DC-OPF Formulation with Price-Sensitive Demand Bids

Junjie Sun and Leigh Tesfatsion

Last Revised: 19 March 2008

1 Cost and Demand Function Representations

Generator i's total cost function:

$$\mathrm{TC}_i(p_{Gi}) \;=\; a_i \cdot p_{Gi} \;+\; b_i \cdot p_{Gi}^2 \;+\; \mathrm{FCost}_i \tag{1}$$

Generator i's total variable cost function:

$$\mathrm{TVC}_i(p_{Gi}) \;=\; a_i \cdot p_{Gi} \;+\; b_i \cdot p_{Gi}^2 \tag{2}$$

Generator i's marginal cost function (supply offer schedule):

$$\mathrm{MC}_i(p_{Gi}) \;=\; a_i \;+\; 2 \cdot b_i \cdot p_{Gi} \tag{3}$$

LSE j's demand bid p_{Lj} consists of two parts: a fixed demand bid p_{Lj}^F and a price-sensitive demand bid p_{Lj}^S, i.e.,

$$p_{Lj} \;=\; p_{Lj}^F \;+\; p_{Lj}^S \tag{4}$$

LSE j's price-sensitive demand bid function expressing maximum willingness to pay as a function of the demanded quantity p_{Lj}^S:

$$\mathrm{D}_j(p_{Lj}^S) \;=\; c_j \;-\; 2 \cdot d_j \cdot p_{Lj}^S \tag{5}$$

The gross surplus of LSE j corresponding to its price-sensitive demand bid:[1]

$$\mathrm{GSS}_j(p_{Lj}^S) \;=\; c_j \cdot p_{Lj}^S \;-\; d_j \cdot {p_{Lj}^S}^2 \tag{6}$$

Total net surplus corresponding to price-sensitive demand bids:[2]

$$\mathrm{TNSS}(\mathbf{p}_G, \mathbf{p}_L^S) \;=\; \mathrm{GSS}(\mathbf{p}_L^S) - \mathrm{TVC}(\mathbf{p}_G) \tag{7}$$

[1] The gross surplus of LSE j corresponding to its fixed demand bid is always infinite (vertical demand curve). For this reason, the DC-OPF objective function used by the ISO to determine efficient commitment and dispatch of generation will only take into account LSE gross surplus corresponding to price-sensitive demand bids.

[2] Note that TNSS coincides with the usual measure for total net surplus in the absence of fixed demand bids.

where

$$\mathbf{p}_G = (p_{G1}, p_{G2}, \cdots, p_{GI}) \tag{8}$$

$$\mathbf{p}_L^S = \left(p_{L1}^S, p_{L2}^S, \cdots, p_{LJ}^S\right) \tag{9}$$

$$\mathrm{GSS}(\mathbf{p}_L^S) = \sum_{j=1}^{J} \mathrm{GSS}_j(p_{Lj}^S) \tag{10}$$

$$\mathrm{TVC}(\mathbf{p}_G) = \sum_{i=1}^{I} \mathrm{TVC}_i(p_{Gi}) \tag{11}$$

$$\tag{12}$$

Total net cost function corresponding to price-sensitive demand bids:

$$\mathrm{TNCS}(\mathbf{p}_G, \mathbf{p}_L^S) \ = \ -\,\mathrm{TNSS}(\mathbf{p}_G, \mathbf{p}_L^S) \tag{13}$$

2 DC-OPF Problem in Structural Form

A commonly used representation for an hourly DC-OPF problem with price-sensitive load bids is to minimize total net costs corresponding to the price-sensitive demand (TNCS) subject to various transmission constraints. As explained at length in Sun and Tesfation (2007), it is useful to modify the objective function for this standard DC-OPF problem to include a soft penalty function for large voltage angle deviations.

