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Course Site for Econ 502 (M.S. Macroeconomic Theory:)
https://www2.econ.iastate.edu/tesfatsi/Syl502.htm
Elements of Dynamic Economic Modeling:
https://www2.econ.iastate.edu/tesfatsi/DynamicEconomicModelingBasics.WPVersion.pdf
The Basic Solow-Swan Descriptive Growth Model:
https://www2.econ.iastate.edu/tesfatsi/SolowMod.pdf
A Simple Illustrative Optimal Growth Model:
https://www2.econ.iastate.edu/tesfatsi/OptGrow.pdf

A SIMPLE ILLUSTRATIVE OPTIMAL GROWTH MODEL

A. From Descriptive to Optimal Growth

Recall the following relation describing the change in the capital-labor ratio $k(t) \equiv K(t)/L(t)$ for the per-capita version of the *basic Solow-Swan descriptive growth (BSSDG)* model developed in an earlier section of the course:

$$Dk_{+}(t) = f(k(t)) - [g+\delta]k(t) - c(t)\ ,\ t \geq 0\ \ . \tag{1}$$

Here $c(t)$ denotes per capita consumption at time t, $g > 0$ denotes the growth rate of labor, $\delta \geq 0$ denotes the capital depreciation rate, and $f(k)$ denotes the per-capita production function satisfying the standard neoclassical production function assumptions in per-capita form.[1]

Let $\theta \equiv [g+\delta] > 0$, and let $i(t)$ denote per capita gross investment in period t, i.e.,

$$i(t)\ =\ [D_{+}K(t) + \delta K(t)]/L(t)\ =\ [D_{+}k(t) + \theta k(t)]. \tag{2}$$

Relation (1) can then be re-expressed as a "production possibility frontier" tradeoff between consumption and gross investment taking the output level $f(k(t))$ as given:

$$c(t) + i(t)\ =\ f(k(t))\ . \tag{3}$$

[1] More precisely, $f(0) = 0$, $f(k)$ is continuous over all $k \geq 0$ and twice continuously differentiable over all $k > 0$ with $f'(k) > 0$ and $f''(k) < 0$, $\lim_{k\to\infty} f'(k) = 0$, and $\lim_{k\to 0} f'(k) = \infty$.

Such a trade-off curve for time t might be as depicted in Fig. 1.

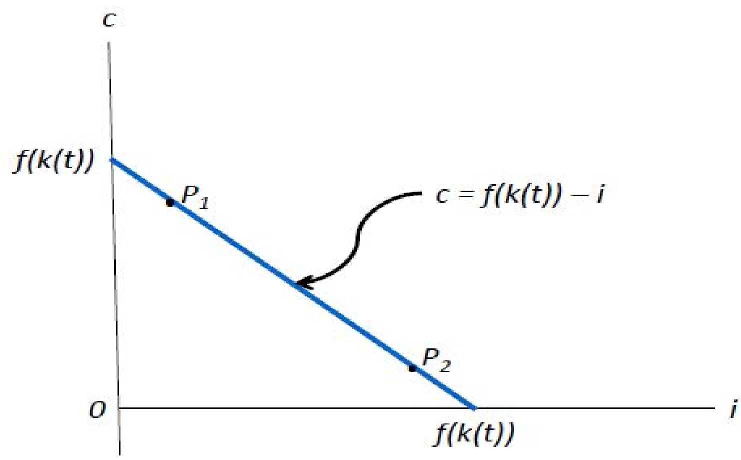

Figure 1: Investment-consumption trade-offs at time t, conditional on $k(t)$

For the BSSDG model, a point on this time-t consumption-investment production possibility frontier is determined by the direct specification of a consumption demand function for $c(t)$. For example, we considered the particular case in which consumption demand was assumed to be a fixed proportion $[1 - s]$ of net income $[y - \delta k]$. More generally, however, the relevant trade-off for society at any time t might not be time-t consumption versus time-t investment, but rather the *utility* of an addition to time-t consumption versus the *utility* of an addition to time-t investment.

A completely myopic society interested only in the maximization of current instantaneous utility would never invest, because investment does not directly enter the instantaneous utility function. However, if current consumption is increased at the expense of current investment, the maximum attainable level of consumption in each future period is decreased. For example, referring to Fig. 1, the relatively high consumption point P_1 at time t looks more desirable than the low consumption point P_2 if the only concern is currently attainable utility. Yet the choice of P_1 rather than P_2 results in lower future capital levels and hence a contraction inward of all future consumption-investment trade-off curves. Consequently, the utility attainable from *future* consumption is decreased.

In the "optimal growth" version of the BSSDG model, the choice of consumption in any period t is assumed to take into account the affects of this choice on the utility attainable from *future* consumption. A simple illustration of an optimal growth model will now be given.

B. The Basic Optimal Growth Problem as a Social Planning Problem

B.1 Overview

Consider an economy managed by a social planner that exists over the time interval $[0, T]$. The economy has an initial endowment of capital that depreciates at a fixed positive rate and an initial population of one-period-lived agents, each with the same labor endowment and preferences. The population of these agents is growing at a fixed positive rate, implying aggregate labor is growing at this same fixed positive rate. Each agent is assumed to supply their entire labor endowment inelastically in each period.

Production possibilities in the economy are represented by an aggregate production function exhibiting constant returns to scale, and agents are assumed to attain utility from the consumption of the output from this production. Since labor is supplied inelastically, leisure does not appear as an argument of the utility function; agents do not make trade-offs between output consumption and consumption of leisure. Finally, there is a given capital-labor ratio target for the final time T.

The basic optimal growth problem faced by the social planner is then as follows: Allocate production in each period between consumption and capital investment so as to maximize the discounted sum of the utility levels achieved by representative agents in each successive generation of agents while satisfying the given capital-labor ratio target for the final time T.

