

# DC-Optimal Power Flow and LMP Determination in the AMES Test Bed

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# Presentation Outline

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- Introduction
- Double auction basics for energy markets
  - Supply, demand, & market equilibrium
  - Net surplus extraction
- Market efficiency vs. social welfare: Implications for independent system operators in energy markets
- Illustrative AMES Test Bed experiments for a 5-bus test case with learning generators

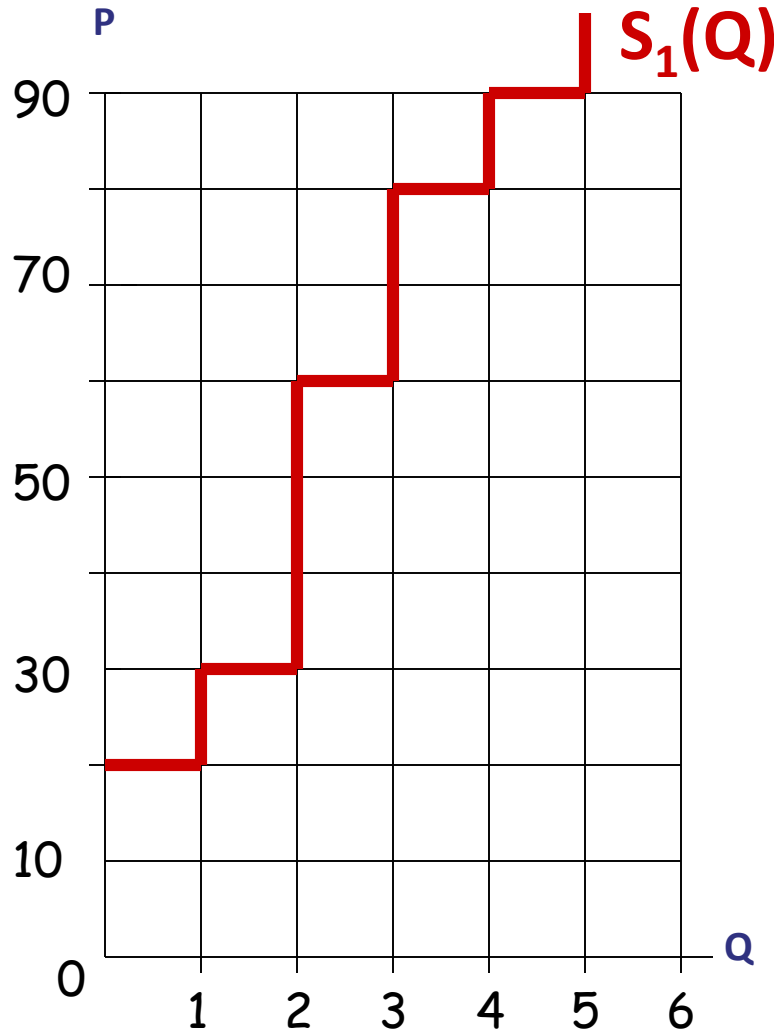
# Introduction

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- ◆ In many regions of U.S., wholesale electric energy -- measured in megawatt-hours (MWh) -- is transacted in “day-ahead” markets designed as double auctions.
- ◆ **Double Auction** = A centrally-cleared market in which sellers make supply offers & buyers make demand bids.
- ◆ After review of basic double auction concepts, efficiency & welfare issues arising from use of double auctions for centrally-managed day-ahead markets for energy will be discussed.

## DOUBLE-AUCTION BASICS: EXAMPLE

Seller 1's Supply Offer:  $P = S_1(Q)$ , where  $P = \text{Price}$  and  $Q = \text{Quantity}$



$Q = \text{Quantity}$  of specialty apples (in bushels)  
 $P = \text{Price}$  of specialty apples (\$ per bushel)

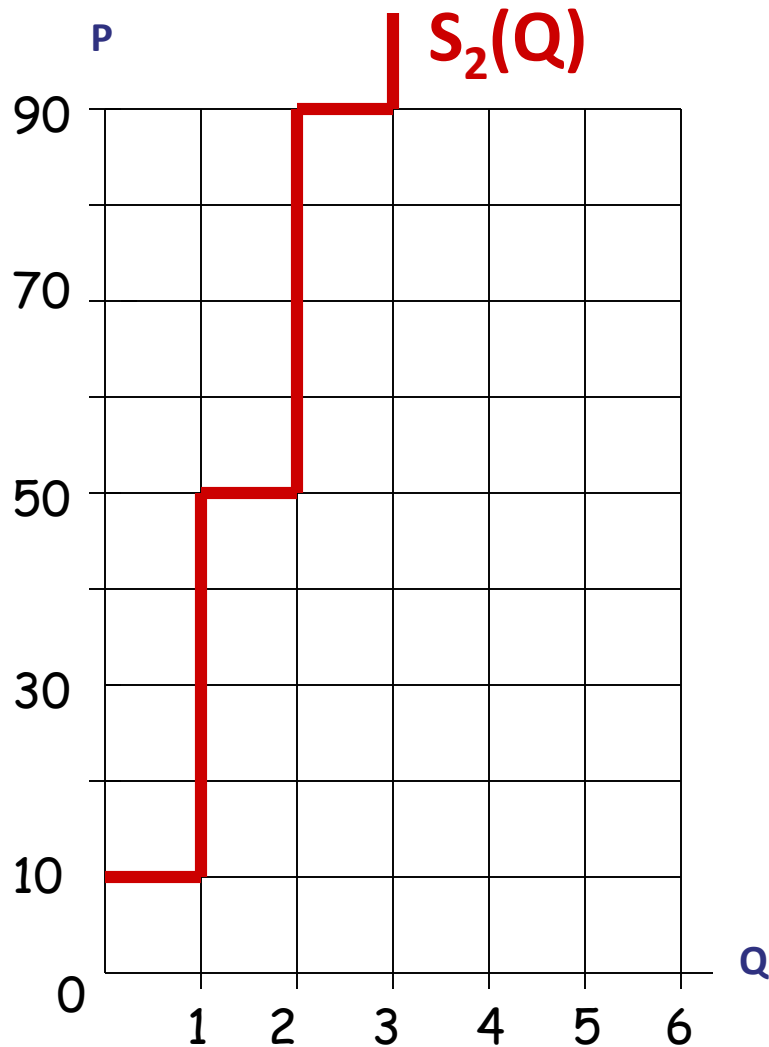
For each  $Q$ :  $P=S_1(Q)$  is Seller 1's **minimum acceptable sale price** for the "last" bushel it supplies at  $Q$ .

Bushels $Q$	Price $P = S_1(Q)$
1	\$20
2	\$30
3	\$60
4	\$80
5	\$90
6	$\infty$

5 bushels = Seller  $S_1$ 's max possible supply.

**Note:** "Minimum acceptable sale price" is also called a "(sale) reservation value"

## Seller 2's Supply Offer: $P = S_2(Q)$ , where $P = \text{Price}$ and $Q = \text{Quantity}$



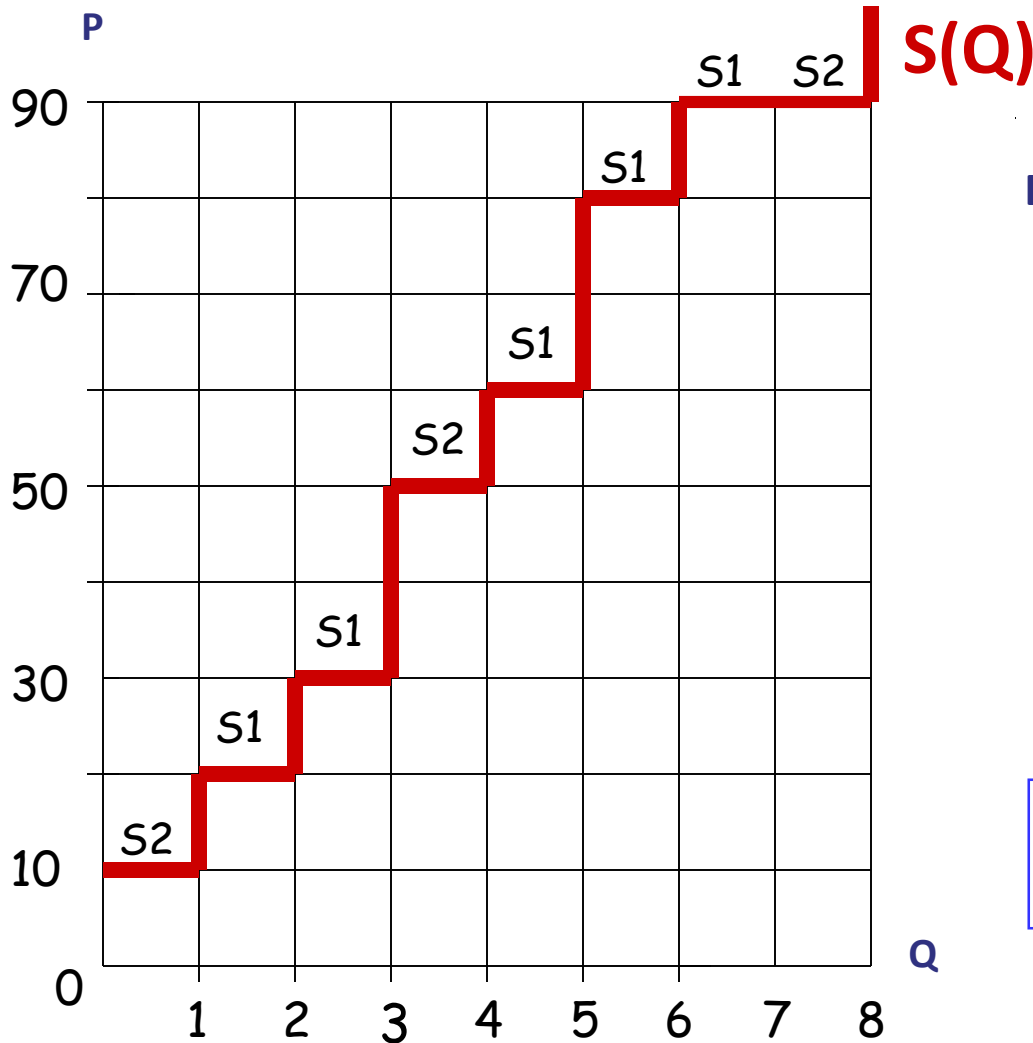
For each  $Q$ :  $P = S_2(Q)$  is Seller 2's *minimum acceptable sale price* for the last bushel it supplies at  $Q$ .

**Bushels  $Q$**     **Price  $P = S_2(Q)$**

1	\$10
2	\$50
3	\$90
4	$\infty$

3 bushels = Seller  $S_2$ 's  
max possible supply.

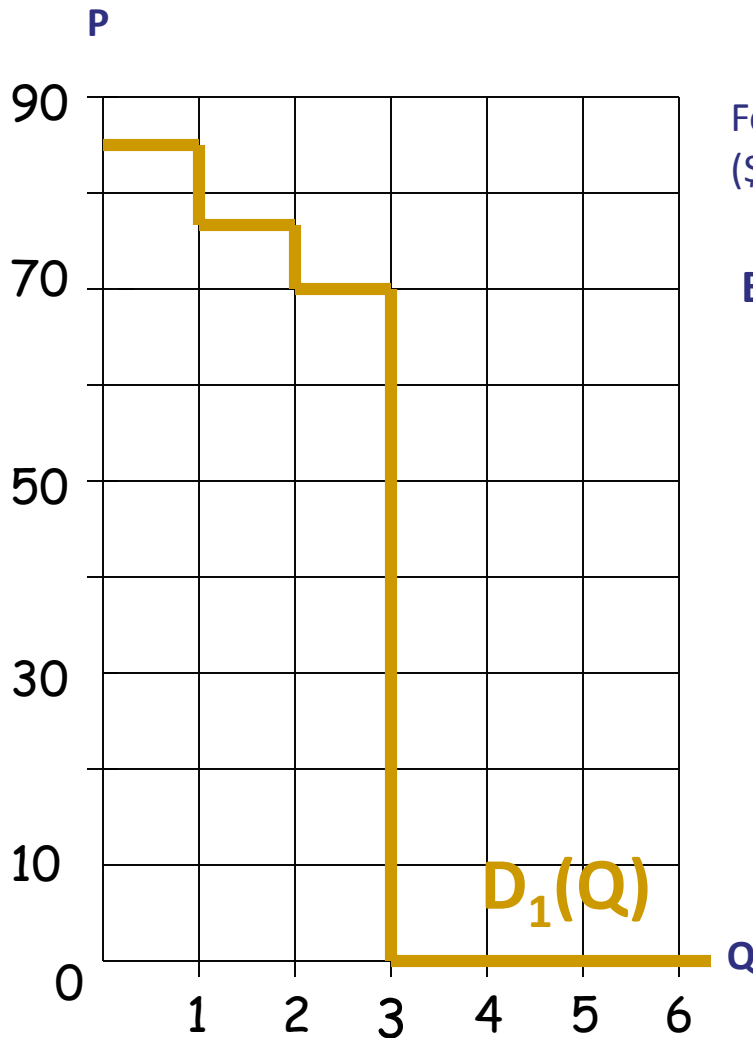
# Total System (Inverse) Supply Function: $P = S(Q)$



Bushels $Q$	Price $P = S(Q)$
1	\$10 (S2)
2	\$20 (S1)
3	\$30 (S1)
4	\$50 (S2)
5	\$60 (S1)
6	\$80 (S1)
7	\$90 (S1/S2)
8	\$90 (S2/S1)
9	$\infty$

Max possible total market supply  
= 8 bushels of apples.

# Buyer 1's Demand Bid: $P = D_1(Q)$ , where $P = \text{Price}$ and $Q = \text{Quantity}$



For each  $Q$ :  $P = D_1(Q)$  is Buyer 1's **max purchase price** (\$/bushel) for the last bushel it purchases at  $Q$ .

