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Course Site for Econ 502 (M.S. Macroeconomic Theory:)

<https://www2.econ.iastate.edu/tesfatsi/Syl502.htm>

Elements of Dynamic Economic Modeling:

<https://www2.econ.iastate.edu/tesfatsi/DynamicEconomicModelingBasics.WPVersion.pdf>

The Basic Solow-Swan Descriptive Growth Model:

<https://www2.econ.iastate.edu/tesfatsi/SolowMod.pdf>

The Basic Optimal Growth Model:

<https://www2.econ.iastate.edu/tesfatsi/OptGrow.pdf>

THE BASIC SOLOW-SWAN DESCRIPTIVE GROWTH MODEL

1 Introduction

These notes provide a concise rigorous introduction to the basic descriptive growth model independently developed by Robert Solow (1956) and Trevor Swan (1956). The key issue addressed by the Solow-Swan descriptive growth model is the extent to which full employment of all productive resources can be maintained in the long run, given the capacity-creating effects of net investment.

The Solow-Swan descriptive growth model provides the basic foundation for modern macroeconomic modeling approaches, including optimal growth theory, overlapping generations models, real business cycle theory, and dynamic stochastic general equilibrium (DSGE) modeling. Nevertheless, many criticisms of the model remain on both theoretical and empirical grounds. We will touch on some of these criticisms in the concluding section of these notes and in subsequent course readings.

After presentation of the basic model, the following question is posed: To what extent are the model's predictions consistent with regularities in empirical macroeconomic data? As detailed in Barro and Sala-i-Martin (2003, Chapter 1), data gathered on many countries up through 2003 were observed to be roughly consistent with the following *Stylized Facts of Growth* originally developed by Nicholas Kaldor in the 1960s.

[SF1] The ratio K/Y of physical capital K to output Y is nearly constant over time.

[SF2] The shares of labor and physical capital in national income, wL/Y and rK/Y , are nearly constant over time, where w and r denote the real wage rate of labor and the real rental rate of capital, respectively.¹

[SF3] Per capita output $y = Y/L$ grows over time without a tendency to converge to a constant value.

[SF4] Per-capita physical capital $k = K/L$ grows over time without a tendency to converge to a constant value.

[SF5] The growth rate D_+y/y of $y = Y/L$ differs substantially across countries.

As it turns out, the basic Solow-Swan descriptive growth model is *not* consistent with stylized facts [SF3] and [SF4], above. In subsequent work, however, Solow extended his basic model to include “labor-augmenting technological change” in such a manner that consistency with stylized facts [SF1] through [SF4] was attained. In addition, Solow also developed his now famous “growth accounting equation” that decomposes the growth rate in real GDP by source. We will cover both of these developments in later sections of these notes.

2 Basic Solow-Swan Descriptive Growth Model in Level Form

Suppose at each time t in some specified time interval $[0, T]$ an economy produces an output, $Y(t)$, using capital and labor inputs $K(t)$ and $L(t)$ in a production technology described by an *aggregate production function*

$$Y = F(K, L) . \quad (1)$$

Consumption at time t is denoted by $C(t)$. The capital stock $K(t)$ is assumed to depreciate at a constant nonnegative rate δ , so that gross investment $I(t)$ at time t is given by

$$I(t) = D_+K(t) + \delta K(t) . \quad (2)$$

¹We now know that the share of labor in U.S. national income has in fact been declining since the mid-1970s, in violation of SF2, first gradually and then (since 2000) at a fairly steep rate; see Elsby et al. (2013). In a well-researched study dubbed “disturbingly persuasive” by the Financial Times, Ford (2015) connects this fall in labor share to the accelerating use of artificially intelligent machines as a substitute for labor.

The time- t output $Y(t)$ is assumed to be divided between consumption and (gross) investment, the usual “national income accounting identity” for an economy without a government or foreign sector. Finally, the labor supply $L(t)$ is assumed to grow at a constant positive rate g . The relations described above can be formalized analytically as follows:

$$Y(t) = F(K(t), L(t)) ; \quad (3)$$

$$I(t) = D_+K(t) + \delta K(t) ; \quad (4)$$

$$Y(t) = C(t) + I(t) ; \quad (5)$$

$$D_+L(t) = gL(t) . \quad (6)$$

TECHNICAL REMARK: In this and following sections, time-derivatives are generally assumed to be *right* derivatives, denoted by D_+ . The *right derivative* of a function $H:R \rightarrow R$ at a point t is defined to be

$$D_+H(t) = \lim_{s \rightarrow t, s > t} \left[\frac{H(s) - H(t)}{s - t} \right] . \quad (7)$$

In contrast, the *derivative* of $H(\cdot)$ at t is defined as follows:

$$DH(t) = \lim_{s \rightarrow t} \left[\frac{H(s) - H(t)}{s - t} \right] . \quad (8)$$

Comparing (7) with (8), one sees for the right derivative that the ratio $[H(s) - H(t)]/[s - t]$ is only required to converge as s approaches t “from the right,” i.e., for values $s > t$, whereas for the ordinary derivative this same ratio is required to converge as s approaches t from *any* feasible direction. As will be seen below, imposing only right differentiability on rates of change for time- t predetermined variables permits discontinuous jumps in these rates of change. Consequently, these rates of change are free to move as time- t endogenous variables, unconstrained by past variable realizations. In contrast, requiring these rates of change to be ordinary derivatives as in (8) would reduce them to time- t *predetermined* variables, i.e., variables determined by past realizations, since the existence of an ordinary derivative for a variable at a point requires the equality of its right and left derivatives at this point. For more discussion of this point, see Tesfatsion (2016).

The simple model described above is a *one-sector model*, in the sense that it incorporates only one produced good, Y .² Capital is simply a stockpile of the one produced good. Notice

²Leisure is not traditionally thought of as a “produced good.”

that the labor market is *implicitly* assumed to clear. That is, the labor demanded—namely, the labor quantity appearing in the production function—is simply assumed to be equal to the labor supply $L(t)$. Also, capital is assumed to be fully utilized.

So far, no restrictions have been placed on the production function (1). In very early work on growth (e.g., in the work of Domar and Harrod), this production function was often assumed to take the form of a *fixed-coefficient production function*

$$Y = F(K, L) \equiv \min\left\{\frac{K}{v}, \frac{L}{u}\right\}, \quad (9)$$

where v and u are fixed positive coefficients having dimensions “units of K per unit of Y ” and “units of L per unit of Y ,” respectively. By construction, then, one unit of Y is obtained by using v units of K and u units of L . Moreover, (v, u) is the most efficient input configuration for producing one unit of Y in the following sense: Given $(K, L) = (v, u)$, any additional usage of K (keeping L equal to u), or any additional usage of L (keeping K equal to v), simply wastes input resources since it results in no additional output.

More generally, as shown in Fig. 1, it follows from (9) that efficient production (i.e., non-wastage of K or L) requires $L/u = K/v$, or $L = [u/v]K$. Therefore, given any desired production level \hat{Y} , the efficient production of \hat{Y} requires $\hat{Y}v$ units of K and $\hat{Y}u$ units of L . As depicted in Fig. 1, the isoquants for the fixed coefficient production function (9) are thus rectangular with kink points along the line $L = [u/v]K$. Notice, also, that this production function exhibits *constant returns to scale*; that is,

$$bY = F(bK, bL), \text{ for all } b > 0. \quad (10)$$

As a generalization of (9), suppose that two distinct fixed coefficient production processes are available to firms, as follows:

$$Y = \min\left\{\frac{K}{v_1}, \frac{L}{u_1}\right\}; \quad (11)$$

$$Y = \min\left\{\frac{K}{v_2}, \frac{L}{u_2}\right\}. \quad (12)$$

To rule out trivial dominance of one process over the other, suppose that the first production process uses less capital but more labor than the second process per each unit of Y ; i.e., suppose that $v_2 > v_1$ and $u_2 < u_1$. What do the production isoquants for firms look like now?

