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A Note on Dependence between a Venture and a Current Prospect*

Some sufficient conditions that a random variable be positively correlated with every strictly decreasing function of a second random variable are developed and applied to the problem of choosing the optimal amount of an uncertain venture. Two of the conditions generalize conditions previously employed by Samuelson [5] and Scheffman [6].

1. Introduction

Consider a decision problem of the form

$$\max_{\alpha \in A} \eta(\alpha) = E\varphi(X + \alpha Y), \tag{1.1}$$

where X is a random variable representing a decision maker's current prospect, Y is a random variable representing a possible venture, and φ is his utility function for future wealth.

The current prospect reflects the decision maker's possible values of future wealth if he proceeds with his present plans, commitments, business undertakings, investments, etc. The venture Y is a prospective security purchase or sale, business deal, insurance policy, or other project that, if undertaken, may influence future wealth. If α is the amount of the venture undertaken, $X + \alpha Y$ becomes the decision maker's new prospect.[1] A represents possible emounts of the venture.

A is determined by the circumstances of the particular venture under consideration. A purchase of common stock could be any nonnegative integral number of shares up to the limit of the decision maker's resources. Stock options or commodity futures could be bought or sold so α could be

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[1] Direct applicability of this simple model is limited by several of the assumptions. It is a traditional two-period model with just one prospective venture. Leland [4] has shown that familiar conditions for a maximum apply if α, Y are interpreted as vectors. Fama [1] has shown that, for some problems, the two-period model can be embedded in a multiperiod model. For some business ventures, the linearity and additivity assumptions may not be appropriate; but it may sometimes be possible to approximate a single nonadditive or nonlinear venture by several appropriately restricted linear and additive ventures.

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positive or negative. Joining a partnership might require a specified investment so A would consist of two points: 0 and the specified investment. It is convenient to initially assume that α might be any real number and subsequently consider restrictions that might be imposed in particular applications (see [2, p. 10]). Unless otherwise noted, A is assumed equal to R, the real line.

If φ is differentiable and concave (risk aversion) and if $E|\varphi(X+\alpha Y)|$, $E|Y\varphi'(X+\alpha Y)|$ are finite then η is differentiable [3, p. 3] and

$$\eta'(\alpha) = EY\varphi'(X+\alpha Y). \tag{1.2}$$

φ strictly concave implies η strictly concave in which case the optimal (maximizing) value of α, say $\hat{\alpha}$, is unique if it exists. $\hat{\alpha}$ exists if $\lim_{x\to\infty}\varphi'(x)=0$ and $P(Y>0)>0$ and $P(Y<0)>0$ (neither Y nor $-Y$ is a sure thing) [2, p. 10].

One is interested in relating $\hat{\alpha}$ to properties of the initial prospect and the venture that may sometimes be determinable in practical situations. With strict concavity, hereinafter assumed, $\hat{\alpha}$ uniquely solves $\eta'(\alpha)=0$. $\eta(\alpha)$ has the form of an inverted U, so one way to investigate the general location of $\hat{\alpha}$ is to try to determine the sign of $\eta'(\alpha)$ for interesting values of α. If $\eta'(\bar{\alpha})>0$ for a chosen $\bar{\alpha}$ then $\hat{\alpha}>\bar{\alpha}$ since $\eta(\alpha)$ must level off to the right of $\bar{\alpha}$. Thus

$$\hat{\alpha} \gtreqless \bar{\alpha} \Leftrightarrow \eta'(\bar{\alpha}) \gtreqless 0. \tag{1.3}$$

Thus determining the sign of $\eta'(0)$, marginal expected utility at the origin, indicates the sign of $\hat{\alpha}$. If the decision maker can either buy or sell short, a negative $\hat{\alpha}$ would indicate the latter. In cases where only nonnegative amounts of the venture are feasible, negative $\hat{\alpha}$ indicates that he would retain his current prospect. Recalling (1.2)

$$\eta'(0) = EY\varphi'(X) = (EY)(E\varphi'(X)) + \operatorname{Cov}(Y, \varphi'(X)). \tag{1.4}$$

