dispersion of the probability distributions $p_n(\omega)$ in determining foresight sensitivity are thus clarified.

Finally, the results of a foresight sensitivity study for a special 3-period basic investment model (S) are summarized in section 4. Final-period expected utility was found to be a monotone increasing function of the $M$-period foresight range incorporated into initial period investment, $1 \leqq M \leqq 3$. This result was expected, since period-by-period returns in the basic investment model are positively correlated, given any monotone increasing specifications for the myopic intermediate-period utility functions $U_n(x)$. See Tesfatsion (1980a).

Nevertheless, it was also found that a 2-period foresight range for initial period investment yielded optimal final-period expected utility to at least four decimal places for final-period $\varphi^*$ utility of wealth functions of the form $\varphi(x) = a - c_1 \exp(-d_1 x) - c_2 \exp(-d_2 x)$. This result was not expected in view of the shortness of the time horizon. It is interesting to note that the utility of wealth function fitted by Hildreth (1979) to the responses of Minnesota farmers facing hypothetical risk situations took precisely this form, i.e., $\varphi(x) = -\exp(-0.00006x) - 0.018 \exp(-0.0002x)$.

Similar simulation results, not reported, were obtained for a 4-period special basic investment model. The $\varphi^*$ class of utility functions $\varphi(x) = a + bx - \sum c_k \exp(-d_k x)$ thus seems to have special and interesting foresight sensitivity properties. A more extensive investigation of the $\varphi^*$ class of utility functions is currently being undertaken, and comparative foresight sensitivity studies of alternative classes of decreasingly risk averse utility functions are planned.

## Appendix: Proof outlines

### *A.1. Proof of Theorem 3.1*

By definition, $F_N(x)=E_N[\varphi \circ f_N(\omega, v_N^o(x), x)]$. By compactness of $\Omega$ and twice continuous differentiability of $\varphi(\cdot)$ and $f_N(\omega, \cdot)$, it follows by a Lebesque dominated convergence argument that expectation and differentiation operations can be interchanged for $\varphi \circ f_N$, thus condition (9a) and the negative definiteness of $D_N(x)$ in (8c) hold by assumption (A.4). It follows by an implicit function argument that $v_N^o(x)$ is continuously differentiable at $x$, and satisfies (8). The derivatives (9b) and (9c) then follow by direct calculation. The analogous results for $n<N$ are similarly obtained. Given (A.5), the thrice continuous differentiability of each $F_n(\cdot)$ follows by induction from (8) and (9c). Q.E.D.

### *A.2. Proof of Theorem 3.2*

Given (A.3) and (A.6) , it follows by Tesfatsion (1980a, thm. 3.6) that each indirect utility function $F_n(\cdot)$ for the basic model is strictly increasing.

To establish concavity, let $t \in [0,1]$ and $x^1, x^2 \in X$ be given. Let $v^1$ and $v^2$ attain $F_N(x^1)$ and $F_N(x^2)$, respectively, and define $x^t \equiv tx^1+[1-t]x^2$ and $v^t \equiv tv^1+[1-t]v^2$. Using assumptions (A.7) and (A.8),

$$\begin{aligned} F_N(x^t) &\geqq E_N\varphi(f_N(\omega, v^t, x^t)) \\ &\geqq E_N\varphi(tf_N(\omega, v^1, x^1)+[1-t]f_N(\omega, v^2, x^2)) \\ &\geqq tE_N\varphi(f_N(\omega, v^1, x^1))+[1-t]E_N\varphi(f_N(\omega, v^2, x^2)) \\ &= tF_N(x^1)+[1-t]F_N(x^2), \end{aligned} \tag{a.1}$$

hence $F_N(\cdot)$ is concave. It follows by induction that $F_N(\cdot)$ is concave for each $n \in (0,\ldots,N)$. Moreover, it is clear from (a.1) that strict concavity will hold for each $F_n(\cdot)$ if $\varphi(\cdot)$ is strictly concave. Continuity of the function $F_n(\cdot)$ over $X$ then follows from the assumed openness of $X$. Q.E.D.