The resulting modified DC-OPF problem formulation is as follows, where all endogenous and exogenous variables are defined as in Tables (1) and (2):

Minimize

$$\mathrm{TNCS}(\mathbf{p}_G, \mathbf{p}_L^S) \ + \ \pi \left[\sum_{km \in BR} [\delta_k - \delta_m]^2 \right] \tag{14}$$

with respect to real power generation levels, real power price-sensitive loads, and voltage angles

$$p_{Gi}, \ i = 1, ..., I; \ p_{Lj}^S, \ j = 1, ..., J; \ \delta_k, \ k = 1, ..., K$$

subject to:

Real power balance constraint for each node $\mathbf{k = 1, ..., K}$:

$$\sum_{i \in I_k} p_{Gi} \ - \ \sum_{j \in J_k} p_{Lj} \ - \ \sum_{km \,\mathrm{or}\, mk \ \in \ BR} P_{km} \ = \ 0 \tag{15}$$

where

$$p_{Lj} \ = \ p_{Lj}^F \ + \ p_{Lj}^S \tag{16}$$

$$P_{km} = B_{km} \, [\delta_k - \delta_m] \tag{17}$$

Alternatively,

$$\sum_{i \in I_k} p_{Gi} \; - \; \sum_{j \in J_k} p^S_{Lj} \; - \sum_{km \text{ or } mk \; \in \; BR} P_{km} \; = \; \sum_{j \in J_k} p^F_{Lj} \tag{18}$$

Real power thermal constraint for each branch km $\in$ BR:

$$|P_{km}| \quad \leq \quad P^U_{km} \tag{19}$$

Real power operating capacity constraints for each Generator $\mathbf{i = 1, .., I}$:

$$Cap^L_i \quad \leq \quad p_{Gi} \quad \leq \quad Cap^U_i \tag{20}$$

Real power price-sensitive load constraints for each LSE $\mathbf{j = 1, .., J}$:

$$SLoad^L_j \quad \leq \quad p^S_{Lj} \quad \leq \quad SLoad^U_j \tag{21}$$

Voltage angle setting at reference node 1:

$$\delta_1 \quad = \quad 0 \tag{22}$$

3 DC-OPF Problem in Matrix Form

3.1 General Matrix Formulation

Let δ_1 be set to zero everywhere in the DC-OPF problem presented in the previous section 2, in accordance with constraint (22). The general matrix depiction for the resulting reduced-form DC-OPF problem can then be expressed as follows:

Minimize

$$\mathbf{f(x) = \frac{1}{2} x^T G x + a^T x} \tag{23}$$

with respect to

$$\mathbf{x} = \begin{bmatrix} p_{G1} & \dots & p_{GI} & p^S_{L1} & \dots & p^S_{LJ} & \delta_2 & \dots & \delta_K \end{bmatrix}^{\mathbf{T}}_{(I+J+K-1)\times 1}$$

subject to

$$\mathbf{C^T_{eq} x = b_{eq}} \tag{24}$$

$$\mathbf{C^T_{iq} x \geq b_{iq}} \tag{25}$$

Given this general matrix formulation, the problem is now to find the specific matrix and vector representations $\mathbf{a}$ and $\mathbf{G}$ for the objective function, $\mathbf{C_{eq}}$ and $\mathbf{b_{eq}}$ for the equality constraints, and $\mathbf{C_{iq}}$ and $\mathbf{b_{iq}}$ for the inequality constraints.

3.2 Objective Function Representation

First, the vector $\mathbf{a^T}$ in the objective function is given by

$$\mathbf{a^T} = \left[\begin{array}{ccccccccc} a_1 & \cdots & a_I & -c_1 & \cdots & -c_J & 0 & \cdots & 0 \end{array} \right]_{1\times(I+J+K-1)}$$

Next, the positive definite matrix $\mathbf{G}$ in the objective function is given by

$$\mathbf{G} = \text{blockDiag}\left[\begin{array}{cc} \mathbf{U} & \mathbf{W_{rr}} \end{array} \right] = \left[\begin{array}{cc} \mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{W_{rr}} \end{array} \right]_{(I+J+K-1)\times(I+J+K-1)} \tag{26}$$

where

$$\mathbf{U} = \text{diag}\left[\begin{array}{cccccc} 2b_1 & \cdots & 2b_I & 2d_1 & \cdots & 2d_J \end{array} \right]_{(I+J)\times(I+J)} \tag{27}$$

$$\mathbf{W_{rr}} = 2\pi \left[\begin{array}{cccc} \sum_{k\neq 2} \mathbb{E}_{k2} & -\mathbb{E}_{23} & \cdots & -\mathbb{E}_{2K} \\ -\mathbb{E}_{32} & \sum_{k\neq 3} \mathbb{E}_{k3} & \cdots & -\mathbb{E}_{3K} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathbb{E}_{K2} & -\mathbb{E}_{K3} & \cdots & \sum_{k\neq K} \mathbb{E}_{kK} \end{array} \right]_{(K-1)\times(K-1)} \tag{28}$$