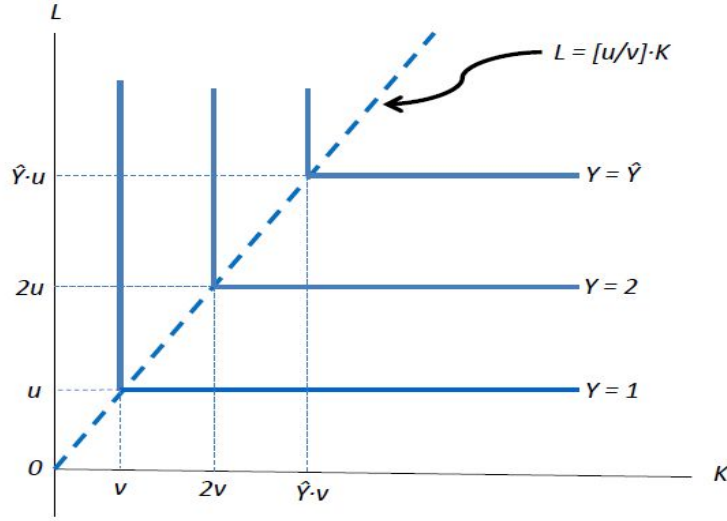


Figure 1: Isoquants for a fixed-coefficient production function

Suppose, for example, that a firm wishes to produce \hat{Y} units of Y , and is able to use any combination of the two production processes (11) and (12) to accomplish this production. Using only production process 1, an efficient firm would use $A = (\hat{Y}v_1, \hat{Y}u_1)$ units of K and L to produce \hat{Y} units of Y . Using only production process 2, an efficient firm would use $B = (\hat{Y}v_2, \hat{Y}u_2)$ units of K and L to produce \hat{Y} units of Y . On the other hand, the firm can also divide the production of \hat{Y} between the two production processes in any proportion. For example, for any given $\lambda \in [0, 1]$, the firm can produce $\lambda\hat{Y}$ units of Y using production process 1 and $[1 - \lambda]\hat{Y}$ units of Y using production process 2. The efficient input requirements for this division of production between processes 1 and 2 are given by

$$C = \lambda A + [1 - \lambda]B = (\hat{Y}v_c, \hat{Y}u_c), \quad (13)$$

where $v_c = \lambda v_1 + [1 - \lambda]v_2$ and $u_c = \lambda u_1 + [1 - \lambda]u_2$. By construction, point C lies on the line segment connecting A to B ; and all such points C are on the isoquant corresponding to the output level \hat{Y} . It follows that this isoquant takes the form of a kinked line segment as depicted in Fig. 2(a). Note that production still exhibits constant returns to scale.

As more and more fixed coefficient production processes are made available to the firm, the firm has a greater opportunity to substitute K for L in the production process, i.e., to choose among different combinations of capital and labor to produce any given quantity of good. Reflecting this increased input substitutability, the firm's isoquants become increas-

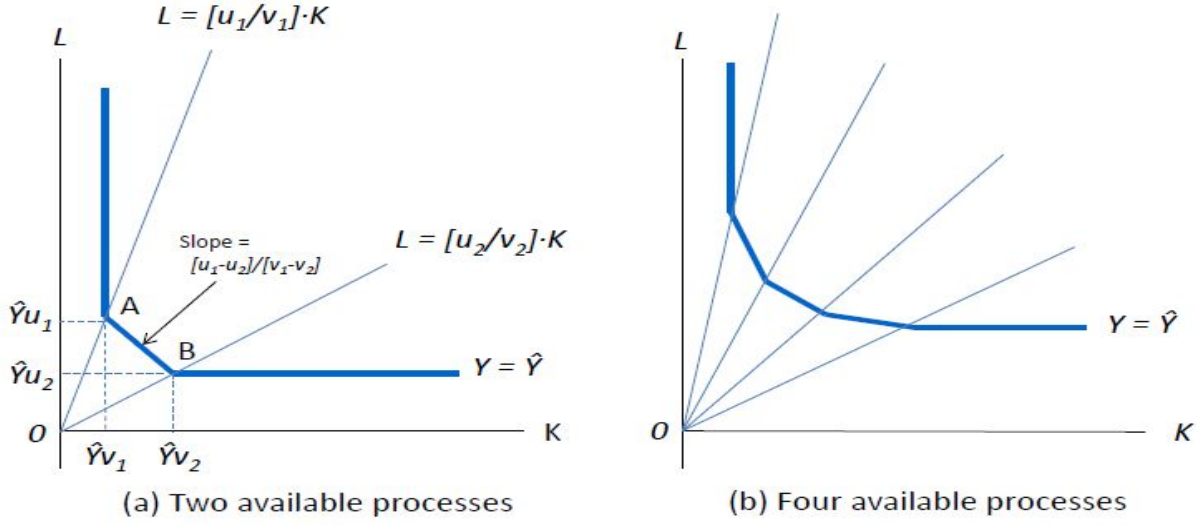


Figure 2: Isoquants for convex combinations of fixed-coefficient production processes

ingly “bowl-shaped” with constant returns to scale continuing to hold; see Fig. 2(b). In the limit, as the number of available production processes approaches infinity, the isoquants for the firm take on the usual neoclassical form of a smooth downward-sloping convex curve.

The Solow-Swan growth model takes this limiting production process as its starting point. More precisely, it is assumed that the production function $Y = F(K, L)$ in (1) satisfies the following *Standard Neoclassical Production Function Assumptions in Level Form*:³

1. $F(K, L)$ exhibits constant returns to scale;
2. $F(K, L)$ is continuous over $(K, L) \geq 0$;
3. $F(K, L)$ is twice continuously differentiable and concave, with $F_{KK}(K, L) < 0$, over all $(K, L) > 0$;
4. $F_K(K, L) > 0$ and $F_L(K, L) > 0$ for all $(K, L) > 0$;
5. $F(0, L) = 0$ for all $L \geq 0$;

³Given properties 1, 2, and 6, it can be shown that property 5 holds if $F_L(K, L) \rightarrow 0$ as $L \rightarrow \infty$ for each $K > 0$; see Barro and Sala-i-Martin (2003, p. 77). To see this, use $F(K, L)/L = F(K/L, 1)$ converges to $F(0, 1)$ as $L \rightarrow \infty$, but $F(K, L)/L$ approaches 0 as $L \rightarrow \infty$ either because $F(K, L)$ remains bounded as $L \rightarrow \infty$ or by L'Hospital's Rule. Hence $F(0, 1) = 0 = LF(0, 1) = F(0, L)$ for all $L \geq 0$. Some textbooks impose this condition on $F_L(K, L)$ instead of assuming property 5.

6. (Inada Conditions in Level Form):⁴ For each $L > 0$, $F_K(K, L) \rightarrow \infty$ as $K \rightarrow 0$, and $F_K(K, L) \rightarrow 0$ as $K \rightarrow \infty$.

An example of a production function satisfying all of these conditions is the well-known *Cobb-Douglas production function*:

$$Y = F(K, L) = K^\alpha L^{1-\alpha}, \quad 0 < \alpha < 1. \quad (14)$$

So far, no behavioral motivation has been given for the determination of the consumption level $C(t)$. The descriptive growth literature generally follows Keynes in postulating that consumption is determined as a fixed proportion $[1 - s]$ of net income (i.e., income Y net of capital depreciation δK), where s denotes an exogenously given marginal (and average) propensity to save. Thus, suppose that consumers behave in accordance with the following consumption function:

$$C(t) = [1 - s][Y(t) - \delta K(t)]. \quad (15)$$

Remark: Equation (15) presumes that capital depreciation $\delta K(t)$ is subtracted from total income $Y(t)$ and consumption is then determined as a fraction $[1 - s]$ of the remaining income $[Y(t) - \delta K(t)]$. As will be seen below, consumer saving $s[Y(t) - \delta K(t)]$ then finances *net* capital investment $D_+K(t)$. An alternative possibility (seen in some textbooks on economic growth) is that consumption is determined as a fraction $[1 - s]$ of total income $Y(t)$ and consumer saving $sY(t)$ then finances *gross* capital investment $D_+K(t) + \delta K(t)$. The dynamics of the Solow-Swan model are qualitatively the same under either treatment.

By appending the consumption function (15) to the previously derived model equations (3) through (6), an implicit assumption is being made that the market for consumption good continuously clears; for the *same* variable $C(t)$ is used both in the per capita national income accounting identity (5) to denote the realized supply of consumption goods and in relation (15) to denote the demand for consumption goods.

⁴The first systematic study of the role played by these conditions in ensuring the existence and uniqueness of stationary solutions in neoclassical growth models is typically attributed to Ken-ichi Inada (1963).