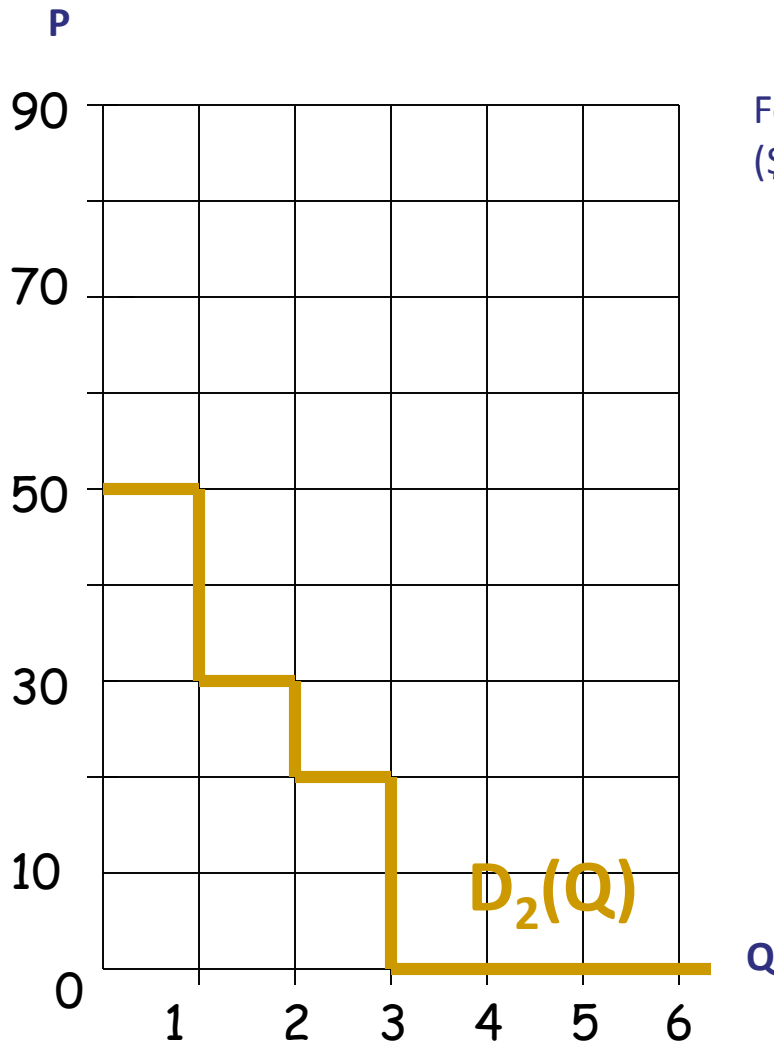
**Bushels  $Q$**       **Price  $P = D_1(Q)$**

1	\$84
2	\$76
3	\$70
4	\$ 0

Buyer 1's demand for apples is "satiated" at 3 bushels.

**Note:** "Maximum purchase price"  $\equiv$  "maximum willingness to pay" is also called a "(purchase) reservation value."

**Buyer 2's Demand Bid:  $P = D_2(Q)$ , where  $P = \text{Price}$  and  $Q = \text{Quantity}$**



For each Q:  $P = D_2(Q)$  is Buyer 2's **max purchase price** (\$/bushel) for the last bushel it purchases at Q.

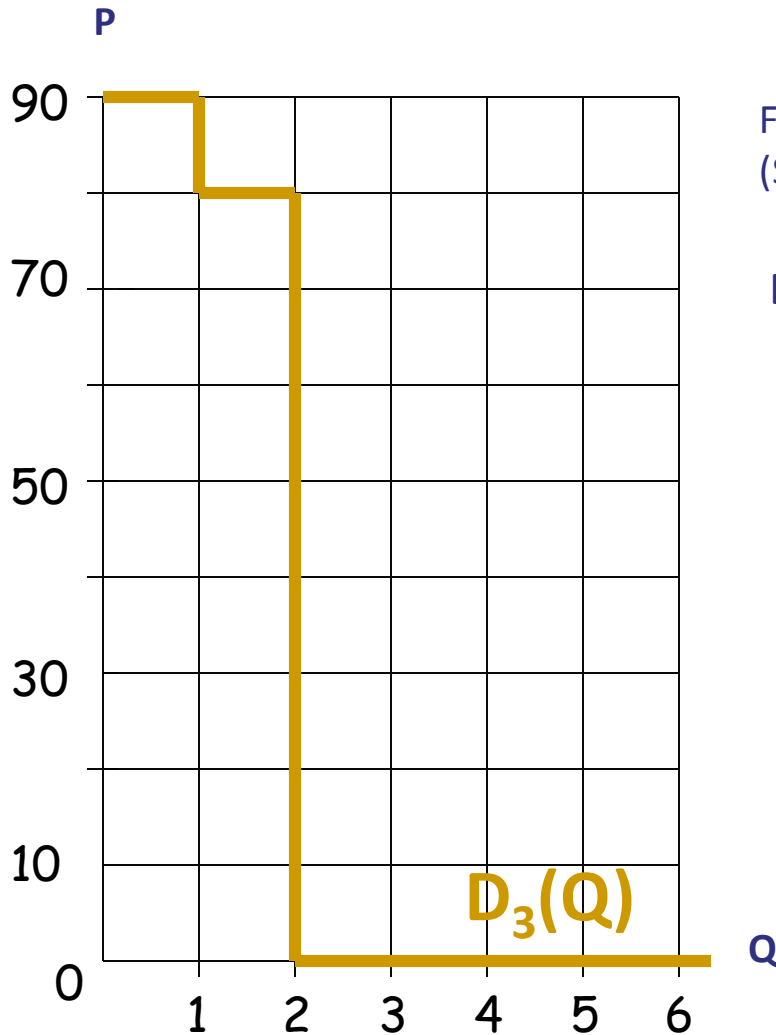
**Bushels Q    Price P =  $D_2(Q)$**

1	\$50
2	\$30
3	\$20
4	\$ 0

Buyer 2's demand for apples is "satiated" at 3 bushels.



## Buyer 3's Demand Bid: $P = D_3(Q)$ , where $P$ =Price and $Q$ = Quantity



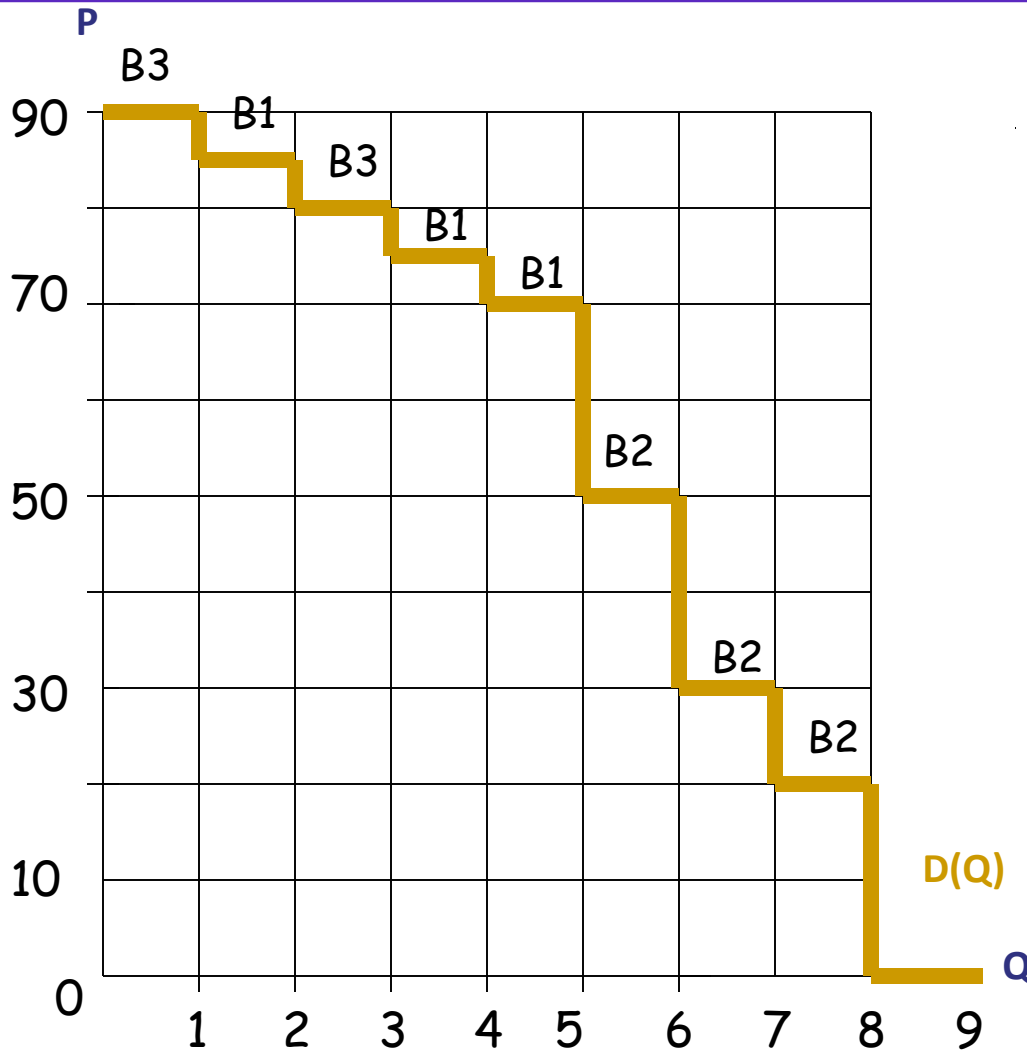
For each Q:  $P = D_3(Q)$  is Buyer 3's *max purchase price* (\$/bushel) for the last bushel it purchases at Q

**Bushels Q**      **Price  $P = D_3(Q)$**

1	\$90
2	\$80
3	\$ 0

Buyer 3's demand for apples is "satiated" at 2 bushels.

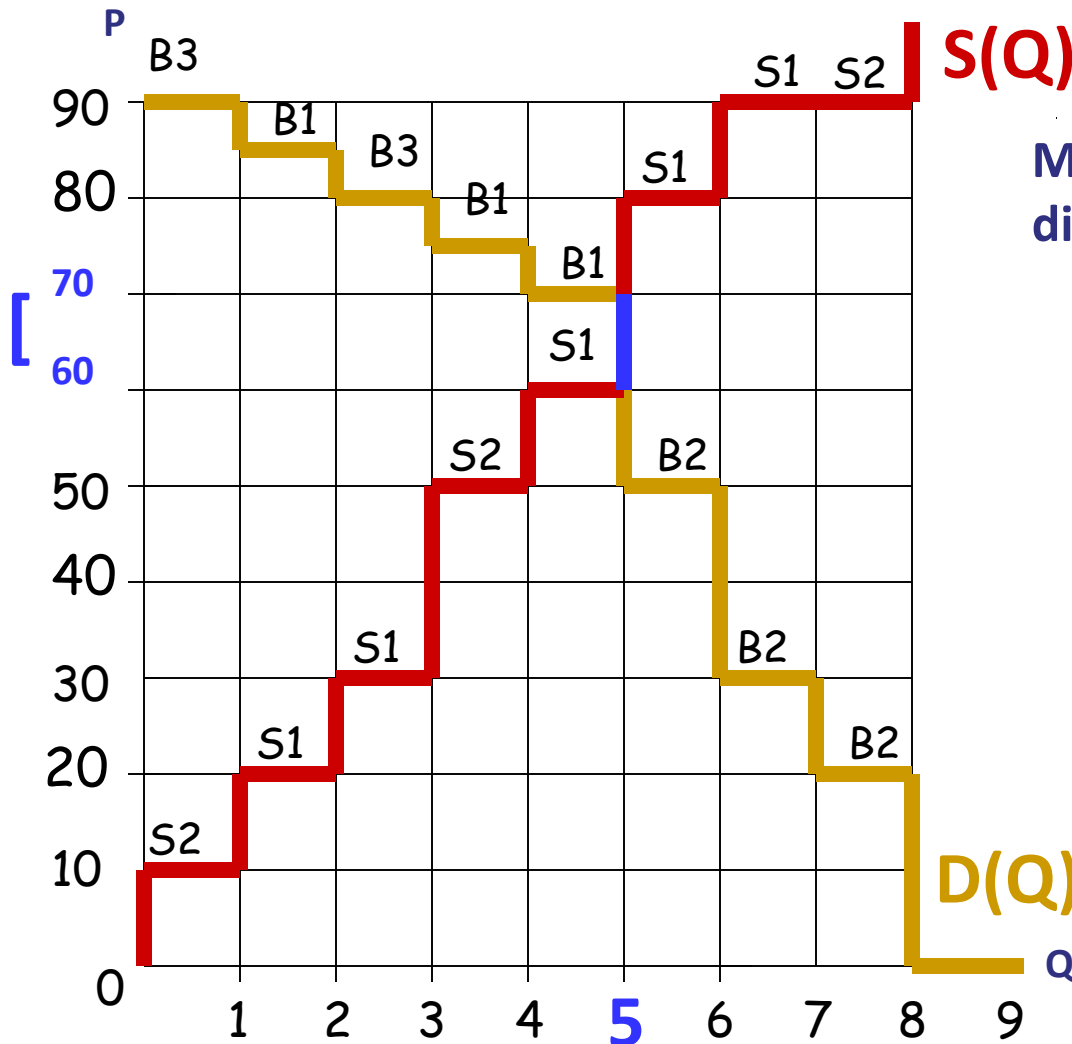
# Total System (Inverse) Demand Function: $P = D(Q)$



Bushels $Q$	Price $P = D(Q)$
1	\$90 (B3)
2	\$84 (B1)
3	\$80 (B3)
4	\$76 (B1)
5	\$70 (B1)
6	\$50 (B2)
7	\$30 (B2)
8	\$20 (B2)
9	\$ 0

# Competitive Market Clearing (CMC) Points

Points  $(Q,P)$  where the aggregate supply curve  $P = S(Q)$  intersects the aggregate demand curve  $P = D(Q)$ :  $P = S(Q) = D(Q)$



Multiple CMC points  $(Q^*, P^*)$  with different CMC prices  $P^*$ :