$$\mathbb{E} = \left[\begin{array}{ccccc} 0 & \mathbb{I}(1\leftrightarrow 2) & \mathbb{I}(1\leftrightarrow 3) & \cdots & \mathbb{I}(1\leftrightarrow K) \\ \mathbb{I}(2\leftrightarrow 1) & 0 & \mathbb{I}(2\leftrightarrow 3) & \cdots & \mathbb{I}(2\leftrightarrow K) \\ \mathbb{I}(3\leftrightarrow 1) & \mathbb{I}(3\leftrightarrow 2) & 0 & \cdots & \mathbb{I}(3\leftrightarrow K) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{I}(K\leftrightarrow 1) & \mathbb{I}(K\leftrightarrow 2) & \mathbb{I}(K\leftrightarrow 3) & \cdots & 0 \end{array} \right]_{K\times K} \tag{29}$$

$$\mathbb{I}(k\leftrightarrow m) = \begin{cases} 1 & \text{if either } km \text{ or } mk \in BR \\ 0 & \text{otherwise} \end{cases}$$

3.3 Equality Constraints Representation

Then, the equality constraint matrix $\mathbf{C^T_{eq}}$ takes the form:

$$\mathbf{C^T_{eq}} = \left[\begin{array}{ccc} \mathbf{II} & -\mathbf{JJ} & -\mathbf{B_r'^T} \end{array} \right]_{K\times(I+J+K-1)}$$

where

$$\mathbf{II} = \left[\begin{array}{cccc} \mathbb{I}(1\in I_1) & \mathbb{I}(2\in I_1) & \cdots & \mathbb{I}(I\in I_1) \\ \mathbb{I}(1\in I_2) & \mathbb{I}(2\in I_2) & \cdots & \mathbb{I}(I\in I_2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{I}(1\in I_K) & \mathbb{I}(2\in I_K) & \cdots & \mathbb{I}(I\in I_K) \end{array} \right]_{K\times I} \tag{30}$$

$$\mathbb{I}(i\in I_k) = \begin{cases} 1 & \text{if } i\in I_k \\ 0 & \text{if } i\notin I_k \end{cases}$$

$$\mathbf{JJ} = \begin{bmatrix} \mathbb{I}(1 \in J_1) & \mathbb{I}(2 \in J_1) & \cdots & \mathbb{I}(J \in J_1) \\ \mathbb{I}(1 \in J_2) & \mathbb{I}(2 \in J_2) & \cdots & \mathbb{I}(J \in J_2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{I}(1 \in J_K) & \mathbb{I}(2 \in J_K) & \cdots & \mathbb{I}(J \in J_K) \end{bmatrix}_{K \times J} \tag{31}$$

$$\mathbb{I}(j \in J_k) = \begin{cases} 1 & \text{if } j \in J_k \\ 0 & \text{if } j \notin J_k \end{cases}$$

$$\mathbf{B'_r} = \begin{bmatrix} -B_{21} & \sum_{k \neq 2} B_{k2} & -B_{23} & \cdots & -B_{2K} \\ -B_{31} & -B_{32} & \sum_{k \neq 3} B_{k3} & \cdots & -B_{3K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -B_{K1} & -B_{K2} & -B_{K3} & \cdots & \sum_{k \neq K} B_{kK} \end{bmatrix}_{(K-1) \times K} \tag{32}$$

$$B_{km} = \begin{cases} \frac{1}{x_{km}} > 0 & \text{if } km \text{ or } mk \in BR \\ 0 & \text{otherwise} \end{cases}$$

The associated equality constraint vector $\mathbf{b_{eq}}$ takes the form:

$$\mathbf{b_{eq}} = \begin{bmatrix} \sum_{j \in J_1} p^F_{Lj} & \sum_{j \in J_2} p^F_{Lj} & \cdots & \sum_{j \in J_K} p^F_{Lj} \end{bmatrix}^{\mathbf{T}}_{K \times 1}$$

3.4 Inequality Constraints Representation

Finally, the inequality constraint matrix $\mathbf{C_{iq}}$ takes the form as follows.