### *A.3. Proof of Lemma 3.1*

Letting primes denote differentiation with respect to $x$, $h'(x) = \int_\Omega g'(\omega,x)\mu(\mathrm{d}\omega)$, $h''(x)=\int_\Omega g''(\omega,x)\mu(\mathrm{d}x)$, and $h'''(x)=\int_\Omega g'''(\omega,x)\mu(\mathrm{d}\omega)$. For any $x \in X$, $R_h'(x) \leqq 0$ if and only if $h'(x)h'''(x) \geqq (h''(x))^2$. However,

$$R_g'(x,\omega) \leqq 0 \ \mu-\text{a.e.}\omega \Rightarrow g'(\omega,x)g'''(\omega,x) \geqq (g''(\omega,x))^2 \ \mu-\text{a.e.}\omega.$$

$$\Rightarrow \sqrt{g'(\omega,x)g'''(\omega,x)} \geqq |g''(\omega,x)| \ \mu-\text{a.e.}\omega.$$

$$\Rightarrow \left(\int_\Omega \sqrt{g'(\omega,x)g'''(\omega,x)}\mu(\mathrm{d}\omega)\right)^2 \geqq \left(\int_\Omega |g''(\omega,x)|\mu(\mathrm{d}\omega)\right)^2$$

$$= [h''(x)]^2; \tag{a.2}$$

and, by Hölder's Inequality,

$$h'(x)h'''(x) \geqq \left(\int_\Omega \sqrt{g'(\omega,x)g'''(\omega,x)}\mu(\mathrm{d}\omega)\right)^2, \tag{a.3}$$

strict inequality holding in (a.3) unless (12) holds in the statement of Lemma 3.1. Q.E.D.

### *A.4. Proof of Lemma 3.2*

To establish the desired conclusion, it suffices to prove that $s(x)$ is a thrice differentiable strictly increasing function satisfying $s'''(x)s'(x) \geqq [s''(x)]^2$ for each $x \in X$, strict inequality holding if $R_h(v,x)$ is a strictly decreasing function of $x$.

By assumption (14), $h_v(v(x),x) = h_{vx}(v(x),x) = h_{vxx}(v(x),x) = 0$ for all $x \in X$. Since $h(\cdot)$ is thrice continuously differentiable, all partials up to the third order are interchangeable. By assumption, $v(x)$ is continuous and piecewise differentiable. Thus $s'(x) = h_x(v(x),x), s''(x) = h_{xx}(v(x),x)$, and $s'''(x) = h_{xxx}(v(x),x)$; and the desired conclusion follows immediately from the assumptions on $R_h(v,x)$. Q.E.D.

### *A.5. Proof of Theorem 3.3*

By Theorem 3.1 and Theorem 3.2, $F_n' > 0$ and $F_n'' \leqq 0$, and hence $R_{F_n} \geqq 0$ for each $n \in \{0,\dots,N\}$. It remains to show that $R_{F_n}' \leqq 0$.

Define a thrice continuously differentiable function $h: V \times X \to R$, strictly increasing in $x$, by $h(v,x) \equiv E_N[\varphi \circ f_N(\omega,v,x)]$; and, for each $v \in V$, define a function $gv: \Omega \times X \to R$, continuous over $\Omega$ for each $x \in X$ and thrice continuously differentiable and strictly increasing over $X$ for each $\omega \in \Omega$, by $gv(\omega,x) \equiv \varphi \circ f_N(\omega,v,x)$.