To determine the common sign of $\eta'(0)$ and $\hat{\alpha}$, note that a decision maker's normal preference for higher income implies that φ' and therefore $E\varphi'$ is positive. Thus, if EY and $\operatorname{Cov}(Y, \varphi'(X))$ agree in sign, $\eta'(0)$ and $\hat{\alpha}$ will also have this sign. If EY and $\operatorname{Cov}(Y, \varphi'(X))$ differ in sign one has to know further particulars of the utility function and the joint distribution of X, Y to determine the sign of $\hat{\alpha}$.

If X and Y are stochastically independent, then $\operatorname{Cov}(Y, \varphi'(X))=0$ and $\hat{\alpha}$ agrees in sign with EY. The main purpose of this note is to indicate several conditions sufficient to determine the sign of $\operatorname{Cov}(Y, \varphi'(X))$ when the initial prospect and venture are not independent. The conditions and some logical relations among them are given in Section 2. Some hypothetical applications are cited in Section 3.

2. Sufficient Conditions for Positive Covariance

Under risk aversion, φ' is decreasing so Condition (i) below makes Cov $(Y, \varphi'(X))$ positive. Each of the other conditions is shown to imply Condition (i) and relations among the other conditions are explored. As is customary if F, G are probability distribution functions, $F < G$ is defined to mean $F(x) \leqslant G(x)\ \forall x \in R$ and $F \neq G$.

THEOREM 1. *Let X, Y be nondegenerate random variables with finite means and variances and with distribution functions F_X, F_Y. The following implications hold among the conditions listed below*: (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (ii), (v) $\Rightarrow$ (iii), (vi) $\Rightarrow$ (ii), (vii) $\Rightarrow$ (i), (viii) $\Rightarrow$ (vii), (viii) $\Rightarrow$ (v). *Statements about conditional expectations and distributions in* (ii) *through* (v) *are to be understood to hold* a.s. F_X.

(i) *Y is positively correlated with any strictly decreasing function of X with finite second moment.*

(ii) $\exists \tilde{x} \ni [x < \tilde{x} \Rightarrow E(Y \mid X = x) > EY]$,

$$[x > \tilde{x} \Rightarrow E(Y \mid X = x) < EY].$$

(iii) *$E(Y \mid X = x)$ is a strictly decreasing function of x.*

(iv) $\exists \tilde{x} \ni [F_{Y \mid X=x} \gtreqless F_Y \Leftrightarrow x \gtreqless \tilde{x}]$.

(v) $\forall \tilde{x}, \hat{x} \in$ *support* F_X *with* $\tilde{x} < \hat{x}$,

$$F_{Y \mid X=\tilde{x}} < F_{Y \mid X=\hat{x}}\,.$$

(vi) $\exists \tilde{x} \ni [Y \gtreqless EY \Leftrightarrow X \gtreqless \tilde{x}]$.

(vii) *$Y = f(W, V)$ and $X = g(W, Z)$ where W is a nontrivial random variable; V and Z are random mappings; W, V, Z are independent; $f(\cdot, \cdot)$ is strictly increasing in its first argument; $g(\cdot, \cdot)$ is strictly decreasing in its first argument.*

(viii) *$Y = f(W, V)$ and $X = g(W)$ where W, V, f are as in* (vii) *and g is strictly decreasing.*

Proof. (ii) $\Rightarrow$ (i). Let $\gamma : R \to R$ be strictly decreasing and $Y^* = Y - EY$. Then

$$\begin{aligned}\operatorname{Cov}(Y, \gamma(X)) = EY^* \gamma(X) &= \int_{x<\tilde{x}} E(Y^* \mid X = x)\, \gamma(x)\, dF_X \\ &\quad + \int_{x \geqslant \tilde{x}} E(Y^* \mid X = x)\, \gamma(x)\, dF_X \\ &> \gamma(\tilde{x}) \int E(Y^* \mid X = x)\, dF_X = \gamma(\tilde{x})\, EY^* = 0.\end{aligned}$$

(iii) $\Rightarrow$ (ii). Choose a version of $E(Y \mid X = x)$ that is strictly decreasing. Let $\tilde{x} = \sup\{x : E(Y \mid X = x) > EY\}$.