Basic Solow-Swan Descriptive Growth Model in Level Form

Model Equations: $t \geq 0$:

$$Y(t) = F(K(t), L(t)) \quad (16)$$

$$I(t) = D_+K(t) + \delta K(t) \quad (17)$$

$$Y(t) = C(t) + I(t) \quad (18)$$

$$C(t) = [1 - s][Y(t) - \delta K(t)] \quad (19)$$

$$D_+L(t) = g \cdot L(t) \quad (20)$$

Classification of Variables:

Time- t Endogenous Variables ($t \geq 0$): $Y(t)$, $I(t)$, $C(t)$, $D_+K(t)$, $D_+L(t)$

Time- t Predetermined (State) Variables ($t > 0$):

$$K(t) = \int_0^t D_+K(\tau) d\tau + K(0) \quad (21)$$

$$L(t) = \int_0^t D_+L(\tau) d\tau + L(0) \quad (22)$$

Admissible Exogenous Variables and Functional Forms:

$K(0)$, $L(0)$, s , δ , and g , satisfying $0 < K(0)$, $0 < L(0)$, $0 < s < 1$, $0 \leq \delta$, and $0 < g$, plus a function $F(K, L)$ that satisfies the *Standard Neoclassical Production Function Assumptions in Level Form*

3 Basic Solow-Swan Descriptive Growth Model in Per-Capita Form

Given a constant-returns-to-scale production function (1), the level model described by equations (16) through (20) can be transformed into a “per capita” model. Let per-capita output,

capital, gross investment, and consumption be denoted, respectively, by

$$y(t) = Y(t)/L(t) ; \quad (23)$$

$$i(t) = I(t)/L(t) ; \quad (24)$$

$$k(t) = K(t)/L(t) ; \quad (25)$$

$$c(t) = C(t)/L(t) . \quad (26)$$

Using the constant returns to scale relation (10) with $b = [1/L(t)]$, note that

$$F(K(t), L(t))/L(t) = F(K(t)/L(t), 1) = F(k(t), 1) . \quad (27)$$

Given the restrictions on the original production function $F(\cdot)$, it can be shown that the per capita production function $f:[0, +\infty) \rightarrow R$ defined by $f(k) = F(k, 1)$ is strictly increasing and strictly concave over $k \geq 0$ and differentiable over $k > 0$.⁵ An example of such a function is the well-known per capita Cobb-Douglas production function,

$$y = f(k) = k^\alpha , \quad 0 < \alpha < 1 . \quad (28)$$

Dividing equation (16) by $L(t)$, and using the notation introduced above, one obtains

$$y(t) = f(k(t)) . \quad (29)$$

Dividing equation (17) by $L(t)$, per capita gross investment $i(t)$ is given by

$$i(t) = \frac{D_+K(t) + \delta K(t)}{L(t)} . \quad (30)$$

Finally, dividing equation (18) by $L(t)$, one obtains

$$f(k(t)) = i(t) + c(t) . \quad (31)$$

Recall that the (percentage) rate of change of a ratio is the difference in the (percentage) rates of change of the numerator and denominator terms. Thus,

$$D_+k(t)/k(t) = D_+K(t)/K(t) - D_+L(t)/L(t) . \quad (32)$$

⁵To see this, take the first and second derivatives of $F(K, L) = Lf(k)$ with respect to K .

Combining (6), (25), and (32), one obtains

$$D_+K(t)/L(t) = D_+k(t) + gk(t) . \quad (33)$$

Thus, using equation (30) to substitute out for $i(t)$ in equation (31), using (33) to substitute out for $D_+K(t)/L(t)$, and rearranging terms, one obtains the following differential equation for the per capita capital stock:

$$D_+k(t) = f(k(t)) - [g + \delta]k(t) - c(t) . \quad (34)$$

The investment-consumption trade-off facing the economy at time t , conditional on $k(t)$, is depicted in Fig. 3. The economy must choose between consuming resources now, and investing resources now in order to enhance consumption later. A choice for more consumption today, hence less investment, results in a smaller capital stock tomorrow, and hence a tradeoff line tomorrow that lies closer to the origin. More generally, the choice of consumption today affects the placement of the tradeoff line for all future periods. A specification of a behavioral relation for consumption is in effect a rule for choosing a point on the investment-consumption tradeoff line in each period t .

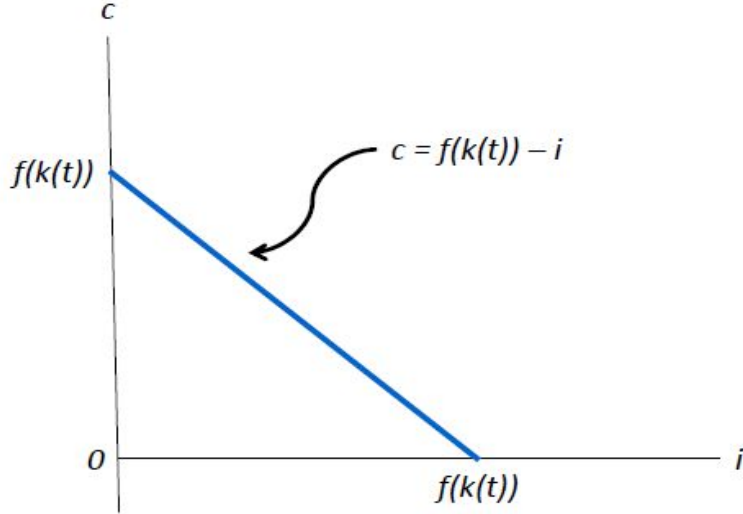


Figure 3: Investment-consumption trade-off at time t , conditional on $k(t)$

As seen in (19), consumption is determined as a fixed proportion $[1 - s]$ of net income (i.e., income net of capital depreciation), where s denotes an exogenously given marginal

(and average) propensity to save. The per-capita form of (19) is given by

$$c(t) = [1 - s][y(t) - \delta k(t)] . \quad (35)$$

The fact that consumption is directly specified (“described”) in (35) rather than derived as the solution to an optimization problem is why the Solow-Swan growth model is referred to as a *descriptive* growth model rather than an *optimal* growth model.

Substituting relation (35) into relation (34), and letting $\lambda = [g + s\delta]$ for ease of notation, one obtains the following fundamental equation describing the growth of per capita capital:

$$D_+k(t) = sf(k(t)) - \lambda k(t) . \quad (36)$$

Finally, given (35), note that per capita (net) savings $s(t)$ take the form

$$s(t) = [y(t) - \delta k(t)] - c(t) = s[y(t) - \delta k(t)] . \quad (37)$$

It is common in the descriptive growth literature to work with per capita savings rather than with per capita consumption.

The *per capita Solow-Swan descriptive growth model* will now be stated in summary form.

Basic Solow-Swan Descriptive Growth Model in Per-Capita Form:

Model Equations: For each time $t \geq 0$,

$$y(t) = f(k(t)) ; \quad (38)$$

$$s(t) = s[y(t) - \delta k(t)] ; \quad (39)$$

$$D_+k(t) = sf(k(t)) - \lambda k(t) . \quad (40)$$

Classification of Variables:

Time-t Endogenous Variables ($t \geq 0$): $y(t)$, $s(t)$, $D_+k(t)$;

Time-t Predetermined (State) Variable ($t > 0$): $k(t) = \int_0^t D_+k(\tau)d\tau + k(0)$;

Admissible Exogenous Variables and Functional Forms:

$k(0)$, s , and $\lambda = [g + s\delta]$, with $0 < k(0)$, $0 < s < 1$, $0 < g$, and $0 \leq \delta$;

Also, $f(k) \equiv F(k, 1)$ satisfies the following *Standard Neoclassical Production Function Assumptions in Per-Capita Form*: $f(k)$ is continuous over $k \geq 0$, and $f(k)$ is twice continuously differentiable with $f'(k) > 0$ and $f''(k) < 0$ over $k > 0$; $f(0) = 0$; and $f'(k) \rightarrow +\infty$ as $k \rightarrow 0$ and $f'(k) \rightarrow 0$ as $k \rightarrow +\infty$.

REMARK: The admissibility conditions assumed here for the per-capita production function $f(k)$ follow from the admissibility conditions earlier assumed for the production function $F(K, L)$ in level form. To see this, use the fact that

$$F(K, L) = Lf(k), \text{ where } k = K/L, \quad (41)$$

which implies that

$$F_K(K, L) = \partial F(K, L)/\partial K = f'(k); \quad (42)$$

$$F_L(K, L) = \partial F(K, L)/\partial L = f(k) - f'(k)k; \quad (43)$$

$$F_{KK}(K, L) = \partial^2 F(K, L)/\partial K^2 = f''(k)/L. \quad (44)$$

The reason for imposing these conditions on the production function will become apparent in the following discussion on the existence, uniqueness, and stability of stationary solutions for the per-capita Solow-Swan descriptive growth model.

4 Existence, Uniqueness, and Stability of Stationary Solutions for the Per-Capita Growth Model

A state-space model in initial value form will be said to have a *Basic Causal System (BCS)* if it is possible in each period t to substitute out for all period- t endogenous variables except for the rate of change $D_+x(t)$ of the period- t state vector $x(t)$, resulting in a reduced-form model of the form $D_+x(t) = f(\alpha(t), x(t))$ where $\alpha(t)$ consists only of period- t exogenous variables. See Tesfatsion[Section 4.6](2016) for additional discussion of the BCS concept.

Given any admissible specifications for s , λ , and $f(\cdot)$, the BCS for the per capita Solow-Swan descriptive growth model can be represented in the following form:

Basic Causal System:

$$\begin{aligned} D_+k(t) &= sf(k(t)) - \lambda k(t) , \quad t \geq 0 ; \\ k(0) &= u . \end{aligned} \tag{45}$$

Classification of Variables:

Time-t Endogenous Variable ($t \geq 0$): $D_+k(t)$

Time-t Predetermined (State) Variable ($t > 0$): $k(t) = \int_0^t D_+k(\tau)d\tau + k(0)$

Admissible Exogenous Variables and Functional Forms:

u , s , and $\lambda = [g + s\delta]$, with $0 < u$, $0 < s < 1$, $0 < g$, and $0 \leq \delta$.