$$\mathbf{C^T_{iq}} = \begin{bmatrix} \mathbf{MatrixT} \\ \hline \mathbf{MatrixG} \\ \hline \mathbf{MatrixL} \end{bmatrix} = \begin{bmatrix} \mathbf{O_{NI}} & \mathbf{O_{NJ}} & \mathbf{Z}\mathbb{A}_\mathbf{r} \\ \mathbf{O_{NI}} & \mathbf{O_{NJ}} & -\mathbf{Z}\mathbb{A}_\mathbf{r} \\ \hline \mathbf{I_{II}} & \mathbf{O_{IJ}} & \mathbf{O_{IK}} \\ -\mathbf{I_{II}} & \mathbf{O_{IJ}} & \mathbf{O_{IK}} \\ \hline \mathbf{O_{JI}} & \mathbf{I_{JJ}} & \mathbf{O_{JK}} \\ \mathbf{O_{JI}} & -\mathbf{I_{JJ}} & \mathbf{O_{JK}} \end{bmatrix}_{(2N+2I+2J) \times (I+J+K-1)}$$

where $\mathbf{O_{NI}}$ is an $N \times I$ zero matrix, $\mathbf{O_{NJ}}$ is an $N \times J$ zero matrix, $\mathbf{O_{IJ}}$ is an $I \times J$ zero matrix, $\mathbf{O_{IK}}$ is an $I \times (K-1)$ zero matrix, $\mathbf{O_{JI}}$ is a $J \times I$ zero matrix, and $\mathbf{O_{JK}}$ is a $J \times (K-1)$ zero matrix; $\mathbf{I_{II}}$ is an $I \times I$ identity matrix and $\mathbf{I_{JJ}}$ is a $J \times J$ identity matrix; and matrices $\mathbf{Z}$ and $\mathbb{A}_\mathbf{r}$ are defined as follows.

Let **BI** denote the listing of the N physically distinct branches $km \in BR$ constituting the transmission grid, lexicographically sorted as in a dictionary from lower to higher numbered nodes. Let $\mathbf{BI}_n$ denote the nth branch listed in **BI**. Then the *adjacency matrix* $\mathbb{A}$ with entries of 1 for the "from" node and −1 for the "to" node can be expressed as follows:

$$\mathbb{A} = \begin{bmatrix} \mathbb{J}(1, \mathbf{BI}_1) & \mathbb{J}(2, \mathbf{BI}_1) & \cdots & \mathbb{J}(K, \mathbf{BI}_1) \\ \mathbb{J}(1, \mathbf{BI}_2) & \mathbb{J}(2, \mathbf{BI}_2) & \cdots & \mathbb{J}(K, \mathbf{BI}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{J}(1, \mathbf{BI}_N) & \mathbb{J}(2, \mathbf{BI}_N) & \cdots & \mathbb{J}(K, \mathbf{BI}_N) \end{bmatrix}_{N \times K} \tag{33}$$

where $\mathbb{J}(\cdot)$ is an indicator function defined as:

$$\mathbb{J}(k, \mathbf{BI}_n) = \begin{cases} +1 & \text{if } \mathbf{BI}_n \text{ takes the form } km \in BR \text{ for some node } m > k \\ -1 & \text{if } \mathbf{BI}_n \text{ takes the form } mk \in BR \text{ for some node } m < k \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for all nodes } k = 1, ..., K \text{ and for all branches } n = 1, ..., N$$

Let the *reduced adjacency matrix* $\mathbb{A}_\mathbf{r}$ be defined as $\mathbb{A}$ with its first column deleted. Thus, $\mathbb{A}_\mathbf{r}$ is expressed as

$$\mathbb{A}_\mathbf{r} = \begin{bmatrix} \mathbb{J}(2, \mathbf{BI}_1) & \cdots & \mathbb{J}(K, \mathbf{BI}_1) \\ \mathbb{J}(2, \mathbf{BI}_2) & \cdots & \mathbb{J}(K, \mathbf{BI}_2) \\ \vdots & \ddots & \vdots \\ \mathbb{J}(2, \mathbf{BI}_N) & \cdots & \mathbb{J}(K, \mathbf{BI}_N) \end{bmatrix}_{N \times (K-1)} \tag{34}$$

The matrix $\mathbf{Z}$ is defined as the diagonal matrix whose diagonal entries give the B_{km} values for all distinct connected branches $km \in BR$ ordered as in BI. That is,

$$\mathbf{Z} = \text{diag} \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_N \end{bmatrix}_{N \times N} \tag{35}$$

where $Z_n = B_{km}$ if BI_n (the nth element of BI) corresponds to branch $km \in BR$.