(iv) $\Rightarrow$ (ii). The following lemma is proved in [7] and modifies an earlier lemma by Hanoch and Levy.

LEMMA. *If F, G are distribution functions and $\theta : R \to R$ is continuous, nondecreasing, and $\int \theta dF < \infty$, $\int \theta dG < \infty$, then*

$$\int \theta dF - \int \theta dG = \int (G - F)\, d\theta.$$

Suppose $x > \tilde{x}$. Then $F_{Y|X=x} > F_Y$ a.s. *and*

$$E(Y \mid X = x) - EY \overset{\text{a.s.}}{=} \int y dF_{Y|X=x} - \int y dF_Y$$

$$= \int (F_Y(y) - F_{Y|X=x}(y))\, dy < 0.$$

(v) $\Rightarrow$ (iii). Similar use of the above lemma. That (v) $\Rightarrow$ (iii) is essentially the same as a proposition of Scheffman [6, p. 15].

(vi) $\Rightarrow$ (ii).

$$E(Y \mid X = x) \overset{\text{a.s.}}{=} \int y dF_{Y|X=x} \qquad \text{and for } x < (\text{resp. } >)\ \tilde{x},\ y > (<)\ EY.$$

(vii) $\Rightarrow$ (i). Let $\gamma : R \to R$ be any strictly decreasing function. Define $h(W, Z) = \gamma(g(W, Z))$. Clearly h is strictly increasing in its first argument.

Define $\bar{f}(w) = Ef(w, V)$ and $\bar{h}(w) = Eh(w, Z)$. $\bar{f}$ and $\bar{h}$ are strictly increasing. Without loss of generality let $EY = 0$. Define $w_0 = \inf\{w \mid \bar{f}(w) > 0\}$. Then

$$\begin{aligned}\operatorname{Cov}(Y, \gamma(X)) &= E(Y\gamma(X)) = E(f(W, V)\, h(W, Z)) \\ &= \int \bar{f}(w)\, \bar{h}(w)\, dF_W(w) = \int_{w \geqslant w_0} \bar{f}(w)\, \bar{h}(w)\, dF_W(w) \\ &\quad + \int_{w < w_0} \bar{f}(w)\, \bar{h}(w)\, dF_W(w) \\ &> \bar{h}(w_0) \int \bar{f}(w)\, dF_W(w) = \bar{h}(w_0)\, EY = 0.\end{aligned}$$

This result generalizes another of Scheffman's lemmas [6, p. 14].

(viii) $\Rightarrow$ (vii). Obvious since (viii) may be regarded as the special case of (vii) in which Z is constant.

(viii) $\Rightarrow$ (v). Since g is strictly decreasing, g^{-1} exists and decreases strictly. Write $W = g^{-1}(X)$ and $Y = f(g^{-1}(X), V) = h(X, V)$ where h is strictly decreasing in its first argument. For any $\tilde{x} < \hat{x}$, $E(Y \mid X = \tilde{x}) =^{\text{a.s.}} Eh(\tilde{x}, V) > Eh(\hat{x}, V) =^{\text{a.s.}} E(Y \mid X = \hat{x})$ since $h(\tilde{x}, v) > h(\hat{x}, v)$ for all v. For any y, $F_{Y|X=\hat{x}}(y) =^{\text{a.s.}} P_V(\{v : h(\tilde{x}, v) \leqslant y\}) \leqslant P_V(\{v : h(\hat{x}, v) \leqslant y\}) =^{\text{a.s.}} F_{Y|X=\tilde{x}}(y)$ since $\{v : h(\tilde{x}, v) \leqslant y\}\ p\ \{v : h(\hat{x}, v) \leqslant y\}$. The distributions cannot be equal since it was shown that $E(Y \mid X = \tilde{x}) > E(Y \mid X = \hat{x})$ a.s.