The first economist to investigate this basic optimal growth problem was apparently Frank Ramsey (1928). His work was largely ignored at the time by the economics profession, presumably due both to its technical nature and to the disruption caused by the Great

Depression and World War II. Interest in the basic optimal growth problem was revived in the nineteen fifties, notably by Jan Tinbergen, and it was solved by Tjalling C. Koopmans and David Cass. For extensive bibliographical references to this early literature, see Burmeister and Dobell (1970).

B.2 Analytical Formulation

Suppose the welfare of a representative one-period-lived agent in the economy is measured by a utility function $u{:}R_{++} \to R$ that is twice continuously differentiable with $u' > 0$ and $u'' < 0$. The *discount rate* measuring the degree to which future utility is discounted relative to current utility is given by an exogenous constant $\rho \geq 0$.

Let $\theta \equiv [g + \delta]$, where $g > 0$ denotes an exogenously given growth rate of labor and $\delta \geq 0$ denotes an exogenously given capital depreciation rate. Let k_0 and k_T denote exogenously given values for the initial and final per-capita capital ratios (equivalently, capital-labor ratios), the first value interpreted as an *historically* given value and the second value interpreted as a *target* value the society desires to attain in period T. Finally, let the sequences of real-valued per-capita capital and consumption levels over $[0, T]$ be denoted by

$$\mathbf{k} \;=\; (k(t) : t \in [0, T]) \;; \tag{4}$$

$$\mathbf{c} \;=\; (c(t) : t \in [0, T]) \;. \tag{5}$$

Consider the following intertemporal utility maximization problem:

$$\max_{\mathbf{c},\mathbf{k}} \int_0^T u(c(t)) e^{-\rho t} dt \tag{6}$$

subject to

$$c(t) \;=\; f(k(t)) - \theta k(t) - D_+ k(t) \;, \quad 0 \leq t \leq T \;; \tag{7}$$

$$k(0) \;=\; k_0 \;; \tag{8}$$

$$k(T) \;=\; k_T \;. \tag{9}$$

Problem (6)-(9) will now be expressed in a more compact reduced form.

Let $\mathbf{K}$ denote the collection of all twice differentiable[2] functions $\mathbf{k}$ taking the form $k{:}[0,T] \to R$ with boundary conditions $k(0) = k_0$ and $k(T) = k_T$. In particular, then, given any $\mathbf{k} \in \mathbf{K}$, one can in principle determine a value for $Dk(t)$ as well as for $k(t)$ at each time $t \in [0,T]$. Using the first constraint appearing in (6) to substitute out for $c(t)$ in the objective function, one obtains a representation for this objective function as a function only of $\mathbf{k}$, as follows:

$$J(\mathbf{k}) \;=\; \int_0^T [u(f(k(t)) - \theta k(t) - Dk(t))]e^{-\rho t}dt \; . \tag{10}$$

Problem (6)-(9) then takes the compact form

$$\max_{\mathbf{k}\in\mathbf{K}} J(\mathbf{k}) \; . \tag{11}$$

Under the assumptions set out in Section B.1, the optimization problem (11) is a social planning problem to be solved by some unmodelled policy maker in charge of society.[3] The social welfare function $J(\mathbf{k})$ gives the discounted sum of the instantaneous utilities for representative agents in successive generations $t \in [0,T]$, where each *generation* t consists of $L(t)$ newly born agents with identical tastes. Many commentators — for example, Kirman (1992) — have argued that it is not clear either intuitively or mathematically who the "representative" agent in each generation t is meant to represent. Under what conditions do consumers in the aggregate act as if they were one individual with preferences well represented by a single utility function?[4]

Another issue is the specification of the social discount rate ρ. If $\rho > 0$, later generations are being discounted relative to earlier generations, a form of welfare function that Ramsey

[2]This admissibility restriction on $\mathbf{k}$ is unnecessarily strong. It is made to simplify the analytical treatment below. Given this assumption, note that one can write $Dk(t)$ instead of $D_+k(t)$ for the time-derivative of $k(t)$.

[3]Another possible approach, not pursued here, is to assume $g = 0$ and to interpret $J(\mathbf{k})$ as the total discounted lifetime utility attained by a representative consumer in the society when he selects the per-capita capital sequence $\mathbf{k}$ over his lifetime $[0,T]$.

[4]The more usual form of this question is as follows: Can a flock of birds be modeled as one big bird?

found objectionable. However, if $\rho = 0$ and $T = \infty$, the integral in (10) generally fails to exist (it "blows up"), implying some alternative representation is needed.

If T is finite, the specification of the terminal boundary value k_T is problematic. Presumably k_T then constitutes some form of socially planned bequest to unmodeled future generations, but no guidance is given for setting this bequest.[5] On the other hand, if $T = \infty$, then the only plausible value for k_T from a social planning point of view is $k_T = 0$; for why would society want to leave any positive capital at the end of time?

Often the horizon length T is set equal to $+\infty$ to avoid the problem of specifying k_T. Although a model with an infinite planning horizon is clearly unrealistic, requiring the determination of consumption levels for all generations until the end of time, in some situations it might provide a better benchmark than a finite planning horizon model incorporating an arbitrary terminal boundary condition.

Finally, note that the social preference ordering over the sequences $\mathbf{k}$ induced by $J(\mathbf{k})$ is only invariant up to a positive linear affine transformation of the instantaneous utility function u (i.e., $u(\cdot) \rightarrow au(\cdot) + b$ for some $a > 0$). Thus, the social welfare function $J(\cdot)$ has more cardinality properties than the welfare functions typically appearing in deterministic static micro theory studies.