Let $\text{P}^U_{BI_n} = \text{P}^U_{km}$ if BI_n corresponds to branch $km \in \text{BR}$. The associated inequality constraint vector $\mathbf{b_{iq}}$ can then be expressed as follows:

$$\mathbf{b_{iq}} = \begin{bmatrix} -\mathbf{P^U} & -\mathbf{P^U} & \mathbf{Cap^L} & -\mathbf{Cap^U} & \mathbf{SLoad^L} & -\mathbf{SLoad^U} \end{bmatrix}^\mathbf{T}_{(2N+2I+2J)\times 1}$$

where

$$\mathbf{P^U} = \begin{bmatrix} P^U_{\mathbf{BI}_1} & P^U_{\mathbf{BI}_2} & \cdots & P^U_{\mathbf{BI}_N} \end{bmatrix}^\mathbf{T}_{N \times 1}$$

$$\mathbf{Cap^L} = \begin{bmatrix} Cap^L_1 & Cap^L_2 & \cdots & Cap^L_I \end{bmatrix}^\mathbf{T}_{I \times 1}$$

$$\mathbf{Cap^U} = \begin{bmatrix} Cap^U_1 & Cap^U_2 & \cdots & Cap^U_I \end{bmatrix}^\mathbf{T}_{I \times 1}$$

$$\mathbf{SLoad^L} = \begin{bmatrix} SLoad^L_1 & SLoad^L_2 & \cdots & SLoad^L_J \end{bmatrix}^\mathbf{T}_{J \times 1}$$

$$\mathbf{SLoad^U} = \begin{bmatrix} SLoad^U_1 & SLoad^U_2 & \cdots & SLoad^U_J \end{bmatrix}^\mathbf{T}_{J \times 1}$$

4 Illustrative 5-Node Example

Now consider a five-node case for which the transmission grid is not completely connected; see Figure 1. Let five Generators and three LSEs be distributed across the grid as follows: Generators 1 and 2 are located at node 1; LSE 1 is located at node 2; Generator 3 and LSE 2 are located at node 3; Generator 4 and LSE 3 are located at node 4; and Generator 5 is located node 5.

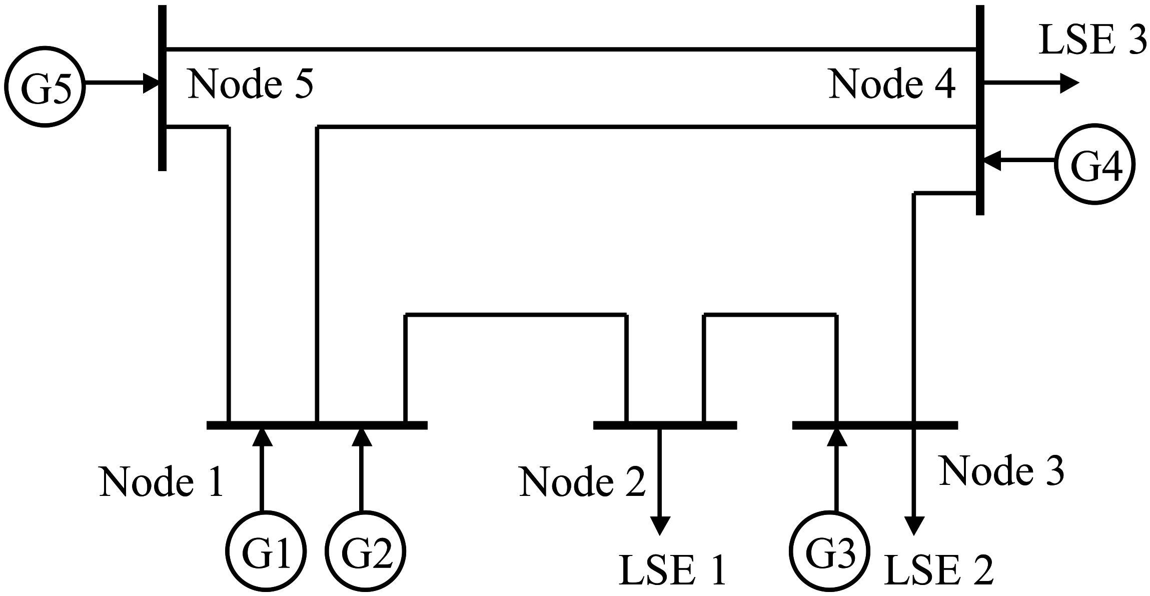

Figure 1: 5-Node Transmission Grid

4.1 5-Node Structural Form

This information implies the following structural configuration for the transmission grid:

$$I = 5;\ J = 3;\ K = 5;\ N = 6$$

$$I_1 = \{\text{G1}, \text{G2}\},\ I_2 = \{\emptyset\},\ I_3 = \{\text{G3}\},\ I_4 = \{\text{G4}\},\ I_5 = \{\text{G5}\};$$

$$J_1 = \{\emptyset\},\ J_2 = \{\text{LSE1}\},\ J_3 = \{\text{LSE2}\},\ J_4 = \{\text{LSE3}\},\ J_5 = \{\emptyset\};$$

$$BR = \{(1,2), (1,4), (1,5), (2,3), (3,4), (4,5)\}$$

The structural DC-OPF problem then takes the following form:

Minimize

$$\begin{aligned} &\textstyle\sum_{i=1}^{5}[a_i \cdot p_{Gi} + b_i \cdot p_{Gi}^2] \ - \textstyle\sum_{j=1}^{3}[c_j \cdot p_{Lj}^S - d_i \cdot {p_{Lj}^S}^2] \\ &+ \pi\Big[[\delta_1-\delta_2]^2 + [\delta_1-\delta_4]^2 + [\delta_1-\delta_5]^2 + [\delta_2-\delta_3]^2 + [\delta_3-\delta_4]^2 + [\delta_4-\delta_5]^2\Big] \end{aligned} \tag{36}$$

with respect to real power generation levels, real power price-sensitive loads, and voltage angles

$$p_{Gi},\ i = 1, ..., 5;\ p_{Lj}^S,\ j = 1, ..., 3;\ \delta_k,\ k = 1, ..., 5$$

subject to:

Real power balance constraint for each node $\mathbf{k = 1, ..., 5}$:

$$\sum_{i \in I_k} p_{Gi} \; - \; \sum_{j \in J_k} p^S_{Lj} \; - \; \sum_{km \text{ or } mk \; \in \; BR} P_{km} \; = \; \sum_{j \in J_k} p^F_{Lj} \tag{37}$$

where

$$P_{km} \;\; = \;\; B_{km} \, [\delta_k - \delta_m] \tag{38}$$

Real power thermal constraint for each branch $\mathbf{km \in BR}$:

$$|P_{km}| \;\; \leq \;\; P^U_{km} \tag{39}$$

Real power operating capacity constraints for each Generator $\mathbf{i = 1, .., 5}$:

$$Cap^L_i \;\; \leq \;\; p_{Gi} \;\; \leq \;\; Cap^U_i \tag{40}$$

Real power price-sensitive load constraints for each LSE $\mathbf{j = 1, .., 3}$:

$$SLoad^L_j \;\; \leq \;\; p^S_{Lj} \;\; \leq \;\; SLoad^U_j \tag{41}$$

Voltage angle setting at reference node 1:

$$\delta_1 \;\; = \;\; 0 \tag{42}$$

4.2 5-Node Objective Function Representation

First, the solution vector $\mathbf{x}$ takes the form

$$\mathbf{x} = \left[\; p_{G1} \;\; p_{G2} \;\; p_{G3} \;\; p_{G4} \;\; p_{G5} \;\; p^S_{L1} \;\; p^S_{L2} \;\; p^S_{L3} \;\; \delta_2 \;\; \delta_3 \;\; \delta_4 \;\; \delta_5 \; \right]^{\mathbf{T}}_{12\times 1}$$

The vector $\mathbf{a^T}$ in the objective function is given by

$$\mathbf{a^T} = \left[\; a_1 \;\; a_2 \;\; a_3 \;\; a_4 \;\; a_5 \;\; -c_1 \;\; -c_2 \;\; -c_3 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \; \right]_{1\times 12}$$