Summarizing, and taking account of transitivity of $\Rightarrow$, we have

COROLLARY 1. *For the conditions of Theorem* 1,

$$
\begin{aligned}
&\text{(ii)} \Rightarrow \text{(i)};\\
&\text{(iii)} \Rightarrow \text{(ii), (i)};\\
&\text{(iv)} \Rightarrow \text{(ii), (i)};\\
&\text{(v)} \Rightarrow \text{(iii), (ii), (i)};\\
&\text{(vi)} \Rightarrow \text{(ii), (i)};\\
&\text{(vii)} \Rightarrow \text{(i)};\\
&\text{(viii)} \Rightarrow \text{(vii), (v), (iii), (ii), (i)}.
\end{aligned}
$$

Whether other implications might exist is a natural question answered by

THEOREM 2. *Consider random variables X, Y and Conditions* (i) *through* (viii) *as in Theorem* 1. *The implications listed in Corollary* 1 *are the only valid implications among these conditions.*

Proof. (viii) $\not\Rightarrow$ (vi), (viii) $\not\Rightarrow$ (iv).

(a) Denote the probability space on which the random variables are defined by $(\Omega, \mathscr{F}, P)$. Let $\Omega = \{1, 2, 3, 4\}$ with respective probabilities 0.3, 0.2, 0.2, 0.2. Let $W(1) = W(3) = 0$, $W(2) = W(4) = 1$, $V(3) = V(4) = 0$, $V(1) = V(2) = 1$. Let $X = -W$ and $Y = W + 2V$. A little arithmetic verifies that (viii) holds but neither (vi) nor (iv) hold.

(vii) implies only (i).

(b) Let W, V, Z of Condition (vii) each take the value -1 with probability $\frac{1}{2}$ and 1 with probability $\frac{1}{2}$ and be independent. Let $Y = V + \epsilon W$, $X = Z - \epsilon W$, where $0 < \epsilon < 1$. (vii) is satisfied. Using independence of W, V, Z a simple calculation yields

$$
\begin{aligned}
E(Y \mid X = -1 - \epsilon) &= \epsilon,\\
E(Y \mid X = -1 + \epsilon) &= -\epsilon,\\
E(Y \mid X = 1 - \epsilon) &= \epsilon,\\
E(Y \mid X = 1 + \epsilon) &= -\epsilon,
\end{aligned}
$$

which violate (ii). (vii) cannot imply (iii) since (iii) implies (ii). Similarly (vii) cannot imply (iv), (v), (vi), or (viii) since each of these implies (ii).

(vi) $\not\Rightarrow$ (viii), (vii), (v), (iv), or (iii).

(c) Let $\Omega = \{1, 2, 3, 4\}$ with $P(\omega\} = \frac{1}{4} \; \forall \; \omega$; $X(\omega) = \omega \forall \omega$; $Y(1) = 11$, $Y(2) = Y(4) = 0$, $Y(3) = 1$. Then $EY = 3$ and $\tilde{x} = 1\frac{1}{2}$ makes (vi) satisfied. However, (iii), (iv), (v), (vii), (viii) are violated. To see that (vii) and (viii) are not satisfied note that any Z independent of Y would have to be a constant as would any V independent of X. Thus to satisfy (vii) or (viii) there would have to be a W monotonically related to both X and Y. But this is impossible since X is monotonic on Ω and Y is not.

(v) $\not\Rightarrow$ (viii), (vii), (vi), or (iv).

(d) Let $\Omega = \{1, 2, 3, 4, 5\}$ with $P(\omega) = 0.2 \forall \omega$; $Y(\omega) = \omega$; $X(1) = X(3) = 1$, $X(2) = X(4) = 0$, $X(5) = -1$. (v) is satisfied. By an argument similar to that in (c), (vii) is not satisfied. Since (viii) $\Rightarrow$ (vii), (v) $\not\Rightarrow$ (vii) it follows that (v) $\not\Rightarrow$ (viii). Since (viii) $\Rightarrow$ (v), (viii) $\not\Rightarrow$ (iv) it follows that (v) $\not\Rightarrow$ (iv); a similar argument shows (v) $\not\Rightarrow$ (vi).