As elaborated in Aghion and Durlauf (2005), the basic optimal growth problem (11) has been generalized in numerous directions since the nineteen sixties. Some economists have extended the model to include constraints imposed on growth by limited resource availability and ecological deterioration. Others, following the influential papers by Romer (1986, 1994) and Lucas (1988), have explored "endogenous growth" extensions of the model in which the long-run growth rate of the economy is endogenously determined by the physical and

[5]In more general versions of the optimal growth problem, discussed below in Section G, the social planner is assumed to make an optimal (utility maximizing) choice for both T and k_T over some specified set B of possible terminal boundary points (T, k_T). The first order necessary conditions for this choice to be optimal are referred to as "transversality conditions."

human capital investment decisions of private agents rather than exogenously determined by postulated growth rates in population and/or technology. Still others have introduced various types of stochastic shocks, along with limited forms of market frictions and price rigidities, in an attempt to achieve improved fits to empirical data. The latter types of models are now typically referred to as *Dynamic Stochastic General Equilibrium (DSGE)* models, sometimes with the additional qualifier "New Keynesian" to indicate the presence of frictions and price stickiness; see Sbordone et al. (2010).

C. Solution Characterization for the Basic Optimal Growth Problem

Recall that a necessary condition for a point y^* in R^n to maximize a continuously differentiable function $F{:}R^n \rightarrow R$ is that the gradient vector

$$\frac{\partial F}{\partial y}(y^*) = (F_1(y^*), \ldots, F_n(y^*)) \tag{12}$$

of $F(\cdot)$ evaluated at y^* vanish. The trick in solving the optimal growth problem (11) (or any "calculus of variations" problem of this type) is to reduce the maximization of $J(\mathbf{k})$ with respect to $\mathbf{k}$ to a finite dimensional maximization problem for which this necessary condition can be applied.

It will now be shown how this can be done. Suppose $\mathbf{k}^*$ is a trajectory in $\mathbf{K}$ which maximizes $J(\mathbf{k})$ over $\mathbf{K}$. Let $h{:}[0,T] \rightarrow R$ denote any twice differentiable function satisfying $h(0) = h(T) = 0$. For each $\epsilon \in R$, consider the *variation* $\mathbf{k}^\epsilon$ of $\mathbf{k}^*$ defined by

$$k^\epsilon(t) \;=\; k^*(t) + \epsilon h(t) \;,\; t \in [0,T]. \tag{13}$$

Note, by construction, that each of these variations is also an element of the admissible set $\mathbf{K}$. Several such variations of $\mathbf{k}^*$ are depicted in Fig. 2.

Now define a function $F{:}R \rightarrow R$ by

$$F(\epsilon) \;=\; J(\mathbf{k}^\epsilon) \;,\; \epsilon \in R \;. \tag{14}$$

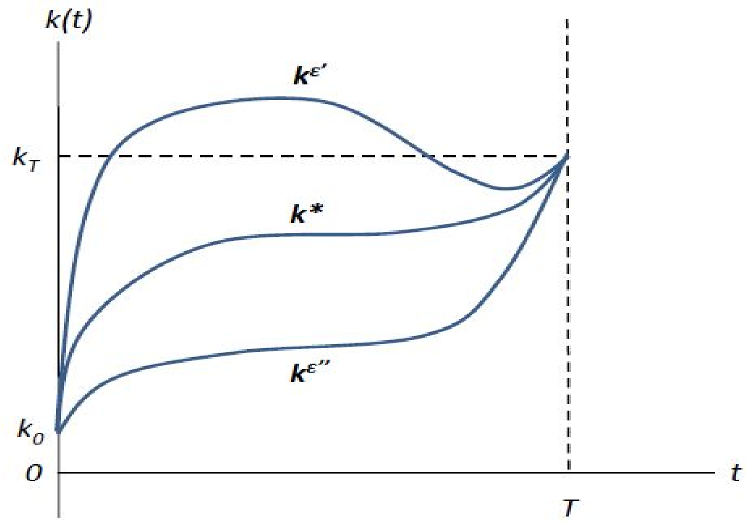

Figure 2: Variations of the optimal solution $\mathbf{k}^*$

As ϵ varies over R, $\mathbf{k}^\epsilon$ varies over a subset of $\mathbf{K}$ which includes $\mathbf{k}^*$, since $\mathbf{k}^0 = \mathbf{k}^*$. By definition of $\mathbf{k}^*$, it must hold that

$$F(0) = \max_{\epsilon \in R} F(\epsilon) \ ; \tag{15}$$

that is, $F(\epsilon)$ attains a maximum at $\epsilon = 0$. It follows that the gradient of $F(\epsilon)$ must vanish at 0:

$$0 = \frac{dF}{d\epsilon}(0) \ . \tag{16}$$

As we shall now see, condition (16) provides the well-known "Euler-Lagrange equation" for the maximization of $J(\mathbf{k})$ over $\mathbf{K}$.

Define a function $I{:}R^3 \to R$ by

$$I(k, Dk, t) \ = \ [u(f(k) - \theta k - Dk)]e^{-\rho t} \ . \tag{17}$$

Note that $I(k(t), Dk(t), t)$ is the integrand of $J(\mathbf{k})$ in (10). Next, define a composite function $f{:}[0, T] \times R \to R$ by

$$f(t, \epsilon) \ = \ I(k^\epsilon(t), Dk^\epsilon(t), t) \ = \ [u(f(k^\epsilon(t)) - \theta k^\epsilon(t) - Dk^\epsilon(t))] \, e^{-\rho t} \ . \tag{18}$$

It then follows from (10) and (14) that

$$F(\epsilon) \;=\; \int_0^T f(t,\epsilon)dt \; . \tag{19}$$

Note also that, by construction, $f(\cdot,\epsilon)$ is continuous over $[0,T]$, with continuous first partial derivative with respect to ϵ given by

$$\frac{\partial f}{\partial \epsilon}(t,\epsilon) \;=\; I_1(k^\epsilon(t), Dk^\epsilon(t), t)\cdot[h(t)] \;+\; I_2(k^\epsilon(t), Dk^\epsilon(t), t)\cdot[Dh(t)] \; . \tag{20}$$