Also, $f(k) \equiv F(k, 1)$ satisfies the following *Standard Neoclassical Production Function Assumptions in Per-Capita Form*: $f(k)$ is continuous over $k \geq 0$, and $f(k)$ is twice continuously differentiable with $f'(k) > 0$ and $f''(k) < 0$ over $k > 0$; $f(0) = 0$; and $f'(k) \rightarrow +\infty$ as $k \rightarrow 0$ and $f'(k) \rightarrow 0$ as $k \rightarrow +\infty$.

By construction, any solution derived for the BCS (45) will be functionally dependent on $(s, \lambda, f(\cdot))$ as well as on u . For ease of exposition, the dependence on $(s, \lambda, f(\cdot))$ will hereafter be suppressed when no changes for these variables are under consideration.

Suppose a unique solution exists for the BCS for each $u > 0$. Let this unique solution be denoted by

$$\mathbf{k}(u) = \{k(t; u) \mid t \geq 0\} , \tag{46}$$

and let

$$\mathbf{V} = \{\mathbf{k}(u) \mid u > 0\} \tag{47}$$

denote the set of all admissible solutions for the BCS, conditional on $(s, \lambda, f(\cdot))$. It will now be shown that \mathbf{V} contains a solution that is “distinguished” from all others in the sense that it characterizes the long-run behavior of the Solow-Swan economy.

A solution $\mathbf{k}(k^*)$ in \mathbf{V} corresponding to a positive initial value k^* for $k(0)$ is said to be a *stationary solution* if

$$k(t; k^*) = k^* , \quad t \geq 0 . \tag{48}$$

A stationary solution thus constitutes a *rest-point* for the per-capita Solow-Swan descriptive growth model in the sense that the state of the modeled economy remains constant over time. Several questions about such solutions will now be addressed. First, does a stationary solution necessarily exist for the BCS? Second, if such a solution exists, is it necessarily unique? Finally, do such stationary solutions have any particular economic meaning apart from being rest points?

It will first be shown that the BCS for the per-capita Solow-Swan descriptive growth model necessarily has a unique admissible stationary solution for any given admissible specification for $(s, \lambda, f(\cdot))$. For notational simplicity, let any stationary solution $\mathbf{k}(k^*)$ in \mathbf{V} be abbreviated by the constant state value $k^* > 0$. It then follows from (48) that a stationary solution $k^* > 0$ exists if and only if the state $k(t)$ takes on the constant value k^* for all $t \geq 0$. By (45), this is true if and only if k^* satisfies

$$0 = sf(k^*) - \lambda k^* . \quad (49)$$

Letting $z = [\lambda/s]k$ and $x = f(k)$, it follows from (49) that $k^* > 0$ is an admissible stationary solution for the BCS if and only if the graphs of z and x as functions of k intersect at k^* ; see Fig. 4.

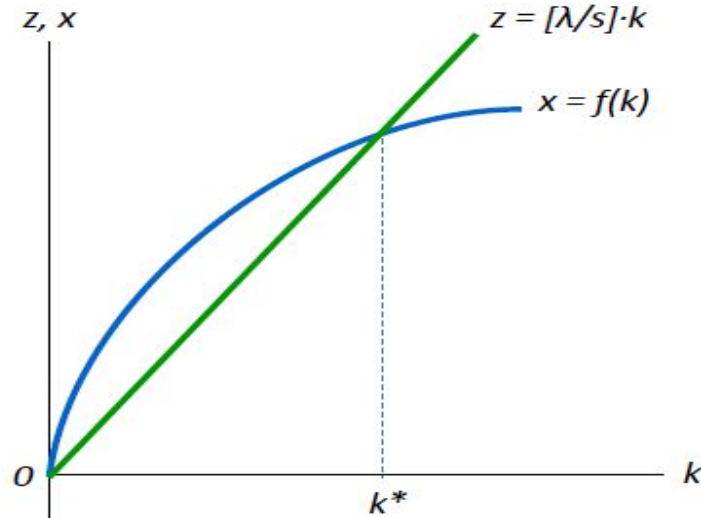


Figure 4: Existence of a unique admissible stationary solution k^* for the basic causal system derived for the per-capita Solow-Swan descriptive growth model, given s , λ , and $f(\cdot)$

In Fig. 4, a unique intersection point k^* is depicted. The existence of a unique intersection

point is a necessary consequence of the conditions imposed on the production function $f(\cdot)$ along with the assumed positivity of s and λ . In particular, whatever positive finite value λ/s takes on, the graph of $x = f(k)$ must rise *above* the graph of $z = [\lambda/s]k$ near 0 since $f(0) = 0$, $f'(k) > 0$ for all $k > 0$, and $f'(k)$ approaches $+\infty$ as k approaches 0. Moreover, the graph of x must eventually cross over the graph of z at some positive k value since $f'(k)$ approaches 0 as k approaches $+\infty$. Finally, the strict concavity of $f(k)$ guarantees that x and z have one and only one intersection point k^* over the interval $k > 0$.⁶

Keep in mind, however, that k^* is by construction a function of the exogenously given specifications $(s, \lambda, f(\cdot))$. This is clear from Fig. 4; for a change in either λ/s or in $f(\cdot)$ affects the graphs of z or x , respectively, which in turn changes the placement of the unique intersection point. For example, recalling that $\lambda = [g + \delta s]$, it is easily seen that an increase in s results in a decreased value for λ/s and hence in an increased value for the intersection point k^* .

What economic interpretation can be provided for the stationary solution k^* ? Recalling that $\lambda = [g + \delta s]$, it follows from (49) that k^* satisfies the relation

$$s/v^* - s\delta = g, \quad (50)$$

where $v^* = k^*/f(k^*)$ denotes the stationary capital-output ratio along the stationary solution path. The right side of (50) gives the *natural rate of growth* for the Solow-Swan economy, i.e., the growth rate of the labor supply $L(t)$. It will now be shown that the left side of (50) gives the *warranted rate of growth* for the Solow-Swan economy, i.e., the rate of growth of the capital stock $K(t)$. Consequently, k^* is the unique admissible initial state value for which the warranted rate of growth equals the natural rate of growth.

Using the basic equations of the Solow-Swan growth model in level form, it can be shown that $D_+K(t) = S(t)$ must hold along any solution path, i.e., net investment $D_+K(t)$ must equal net savings $S(t) = s[Y(t) - \delta K(t)]$. Consequently, at each time t along a solution path

⁶A stationary solution need not exist for the Solow-Swan model if the restrictions on the production function are relaxed to permit the marginal product of capital $f'(k)$ to be bounded away from zero as k approaches $+\infty$. However, the latter assumption has the unattractive implication that labor is nonessential in production, i.e., $F(K, 0) > 0$ for all $K > 0$. Also, multiple stationary solutions become possible for the Solow-Swan model if $f'(k)$ does not decline *monotonically* to zero as k approaches $+\infty$.

it must hold that

$$D_+K(t)/K(t) = S(t)/K(t) = s[Y(t) - \delta K(t)]/K(t) = s/v(t) - s\delta, \quad (51)$$

where $v(t) = k(t)/f(k(t))$. The left side of (50) is a special case of (51) with $k(t) = k^*$. An even easier way to see that the warranted and natural rates of growth must be equal at any stationary solution for the per-capita Solow-Swan growth model is as follows. Note that the existence of a stationary solution k^* implies $k(t) = k^*$ for all $t \geq 0$. Consequently, recalling that $k = K/L$, it holds for each $t \geq 0$ that

$$0 = D_+k(t)/k(t) = D_+K(t)/K(t) - D_+L(t)/L(t) \quad (52)$$

$$= D_+K(t)/K(t) - g. \quad (53)$$

In summary, at any stationary solution for the per-capita Solow-Swan growth model, both $K(t)$ and $L(t)$ are growing at the same constant rate g . Thus, a stationary solution for the per-capita Solow-Swan descriptive growth model corresponds to a (*balanced*) *steady-state growth solution* for the Solow-Swan descriptive growth model in level form, in the sense that the state variables K and L for the level-form model grow at the same constant rate. Since $y = f(k)$, another implication is that — in this stationary solution — the income level $Y(t)$ must be growing at this same constant rate g as well.