(iv) $\not\Rightarrow$ (viii), (vii), (vi), (v), or (iii).

(e) Let $\Omega = \{1, 2, 3, 4\}$ with probabilities $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$. Let $X(0) = 0$, $X(2) = X(3) = 1$, $X(4) = 2$ and let $Y(1) = Y(3) = 1$, $Y(2) = Y(4) = 0$. (iv) holds with $\tilde{x} = 1$. (vii), and therefore (viii), does not hold. Tentatively suppose (vii) holds. Any V independent of X must be constant so we may write $Y = f(W)$, $W = f^{-1}(Y)$, and $X = g(f^{-1}(Y), Z) = h(Y, Z)$ where h is strictly decreasing in its first argument. Let $Z(1) = \tilde{x}$, $Z(3) = \nu$. Then Z independent of Y requires that $Z(2) = \nu$, $Z(4) = \tilde{x}$. But then $h(1, \nu) = h(0, \nu) = 1$ which contradicts the fact that h is strictly decreasing in its first argument.

(f) Let $\Omega = \{1, 2, \ldots, 7\}$ with $P\{\omega\} = \frac{1}{12}$ for $\omega = 1 \cdots 4$, $P\{5\} = \frac{1}{6}$, $P\{6\} = P\{7\} = \frac{1}{4}$. Let $X(1) = X(5) = X(6) = 0$, $X(2) = 1$, $X(3) = X(4) = 2$, $X(7) = -1$; and $Y(1) = Y(4) = 4$, $Y(2) = Y(3) = Y(5) = 0$, $Y(6) = Y(7) = 5$. Then (iv) holds with $\tilde{x} = 0$; but (vi), (v), (iii) do not hold.

(iii) implies only (ii) and (i).

(g) Let $\Omega = \{1, 2, 3\}$, $P(\omega) = \frac{1}{3} \forall \omega$. Let $X(1) = 0$, $X(2) = X(3) = 1$. Let $Y(1) = 1$, $Y(2) = 2$, $Y(3) = -2$. Then (iii) holds but not (v). Since (v) $\Rightarrow$ (iii) and (v) $\not\Rightarrow$ (viii), (vii), (vi), or (iv); it follows that (iii) $\not\Rightarrow$ (viii), (vii), (vi), (iv).

(ii) implies only (i).

Since (iv) $\Rightarrow$ (ii) and (iv) $\not\Rightarrow$ (viii), (vii), (vi), (v), or (iii); (ii) does not imply any of the latter. Since (iii) $\Rightarrow$ (ii) and (iii) $\not\Rightarrow$ (iv); (ii) $\not\Rightarrow$ (iv).

(i) implies none of the others.

Since (vii) $\Rightarrow$ (i) but (vii) $\not\Rightarrow$ (ii), (i) $\not\Rightarrow$ (ii). Since each of the others implies (ii), (i) could not imply any without implying (ii).

Quite a few propositions closely related to those of Theorem 1 may be obtained by reversing or weakening appropriate inequalities and monotonicities in both assumptions and conclusions. For example, (ii') $\Rightarrow$ (i') and (iii*) $\Rightarrow$ (i*) where

(ii') $\exists \tilde{x} \ni [x > \tilde{x} \Rightarrow E(Y \mid X = x) > EY]$, $[x < \tilde{x} \Rightarrow E(Y \mid X = x) < EY]$.

(i') Y is negatively correlated with any strictly decreasing function of X that has finite second moment.

(iii*) $E(Y \mid X = x)$ is a nonincreasing function of x.

(i*) Y is not negatively correlated with any nonincreasing function of X with finite second moment.

Other possible modifications seem reasonably clear and too numerous to try to list.