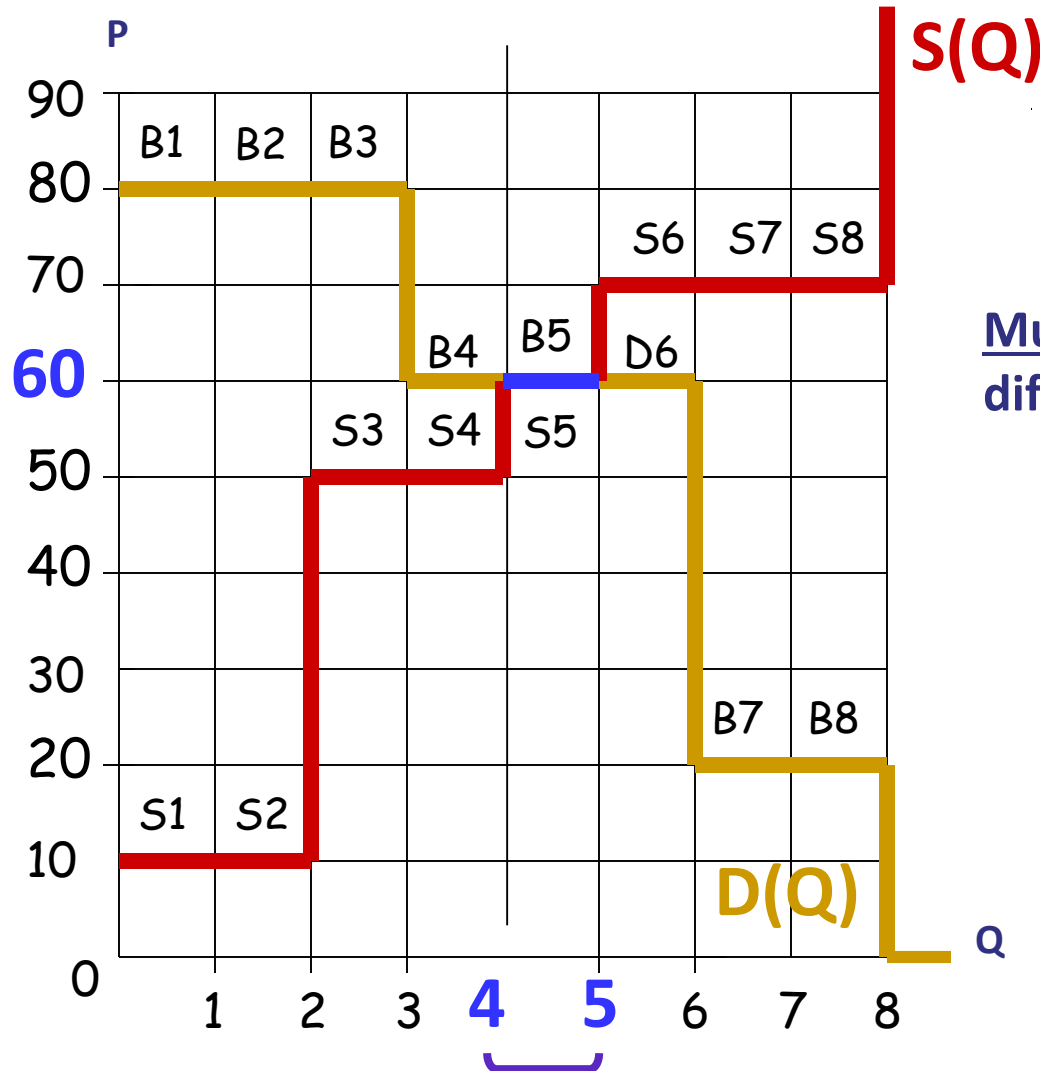
$$Q^*=5, \$60 \leq P^* \leq \$70$$

Bushels Q    Max Buy P    Min Sell P

1	\$90	\$10
2	\$84	\$20
3	\$80	\$30
4	\$76	\$50
5	\$70	\$60
6	\$50	\$80
7	\$30	\$90
8	\$20	\$90
9	0	$\infty$

No bushel sales are possible beyond five bushels !

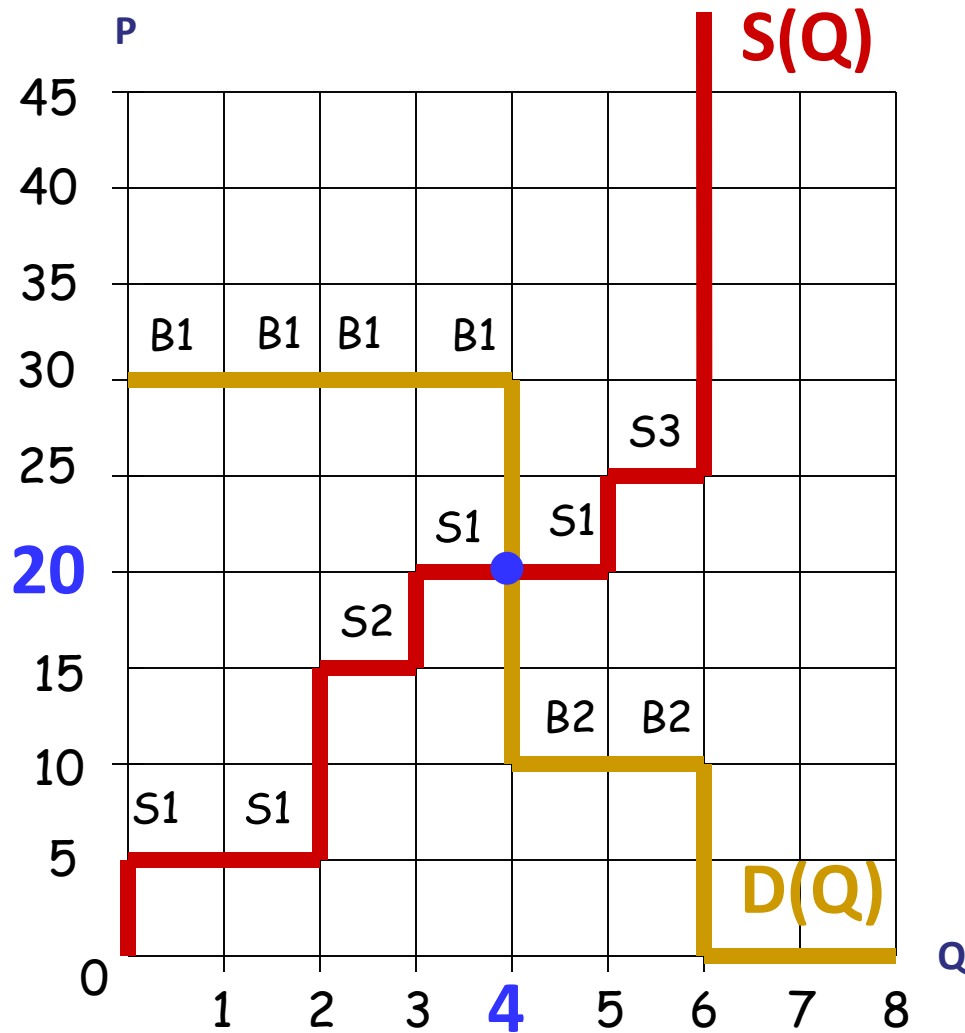
# Can also possibly have multiple CMC points with a range of CMC quantities



Multiple CMC points  $(Q^*, P^*)$  with  
different CMC quantities  $Q^*$ :

$$4 \leq Q^* \leq 5, P^* = \$60$$

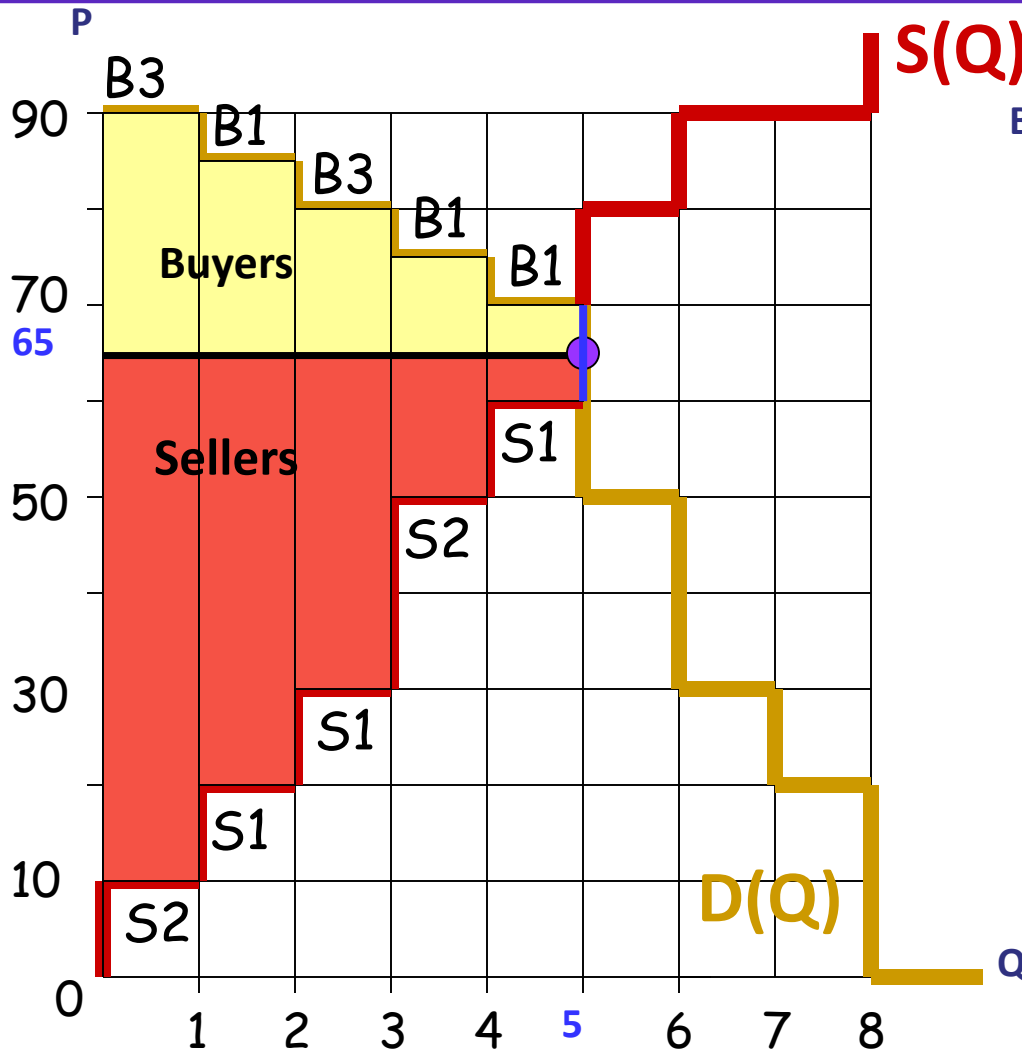
# Can also possibly have a unique CMC point



Unique CMC Point:

$$Q^*=4, P^*=\$20$$

# Seller & Buyer Net Surplus Amounts at CMC Points



Ex 1: CMC Point  $Q^*=5$ ,  $P^*=\$65$

Bushels Q	MaxBPrice	$P^*=65$	BuyNetSur
1	\$90	- \$65	= \$25
2	\$84	- \$65	= \$19
3	\$80	- \$65	= \$15
4	\$76	- \$65	= \$11
5	\$70	- \$65	= \$5

**BUYER NET SURPLUS: \$75**

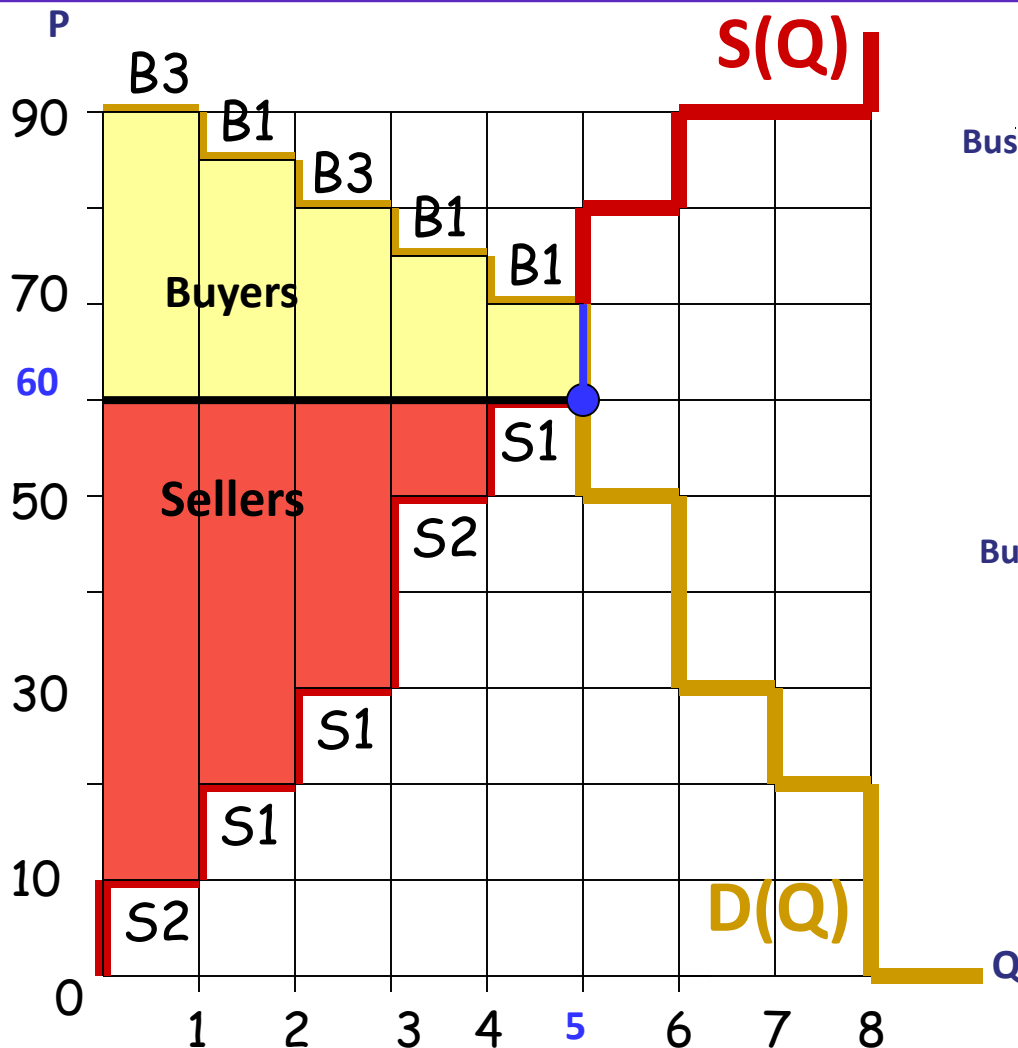
Bushels Q	$P^*=65$	MinSPrice	SellNetSur
1	\$65	- \$10	= \$55
2	\$65	- \$20	= \$45
3	\$65	- \$30	= \$35
4	\$65	- \$50	= \$15
5	\$65	- \$60	= \$5