Can any additional economic interpretation be provided for k^* ? Using “Notes on Differential Equations,” it will first be shown that k^* is “locally stable” relative to \mathbf{V} , meaning that all solutions $\mathbf{k}(u)$ in \mathbf{V} with u sufficiently close to k^* must ultimately converge to k^* over time. Using phase diagram techniques, it will then be shown that k^* is actually “globally stable” relative to \mathbf{V} , meaning that all solutions $\mathbf{k}(u)$ in \mathbf{V} ultimately converge to k^* over time no matter how far the initial state value u is from k^* . Thus, k^* describes the long-run capital-labor ratio for the Solow-Swan economy, regardless of the particular admissible value $u > 0$ specified for the initial state $k(0)$. Recall, however, that k^* by construction is actually a *function* $k^*(s, \lambda, f(\cdot))$ of the exogenously given specifications for $(s, \lambda, f(\cdot))$. Hence, a change in any one of these specifications will lead to a change in the long-run outcome k^* .

Let $(s, \lambda, f(\cdot))$ be any admissible exogenously given specifications, and let $k^* > 0$ denote the unique stationary solution in the admissible solution set \mathbf{V} , conditional on these

specifications. Define a function $\psi: R_+ \rightarrow R$ by

$$\psi(k) = sf(k) - \lambda k, \quad (54)$$

where $R_+ = \{k \in R : k \geq 0\}$. It then follows from the admissibility conditions imposed on $(s, \lambda, f(\cdot))$ and the definition of k^* that $\psi(k)$ satisfies the following properties:⁷

$$\psi(0) = \psi(k^*) = 0; \quad (55)$$

$$\psi'(k) = sf'(k) - \lambda, \text{ with } \psi'(k^*) < 0; \quad (56)$$

$$\psi'(0) = +\infty; \quad (57)$$

$$\psi''(k) = sf''(k) < 0. \quad (58)$$

The graph of $\psi(k)$ is schematically depicted in Fig. 5.

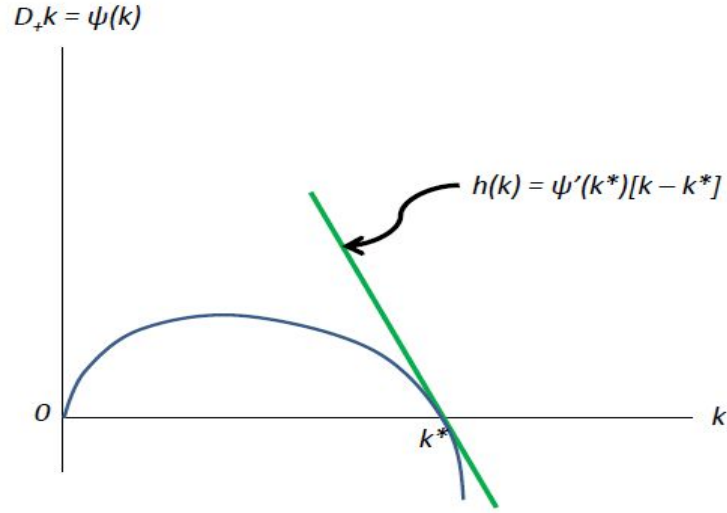


Figure 5: Linear approximation for the basic causal system $D_+k = \psi(k)$

The “linear approximation system” for the BCS is then constructed as follows. Using

⁷To establish that $\psi'(k^*) < 0$, consider the following useful fact that holds for any strictly concave twice differentiable function $h: R \rightarrow R$. By a simple Taylor’s expansion argument, expanding $h(0)$ around $h(k)$, and using $h''(k) < 0$, it can be shown that $h'(k) < [h(k) - h(0)]/k$ at each nonzero k . See any good real analysis textbook for details.

Taylor's Theorem, expand $\psi(k)$ about the stationary solution k^* to get

$$\psi(k) = \psi(k^*) + \psi'(k^*)[k - k^*] + \text{Remainder Term} \quad (59)$$

$$= 0 + \psi'(k^*)[k - k^*] + \text{Remainder Term} \quad (60)$$

$$\approx \psi'(k^*)[k - k^*] \text{ for } k \approx k^* . \quad (61)$$

As seen in Fig. 5, the function $h(k) = \psi'(k^*)[k - k^*]$ is a good linear approximation to the original nonlinear function $\psi(k)$ in a sufficiently small neighborhood of k^* .

Now define the *linear approximation system (LAS)* for the BCS by

$$D_+x(t) = Ax(t) , \quad t \geq 0 ; \quad (62)$$

$$x(0) = w ,$$

where $A = \psi'(k^*) < 0$ and $x(t) = [k(t) - k^*]$. Let $\mathbf{x}(w) = \{x(t; w) : t \geq 0\}$ denote any solution to (62), and let

$$\mathbf{V}^{\text{app}} = \{\mathbf{x}(w) \mid w \in R\} \quad (63)$$

denote the set of all admissible solutions for the LAS (62). Note that 0 is the unique stationary solution for the LAS.

As detailed in the earlier Course Packet reading titled “Notes on Differential Equations,” the stationary solution 0 is stable relative to the admissible solution set \mathbf{V}^{app} if and only if all of the eigenvalues of A have negative real parts. However, as established earlier, A is a negative real number. Hence, trivially, A is its own unique eigenvalue and it clearly has a “negative real part.” Thus 0 is a stable stationary solution for (62) relative to \mathbf{V}^{app} . Moreover, 0 is a stable stationary solution relative to \mathbf{V}^{app} *only* if k^* is a *locally* stable solution for the BCS (45) relative to the admissible solution set \mathbf{V} . This completes the proof of the local stability of k^* .

Although this proof is a useful illustration of local linear approximation techniques for nonlinear differential systems, for the special nonlinear differential system at hand—the one-dimensional BCS for the per capita Solow-Swan descriptive growth model—it is actually possible to use “phase diagram techniques” to establish the *global* stability of k^* relative to \mathbf{V} . Refer to the graph of $D_+k = \psi(k)$ in Fig. 6. The arrowheads on the graph indicate the direction of motion in k at each domain point $k > 0$.

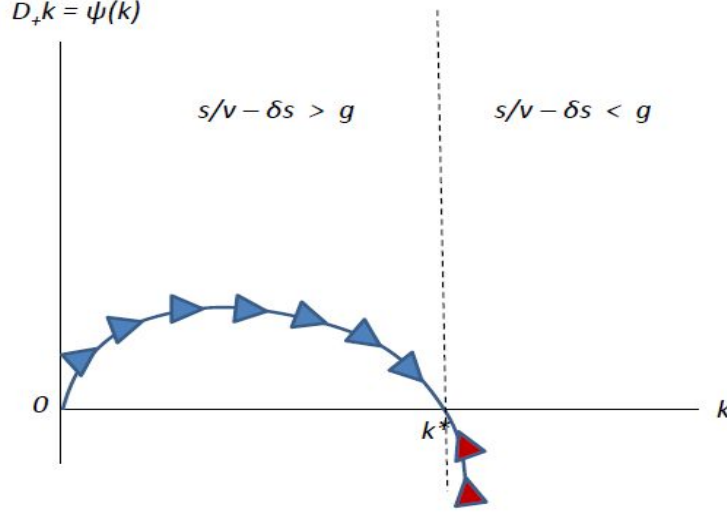


Figure 6: Phase diagram for the basic causal system $D_+k = \psi(k)$, where $v = k/f(k)$

Note, in particular, that $D_+k > 0$ if and only if $k < k^*$, $D_+k = 0$ if and only if $k = k^*$, and $D_+k < 0$ if and only if $k > k^*$. Thus, given any $u > 0$ for the initial state $k(0)$, with $u \neq k^*$, the direction of motion is always in the direction of k^* . It follows that k^* is a globally stable stationary solution relative to the admissible solution set \mathbf{V} .

As a corollary of this argument, note from Fig. 6 that the Solow-Swan descriptive growth model predicts that D_+k tends to be larger for smaller values of k . The latter finding is the source of the prediction that, all else equal, poorer countries should exhibit faster rates of growth than richer countries.

Why, on intuitive economic grounds, does the per capita Solow-Swan model exhibit global stability? Recall from (50) and (51) that a stationary solution k^* is characterized as a point of equality for the warranted rate of growth $[s/v(t)] - \delta s = D_+K(t)/K(t)$ and the natural rate of growth $g = D_+L(t)/L(t)$. Suppose the economy is at a point in time where the warranted rate of growth exceeds the natural rate of growth, so that capital is accumulating at a faster rate than labor. Thus, firms must be substituting capital for labor in the production process, i.e., $k(t)$ must be increasing, which implies (using the conditions imposed on $f(k)$) that the average product of capital $f(k(t))/k(t)$ is decreasing. It follows that $v(t) \equiv k(t)/f(k(t))$ is increasing, hence $[s/v(t)] - \delta s$ is decreasing — i.e., it is tending *back* towards g . Conversely, whenever the economy is at a point where the natural rate of growth exceeds the warranted

rate, a similar economic argument can be given for why the warranted rate should then be increasing. Consequently, as indicated in Fig. 6, the direction of motion in $k(t)$ is always towards a point of equality between the warranted and natural rates of growth, i.e., towards the stationary solution k^* .