Define $R_{f_N} \equiv -(\partial^2 f_N/\partial x^2)/\partial f_N/\partial x$ and $R_{f_N}' \equiv \partial R_{f_N}/\partial x$. Note that $R_{f_N}' \leqq 0$ if and only if condition (15) in Theorem 3.3 holds. Thus, defining the usual absolute risk aversion measure for $gv(\omega, x)$ considered as a function of $x$,

$$R_{gv} \equiv \frac{-(gv)_{xx}}{(gv)_x} = R_\varphi(f_N(\omega, v, x)) \frac{\partial f_N}{\partial x}(\omega, v, x) + R_{f_N}(\omega, v, x), \tag{a.4}$$

it follows by condition (15) and the concavity and monotonicity restrictions on $\varphi$, $R_\varphi$, and $f_N$ that

$$R'_{gv}(\omega, x) = R'_\varphi \left[\frac{\partial f_N}{\partial x}\right]^2 + R_\varphi \frac{\partial^2 f_N}{\partial x^2} + R'_{f_N} \leqq 0, \tag{a.5}$$

for all $(\omega, x) \in \Omega \times X$. Hence, by Lemma 3.1, $R_h(v, x) \equiv -h_{xx}(v, x)/h_x(v, x)$ is a non-increasing function of $x$. It follows by (A.4), Theorem 3.1, and Lemma 3.2, that $R_{F_N}(x) \equiv -F''_N(x)/F'_N(x)$ is a non-increasing function of $x$, where

$$F_N(x) \equiv \sup_{v \in V(N,x)} h(v, x) = \sup_{v \in \text{Int}(V(N,x))} h(v, x) = h(v^\circ_N(x), x) \tag{a.6}$$

is a strictly increasing, concave, thrice continuously differentiable function of $x$. The remainder of the proof for $n < N$ follows by induction on $n$. Q.E.D.

### *A.6. Proof of Corollary 3.1*

In the induction proof for Theorem 3.3, concavity of the indirect utility function $F_{n+1}(\cdot)$ is used to ensure $R_{F_{n+1}} \geqq 0$, hence

$$R'_{gv}(\omega, x) = R'_{F_{n+1}} \left[\frac{\partial f_n}{\partial x}\right]^2 + R_{F_{n+1}} \frac{\partial^2 f_n}{\partial x^2} + R'_{f_n} \leqq 0, \tag{a.7}$$

in the $n$th induction step, given $R'_{F_{n+1}} \leqq 0$, $R'_{f_n} \leqq 0$, and $f_n(\cdot)$ concave in $x$. If $\partial^2 f_n/\partial x^2 \equiv 0$, then the final two terms in (a.7) vanish, and concavity of $F_{n+1}$ is not needed to ensure this inequality. Q.E.D.

### *A.7. Proof of Corollary 3.2*

Under the assumptions of Corollary 3.2, it follows from Theorem 3.1 that

$$S_N(x) = E_N \varphi'' b_N \frac{\partial f_N^T}{\partial v} = -E_N R_\varphi \varphi' b_N \frac{\partial f_N^T}{\partial v} = -c b_N E_N \varphi' \frac{\partial f_N^T}{\partial v} = \mathbf{0}.$$

Thus

$$F_N''(x)=E_N\varphi''b_N^2=E_N[-R_{\varphi}\varphi']b_N^2=-cb_N^2E_N\varphi'.$$

$$F_N'(x)=b_NE_N\varphi'.$$

It follows that

$$R_{F_N}(x)=-[-cb_N^2E_N\varphi']/b_NE_N\varphi'=cb_N\equiv c_N.$$

The remainder of the proof follows by induction on $n$. Q.E.D.

*A.8. Proof of Theorem 3.4*

For any $x\in X$ and $n\in\{0,\ldots,N\}$, define $B:V\to R$ by

$$B(v)\equiv E_nF_{n+1}\circ f_n(\omega,v,x). \tag{a.8}$$

By Theorem 3.1 and Theorem 3.2, $B(\cdot)$ is a strictly concave function satisfying

$$B'(v)=E_nF_{n+1}'(f_n(\omega,v,x))\frac{\partial f_n}{\partial v}(\omega,v,x), \tag{a.9}$$

$$B'(v_n^{\circ}(x)=0. \tag{a.10}$$

Thus, by assumption (18),

$$B'(0)=F_{n+1}'(H_n(x,0))E_n\left[G_n(\omega_n,0,x)+\frac{\partial H_n}{\partial v}(x,0)\right]. \tag{a.11}$$