3. Application

Condition (vii), called negative S-correlation by Scheffman, has been found useful by Samuelson [5] and Scheffman [6] in establishing several theorems on diversification of investlents. Some illustrative applications of other conditions follow. The general assumptions of Section 1 (e.g., the strict concavity of the utility function φ) will be assumed to hold throughout.

Insurance

A decision maker stands to lose an amount $w > 0$ if the event A occurs. In exchange for a premium c, he is offered an insurance policy that will cover this contingent loss. Viewed as a venture the policy can be written $Y = wI_A - c$ where I_A is the indicator function of the event A. Suppose he can also elect partial coverage at a proportionally reduced premium, i.e., he can elect to pay a premium αc, $0 \leqslant \alpha \leqslant 1$, and be reimbursed αw if the loss occurs.

Let Z represent his current prospect other than this possibility of loss. His expacted utility as a function of the chosen coverage α is then

$$\eta(\alpha) = E\varphi(Z - wI_A + \alpha(wI_A - c)) = E\varphi(X + \alpha Y) \tag{3.1}$$

with $X = Z - wI_A$, $Y = wI_A - c$. Assuming Z is independent of A, Condition (vii) of Section 2 is satisfied. To observe the circumstances under which some coverage will be taken, note

$$\eta'(0) = EYE\varphi'(X) + \mathrm{Cov}(Y, \varphi'(X)). \tag{3.2}$$

Since (Theorem 1) (vii) $\Rightarrow$ (i), we know that the covariance is positive. Examining the first term on the right,

$$EY\,E\varphi'(X) = (wP_A - c)\,E\varphi'(X), \tag{3.3}$$

one observes that $E\varphi'$ is always positive and $(WP_A - c)$ is the subjective actuarial value[2] of the policy. From (3.2), (3.3), and (1.3),

$$\hat{\alpha} \gtreqless 0 \Leftrightarrow \eta'(0) \gtreqless 0 \Leftrightarrow (wP_A - c) \gtreqless \frac{-\mathrm{Cov}(Y, \varphi'(X))}{E\varphi'(X)}. \tag{3.4}$$

Since the ratio on the right is known to be negative, it is clear that some coverage will be chosen if the subjective actuarial value is nonnegative or even somewhat negative, so long as

$$c < wP_A + \frac{\mathrm{Cov}(Y, \varphi'(X))}{E\varphi'(X)}. \tag{3.5}$$

Calculation of this upper limit on the premium would, of course, require detailed knowledge of decision maker's utility and subjective probability.

One may also be interested in the circumstances under which full coverage will be taken. By (1.3) this depends on $\eta'(1)$. In this case

$$\begin{aligned} \eta'(1) = EY\varphi'(X + Y) &= E(wI_A - c)\,\varphi'(Z - c) \\ &= E(wI_A - c)\,E\varphi'(Z - c), \end{aligned} \tag{3.6}$$

the final equality following from the independence of Z and A. Thus

$$\hat{\alpha} \gtreqless 1 \Leftrightarrow wP_A - c \gtreqless 0. \tag{3.7}$$

Thus full coverage will be desired if the policy is offered at exactly subjective actuarial value and less (more) than full coverage if the policy offers less (more) than subjective actuarial value.

The results readily extend to more general kinds of coverage. Let W be any pattern of potential loss and let c be a premium covering such a loss. Then $X = Z - W$, $Y = W - c$. Condition (vii) is still satisfied (assuming Z, W independent) and

$$\begin{aligned} &\hat{\alpha} \gtreqless 0 \Leftrightarrow (EW - c) \gtreqless \frac{-\mathrm{Cov}(W, \varphi'(X))}{E\varphi'(X)}, \\ &\hat{\alpha} \gtreqless 1 \Leftrightarrow EW - c \gtreqless 0, \end{aligned} \tag{3.8}$$

[2] The subjective actuarial value could be different for the decision maker and the insurance company if they have different estimates of P_A, or if an uninsured property loss by the decision maker would involve secondary losses—loss of customers, borrowing on unfavorable terms, etc. In the latter case, the actual claim would be less than w.

where $EW - c$ is the subjective actuarial value and $\mathrm{Cov}(W, \varphi'(X))$ is known to be positive.