By construction, the long-run stationary solution value k^* is the solution for equation (50) and hence a function $k^* = k^*(s, \lambda, f(\cdot))$ of the exogenously specified factors $(s, \lambda, f(\cdot))$ appearing in equation (50). Although the solution trajectories for the nonlinear BCS are guaranteed to converge in the long run to $k^*(s, \lambda, f(\cdot))$, given any initial state value $u > 0$, their dependence in the short run on u , s , λ , and $f(\cdot)$ can take a rather complicated form. For example, suppose the per-capita production function $f(\cdot)$ takes the commonly used Cobb-Douglas form

$$f(k) = k^\beta \quad (64)$$

for some $\beta \in (0, 1)$. Given (64), the solution to the BCS (45) is⁸

$$k(t; u) = \left([(u)^{1-\beta} - s/\lambda] e^{-[1-\beta]\lambda t} + s/\lambda \right)^{1/[1-\beta]}, \quad t \geq 0. \quad (65)$$

Note that (65) is a highly nonlinear function of the initial state value u as well as the model parameters (s, λ, β) . Nevertheless, for each given parameter vector (u, s, λ, β) , the solution value (65) for the time t capital-labor ratio $k(t; u)$ converges as t approaches infinity to the stationary solution value

$$k^*(s, \lambda, \beta) = \left(\frac{s}{\lambda} \right)^{1/(1-\beta)}. \quad (66)$$

5 Incorporating Technological Change

Recall the first four stylized growth facts [SF1]-[SF4] listed in Section 1:⁹

⁸Given (64) and $u > 0$, it follows from a simple phase diagram argument that $k(t; u) > 0$ for all $t \geq 0$ along the solution path for the BCS. Introduce the variable transformation $a(t; \hat{u}) = k(t; u)/f(k(t; u)) = k(t; u)^{1-\beta}$, with $\hat{u} = u^{1-\beta}$. Using the BCS, it is straightforward to show that $D_+ a(t; \hat{u})/a(t; \hat{u}) = [1-\beta]D_+ k(t; u)/k(t; u)$. In terms of $a(t; \hat{u})$, the BCS thus reduces to a nonhomogeneous linear differential equation with constant coefficients: $D_+ a(t; \hat{u}) = m a(t; \hat{u}) + b$ with $m = -[1-\beta]\lambda$, $b = [1-\beta]s$, and $a(0; \hat{u}) = \hat{u}$. The solution to this reduced BCS can be represented as the sum of the homogenous solution $a_H(t) = c \cdot \exp(mt)$ (for some constant c) and the particular stationary solution $\bar{a} = s/\lambda$, with $a_H(0) + \bar{a} = c + \bar{a} = \hat{u}$ (implying $c = \hat{u} - \bar{a}$). The reduced BCS solution thus takes the form $a(t; \hat{u}) = [c \exp(-[1-\beta]\lambda t) + s/\lambda]$, where $c = [\hat{u} - s/\lambda]$. The solution (65) for $k(t; u)$ is then directly obtained by an inverse transformation.

⁹The fifth stylized fact will not be addressed in these notes since it involves difficult cross-country empirical estimation issues. See Temple (1999) for a detailed discussion of this fifth stylized fact in relation to the

[SF1] The ratio K/Y of physical capital K to output Y is nearly constant over time.

[SF2] The shares of labor and physical capital in national income, wL/Y and rK/Y , are nearly constant over time, where w and r denote the real wage rate of labor and the real rental rate of capital, respectively;

[SF3] Per capita output $y = Y/L$ grows over time without a tendency to converge to a constant value.

[SF4] Per-capita physical capital $k = K/L$ grows over time without a tendency to converge to a constant value.

It is straightforward to show that the first stylized fact [SF1] is consistent with the long-run predictions of the per-capital Solow-Swan descriptive growth model as developed to date. Moreover, if the real wage w is given by the marginal product of labor $F_L(K, L) = [f(k) - f'(k)k]$ and the real rental rate r is given by the marginal product of capital $F_K(K, L) = f'(k)$, then the second stylized fact [SF2] is also consistent with the long-run predictions of this model.

On the other hand, the stylized facts [SF3] and [SF4] are not consistent with the long-run predictions of the per-capita Solow-Swan descriptive growth model as developed to date. The latter model predicts that y and k will converge to constant levels y^* and k^* over time, a contradiction of [SF3] and [SF4].

Consequently, in order for consistency to be achieved, the production relations have to be modified. In particular, this modification must permit y and k to each vary over time (without convergence) while, at the same time, k/y is essentially constant over time.

As will be seen below, a modification that ensures the model's predictions are in conformity with stylized facts [SF1] through [SF4] while retaining much of its elegantly simple structure is the introduction of labor-augmenting technological change.¹⁰ The aggregate production function for the Solow-Swan descriptive growth model, extended to include labor-

Solow-Swan descriptive growth model.

¹⁰Apart from special circumstances, such as when the production function takes the Cobb-Douglas form, the only type of technological change (at a constant rate) that is compatible with the existence of a stationary solution in per-capita terms for the Solow-Swan descriptive growth model is labor-augmenting technological change. See, for example, Barro and Sala-i-Martin (2003, pp. 51-53).

augmenting technological change, generally takes the form

$$Y = A \cdot F(K, N) , \quad (67)$$

where: Y measures real output (GDP); A is a constant (time-invariant) measure of *total factor productivity (TFP)*; $F(\cdot)$ is a strictly increasing concave function exhibiting constant returns to scale; K denotes capital inputs; and $N = B \cdot L$ denotes effective labor inputs, where B measures the skill level of the labor force (due to embodied technological change) and L measures raw labor inputs. An additional standard assumption is that B and L grow at exogenously given rates μ and g over time.

Suppose the basic Solow-Swan descriptive growth model is appropriately extended to include the production function (67) in place of $F(K, N)$, a defining relation $N = BL$ for effective labor, and a relation $D_+B(t) = \mu B(t)$ determining the growth in labor skill over time. The complete model, hereafter referred to as Model M, is appended at the end of these notes.

As a first step towards establishing that Model M is in conformity with [SF1] through [SF4], consider the per-capita version of this model, hereafter referred to as Model M*, that is also appended at the end of this packet. It will next be shown, step by step, how the three equations for Model M* can be derived from the six equations for Model M using the following definitions that link the two models: $\hat{k}(t) = K(t)/N(t)$; $\hat{y}(t) = Y(t)/N(t)$; $\hat{s}(t) = S(t)/N(t)$; and $f(\hat{k}) = F(\hat{k}, 1)$.

First note that the admissibility conditions for Model M imply that $B(0) > 0$ and $L(0) > 0$, hence $N(0) = B(0)L(0) > 0$. It can then be shown either by direct solution¹¹ of equations (107) and (108), or by a forward recursion argument, that $N(t) = B(t)L(t) > 0$ for all $t \geq 0$.

Now let $t \geq 0$ be given. Divide the first three equations (104),(105), and (106) of Model M by $N(t)$, and use constant returns to scale for $F(K, N)$, to get

$$\hat{y}(t) = Af(\hat{k}(t)) \quad (68)$$

$$\hat{s}(t) = s[Af(\hat{k}(t)) - \delta\hat{k}(t)] \quad (69)$$

$$D_+K(t)/N(t) = s\hat{s}(t) \quad (70)$$

¹¹For any linear homogenous differential equation of the form $D_+x(t) = ax(t)$ with initial condition $x(0) = u$, the solution is given by $x(t) = ue^{at}$. Consequently, if $u > 0$, it follows that $x(t) > 0$ for all $t \geq 0$.

Using equations (107), (108), and (109) of Model M, it follows that

$$\begin{aligned}
\frac{D_+N(t)}{N(t)} &= \frac{D_+[B(t)L(t)]}{B(t)L(t)} \\
&= \frac{D_+B(t)}{B(t)} + \frac{D_+L(t)}{L(t)} \\
&= \mu + g .
\end{aligned} \tag{71}$$

Thus,

$$\begin{aligned}
\frac{D_+\hat{k}(t)}{\hat{k}(t)} &= \frac{D_+K(t)}{K(t)} - \mu - g \\
&= \frac{[\frac{D_+K(t)}{N(t)} - \mu\hat{k}(t) - g\hat{k}(t)]}{\hat{k}(t)}
\end{aligned} \tag{72}$$

Hence,

$$D_+K(t)/N(t) = D_+\hat{k}(t) + [\mu + g]\hat{k}(t) \tag{73}$$

Substituting equation (73) into equation (70), using equation (69), and manipulating terms, one gets

$$D_+\hat{k}(t) = sAf(\hat{k}(t)) - \theta\hat{k}(t) \tag{74}$$

where

$$\theta = [\mu + g + s\delta] \tag{75}$$

Equations (68), (69), and (74) are the three equations of Model M*, as desired.