Since $F_{n+1}'>0$, it follows that

$$v_n(x)\gtreqless 0\Leftrightarrow 0=B'(v_n^{\circ}(x))\gtreqless B'(0)$$

$$\Leftrightarrow 0\gtreqless E_n\left[G_n(\omega_n,0,x)+\frac{\partial H_n}{\partial v}(x,0)\right]. \tag{a.12}$$

Suppose in addition (A.5) holds, $R_{\varphi}'\leqq 0$, and conditions (20) and (21) in the statement of Theorem 3.4 are satisfied. Then it follows from Theorem 3.3

that $R'_{F_{n+1}} \leqq 0$. By monotonicity of $f_n(\cdot)$ with respect to $x$, $H'_n > 0$. Thus, using Theorem 3.1 and (a.12), and letting $d \equiv (\omega_n, v_n^o(x), x)$, $E_n(\partial f_n/\partial v)(\omega, 0, x) = E_n G_n(\omega_n) > 0$ implies $v_n^o(x) > 0$ and

$$\begin{aligned}\frac{\partial v_n^o(x)}{\partial x} &= -D_n(x)^{-1} E_n\left[F''_{n+1}(f_n(d))\frac{\partial f_n}{\partial x}(d)\frac{\partial f_n}{\partial v}(d)\right]\\ &= D_n(x)^{-1} E_n\left[R_{F_{n+1}}(f_n(d))F'_{n+1}(f_n(d))\frac{\partial f_n}{\partial x}(d)\frac{\partial f_n}{\partial v}(d)\right]\\ &= D_n(x)^{-1} H'_n(x) E_n[R_{F_{n+1}}(G_n(\omega_n)v_n^o(x) + H_n(x))\\ &\quad \times F'_{n+1}(f_n(d))G_n(\omega_n)]\\ &\geqq D_n(x)^{-1} H'_n(x) R_{F_{n+1}}(H_n(x)) E_n[F'_{n+1}(f_n(d))G_n(\omega_n)]\\ &= D_n(x)^{-1} H'_n(x) R_{F_{n+1}}(H_n(x)) B'(v_n^o(x)) = 0, \end{aligned} \qquad \text{(a.13)}$$

and conversely if $E_n G_n(\omega_n) < 0$. Finally, it follows immediately from (a.12) that $E_n G_n(\omega_n) = 0 \Rightarrow \partial v_n^o(x)/\partial x \equiv 0$. Q.E.D.

### *A.9. Proof of Theorem 3.5*

Let $B(\cdot)$ be defined by (a.8). As in Theorem 3.4, one obtains

$$v_n^o(x) \gtreqless 0 \Leftrightarrow 0 \lesseqgtr B'(0). \qquad \text{(a.14)}$$

However, by independence of $\omega_n^1$ and $\omega_n^2$,

$$\begin{aligned} B'(0) &= E_n F'_{n+1}(H_n(0,x) + I_n(\omega^1, x))\left[G_n(\omega_n^2, 0, x) + \frac{\partial H_n}{\partial v}(0,x)\right]\\ &= [E_n F'_{n+1}(H_n(0,x) + I_n(\omega^1, x))]\left[E_n G_n(\omega_n^2, 0, x) + \frac{\partial H_n}{\partial v}(0,x)\right]. \end{aligned} \qquad \text{(a.15)}$$

The desired result (24) then follows from (a.14) and (a.15), since $F'_{n+1} > 0$.

To establish (27), first define an auxiliary function $\Psi: R \to R$ by

$$\Psi(y) \equiv E_n^1 F_{n+1}(y + I_n(\omega^1)), \tag{a.16}$$

where superscripts 1 and 2 on $E_n[\cdot]$ will be used to denote expectation with respect to the marginal distributions of $\omega_n^1$ and $\omega_n^2$, respectively. Then

$$B(v) = E_n^2 \Psi(G_n(\omega^2)v + H_n(x)), \tag{a.17}$$

and

$$0 = B'(v_n^o(x)) = E_n^2 \Psi'(G_n(\omega^2)v_n^o(x) + H_n(x))G_n(\omega^2). \tag{a.18}$$