Using (18) and (19), it follows by the interchange theorem[6] that

$$0 = \frac{dF}{d\epsilon}(0) = \int_0^T \left[\frac{\partial f}{\partial \epsilon}(t,0)\right] dt \; . \tag{21}$$

Applying integration by parts[7] to the integral of the right-hand-side term in (20), the integral in (21) can equivalently be expressed as

$$\int_0^T \left[I_1(k^*(t), Dk^*(t), t) - \frac{d}{dt}[I_2(k^*(t), Dk^*(t), t)]\right] h(t)dt \;+\; [I_2(k^*(t), Dk^*(t), t)\cdot h(t)]\,|_0^T \; . \tag{22}$$

The last term in (22) clearly vanishes since $h(0) = h(T) = 0$; hence, it follows from (21) that the integral in (22) must also vanish. Since the function h appearing in this integral is an arbitrarily selected twice differentiable function of the form $h{:}[0,T] \to R$ with $h(0) = h(T) = 0$, it follows by the Fundamental Lemma of the Calculus of Variations[8] that the inner bracketed expression in the integral in (22) must then vanish at each point t in the interval $[0,T]$, i.e.,

$$I_1(k^*(t), Dk^*(t), t) \;=\; \frac{d}{dt}[I_2(k^*(t), Dk^*(t), t)] \; , \quad \forall t \in [0,T] \; . \tag{23}$$

Relation (23) is the famous *Euler-Lagrange equation* for the problem at hand – by construction, it is a necessary condition for a per-capita capital trajectory $\mathbf{k}^* \in \mathbf{K}$ to maximize $J(\mathbf{k})$

[6] The interchange theorem provides sufficient conditions permitting the interchange of integration and differentiation operations. See any basic text on real analysis for a rigorous statement of this theorem.

[7] Given certain regularity conditions satisfied by two functions $f{:}[a,b] \to R$ and $g{:}[a,b] \to R$, one has $\int_a^b fdg = f(b)g(b) - f(a)g(a) - \int_a^b gdf$; see any basic text on real analysis for a rigorous statement of this theorem. For the application at hand, let $f = I_2$, $g = h$, $b = T$, and $a = 0$.

[8] For a rigorous statement of this famous lemma, see Takayama (1985, Chapter 5, p. 414).

over $\mathbf{K}$. Indeed, with $\mathbf{x}$ in place of $\mathbf{k}$, it constitutes the Euler-Lagrange equation for the general *calculus of variations problem*

$$\max_{\mathbf{x}\in\mathbf{X}} \int_0^T I(x(t), Dx(t), t)dt \ , \tag{24}$$

where $I(\cdot)$ is an arbitrary twice continuously differentiable function taking R^3 into R, and $\mathbf{X}$ is the collection of all twice differentiable functions over $[0, T]$ with or without boundary conditions at times 0 and T.

What does the Euler-Lagrange equation (23) reduce to for the optimal growth problem (11)? Using (17), together with the relation

$$c^*(t) = f(k^*(t)) - \theta k^*(t) - Dk^*(t) \ , \tag{25}$$

the Euler-Lagrange equation takes the form

$$Dc^*(t) \ = \ -\frac{u'(c^*(t))}{u''(c^*(t))}[f'(k^*(t)) - \theta - \rho] \ . \tag{26}$$

By construction, the relation (26) with $\mathbf{c}^*$ defined as in (25) is a necessary condition for $\mathbf{k}^*$ to solve the optimal growth problem (11).

Additional necessary conditions for $\mathbf{k}^*$ to solve the basic optimal growth problem can be obtained from the second-order necessary condition that the second derivative of $F(\epsilon)$ be nonpositive at $\epsilon = 0$, implying *local* concavity of $F(\epsilon)$ at $\epsilon = 0$. If the second derivative of $F(\epsilon)$ is non-positive for *all* ϵ, implying $F(\epsilon)$ is a concave function of ϵ, the first order condition (16) is both necessary and sufficient for $\epsilon = 0$ to *globally* maximize $F(\epsilon)$. If the second derivative of $F(\epsilon)$ is *strictly negative* for *all* ϵ, implying that $F(\epsilon)$ is a *strictly* concave function of ϵ, then the first-order condition (16) is necessary and sufficient for $\epsilon = 0$ to be the *unique* point at which $F(\epsilon)$ achieves a global maximum.

It can be shown by straightforward differentiation (using the interchange theorem) that $F(\cdot)$ given by (14) is strictly concave if $u(\cdot)$ and $f(\cdot)$ are twice continuously differentiable functions with $u' > 0$, $u'' < 0$, $f' > 0$, and $f'' < 0$. The following theorem therefore holds:[9]

THEOREM 1: Let $\rho \geq 0$ and $\theta \equiv [g + \delta] > 0$ be given. Suppose the utility function $u{:}R_{++} \rightarrow R$ is twice continuously differentiable with $u' > 0$ and $u'' < 0$. Suppose the production function $f{:}R_{+} \rightarrow R$ is continuous over R_{+} and twice continuously differentiable with $f' > 0$ and $f'' < 0$ over R_{++}. Let $\mathbf{K}$ denote the set of all twice differentiable functions $\mathbf{k}$ of the form $k{:}[0,T] \rightarrow R$ with $k(0) = k_0$ and $k(T) = k_T$. Then, in order for a function $\mathbf{k}^*$ in $\mathbf{K}$ to be the unique solution for the optimal growth problem (11), it is necessary and sufficient that $\mathbf{k}^*$ solve the following system of differential equations:

$$Dk^*(t) \quad = \quad f(k^*(t)) - \theta k^*(t) - c^*(t) \ , \quad t \in [0,T] \ ; \tag{27}$$

$$Dc^*(t) \quad = \quad - \frac{u'(c^*(t))}{u''(c^*(t))}[f'(k^*(t)) - \theta - \rho] \ , \quad t \in [0,T] \ . \tag{28}$$

Note that Theorem 1 does *not* guarantee the *existence* of a solution to the optimal growth problem (11). It only gives necessary and sufficient conditions for existence. It could happen, for example, that conditions (27) and (28) are not satisfied by *any* admissible trajectory $\mathbf{k}$.