A Professional Golfer

A successful golfer has decided to allow his name to appear on a related product. He can take as remuneration a percentage of sales, a specified annual fee, or an appropriate mixture of the two.

Let S be a random variable representing prospective sales and let λ be the percentage he receives if he takes no specified fee. Let h be the fee he receives if he takes no percentage of sales, and assume he can also elect to receive $\alpha\lambda S + (1 - \alpha) h$ for $0 \leqslant \alpha \leqslant 1$.

Let R be his prospective earnings from other sources. The relation of his expected utility to his choice of fraction α is given by

$$\begin{aligned} \eta(\alpha) &= E\varphi(R + \alpha\lambda S + (1 - \alpha) h) \\ &= E\varphi(R + h + \alpha(\lambda S - h)), \end{aligned} \tag{3.9}$$

which has the form of (1.1) with $X = R + h$, $Y = (\lambda S - h)$, $0 \leqslant \alpha \leqslant 1$.

Since outcomes of tournaments may be expected to dominate R and strongly influence S, it seems natural that $E(S \mid R = r)$ and therefore $E(Y \mid X = x)$ is a strictly increasing function.[3] This is the opposite of Condition (iii) and implies that Y is negatively correlated with any decreasing function of X.

To investigate sign $\hat{\alpha}$, write

$$\begin{aligned} \eta'(0) &= EY\, E\varphi'(X) + \mathrm{Cov}(Y, \varphi'(X)) \\ &= (\lambda ES - h)\, E\varphi'(X) + \mathrm{Cov}(Y, \varphi'(X)). \end{aligned} \tag{3.10}$$

Then

$$\hat{\alpha} \gtreqless 0 \Leftrightarrow \lambda ES \gtreqless h - \frac{\mathrm{Cov}(Y, \varphi'(X))}{E\varphi'(X)}. \tag{3.11}$$

Thus, a risk averse expected utility maximizing golfer would take the flat fee unless his expectation of return from sales is sufficiently higher than h. Again, "sufficiently" can be made precise only with additional knowledge of utility and subjective probability in a particular case. Clearly, a still higher margin would be required for him to choose only income from sales.

[3] Other conditions might be cited but it seems to us that (iii) would frequently be particularly easy to elicit. This example is mathematically the same as that of a choice between a certain asset and a risky asset, but the circumstances tell us a useful relation between the uncertain asset and the initial prospect.

Business or Securities

The owner–operator of a manufacturing plant has current prospect R and resources to invest equal to h. He has a chance to become a partner in operating a similar plant in another state. Any resources not used in the partnership will be invested in a fixed-yield riskless security.

Assume $R = r(W, Z)$ where W is an index of conditions in the industry in which the decision maker is engaged and Z represents local circumstances influencing profitability of his plant. r is a strictly increasing function of W. Let $S = s(W, V)$ represent prospective returns from the new plant where V represents relevant circumstances in the new location[4] and s is strictly increasing in its first argument. Then expected utility is

$$\eta(\alpha) = E\varphi(R + hb + \alpha(S - b))$$

where b is return on the riskless security and α is the amount invested in the partnership. In the notation of (1.1), $X = R + hb$, $Y = S - b$.

If Z and V are independent a condition opposite to (vii) holds and Y is negatively correlated with any decreasing function of X. Except for the reasoning that leads to the conclusion of negative correlation, the example is like the golfer example. In the present case

$$\hat{\alpha} \gtreqless 0 \Leftrightarrow ES \gtreqless b - \frac{\mathrm{Cov}(Y, \varphi'(X))}{E\varphi'(X)} \tag{3.12}$$

with the covariance known to be negative.

Clearly the conditions studied here do not answer many of the questions investigators must pose, but we hope the illustrations suggest that noting such a condition may sometimes get an analysis started in a useful direction.

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[4] Recall that Condition (vii) allows Z and V to be random vectors or still more general random mappings.

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