**SELLER NET SURPLUS: \$155**

**Total Net Surplus: \$230**

# A *different* selected CMC point

→ *different* seller & buyer net surplus amounts



Ex 2: CMC Point  $Q^*=5$ ,  $P^*=\$60$

Bushels Q	MaxBuyPrice	$P^*=60$	BuyNetSurplus
1	\$90	- \$60	= \$30
2	\$84	- \$60	= \$24
3	\$80	- \$60	= \$20
4	\$76	- \$60	= \$16
5	\$70	- \$60	= \$10

**BUYER NET SURPLUS: \$100**

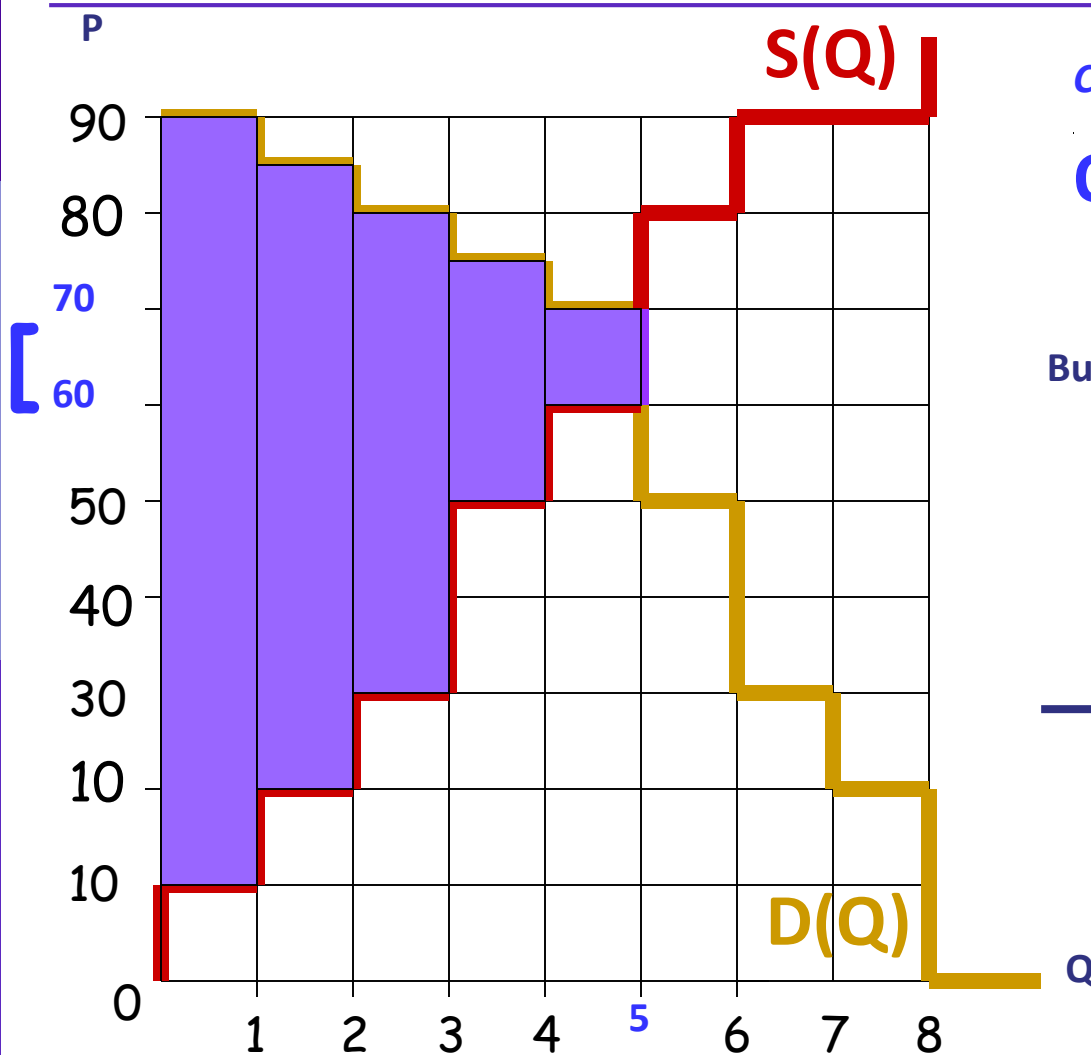
Bushels Q	$P^*=65$	MinSellPrice	SellNetSurplus
1	\$60	- \$10	= \$50
2	\$60	- \$20	= \$40
3	\$60	- \$30	= \$30
4	\$60	- \$50	= \$10
5	\$60	- \$60	= \$0

**SELLER NET SURPLUS: \$130**

**Total Net Surplus: \$230**

# Total Net Surplus at a CMC Point

( If multiple CMC points exist, TNS = same for each point. )



CMC Points:

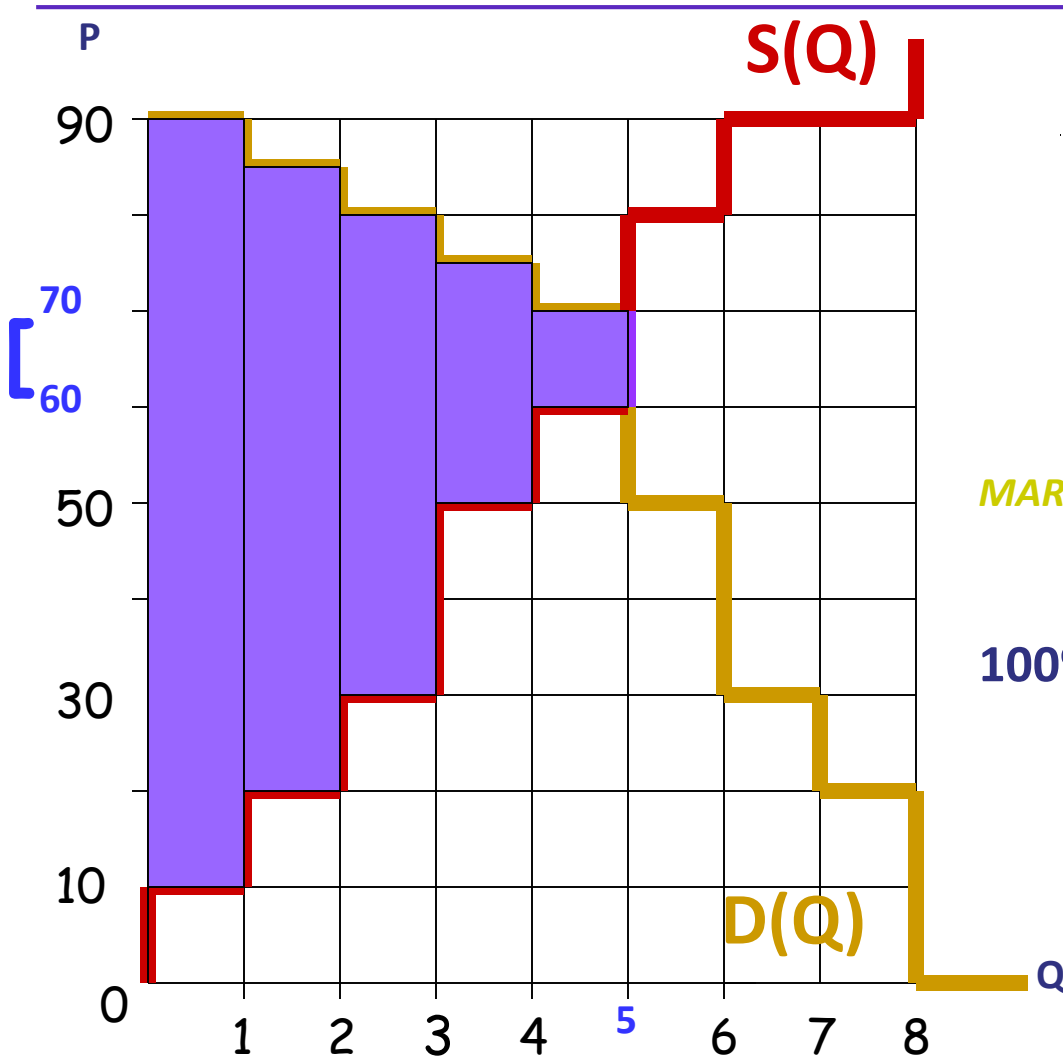
$$Q^*=5, \$60 \leq P^* \leq \$70$$

Bushels Q	MaxBuyP	MinSellP	Net Surplus
1	\$90	- \$10	= \$80
2	\$84	- \$20	= \$64
3	\$80	- \$30	= \$50
4	\$76	- \$50	= \$26
5	\$70	- \$60	= \$10

**TOTAL NET SURPLUS: \$230**



# Standard Measure of Market Efficiency (Non-Wastage of Resources)



*CMC Points:*

$$Q^*=5, \$60 \leq P^* \leq \$70$$

CMC Total Net Surplus

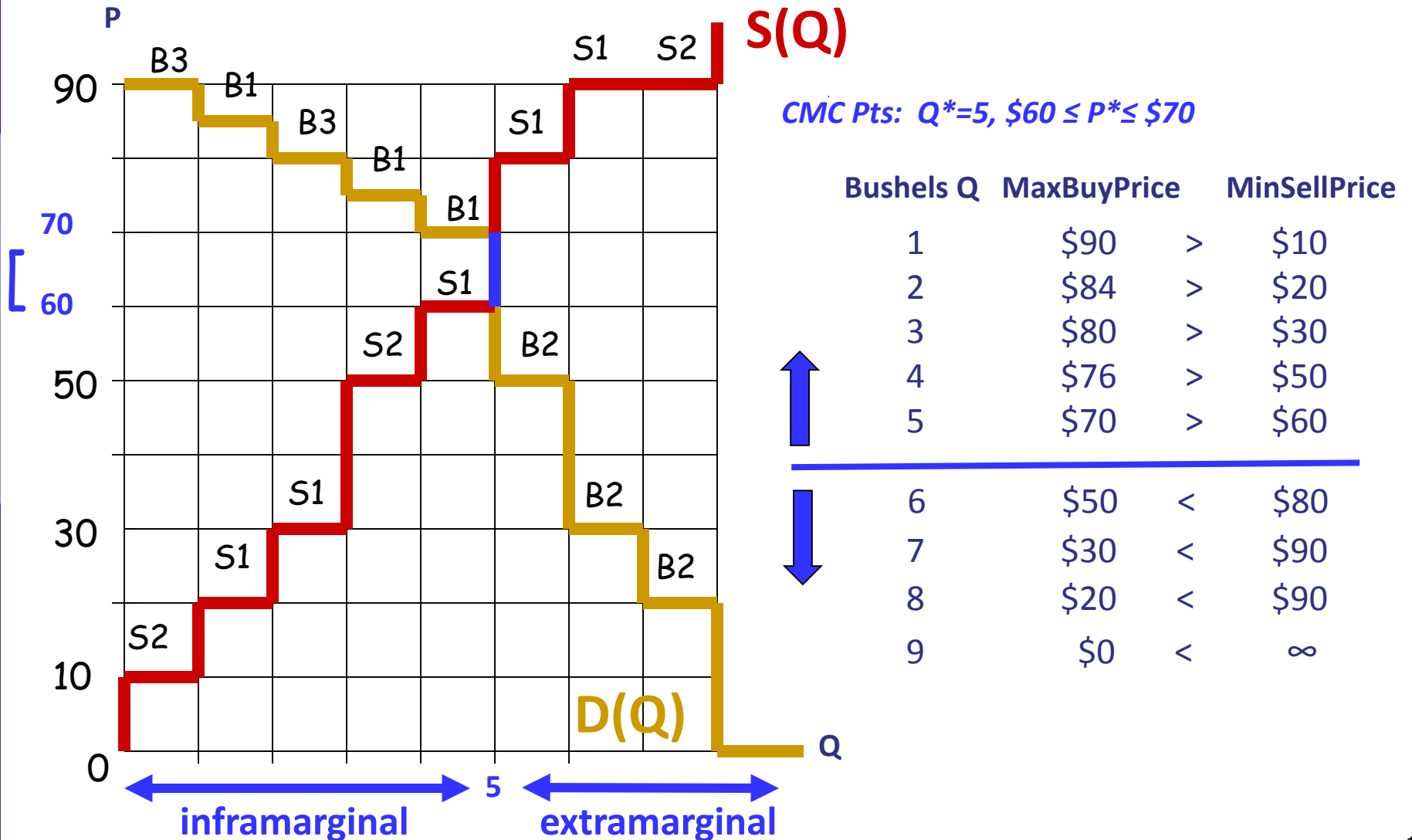
$$= \$230 \text{ (Maximum Possible)}$$

*MARKET EFFICIENCY (ME):*

$$100\% \times \frac{\text{Extracted Total Net Surplus}}{\text{Max Possible Total Net Surplus}}$$