Note, also, that equation (27) coincides with the equation (1) previously derived for the rate of change of the capital-labor ratio k in the per-capita version of the basic Solow-Swan descriptive growth model. However, the characterization (28) for the optimal (utility maximizing) per-capita consumption trajectory is new; it replaces the descriptive growth model specification for per-capita consumption as a proportion of net per-capita income, $c(t) = [1 - s][y(t) - \delta k(t)]$. Using relation (27) to substitute out for $c(t)$ in (28), one obtains the basic Euler-Lagrange equation (23).

[9] For a proof of this famous theorem with accompanying discussion, see Theorem 5.B.4 (page 429) in Takayama (1985).

D. Economic Interpretation of the Solution

Recalling that $\theta = [g + \delta]$, where g is the growth rate of labor and δ is the capital depreciation rate, equation (28) can equivalently be expressed in the following form, sometimes referred to as the "Ramsey-Keynes' Formula":

$$\begin{aligned} f'(k(t)) - \delta \quad &= \quad g \;+\; \rho \;+\; \left(-\frac{u''(c(t))}{u'(c(t))}\right) Dc(t) \\ &= \quad g \;-\; \frac{d}{dt}[\ln(e^{-\rho t}u'(c(t)))] \; . \end{aligned} \tag{29}$$

Relation (29) asserts that the marginal rate of return at time t to investment in per-capita capital, net of depreciation expenditures, must equal the biological growth rate g plus the marginal rate of return at time t to per-capita consumption.

Relation (29) can also be written in an alternative interesting form. Let the rate of growth of consumption at time t be denoted by

$$v(t) \;=\; \frac{Dc(t)}{c(t)} \; , \tag{30}$$

and let the "elasticity of marginal utility" function $e(\cdot)$ be defined by

$$e(c) \;=\; -\; \frac{du'(c)}{dc} \cdot \frac{c}{u'(c)} \; . \tag{31}$$

Then relation (29) can be written as

$$f'(k(t)) - \delta \;=\; g \;+\; \rho \;+\; v(t)e(c(t)) \; . \tag{32}$$

The "optimal" savings rate s for the per-capita version of the BSSDG model is typically taken to be the *golden rule savings rate*, that is, the savings rate that yields the maximum stationary level $\hat{c}$ for long-run per-capita consumption. It follows from relation (1), which gives time-t consumption for the per-capita BSSDG model, that the long-run (stationary) level $\hat{k}$ for per-capita capital that supports the golden-rule consumption level $\hat{c}$ for the per-capita BSSDG model is characterized by the relation $[f'(\hat{k}) - \delta] = g$.

Comparing this finding with relation (32) for the optimal growth model, note that the latter relation provides an explicit expression for the optimal (utility maximizing) time-t rate of return on per-capita capital, $f'(k(t)) - \delta$, at *each* time t along *any* solution path for the optimal growth problem (11). In general, this optimal rate is *endogenously* determined within the model due to the far-right term in (32). However, at any stationary solution $(\bar{c}, \bar{k})$ for the optimal growth model, $v = 0$ in (30), which implies that $[f'(\bar{k}) - \delta] = [g + \rho]$ in (32). Comparing this finding with the finding for the per-capita BSSDG model, it is seen that the discount rate ρ introduced in the optimal growth model (11) can significantly affect both the short-run and the long-run return to capital.

E. Phase Diagram Depiction of Euler-Lagrange Solution Trajectories

Qualitative properties that must be exhibited by any consumption and capital solution trajectories for the Euler-Lagrange differential equations (27) and (28) can be examined by means of a phase diagram analysis. For the moment, we will concentrate on the family of general solutions for this two-equation differential system, ignoring boundary conditions. Throughout this discussion, the following admissibility restrictions on exogenous variables and functional forms will be assumed to hold. Note that these restrictions are slightly stronger than the admissibility restrictions assumed for Theorem 1.

Phase Diagram Admissibility Restrictions:

- $u(\cdot)$ and $f(\cdot)$ are twice continuously differentiable over R_{++}, with $u' > 0$, $u'' < 0$, $f' > 0$, and $f'' < 0$;
- $f(0) = 0$;
- $\lim_{k \to \infty} f'(k) = 0$, and $\lim_{k \to 0} f'(k) = \infty$;
- $\rho > 0$, $g > 0$, and $\delta \geq 0$.

The first step in the construction of the phase diagram for the Euler-Lagrange differential equations (27) and (28) is to graph separately the collection V_k of points (k, c) where $Dk = 0$ and the collection V_c of points (k, c) where $Dc = 0$. Starting with the easiest case, V_c, it follows from (28), from the positivity of $\theta + \rho$, and from the restrictions imposed above on $u(\cdot)$ and $f(\cdot)$, that V_c consists of all points (k, c) such that $k = \bar{k}$, where $\bar{k}$ is the unique solution to $f'(\bar{k}) = \theta + \rho$. Consequently, in graphical terms, V_c is a vertical straight line through the particular k-value $\bar{k}$; see Fig. 3.