Next, the positive definite matrix $\mathbf{G}$ in the objective function is given by

$$\mathbf{G} = \text{blockDiag} \left[\; \mathbf{U} \;\;\; \mathbf{W_{rr}} \; \right] = \begin{bmatrix} \mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{W_{rr}} \end{bmatrix}_{12\times 12} \tag{43}$$

where

$$\mathbf{U} = \text{diag} \left[\; 2b_1 \;\; 2b_2 \;\; 2b_3 \;\; 2b_4 \;\; 2b_5 \;\; 2d_1 \;\; 2d_2 \;\; 2d_3 \; \right]_{8\times 8} \tag{44}$$

$$\mathbf{W_{rr}} = 2\pi \begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}_{4\times4} \tag{45}$$

4.3 5-Node Equality Constraints Representation

Then, the equality constraint matrix $\mathbf{C_{eq}^T}$ takes the form:

$$\mathbf{C_{eq}^T} = \begin{bmatrix} \mathbf{II} & -\mathbf{JJ} & -\mathbf{B_r'^T} \end{bmatrix}_{5\times(12)}$$

where

$$\mathbf{II} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times5} \tag{46}$$

$$\mathbf{JJ} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{5\times3} \tag{47}$$

$$\mathbf{B_r'} = \begin{bmatrix} -B_{21} & B_{21}+B_{23} & -B_{23} & 0 & 0 \\ 0 & -B_{32} & B_{32}+B_{34} & -B_{34} & 0 \\ -B_{41} & 0 & -B_{43} & B_{41}+B_{43}+B_{45} & -B_{45} \\ -B_{51} & 0 & 0 & -B_{54} & B_{51}+B_{54} \end{bmatrix}_{4\times5} \tag{48}$$

The associated equality constraint vector $\mathbf{b_{eq}}$ takes the form:

$$\mathbf{b_{eq}} = \begin{bmatrix} 0 & p_{L1}^F & p_{L2}^F & p_{L3}^F & 0 \end{bmatrix}_{5\times1}^{\mathbf{T}}$$

4.4 5-Node Inequality Constraints Representation

Finally, the inequality constraint matrix $\mathbf{C_{iq}}$ takes the form as follows.

$$\mathbf{C_{iq}^T} = \begin{bmatrix} \mathbf{MatrixT} \\ \hline \mathbf{MatrixG} \\ \hline \mathbf{MatrixL} \end{bmatrix} = \begin{bmatrix} \mathbf{O_{NI}} & \mathbf{O_{NJ}} & \mathbf{Z\mathbb{A}_r} \\ \mathbf{O_{NI}} & \mathbf{O_{NJ}} & -\mathbf{Z\mathbb{A}_r} \\ \hline \mathbf{I_{II}} & \mathbf{O_{IJ}} & \mathbf{O_{IK}} \\ -\mathbf{I_{II}} & \mathbf{O_{IJ}} & \mathbf{O_{IK}} \\ \hline \mathbf{O_{JI}} & \mathbf{I_{JJ}} & \mathbf{O_{JK}} \\ \mathbf{O_{JI}} & -\mathbf{I_{JJ}} & \mathbf{O_{JK}} \end{bmatrix}_{28\times12}$$

where

$$\mathbf{BI} = [(1,2),(1,4),(1,5),(2,3),(3,4),(4,5)]^T_{6\times 1} \tag{49}$$

$$\mathbf{Z} = \text{diag}\left[\begin{array}{cccccc} B_{12} & B_{14} & B_{15} & B_{23} & B_{34} & B_{45} \end{array}\right]_{6\times 6} \tag{50}$$

$$\mathbb{A}_{\mathbf{r}} = \left[\begin{array}{rrrr} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]_{6\times 4} \tag{51}$$

Hence the complete matrix $\mathbf{C^T_{iq}}$ can be found as

$$\mathbf{C^T_{iq}} = \left[\begin{array}{ccccc|ccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_{14} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_{15} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{23} & -B_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{34} & -B_{34} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{45} & -B_{45} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{14} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{15} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_{23} & B_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_{34} & B_{34} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_{45} & B_{45} \\
\hline\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{array}\right]_{28\times 12}$$