It will next be shown that Model M* has a unique admissible stationary solution $\hat{k}^* > 0$, given any admissible specification for $(A, s, \delta, \theta, f(\hat{k}))$.

Let an admissible specification for $(A, s, \delta, \theta, f(\hat{k}))$ in Model M* be given. Given this specification, $\hat{k}^* > 0$ is an admissible stationary solution for Model M* if and only if \hat{k}^* satisfies

$$0 = sAf(\hat{k}^*) - \theta\hat{k}^* \tag{76}$$

Define two functions of \hat{k} as follows: $x(\hat{k}) = sAf(\hat{k})$ and $z(\hat{k}) = \theta\hat{k}$. Then $\hat{k}^* > 0$ is an admissible stationary solution for Model M* if and only if $x(\hat{k})$ and $z(\hat{k})$ intersect at $\hat{k} = \hat{k}^*$.

The admissibility conditions for Model M* imply that $s > 0$, $A > 0$, and $\theta > 0$. They also imply that $x(0) = 0$, $x'(\hat{k}) = sAf'(\hat{k}) > 0$, $x''(\hat{k}) = sAf''(\hat{k}) < 0$, $x'(\hat{k})$ approaches $+\infty$

as \hat{k} approaches 0, and $x'(\hat{k})$ approaches 0 as \hat{k} approaches $+\infty$. It follows that $x(\hat{k})$ and $z(\hat{k})$ have one and only one intersection point \hat{k}^* over the admissible range $\hat{k} > 0$. This could be illustrated using a carefully labeled graph, analogous to Fig. 4 for the original Solow-Swan descriptive growth model in per-capita form.

Next, let an admissible specification $(A, s, \delta, \theta, f(\hat{k}))$ for Model M* be given. Let $\hat{k}^* > 0$ denote the unique admissible stationary solution for Model M* corresponding to this admissible specification, whose existence has just been established. A graphical analysis will now be used to establish the *global stability* of \hat{k}^* relative to the family of all possible admissible solutions for Model M* conditional on the admissible specification $(A, s, \delta, \theta, f(\hat{k}))$.

Define a function $\psi: R_+ \rightarrow R$ by $\psi(\hat{k}) = sAf(\hat{k}) - \theta\hat{k}$. The admissibility conditions imposed on $f(\hat{k})$ in Model M* then imply that

$$\psi(0) = \psi(\hat{k}^*) = 0 \quad (77)$$

$$\psi'(\hat{k}) = sAf'(\hat{k}) - \theta \quad (78)$$

$$\psi'(0) = +\infty \quad (79)$$

$$\psi''(\hat{k}) = sAf''(\hat{k}) < 0 \quad (80)$$

$$\psi'(\hat{k}^*) < \frac{[\psi(\hat{k}^*) - \psi(0)]}{[\hat{k}^* - 0]} = 0, \quad (81)$$

where relation (81) follows from the analysis in Footnote 7. If a plot for $\psi(\hat{k})$ in the R^2 plane is graphically depicted, with “phase diagram” arrows indicating the direction of motion in \hat{k} at each admissible point $(\hat{k}, \psi(\hat{k}))$, the resulting phase diagram will look very much like the one depicted in Fig. 6 for the basic Solow-Swan descriptive growth model in per-capita form. In particular, starting at any admissible initial state value $\hat{k} > 0$, the direction of motion in \hat{k} is always towards the unique stationary solution \hat{k}^* . It follows that \hat{k}^* is globally stable relative to the family of all admissible solutions for Model M*, conditional on the admissible specification $(A, s, \delta, \theta, f(\hat{k}))$.

Using the above findings, it will next be explained how the stylized facts [SF1] through [SF4] are satisfied *in the long run* by any admissible solution for Model M.

Let an admissible specification $(K(0), B(0), L(0), A, s, \delta, \mu, g, F(K, N))$ for Model M be given, which implies an admissible specification $(\hat{k}(0), A, s, \delta, \mu, g, f(\hat{k}))$ for Model M*. By

previous developments,

$$\frac{K(t)}{N(t)} = \hat{k}(t) \rightarrow \hat{k}^* \text{ as } t \rightarrow +\infty \quad (82)$$

It follows that

$$\frac{K(t)}{Y(t)} = \frac{\hat{k}(t)}{\hat{y}(t)} = \frac{\hat{k}(t)}{f(\hat{k}(t))} \rightarrow \frac{\hat{k}^*}{f(\hat{k}^*)} \text{ as } t \rightarrow +\infty \quad (83)$$

Thus, [SF1] holds in the long run for any Model M admissible solution.

Next, recall that $f(\hat{k}) = F(\hat{k}, 1)$, which implies that $ANf(\hat{k}) = AF(K, N)$. Suppose the real wage $w(t)$ is given by the marginal product of raw labor, i.e.,

$$w(t) = \frac{\partial AF(K(t), N(t))}{\partial L} = AF_N(K(t), N(t))B = [Af(\hat{k}(t)) - Af'(\hat{k}(t))\hat{k}(t)]B \quad (84)$$

and the real rental rate $r(t)$ is given by the marginal product of capital, i.e.,

$$r(t) = \frac{\partial AF(K(t), N(t))}{\partial K} = Af'(\hat{k}(t)) \quad (85)$$

Thus,

$$\frac{wL}{Y} = \frac{(w/B) \cdot N}{(Y/N) \cdot N} = \frac{[Af(\hat{k}) - Af'(\hat{k})\hat{k}]}{Af(\hat{k})} \quad (86)$$

$$\frac{rK}{Y} = \frac{r[K/N]}{[Y/N]} = \frac{[Af'(\hat{k})]\hat{k}}{Af(\hat{k})} \quad (87)$$

It then follows from (82), (86), and (87) that stylized fact [SF2] holds in the long run for any admissible solution of Model M.

Another implication of (82) is that

$$\hat{y}(t) = \frac{Y(t)}{N(t)} = \frac{AF(K(t), N(t))}{N(t)} = Af(\hat{k}) \rightarrow Af(\hat{k}^*) \text{ as } t \rightarrow +\infty \quad (88)$$

Using (88) together with (71), it follows that

$$\begin{aligned} \frac{D_+\hat{y}(t)}{\hat{y}(t)} &= \left[\frac{D_+Y(t)}{Y(t)} - \frac{D_+N(t)}{N(t)} \right] \\ &= \left[\frac{D_+Y(t)}{Y(t)} - \mu - g \right] \\ &\rightarrow 0 \text{ as } t \rightarrow +\infty \end{aligned} \quad (89)$$

Note also that (82), together with (71), implies

$$\begin{aligned}
\frac{D_+ \hat{k}(t)}{\hat{k}(t)} &= \left[\frac{D_+ K(t)}{K(t)} - \frac{D_+ N(t)}{N(t)} \right] \\
&= \left[\frac{D_+ K(t)}{K(t)} - \mu - g \right] \\
&\rightarrow 0 \text{ as } t \rightarrow +\infty
\end{aligned} \tag{90}$$

Let $y(t) = Y(t)/L(t)$ and $k(t) = K(t)/L(t)$ denote per-capita output and per-capita capital calculated in terms of *raw* labor $L(t)$. It then follows from (89) that

$$\lim_{t \rightarrow \infty} D_+ y(t)/y(t) = \left[\lim_{t \rightarrow \infty} D_+ Y(t)/Y(t) - \lim_{t \rightarrow \infty} D_+ L(t)/L(t) \right] = \mu > 0 \tag{91}$$

Similarly, it follows from (90) that

$$\lim_{t \rightarrow \infty} D_+ k(t)/k(t) = \left[\lim_{t \rightarrow \infty} D_+ K(t)/K(t) - \lim_{t \rightarrow \infty} D_+ L(t)/L(t) \right] = \mu > 0 \tag{92}$$

Consequently, per-capita output $y(t) = Y(t)/L(t)$ and per-capita capital $k(t) = K(t)/L(t)$ now grow over time without a tendency to converge to constant values, in conformity with stylized facts [SF3] and [SF4]. Note, however, that the long-run growth in $y(t)$ and $k(t)$ is due solely to the exogenously given (and hence unexplained) rate μ of technological progress embodied in the effective labor force $N(t) = B(t) \cdot L(t)$.

6 The Solow Growth Accounting Equation

Another important contribution by Solow is his decomposition of the growth in output by source, resulting in his now-famous “growth accounting equation.” This section derives this famous relation in four steps under successively stronger assumptions. However, it should be noted that even step 1, which presumes the existence of an aggregate production function of the form $Y = AF(K, L)$, involves very strong assumptions; see, e.g., Tesfatsion (2015).

Step One: Suppose A varies over time.