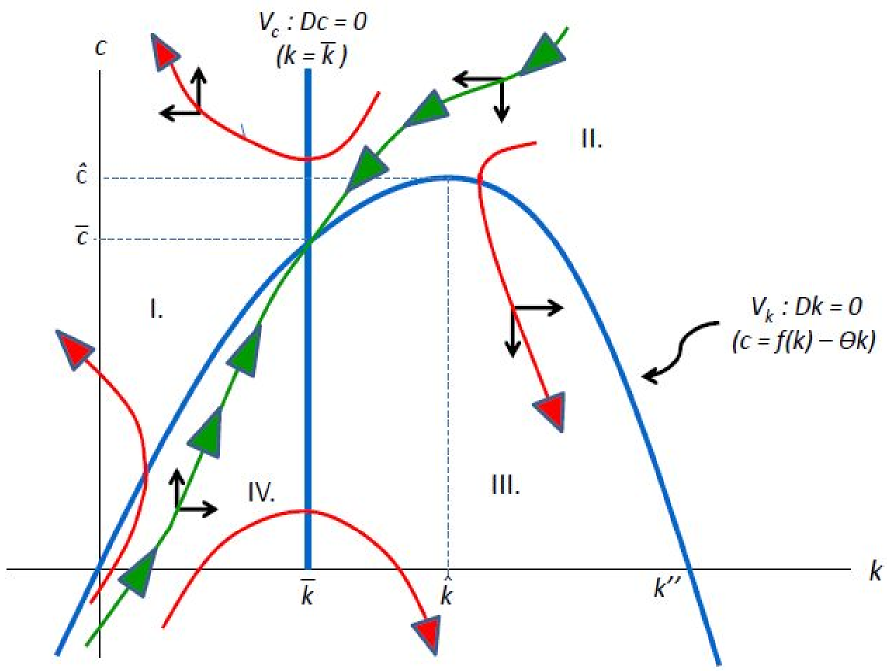

Figure 3: Phase-diagram depiction of solution trajectories for the Euler-Lagrange differential equations (27) and (28) assuming the phase diagram admissibility restrictions hold.

Now consider V_k. Define a function $\psi: R \to R$ by $\psi(k) = f(k) - \theta k$. Then, by (27), V_k consists of all points (k, c) for which $c = \psi(k)$. By the restrictions imposed above on $f(\cdot)$, $\psi(k)$ is a strictly concave function that satisfies $\psi(0) = \psi(k'') = 0$ and $\psi'(0) > 0$, where $k'' > 0$ satisfies $f(k'') = \theta k''$. Moreover, $\psi(k)$ attains its maximum value at the golden rule point $\hat{k}$ satisfying $f'(\hat{k}) = \theta$, where $\hat{k} < k''$. Finally, $\bar{k} < \hat{k}$ if $\rho > 0$ and $\bar{k} = \hat{k}$ if $\rho = 0$. See

Fig. 3 for a depiction of the case in which $\rho > 0$.

By definition of a stationary solution, a constant pair of values $\bar{k}$ and $\bar{c}$ for $k(t)$ and $c(t)$ constitutes a stationary solution for the Euler-Lagrange differential equations (27) and (28) if and only if Dk and Dc both vanish at $(\bar{k}, \bar{c})$, that is, if and only if[10]

$$0 = f(\bar{k}) - \theta\bar{k} - \bar{c} \; ; \tag{33}$$

$$0 = f'(\bar{k}) - \theta - \rho \; . \tag{34}$$

Equivalently, $(\bar{k}, \bar{c})$ is a stationary solution for (27) and (28) if and only if $(\bar{k}, \bar{c})$ lies in the intersection of the sets V_k and V_c.

As noted above, equation (34) has a unique positive solution $\bar{k}$. From equation (33) one then obtains a unique solution for consumption, $\bar{c} = f(\bar{k}) - \theta\bar{k}$, where $0 < \bar{c} < f(\bar{k})$.[11] Note that the capital stock level $K(t)$, the consumption level $C(t)$, the income level $Y(t)$, and the labor force $L(t)$ are all growing at the same constant rate g along the stationary solution path characterized by $(\bar{k}, \bar{c})$.

Now suppose that $(\mathbf{k}, \mathbf{c})$ is any solution for the Euler-Lagrange differential equations (27)

[10]Recall that the utility function $u(c)$ used in these notes is assumed to satisfy $u'(c) > 0$ and $u''(c) < 0$ for all $c > 0$. An example is the *constant absolute risk aversion (CARA)* utility function $u(c) = A - exp(-\beta c)$, where A and β are positive constants. It follows that $-u'(c(t))/u''(c(t))$ on the right side of (28) is positive for all $c > 0$, implying that $Dc(t) = 0$ if and only if condition (34) holds. In contrast, the optimal growth sections of D. Romer (2001) and Barro and Sala-i-Martin (2003) assume a *constant relative risk aversion (CRRA)* utility function, $u(c) = c^{1-\alpha}/[1-\alpha]$ with $0 < \alpha$, for which *relative* risk aversion $-c \cdot u''(c)/u'(c)$ is equal to the constant α for all $c > 0$. Given this CRRA utility function, by multiplying the numerator and denominator of the right side of (28) by $c(t)$ one sees that $Dc(t) = 0$ is then also satisfied when $c(t) = 0$; that is, $Dc(t) = 0$ also holds everywhere along the k-axis. The use of this CRRA utility function thus results in two additional stationary limit points in Fig. 3, at $(0, 0)$ and $(k'', 0)$, and all trajectories entering into quadrant III then converge to $(k'', 0)$ at time t goes to infinity instead of crossing over the k-axis. Thus, the requirement that $c(t)$ remain nonnegative for all t can no longer be used to rule out these trajectories as optimal solutions for the economic growth problem in the case $T = \infty$. On the other hand, all such trajectories result in strictly lower consumption to the consumer at each time t than trajectories lying along the stable manifold and hence cannot be optimal solutions. A formal proof of suboptimality can be given by showing that these quadrant III trajectories fail to satisfy the needed transversality condition at infinity; see Barro and Sala-i-Martin (2003, p. 75.) Thus, the phase diagram changes that result from the use of a CRRA utility function do not materially change any of the conclusions reached in the present notes.