The associated inequality constraint vector $\mathbf{b_{iq}}$ can be expressed as follows:

$$\mathbf{b_{iq}} = \begin{bmatrix} -\mathbf{P^U} & -\mathbf{P^U} & \mathbf{Cap^L} & -\mathbf{Cap^U} & \mathbf{SLoad^L} & -\mathbf{SLoad^U} \end{bmatrix}^{\mathbf{T}}_{28\times1}$$

where

$$\mathbf{P^U} = \begin{bmatrix} P^U_{12} & P^U_{14} & P^U_{15} & P^U_{23} & P^U_{34} & P^U_{45} \end{bmatrix}^{\mathbf{T}}_{6\times1}$$

$$\mathbf{Cap^L} = \begin{bmatrix} Cap^L_1 & Cap^L_2 & Cap^L_3 & Cap^L_4 & Cap^L_5 \end{bmatrix}^{\mathbf{T}}_{5\times1}$$

$$\mathbf{Cap^U} = \begin{bmatrix} Cap^U_1 & Cap^U_2 & Cap^U_3 & Cap^U_4 & Cap^U_5 \end{bmatrix}^{\mathbf{T}}_{5\times1}$$

$$\mathbf{SLoad^L} = \begin{bmatrix} SLoad^L_1 & SLoad^L_2 & SLoad^L_3 \end{bmatrix}^{\mathbf{T}}_{3\times1}$$

$$\mathbf{SLoad^U} = \begin{bmatrix} SLoad^U_1 & SLoad^U_2 & SLoad^U_3 \end{bmatrix}^{\mathbf{T}}_{3\times1}$$

Table 1: DC OPF Admissible Exogenous Variables

Variable	Description	Admissibility Restrictions
K	Total number of transmission grid nodes	$K > 0$
N	Total number of physically distinct network branches	$N > 0$
I	Total number of Generators	$I > 0$
J	Total number of LSEs	$J > 0$
I_k	Set of Generators located at node k	$\text{Card}(\cup_{k=1}^{K} I_k) = I$
J_k	Set of LSEs located at node k	$\text{Card}(\cup_{k=1}^{K} J_k) = J$
S_o	Base apparent power (in three-phase MVAs)	$S_o \geq 1$
V_o	Base voltage (in line-to-line kVs)	$V_o > 0$
V_k	Voltage magnitude (in kVs) at node k	$V_k = V_o,\ k = 1, \ldots, K$
km	Branch connecting nodes k and m (if one exists)	$k \neq m$
BR	Set of all physically distinct branches km, $k < m$	$BR \neq \emptyset$
x_{km}	Reactance (ohms) for branch km	$x_{km} = x_{mk} > 0,\ km \in BR$
B_{km}	$[1/x_{km}]$ for branch km	$B_{km} = B_{mk} > 0,\ km \in BR$
P_{km}^{U}	Thermal limit (MWs) for real power flow on km	$P_{km}^{U} > 0,\ km \in BR$
δ_1	Reference node 1 voltage angle (in radians)	$\delta_1 = 0$
a_i, b_i	Cost coefficients for Generator i	$b_i > 0,\ i = 1, \ldots, I$
Cap_i^L	Lower real power operating capacity for Generator i	$Cap_i^L \geq 0,\ i = 1, \ldots, I$
Cap_i^U	Upper real power operating capacity for Generator i	$Cap_i^U > 0,\ i = 1, \ldots, I$
FCost_i	Fixed costs (hourly prorated) for Generator i	$\text{FCost}_i \geq 0,\ i = 1, \ldots I$
c_j, d_j	Demand coefficients for LSE j	$c_j, d_j > 0,\ j = 1, \ldots, J$
$SLoad_j^L$	Lower real power price-sensitive load limit for LSE j	$SLoad_j^L \geq 0,\ j = 1, \ldots, J$
$SLoad_j^U$	Upper real power price-sensitive load limit for LSE j	$SLoad_j^U \leq c_j/[2d_j], j = 1, \ldots, J$
p_{Lj}^F	Price-insensitive fixed real power load for LSE j	$p_{Lj}^F \geq 0,\ \ j = 1, \ldots, J$

Table 2: DC OPF Endogenous Variables

Variable	Description
p_{Gi}	Real power generation (MWs) supplied by Generator $i = 1, \ldots, I$
p_{Lj}^S	Price-sensitive real power load (MWs) demanded by LSE $j = 1, \ldots, J$
δ_k	Voltage angle (in radians) at node $k = 2, \ldots, K$
P_{km}	Real power (MWs) flowing in branch $km \ \in$ BR