Consider the aggregate production function (67) given by $Y = AF(K, N)$, with $N = B \cdot L =$ effective labor. Suppose $F(K, N)$ is differentiable over the domain $R_{++}^2 = \{(K, N) \in R^2 \mid K > 0, N > 0\}$. Suppose, also, that $A(t)$, $K(t)$, $B(t)$, and $L(t)$ are

right differentiable functions of time t over $t > 0$. Then, taking the (right) total differential of $Y(t) = A(t)F(K(t), N(t))$ at some time $t > 0$, and suppressing time arguments for ease of representation, one has

$$D_+Y = [A \cdot F_K \cdot D_+K] + [A \cdot F_N \cdot D_+N] + [D_+A \cdot F]. \quad (93)$$

Note that constant returns to scale is *not* required to obtain (93). Dividing (93) by $Y = AF(K, N)$, and manipulating terms, one then obtains

$$D_+Y/Y = \beta_K D_+K/K + \beta_N D_+N/N + D_+A/A, \quad (94)$$

where:

$$D_+A/A = \text{Total Factor Productivity (TFP) growth rate}; \quad (95)$$

$$\beta_K = \frac{\partial Y}{\partial K} \cdot \frac{K}{Y} = [A \cdot F_K \cdot K]/Y = \text{elasticity of output with respect to K}; \quad (96)$$

$$\beta_N = \frac{\partial Y}{\partial N} \cdot \frac{N}{Y} = [A \cdot F_N \cdot N]/Y = \text{elasticity of output with respect to N}. \quad (97)$$

Step Two: Suppose, also, that input markets are competitive.

Suppose, in addition, that input markets are perfectly competitive in the sense that the real rental rate $r = R/P$ equals the marginal product of capital $A \cdot F_K$ and the real wage $w = W/P$ equals the marginal product of raw labor $A \cdot F_N \cdot B$. In this case (94) reduces to the famous *Solow growth accounting equation* with

$$\beta_K = r \cdot K/Y = \text{capital share of GDP}; \quad (98)$$

$$\beta_N = w \cdot L/Y = \text{labor share of GDP}. \quad (99)$$

Step Three: Suppose the production function exhibits constant returns.

Suppose, in addition, that $F(K, N)$ exhibits constant returns to scale. It then satisfies the *Euler Theorem (exhaustion of product)*¹² $F(K, N) = F_K \cdot K + F_N \cdot N$. In this case the coefficients (98) and (99) satisfy $\beta_K + \beta_N = 1$.

Step Four: Suppose the production function has a Cobb-Douglas form.

¹²Recall definition (10) for constant returns to scale. Replacing L by N in (10), differentiating each side with respect to b , and then setting b equal to 1, the result is $F(K, N) = F_K(K, N) \cdot K + F_N(K, N) \cdot N$.

Finally, suppose in addition that the production function takes a Cobb-Douglas form. That is, suppose

$$Y = AF(K, N) = A \cdot K^\alpha \cdot N^{[1-\alpha]} \quad , \quad 0 < \alpha < 1 \quad (100)$$

In this case it is straightforward to show that $\beta_K = \alpha$ and $\beta_N = [1 - \alpha]$. Thus, relation (94) reduces to

$$D_+Y/Y = \alpha \cdot D_+K/K + [1 - \alpha] \cdot D_+N/N + D_+A/A \quad , \quad (101)$$

or equivalently,

$$[D_+Y/Y - D_+N/N] = \alpha \cdot [D_+K/K - D_+N/N] + D_+A/A \quad . \quad (102)$$

Defining $\hat{y} \equiv Y/N$ and $\hat{k} \equiv K/N$, relation (102) can equivalently be expressed as follows:

$$\frac{D_+\hat{y}}{\hat{y}} = \alpha \cdot \frac{D_+\hat{k}}{\hat{k}} + \frac{D_+A}{A} \quad . \quad (103)$$

7 Concluding Remarks

Many studies have attempted to estimate a relation such as (94) or (101) in order to measure the separate contributions of K , N , and A to the growth rate in per capita real GDP. Typically, these studies have found that estimates for DA/A are relatively large and strongly positively correlated with DY/Y . Moreover, in tests of (101), the capital share α is consistently found to be only about 1/3, implying that very large changes in capital are needed to have any significant effect on the growth rate of $Y(t)$.

Consequently, in the Solow-Swan descriptive growth model with labor-augmenting technological change, the growth in $Y(t)$ is largely driven by forces not explained within the model itself. These forces are the growth rate of raw labor, $L(t)$, and technological change as embodied in the exogenously determined growth rates for labor skill $B(t)$ and total factor productivity $A(t)$.

It is for this reason that many “endogenous growth” theorists such as Gary Becker, Paul Romer, Gene Grossman, and Elhanan Helpman, among others, claim that the Solow-Swan descriptive growth model with or without labor-augmenting technological change is an *exogenous growth* model that does not actually explain long-run growth. See, for example, Aghion and Howitt (2008).

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MODEL M
SOLOW-SWAN DESCRIPTIVE GROWTH MODEL WITH
LABOR-AUGMENTING TECHNOLOGICAL CHANGE

Model Equations: $t \geq 0$:

$$Y(t) = AF(K(t), N(t)) \quad (104)$$

$$S(t) = s \cdot [Y(t) - \delta K(t)] \quad (105)$$

$$D_+K(t) = S(t) \quad (106)$$

$$D_+B(t) = \mu \cdot B(t) \quad (107)$$

$$D_+L(t) = g \cdot L(t) \quad (108)$$

$$N(t) = B(t)L(t) \quad (109)$$

Classification of Variables:

Time- t Endogenous Variables ($t \geq 0$): $Y(t)$, $S(t)$, $D_+K(t)$, $D_+B(t)$, $D_+L(t)$, $N(t)$

Time- t Predetermined (State) Variables ($t > 0$):

$$K(t) = \int_0^t D_+K(\tau) d\tau + K(0) \quad (110)$$

$$B(t) = \int_0^t D_+B(\tau) d\tau + B(0) \quad (111)$$

$$L(t) = \int_0^t D_+L(\tau) d\tau + L(0) \quad (112)$$

Admissible Exogenous Variables and Functional Forms:

$K(0)$, $B(0)$, $L(0)$, A , s , δ , μ , and g , satisfying $0 < K(0)$, $0 < B(0)$, $0 < L(0)$, $0 < A$, $0 < s < 1$, $0 \leq \delta$, $0 < \mu$, and $0 < g$, plus a function $F(K, N)$ that satisfies the following *Standard Neoclassical Production Function Assumptions in Level Form*:

- a. $F(K, N)$ exhibits constant returns to scale;
- b. $F(K, N)$ is continuous over $(K, N) \geq 0$;
- c. $F(K, N)$ is twice continuously differentiable and concave, with $F_{KK}(K, N) < 0$, over all $(K, N) > 0$;
- d. $F_K(K, N) > 0$ and $F_L(K, N) > 0$ for all $(K, N) > 0$;
- e. $F(0, N) = 0$ for all $N \geq 0$;
- f. [Inada Conditions] For each $N > 0$, $F_K(K, N) \rightarrow +\infty$ as $K \rightarrow 0$ and $F_K(K, N) \rightarrow 0$ as $K \rightarrow +\infty$.

MODEL M*
PER-CAPITA VERSION OF MODEL M

Model Equations: For each time $t \geq 0$,

$$\hat{y}(t) = Af(\hat{k}(t)) ; \quad (113)$$

$$\hat{s}(t) = s[\hat{y}(t) - \delta\hat{k}(t)] ; \quad (114)$$

$$D_+\hat{k}(t) = sAf(\hat{k}(t)) - \theta\hat{k}(t) . \quad (115)$$

Classification of Variables:

Time-t Endogenous Variables ($t \geq 0$): $\hat{y}(t)$, $\hat{s}(t)$, $D_+\hat{k}(t)$;

Time-t Predetermined (State) Variable ($t > 0$):

$$\hat{k}(t) = \int_0^t D_+\hat{k}(\tau)d\tau + \hat{k}(0) \quad (116)$$

Admissible Exogenous Variables and Functional Forms:

$\hat{k}(0)$, A , s , δ , and $\theta = [\mu + g + s\delta]$, where $0 < \hat{k}(0)$, $0 < A$, $0 < s < 1$, $0 \leq \delta$, $0 < \mu$, and $0 < g$.

Also, $f(\hat{k}) \equiv F(\hat{k}, 1)$ satisfies the following *Standard Neoclassical Production Function Assumptions in Per-Capita Form*: $f(\hat{k})$ is continuous over $\hat{k} \geq 0$, and $f(\hat{k})$ is twice continuously differentiable with $f'(\hat{k}) > 0$ and $f''(\hat{k}) < 0$ over $\hat{k} > 0$; $f(0) = 0$; and $f'(\hat{k}) \rightarrow +\infty$ as $\hat{k} \rightarrow 0$ and $f'(\hat{k}) \rightarrow 0$ as $\hat{k} \rightarrow +\infty$.