**How can ME be less than 100% ?**

# Inframarginal vs. Extramarginal Quantity Units at CMC Points



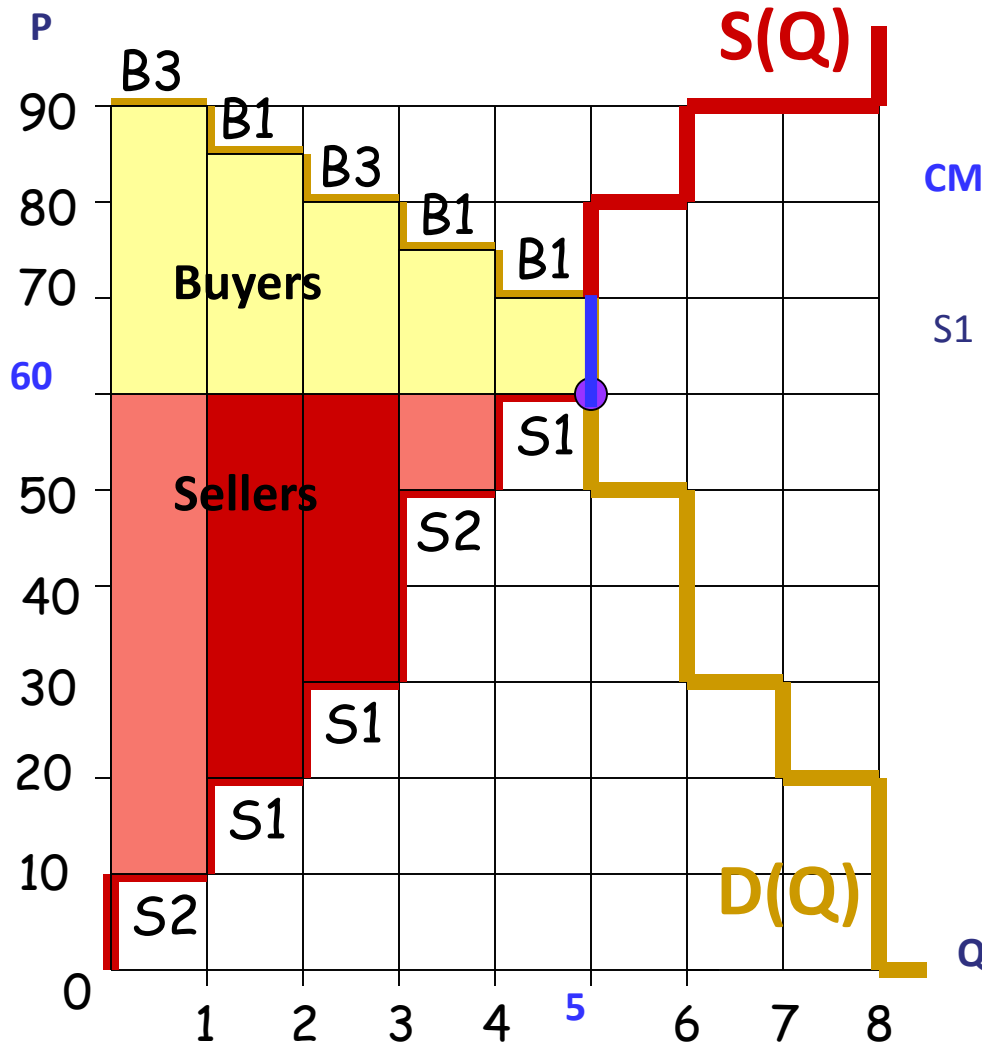
## Market Efficiency < 100% can arise if ...

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- ◆ some **inframarginal** quantity unit **fails to trade**
  - E.g., physical capacity withholding (“**market power**”<sup>\*</sup>)
- ◆ some **extramarginal** quantity unit **is traded**
  - a more costly unit is sold in place of a less costly unit (“out-of-merit-order dispatch”)
  - and/or a less valued unit is purchased in place of a more valued unit (“out-of-merit-order purchase”)

<sup>\*</sup> **Market Power:** Ability of a seller or buyer to extract more net surplus from a market than they would achieve at a CMC point.

# Example: Exercise of market power by Seller S1 that results in ME < 100%



CMC Point:  $Q^*=5, P^*=\$60$

S1 Net Surplus at CMC Point:

$$\$60 - \$20 = \$40$$

$$\$60 - \$30 = \$30$$

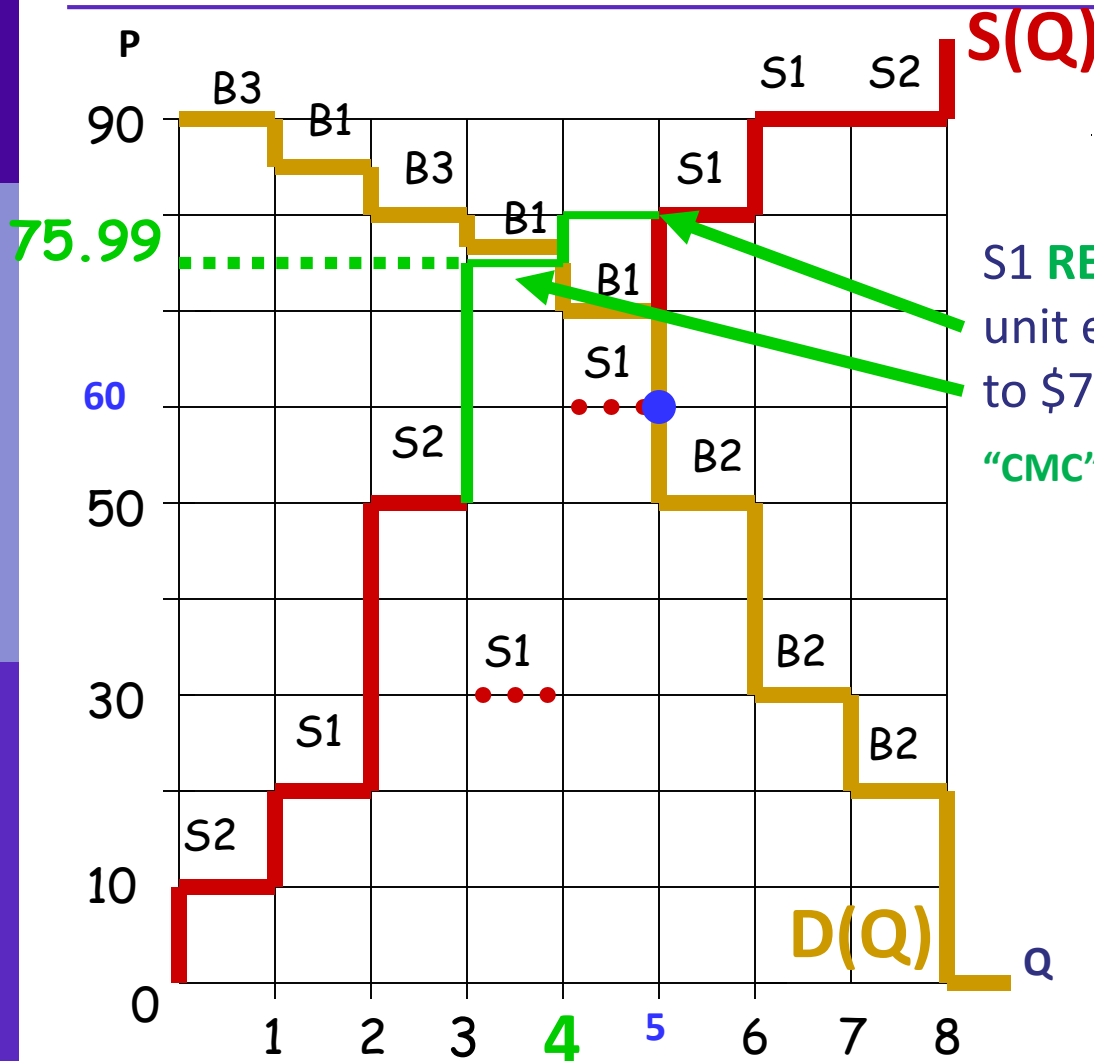
$$\$60 - \$60 = \$0$$

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**S1 Net Surplus = \$70**

**Total Net Surplus: \$230**

# Example: $ME < 100\%$ ... Continued



CMC Point:  $Q^*=5, P^*=\$60$

S1's CMC Net Surplus = \$70

S1 **REPORTS** a max sale price on his 3rd unit equal to \$80 & on his 2nd unit equal to \$75.99.

"CMC" Point:  $Q'=4, P' \cong \$76$

At new "CMC" point, S1 only sells its first 2 units, but **S1's net surplus increases to  $\cong \$102 = [\$56 + \$46]$**

Extracted total net surplus **DECREASES FROM 230 TO 220** because inframarginal 5<sup>th</sup> unit now fails to sell.

# Market Efficiency vs. Social Welfare

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- ◆ **Efficiency** for one market at one time point is a very narrow measure of resource non-wastage.
- ◆ Ideally, **social** efficiency should be measured by resource non-wastage across **all** markets and across **all** current and future time periods.
- ◆ Moreover, economists measure **social welfare** in terms of the **“utility” (well-being) of people** in their roles as consumers/users of final goods and services.
- ◆ **Social efficiency** is **necessary but not sufficient** for the optimization of **social welfare**.

# Market Efficiency, Social Welfare, and the Extraction of Net Surplus by “Third Parties”

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- ◆ Suppose [price  $P_S$  paid to a seller] < [price  $P_B$  charged to a buyer] for some quantity unit sold in a market

➔ **Net surplus  $[P_B - P_S]$  is extracted by some type of “third party”**

*Examples:* Gov't tax revenues; ISO net surplus extractions that result from grid congestion in **Day-Ahead Markets (DAMs)** for grid-delivered energy (MWh) settled by means of **Locational Marginal Prices LMP(b,H)** (\$/MWh) conditional on grid delivery location  $b$  and operating hour  $H$ .

- ◆ “First order effect” of this **third-party extracted net surplus** is a decrease in the net surplus going to sellers & buyers.
- ◆ **Social efficiency/welfare implications** of this third-part extracted net surplus depend on precisely how it is extracted and to what uses it is subsequently put.

# AMES DC-OPF Formulation

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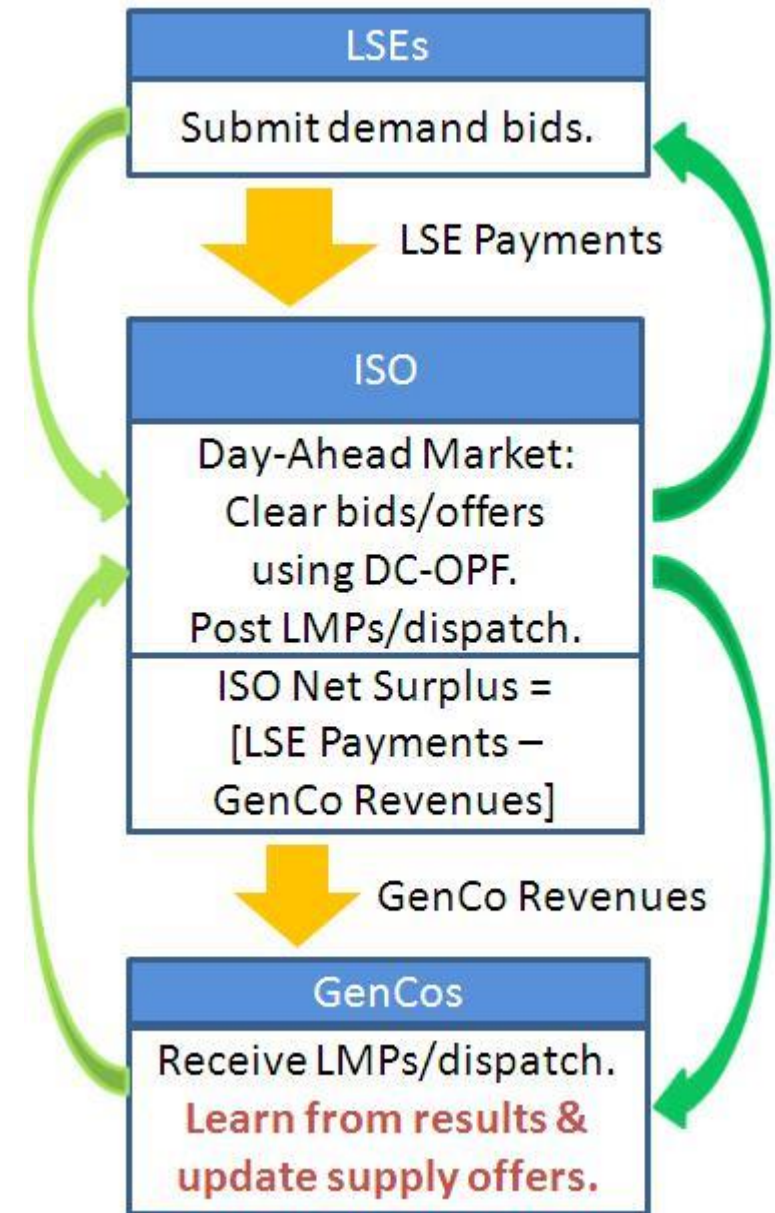
## ***Caution:*** Notation Switch

- P (in MWs) now denotes amounts of power
- $LMP_{k,T}$  (\$/MWh) = Locational Marginal Price at bus k for operating period T, roughly defined as the least cost of maintaining one additional MW of generated power at bus k during operating period T.

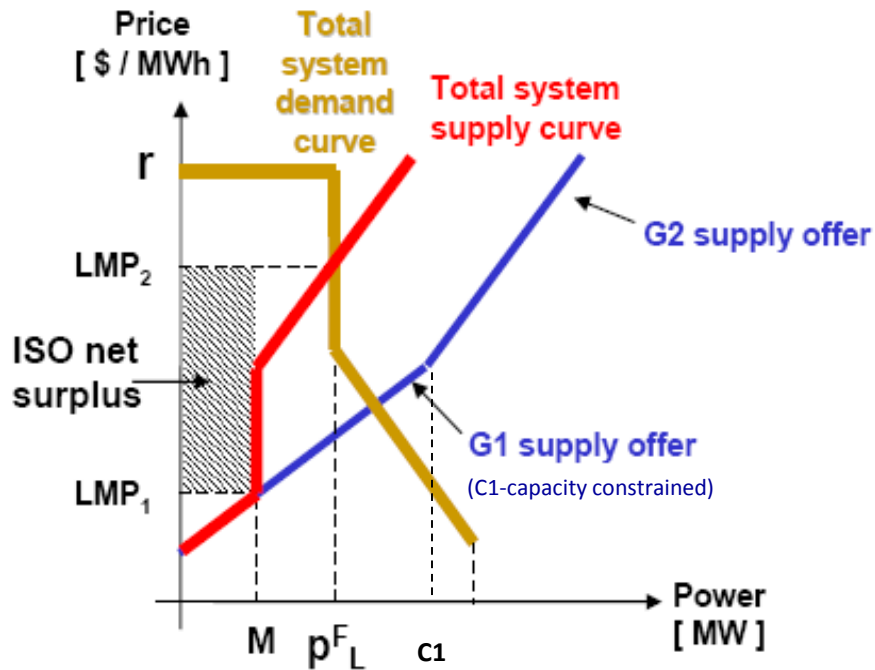
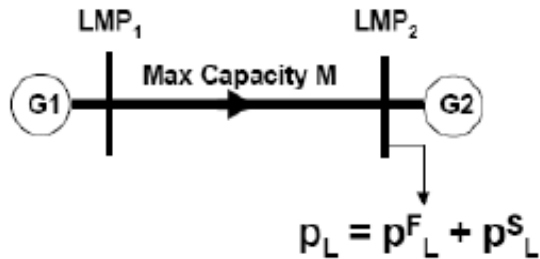


Discussion of double auctions, market efficiency, & social welfare specialized to an ISO managed **Day-Ahead Market (DAM)** for grid-delivered energy (MWh) with LMP settlements (\$/MWh):

Day-ahead market activities on a typical operating day D



# ISO goal is to maximize Total Net Surplus (TNS) subject to system constraints: A Two-Bus Example (Adapted from Harold Salazar, ISU ECpE M.S. Thesis, 2008)



Given the line capacity limit  $M$ , the cleared LSE load at bus 2 =  $p_L^F$ . The LSE receives price  $r$  (\$/MWh) for the resale of  $p_L^F$  at the retail level.