[11]Since $\bar{k} > 0$ and $\theta > 0$, it follows from (33) that $\bar{c} = f(\bar{k}) - \theta\bar{k} < f(\bar{k})$, and $\bar{c} > 0$ if and only if $f(\bar{k})/\bar{k} > \theta$. Since $f(\cdot)$ is a strictly concave function satisfying $f(0) = 0$, one has $f(k)/k > f'(k)$ for each k. It follows from (34) and the admissibility condition $\rho \geq 0$ that $f(\bar{k})/\bar{k} > f'(\bar{k}) = \theta + \rho \geq \theta > 0$.

and (28). Using the definition of $(\bar{k}, \bar{c})$, it can be shown that

$$Dc(t) \quad > 0 \text{ if and only if } [f'(k(t)) - \theta - \rho] > 0 \text{ if and only if } k(t) < \bar{k} \; ; \tag{35}$$

$$Dc(t) \quad = 0 \text{ if and only if } [f'(k(t)) - \theta - \rho] = 0 \text{ if and only if } k(t) = \bar{k} \; ; \tag{36}$$

$$Dc(t) \quad < 0 \text{ if and only if } [f'(k(t) - \theta - \rho] < 0 \text{ if and only if } k(t) > \bar{k} \; . \tag{37}$$

Similarly,

$$Dk(t) \quad > 0 \text{ if and only if } c(t) < f(k(t)) - \theta k(t) \; ; \tag{38}$$

$$Dk(t) \quad = 0 \text{ if and only if } c(t) = f(k(t)) - \theta k(t) \; ; \tag{39}$$

$$Dk(t) \quad < 0 \text{ if and only if } c(t) > f(k(t)) - \theta k(t) \; . \tag{40}$$

As illustrated in Fig. 3 for $\rho > 0$, the k-c plane can thus be partitioned into the four regions I, II, III, and IV in which $(Dk(t), Dc(t))$ takes on the four distinct sign configurations (-,+), (-,-), (+,-), and (+,+), respectively. These four regions are simply the four partition cells created by the graphs of V_k and V_c in the $k - c$ plane.

If the trajectory $(\mathbf{k}, \mathbf{c})$ in the k-c plane ever intersects the unique stationary point $(\bar{k}, \bar{c})$, all motion in k and c ceases. At all other points in the k-c plane, either $Dk \neq 0$ or $Dc \neq 0$. The instantaneous directional change of the trajectory $(\mathbf{k}, \mathbf{c})$ as it hits a boundary of region I, II, III, or IV is determined by the nonzero component of (Dk, Dc) at that point. Thereafter it is determined by a vector sum of $(Dk, 0)$ and $(0, Dc)$. This is indicated in Fig. 3 by the perpendicular and horizontal arrows appearing in each region which indicate the directions of change in k and c in the region.

In particular, note that any trajectory that enters region I remains in region I forever, and similarly for region III. In contrast, every trajectory initiating in regions II or IV eventually enters and remains in either region I or III for sufficiently large horizon length T, *unless* it converges to the unique stationary point $(\bar{k}, \bar{c})$. Indeed, as indicated in Fig. 3, there are unique trajectories—often together referred to as the *stable manifold*—that approach the stationary point from the left and from the right. More precisely, for each admissible level k for per-capita capital, there is a unique corresponding level $c(k)$ for per-capita consumption such that, given $k(t) = k$ and $c(t) = c(k)$ at any time t, the economy henceforth proceeds along the stable manifold toward the stationary point. Given an infinite horizon economy, $(k(t), c(t))$ ultimately converges to $(\bar{k}, \bar{c})$. Note, also, that *all* solution trajectories in Fig. 3 are depicted as arching toward the stationary solution, a feature referred to as the *turnpike property* of the optimal growth model (11).

F. Solution Trajectories with Boundary Conditions Imposed

So far we have said nothing about the boundary conditions $k(0) = k_0$ and $k(T) = k_T$ appearing in the optimal growth problem (11). If $T = \infty$, the only economically meaningful solution trajectories are those that converge to the stationary solution point $(\bar{k}, \bar{c})$; for all other solution trajectories eventually either blow up or go negative. [Note that a solution trajectory cannot converge to any limit point in the $k - c$ plane except $(\bar{k}, \bar{c})$, for any such limit point must by construction be a stationary solution for (27) and (28); and limit cycles are ruled out by (35) through (40).] Consequently, if $T = \infty$, then either the terminal boundary condition for per-capita capital coincides with $\bar{k}$ or no economically meaningful optimal solution exists.

Suppose T is finite, and suppose k_T is an arbitrary positive terminal boundary value for $k(T)$. In Fig. 3, given any initial value k_0 for $k(0)$, there may exist many different trajectories that traverse from k_0 to k_T while satisfying the Euler-Lagrange equations (27) and (28). However, given the regularity conditions that have been imposed on preferences and technology, there is at most one trajectory that takes on the value k_T *at the desired time* T. Such a trajectory cannot be determined from Fig. 3, alone, since actual motion through time is not represented in this phase diagram. Moreover, there may be no trajectories that traverse from k_0 to k_T in the finite time from 0 to T. For example, if k_T is extremely large relative to the initial per capita capital level k_0, it might not be technologically feasible for the economy to reach k_T by time T. In this case the optimal growth problem (11) with initial and terminal boundary conditions k_0 and k_T has no solution.