By Theorem 3.3, $R'_{F_{n+1}} \leqq 0$. It follows by Lemma 3.1 that $R'_\Psi \leqq 0$. By monotonicity of $f_n(\cdot)$ with respect to $x$, $H'_n > 0$. Thus, using Theorem 3.1 and (24), and letting $d \equiv (\omega_n, v_n^o(x), x)$ and $e \equiv G_n(\omega_n^2)v_n^o(x) + H_n(x)$, $E_n^2 G_n(\omega_n^2) > 0$ implies $v_n^o(x) > 0$ and

$$\begin{aligned}
\frac{\partial v_n^o(x)}{\partial x} &= -D_n(x)^{-1} E_n\left[ F''_{n+1}(f_n(d)) \frac{\partial f_n}{\partial x}(d) \frac{\partial f_n}{\partial v}(d) \right] \\
&= -D_n(x)^{-1} E_n F''_{n+1}(G_n(\omega_n^2)v_n^o(x) + H_n(x) + I_n(\omega_n^1)) \\
&\quad \times H'_n(x) G_n(\omega_n^2) \\
&= -D_n(x)^{-1} H'_n(x) E_n^2 \Psi''(G_n(\omega_n^2)v_n^o(x) + H_n(x)) G_n(\omega_n^2) \\
&= D_n(x)^{-1} H'_n(x) E_n^2 R_\Psi(G_n(\omega_n^2)v_n^o(x) + H_n(x)) \Psi'(e) G_n(\omega_n^2) \\
&\geqq D_n(x)^{-1} H'_n(x) R_\Psi(H_n(x)) E_n^2 \Psi'(e) G_n(\omega_n^2) \\
&= D_n(x)^{-1} H'_n(x) R_\Psi(H_n(x)) B'(v_n^o(x)) = 0,
\end{aligned}$$

and conversely if $E_n^2 G_n(\omega_n^2) < 0$. Finally, it follows immediately from (24) that $E_n^2 G_n(\omega_n^2) = 0$ implies $\partial v_n^o(x)/\partial x \equiv 0$. Q.E.D.

### *A.10. Proof of Theorem 3.6*

Theorem 3.6 follows immediately from Theorem 17.12 in Hinderer (1970, p. 116), once the basic model (4) is identified as a special case of Hinderer's model. The principal correspondences are $S \equiv \Omega$, $A \equiv V$,

$$r_n(h, v) \equiv \begin{cases} 0 & \text{if} \quad n \neq N, \\ E_N \varphi \circ f_N(\omega_N, v, t_N(h)) & \text{if} \quad n = N, \end{cases}$$

$$q_n(h, v, \omega) \equiv p_n(\omega), \qquad 0 \leqq n \leqq N,$$

and

$$\Pi_n(h, v) \equiv \begin{cases} 1 & \text{if} \quad v = v_n(t_n(h)), \qquad 0 \leqq n \leqq N, \\ 0 & \text{otherwise}, \end{cases}$$

for all $(h, v, \omega) \in H_n \times V \times \Omega$, where $H_n$ is defined by (28) and $t_n(h)$ is defined by (30). See Hinderer (1970, p. 12, remark 6, pp. 78–81, p. 118). Q.E.D.

### *A.11. Proof of Theorem 4.1*

The proof of Theorem 4.1 follows from Theorem 3.1 and Tesfatsion (1980a, thm. 4.2) by defining $H_{n+1} \equiv F_{n+1} \circ U^{-1}$ and $W_n(\omega, v, x) \equiv U \circ f_n(\omega, v, x)$, and noting that

$$H_{n+1} \circ W_n = F_{n+1} \circ f_n,$$

$$H'_{n+1}(x) = F'_{n+1}(U^{-1\prime}(x))U^{-1\prime}(x) > 0, \qquad x \in X,$$

and

$$H''_{n+1}(U(y)) = \frac{F'_{n+1}(y)}{[U'(y)]^2}[R_U(y) - R_{F_{n+1}}(y)], \qquad y \in Y, \quad x \in X.$$

Q.E.D.

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