$M$  units of  $p_L^F$  are supplied by GenCo G1 at bus 1 at price  $LMP_1$  (\$/MWh); the line capacity limit  $M$  prevents G1 from supplying any additional units. Remaining  $[p_L^F - M]$  units are supplied by GenCo 2 at bus 2 at the higher price  $LMP_2$  (\$/MWh). The LSE at bus 2 pays  $LMP_2$  for each unit of  $p_L^F$ .

As a result of these transactions, the ISO collects "ISO Net Surplus" defined as follows:

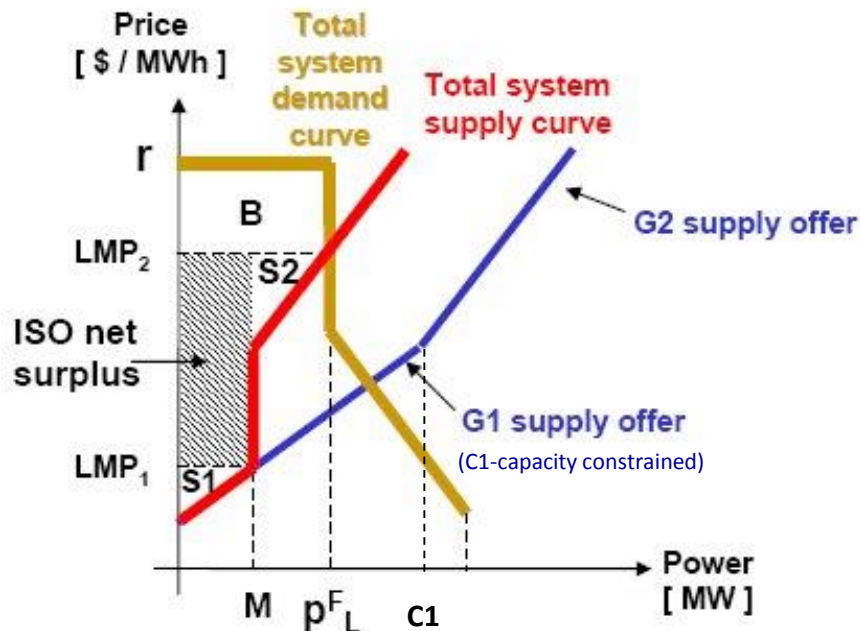
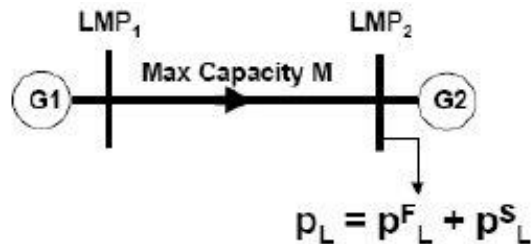
## ISO Net Surplus

$$= [ \text{LSE Payments} - \text{GenCo Revenues} ]$$

$$= LMP_2 \times p_L^F - M \times LMP_1 - [p_L^F - M] \times LMP_2$$

$$= M \times [ LMP_2 - LMP_1 ] = [\text{Shaded Figure Area}]$$

# Two-Bus Example ... Continued



## ISO Net Surplus:

$$\text{Area INS} =: M \times [LMP_2 - LMP_1]$$

## GenCo Net Surplus:

$$\text{Area S1} + \text{Area S2}$$

## LSE Net Surplus:

$$\text{Area B} =: p_L^F \times [r - LMP_2]$$

## Total Net Surplus:

$$\text{TNS} = [\text{INS} + \text{S1} + \text{S2} + \text{B}]$$

## ISO Optimization Objective:

Maximize **TNS** subject to system constraints.

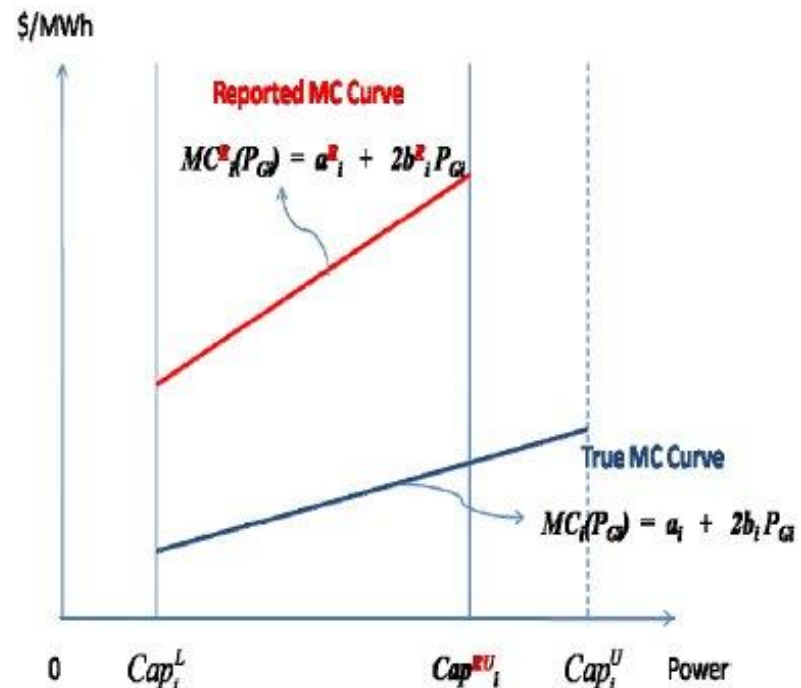
# AMES GenCo Supply Offers

Hourly supply offer for each GenCo  $i$  = **Reported** linear marginal cost function over a **reported** operating capacity interval for real power  $p_{Gi}$  (in MWs):

$$MC_i^R(p_{Gi}) = a_i^R + 2b_i^R p_{Gi}$$

$$Cap_i^L \leq p_{Gi} \leq Cap_i^{RU}$$

GenCos can learn to report **higher-than-true** marginal costs and/or to report **lower-than-true** maximum capacity.



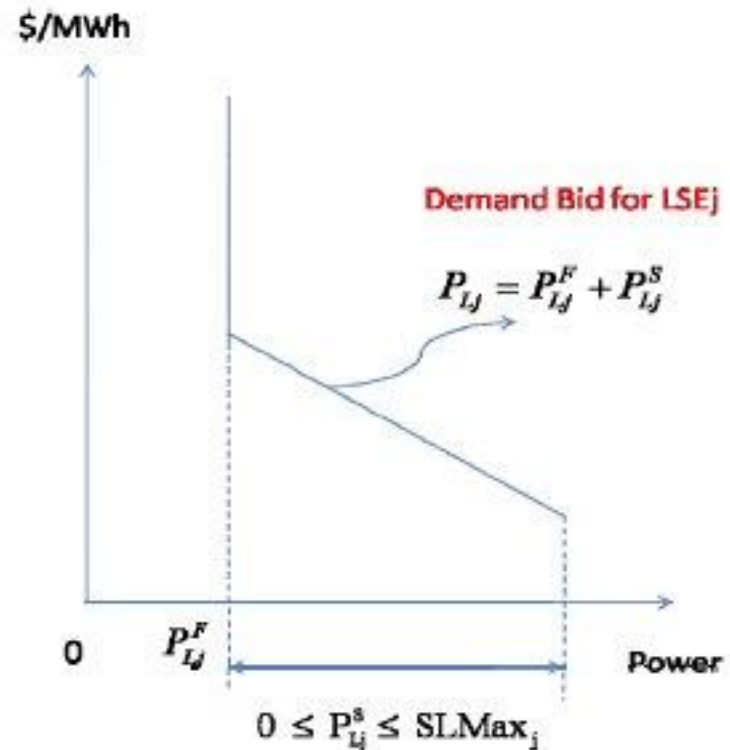
# AMES LSE Demand Bids

Hourly demand bid for each LSE  $j$  = **Fixed demand bid** + **Price-sensitive demand bid**

- Fixed demand bid =  $p_{Lj}^F$  (MWs)
- Price-sensitive demand bid = Inverse demand function for real power  $p_{Lj}^S$  (MWs) over a purchase capacity interval:

$$F_j(p_{Lj}^S) = c_j - 2d_j p_{Lj}^S$$

$$0 \leq p_{Lj}^S \leq \text{SLMax}_j$$

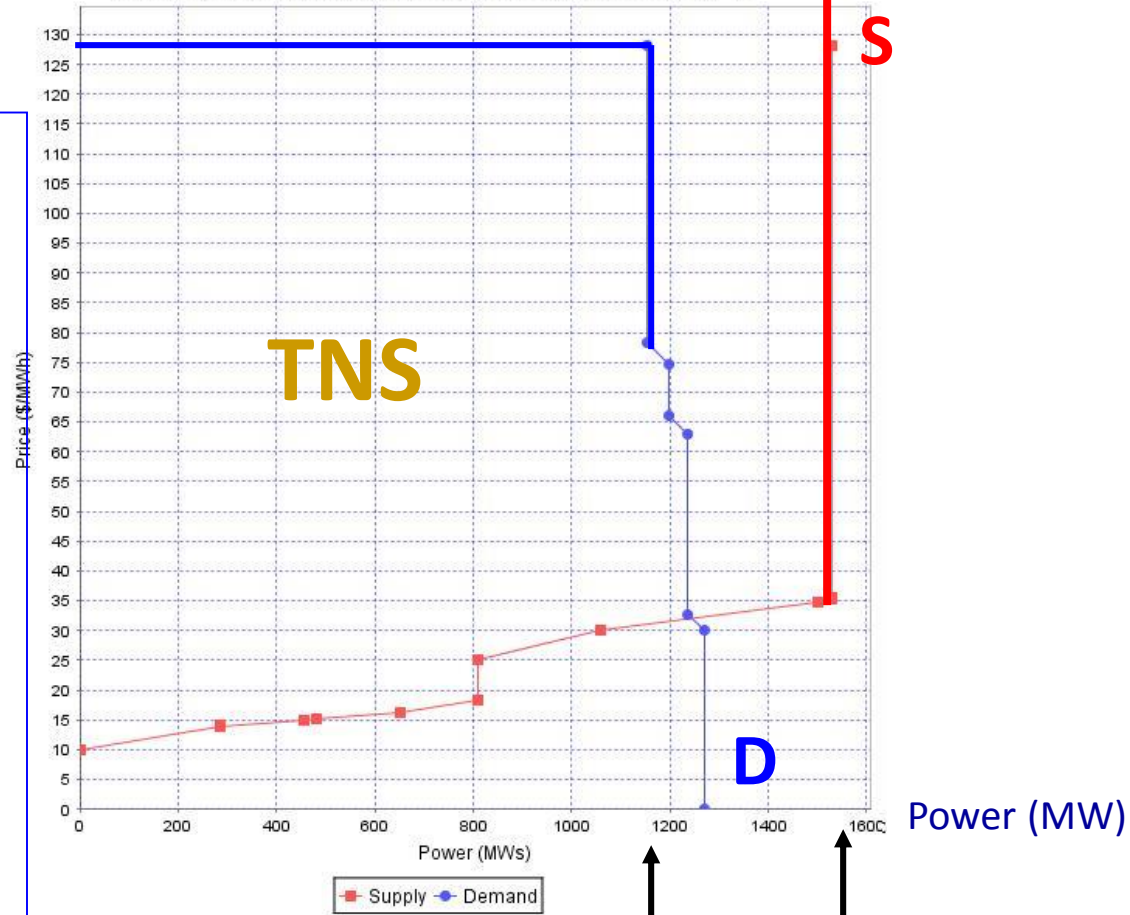


# AMES Illustration: **Total Net Surplus (TNS)** in Hour 17 for 5-Bus Test Case with 5 GenCos and 3 LSEs

\$/MWh

**r**

**True Total Supply and Demand Curves at Hour 17**



**r = Fixed price paid to LSEs by the LSEs' retail customers with flat-price contracts**

**= LSEs' max willingness to pay for each MW of their fixed demand  $p^F$  in wholesale power market**

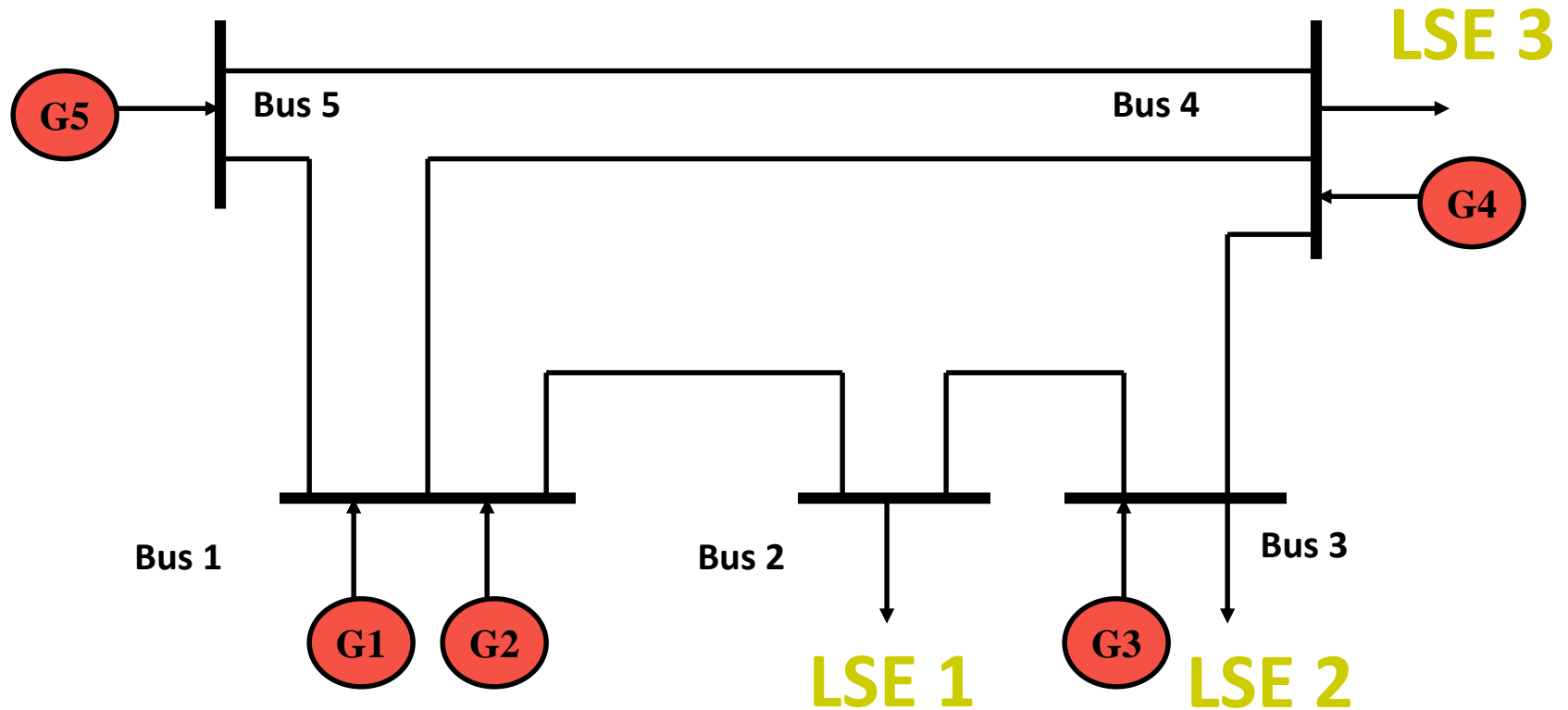
$p^F$

Max Total GenCo Capacity

# ISO Net Surplus Experiments (Li/Tesfatsion, 2009)

(Experiments run with AMES Wholesale Power Market Test Bed)

Five GenCo sellers G1,...,G5 and three LSE buyers LSE 1, LSE 2, LSE 3



# R Measure for Demand-Bid Price Sensitivity

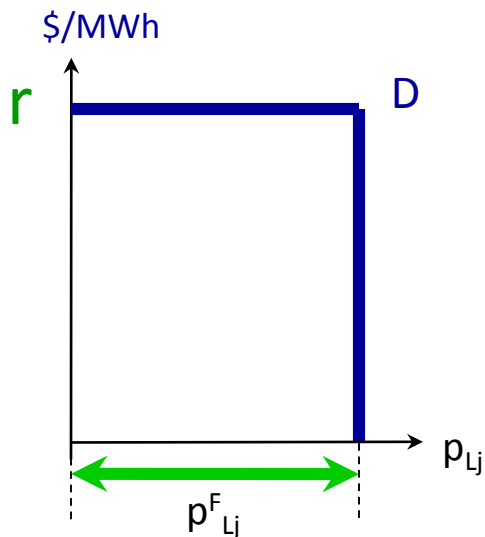
**Note:** In actual U.S. ISO energy regions, price-sensitivity  $R \cong 0.01$

For LSE  $j$  in Hour  $H$ :

$p_{Lj}^F$  = Fixed demand for real power (MWs)

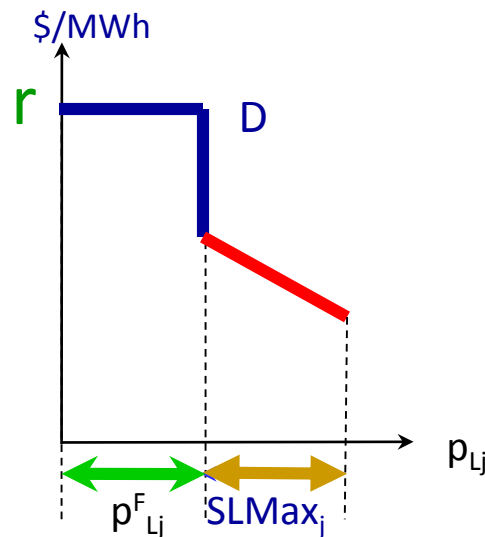
$SLMax_j$  = Maximum potential price-sensitive demand (MWs)

$$R = SLMax_j / [ p_{Lj}^F + SLMax_j ]$$

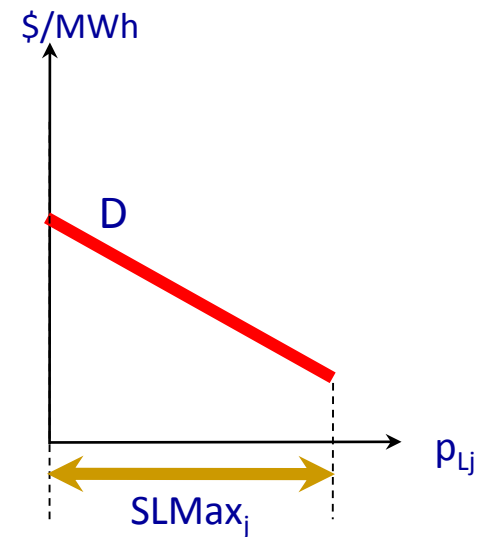


$R=0.0$

(100% Fixed Demand)



$R=0.5$



$R=1.0$

(100% Price-Sensitive Demand)

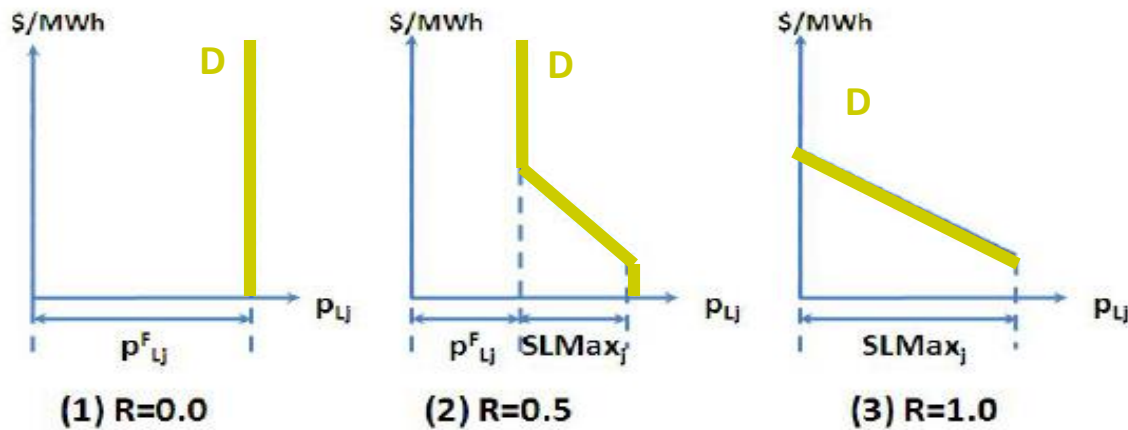


# Experimental Outcomes: Varied Price-Sensitivity for Demand Bids

Demand bid for LSE  $j$  (MW):

Fixed demand bid  $p_{Lj}^F$  + Price-sensitive demand bid  $p_{Lj}^S$ ,

where  $0 \leq p_{Lj}^S \leq SLMax_j$

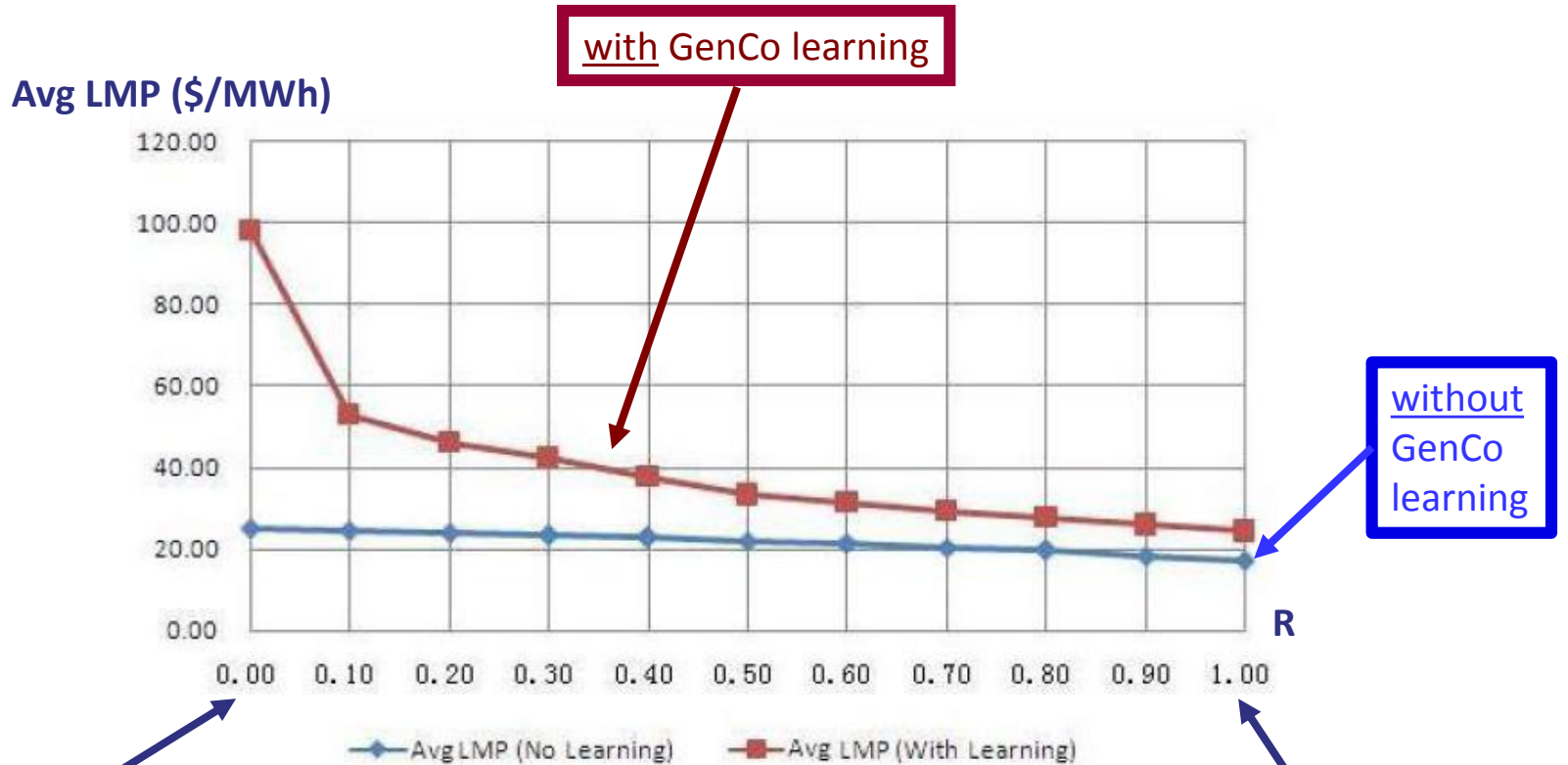


100% fixed demand

100% price-sensitive demand

# Average LMP Outcomes on Day 1000

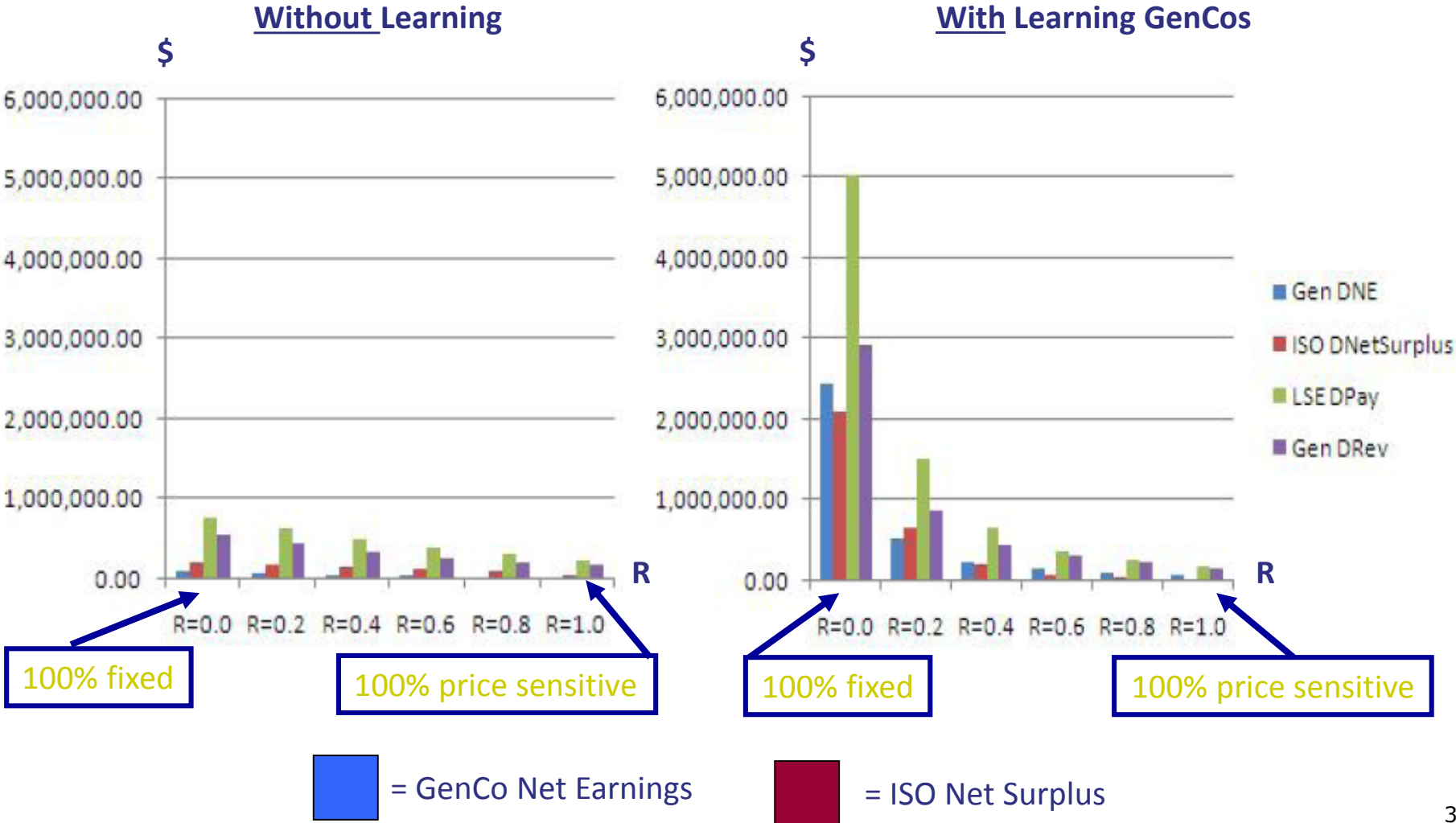
(under varied **GenCo learning** & **LSE demand price-sensitivity** treatments)



R=0: LSE demand is 100% fixed

R=1: LSE demand is 100% price sensitive

# Average ISO Net Surplus Outcomes on Day 1000 for varied learning & demand treatments



# ISO Net Surplus, Market Efficiency, and Social Welfare

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- ◆ Two-bus example and experimental findings suggest ISO net surplus extractions can be **substantial**, and can **dramatically increase** with:
  - *decreases* in price sensitivity of demand
  - *increases* in GenCo learning ability resulting in the reporting of supply offers at higher-than-true costs (especially profitable in presence of fixed demand)
- ◆ **Important Issue:** How to ensure ISO financial incentives are properly aligned with goal of ensuring market efficiency/soc welfare?

# AMES Calculation of TNS: General Form

## (Note LMPs cancel out of TNS expression!)

**Total Net Surplus for Hour H of Day D+1, based on Day D Supply Offers and Demand Bids:**

$$TNS(H, D)$$

$$= \text{LSENetSur}(H, D) + \text{GenNetSur}(H, D) + \text{ISONetSur}(H, D)$$

$$= \sum_{j=1}^J GS_j(H, D) - \sum_{i=1}^I [C_i^a(H, D)]$$

where

$$GS_j(H, D) = [r \cdot p_{Lj}^F(H, D) + \int_0^{p_{Lj}^S(H, D)} F_{jHD}(p) dp]$$

$$C_i^a(H, D) = \int_0^{p_{Gi}(H, D)} MC_i(p) dp$$

LSE j's gross surplus from its retail fixed demand sales

LSE j's gross surplus from its retail price-sensitive demand sales

GenCo i's total avoidable costs of production

## AMES Basic DC-OPF Formulation:

SI unit representation for AMES ISO's DC-OPF problem for hour H of day D+1, solved on day D.

DC-OPF formulation is derived from AC-OPF under three assumptions:

(a) Resistance on each branch  $km = 0$

(b) Voltage magnitude at each bus  $k =$  base voltage  $V_o$

(c) Voltage angle difference  $d_{km} =: [\delta_k - \delta_m]$  across each branch  $km$  is close to zero, implying that  $\cos(d_{km}) \cong 1$  and  $\sin(d_{km}) \cong d_{km}$  in amplitude.

$$\max \text{TNS}^R \quad (15)$$

with respect to LSE real-power price-sensitive demands, GenCo real-power generation levels, and voltage angles

$$p_{Lj}^S, j = 1, \dots, J; p_{Gi}, i = 1, \dots, I; \delta_k, k = 1, \dots, K \quad (16)$$

subject to

(i) a real-power balance constraint for each bus  $k=1, \dots, K$ :

$$\sum_{i \in I_k} p_{Gi} - \sum_{j \in J_k} p_{Lj}^S - \sum_{km} P_{km} = \sum_{j \in J_k} p_{Lj}^E \quad (17)$$

where, letting  $x_{km}$  (ohms) denote reactance for branch  $km$ , and  $V_o$  denote the base voltage (in line-to-line kV),

$$P_{km} = [V_o]^2 \cdot [1/x_{km}] \cdot [\delta_k - \delta_m]$$

(ii) a limit on real-power flow for each branch  $km$ :

$$|P_{km}| \leq P_{km}^U \quad (18)$$

(iii) a real-power operating capacity interval for each GenCo  $i = 1, \dots, I$ :

$$\text{Cap}_i^L \leq p_{Gi} \leq \text{Cap}_i^U \quad (19)$$

(iv) a real-power purchase capacity interval for price-sensitive demand for each LSE  $j = 1, \dots, J$ :

$$0 \leq p_{Lj}^S \leq \text{SLMax}_j \quad (20)$$

(v) and a voltage angle setting at angle reference bus 1:

$$\delta_1 = 0 \quad (21)$$

$\text{TNS}^R =$  Total Net Surplus based on reported GenCo marginal cost functions rather than true GenCo marginal cost functions.

Lagrange multiplier (or "shadow price") solution for the bus- $k$  balance constraint (17) gives  $\text{LMP}_k$  at bus  $k$

# AMES DC-OPF problem is a special type of GNPP, and LMPs are Lagrange Multiplier Solutions for this GNPP

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## General Nonlinear Programming Problem (GNPP):

- $\mathbf{x}$  =  $n \times 1$  choice vector;
- $\mathbf{c}$  =  $m \times 1$  vector &  $\mathbf{d}$  =  $s \times 1$  vector (constraint constants)
- $f(\mathbf{x})$  maps  $\mathbf{x}$  into  $\mathbb{R}$  (all real numbers)
- $\mathbf{h}(\mathbf{x})$  maps  $\mathbf{x}$  into  $\mathbb{R}^m$  (all  $m$ -dimensional vectors)
- $\mathbf{z}(\mathbf{x})$  maps  $\mathbf{x}$  into  $\mathbb{R}^s$  (all  $s$ -dimensional vectors)

**GNPP:** Minimize  $f(\mathbf{x})$  with respect to  $\mathbf{x}$  subject to

$$\mathbf{h}(\mathbf{x}) = \mathbf{c} \quad (\text{e.g., DC-OPF bus balance constraints})$$

$$\mathbf{z}(\mathbf{x}) \geq \mathbf{d} \quad (\text{e.g., DC-OPF branch constraints \& GenCo capacity constraints})$$

# AME DC-OPF as a GNPP ... Continued

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- Define the *Lagrangian Function* as

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{c}, \mathbf{d}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T[\mathbf{c} - \mathbf{h}(\mathbf{x})] + \boldsymbol{\mu}^T[\mathbf{d} - \mathbf{z}(\mathbf{x})]$$

- Assume *Kuhn-Tucker Constraint Qualification (KTCQ)* holds at  $\mathbf{x}^*$ , roughly stated as follows:

The true set of feasible directions at  $\mathbf{x}^*$

= Set of feasible directions at  $\mathbf{x}^*$  assuming a linearized set of constraints in place of original set of constraints.



# AMES DC-OPF as a GNPP ... Continued

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- Given KTCQ, the *First-Order Necessary Conditions (FONC)* for  $\mathbf{x}^*$  to solve (GNPP) are: There exist vectors  $\boldsymbol{\lambda}^*$  and  $\boldsymbol{\mu}^*$  of *Lagrange multipliers (or “shadow prices”)* such that  $(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$  satisfies:

$$\begin{aligned} 0 &= \nabla_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \mathbf{c}, \mathbf{d}) \\ &= [ \nabla_{\mathbf{x}} f(\mathbf{x}^*) - \boldsymbol{\lambda}^{*\top} \bullet \nabla_{\mathbf{x}} h(\mathbf{x}^*) - \boldsymbol{\mu}^{*\top} \bullet \nabla_{\mathbf{x}} z(\mathbf{x}^*) ] ; \end{aligned}$$

$$h(\mathbf{x}^*) = \mathbf{c} ;$$

$$z(\mathbf{x}^*) \geq \mathbf{d}; \boldsymbol{\mu}^{*\top} \cdot [\mathbf{d} - z(\mathbf{x}^*)] = 0; \boldsymbol{\mu}^* \geq \mathbf{0}$$

- ★ These FONC are often referred to as the *Karush-Kuhn-Tucker (KKT) conditions*.

# Solution as a Function of (c,d)

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By construction, the components of the solution vector  $(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$  are functions of the constraint constant vectors  $\mathbf{c}$  and  $\mathbf{d}$

- $\mathbf{x}^* = \mathbf{x}(\mathbf{c}, \mathbf{d})$
- $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}(\mathbf{c}, \mathbf{d})$
- $\boldsymbol{\mu}^* = \boldsymbol{\mu}(\mathbf{c}, \mathbf{d})$

# GNPP Lagrange Multipliers as Shadow Prices

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**Given certain additional regularity conditions...**

- The solution  $\boldsymbol{\lambda}^*$  for the  $m \times 1$  multiplier vector  $\boldsymbol{\lambda}$  is the derivative of the minimized value  $f(\mathbf{x}^*)$  of the objective function  $f(\mathbf{x})$  with respect to the constraint vector  $\mathbf{c}$ , all other problem data remaining the same.

$$\partial f(\mathbf{x}^*)/\partial \mathbf{c} = \partial f(\mathbf{x}(\mathbf{c}, \mathbf{d}))/\partial \mathbf{c} = \boldsymbol{\lambda}^{*\top}$$

# GNPP Lagrange Multipliers as Shadow Prices ...

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**Given certain additional regularity conditions...**

- The solution  $\boldsymbol{\mu}^*$  for the  $s \times 1$  multiplier vector  $\boldsymbol{\mu}$  is the derivative of the minimized value  $f(\mathbf{x}^*)$  of the objective function  $f(\mathbf{x})$  with respect to the constraint vector  $\mathbf{d}$ , all other problem data remaining the same.

$$0 \leq \partial f(\mathbf{x}^*) / \partial \mathbf{d} = \partial f(\mathbf{x}(\mathbf{c}, \mathbf{d})) / \partial \mathbf{d} = \boldsymbol{\mu}^{*\top}$$

# GNPP Lagrange Multipliers as Shadow Prices ...

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## Consequently...

- The solution  $\lambda^*$  for the multiplier vector  $\lambda$  thus essentially gives the **prices (values)** associated with unit changes in the components of the constraint vector  $\mathbf{c}$ , all other problem data remaining the same.
- The solution  $\mu^*$  for the multiplier vector  $\mu$  thus essentially gives the **prices (values)** associated with unit changes in the components of the constraint vector  $\mathbf{d}$ , all other problem data remaining the same.
- Each component of  $\lambda^*$  can take on **any sign**
- Each component of  $\mu^*$  must be **nonnegative**

# Counterpart to Constraint Vector $c$ for AMES DC-OPF?

## AMES DC-OPF Has $K$ Equality Constraints:

(i) a real-power balance constraint for each bus  $k=1,\dots,K$ :

$$\sum_{i \in I_k} p_{Gi} - \sum_{j \in J_k} p_{Lj}^S - \sum_{km} P_{km} = \sum_{j \in J_k} p_{Lj}^F \quad (17)$$

Fixed demand  
of LSE  $j$

Index set for LSEs  
located at bus  $k$

Below is the  $k$ th Component of  $K \times 1$  Constraint Vector  $c$  :

$$\sum_{j \in J_k} p_{Lj}^F = FD_k = \text{Total Fixed Demand at Bus } k$$

# LMP as Lagrange Multiplier

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- **TNS\*(H,D)** = Maximized Value of TNS(H,D) from the ISO's DC-OPF solution on Day D for hour H of the day-ahead market on Day D+1
- **LMP<sub>k</sub>(H,D)** = Least cost of servicing one additional MW of fixed demand at bus k during hour H of day-ahead market on day D+1

$$\mathbf{LMP}_k(\mathbf{H},\mathbf{D}) = \frac{\partial \mathbf{TNS}^*(\mathbf{H},\mathbf{D})}{\partial \mathbf{FD}_k}$$

# Online Resources

## ❑ Notes on DC-OPF Formulation in AMES

<https://www2.econ.iastate.edu/tesfatsi/DCOPFInAMES.LT.pdf>

## ❑ AMES Wholesale Power Market Testbed

<https://www2.econ.iastate.edu/tesfatsi/AMESMarketHome.htm>

## ❑ Market Basics for Price-Setting Agents

<https://www2.econ.iastate.edu/tesfatsi/MBasics.SlidesIncluded.pdf>

## ❑ Optimization Basics for Electric Power Markets

<https://www2.econ.iastate.edu/tesfatsi/OptimizationBasics.LT458.pdf>

## ❑ Power Market Trading with Transmission Constraints

<https://www2.econ.iastate.edu/classes/econ458/tesfatsion/OPFTransConstraintsLMP.KS6.1-6.3.2.9.pdf>



# Online Resources ... Continued

- L. Tesfatsion (2009), **“Auction Basics for Wholesale Power Markets: Objectives & Pricing Rules,”** *IEEE PES General Meeting Proceedings*, July.  
<https://www2.econ.iastate.edu/tesfatsi/AuctionTalk.LT.pdf> (Slide-Set)  
<https://www2.econ.iastate.edu/tesfatsi/AuctionBasics.IEEEPES2009.LT.pdf> (Paper)
  
- H. Li & L. Tesfatsion (2011), **“ISO Net Surplus Collection and Allocation in Wholesale Power Markets Under Locational Marginal Pricing,”** *IEEE Transactions on Power Systems*, Vol. 26, No. 2, pp 627-641.  
<https://www2.econ.iastate.edu/tesfatsi/ISONetSurplus.WP09015.pdf>