The point $(\hat{k}, \hat{c})$ depicted on the phase diagram is referred to in the literature as the *golden rule point* for the basic optimal growth model (11) because, by construction, $\hat{c}$ yields the greatest *constant* level of per-capita utility $u(\hat{c})$ that could be sustained for all generations. In particular, using (27), $\hat{c}$ is supported by the per-capita capital level $\hat{k}$ satisfying $f'(\hat{k}) = \theta$, implying that $\hat{c}$ yields the largest possible value for $c = [f(k) - \theta k]$. Note, however, that if $\rho > 0$, then the golden rule point $(\hat{k}, \hat{c})$ is not a solution for the optimal growth problem (6) even if the boundary conditions $k(0) = k_0$ and $k(T) = k_T$ are ignored. This follows since, by construction, the golden rule point $(\hat{k}, \hat{c})$ does *not* satisfy the Euler-Lagrange equation (28) if $\rho > 0$.

If the discount rate ρ is strictly positive, the unique admissible stationary solution $(\bar{k}, \bar{c})$ for equations (27) and (28) lies strictly to the left of the point $(\hat{k}, \hat{c})$ along V_k. As earlier explained, if problem (11) entails an infinite planning horizon $T = \infty$, an initial boundary condition $k(0) = k_0 > 0$, and a terminal boundary condition $k(T) = k_T = \bar{k}$ at $T = \infty$, then the unique economically meaningful solution to (11) is given by that portion of the stable manifold that traverses from $(k_0, c(k_0))$ to the stationary point $(\bar{k}, \bar{c})$. Consequently, the optimal solution entails convergence to a limit point which yields strictly *lower* per capita consumption than the golden rule point. How can this be?

The answer to this seeming paradox is that utility along the optimal solution path is *discounted* over time if $\rho > 0$. That is, consumption for earlier generations is weighted more heavily than consumption for later generations. For example, under the conditions of the previous paragraph, suppose $k_0 = \hat{k}$. Then the “impatient” social planner would choose a trajectory that starts at $(\hat{k}, c(\hat{k}))$, so the economy is on the stable manifold. Note, however, that $c(\hat{k}) > \hat{c}$. Moreover, the economy eventually converges to $(\bar{k}, \bar{c})$, where $\bar{c}$ is a strictly lower consumption level than $\hat{c}$. Consequently, even though it is technologically feasible to support the consumption level $\hat{c}$ in each period t, the optimal growth solution entails a consumption level that starts higher than $\hat{c}$ at time 0 and that strictly declines over time to a level that is lower than $\hat{c}$.

G. Generalization: Transversality Conditions

So far we have assumed that initial and terminal boundary conditions $(0, k_0)$ and (T, k_T) are given exogenously for the optimal growth problem. More generally, these initial and terminal points could be included as additional choice variables for the social planner.

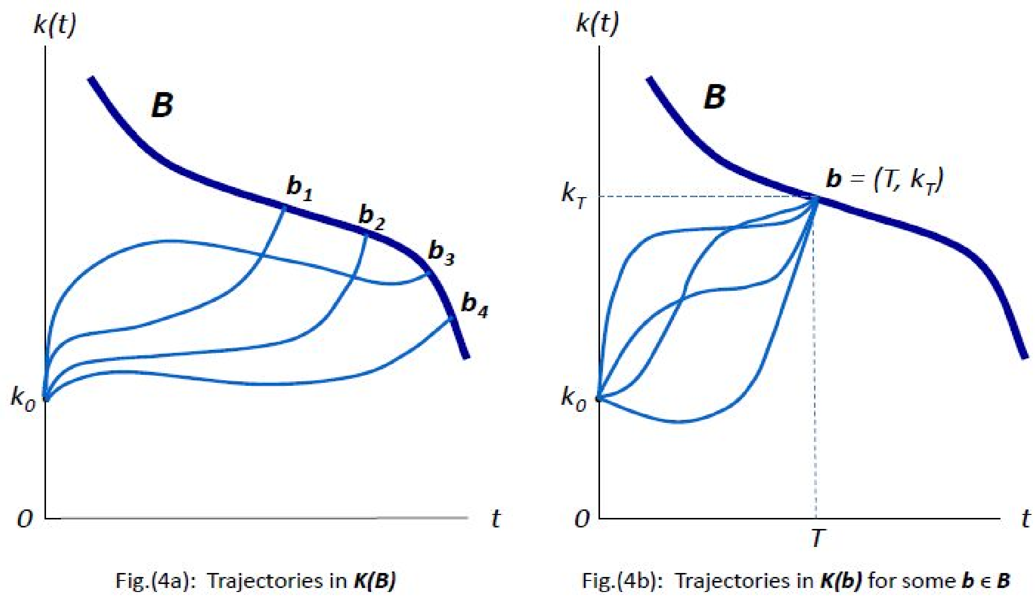

Figure 4: Two-stage method for determination of transversality conditions

For example, suppose the initial boundary conditions are given—the economy starts at time 0 at some exogenously determined level k_0 for $k(0)$—but the only restriction on the terminal time T and the terminal per-capita capital level $k(T)$ is that $(T, k(T))$ must lie in some exogenously given subset B of R^2_+. The optimal growth problem thus takes the form

$$\max_{\mathbf{k} \in \mathbf{K}(B)} J(\mathbf{k}) \ , \tag{41}$$

where $\mathbf{K}(B)$ denotes the set of all twice differentiable functions of the form $k:[0, b_1] \rightarrow R$ with $k(0) = k_0$ and $(b_1, k(b_1)) \in B$. See Fig. 4(a).

In principle, the extended optimal growth problem (41) can be approached in two stages. First, for each given $b = (b_1, b_2) \in B$, define $\mathbf{K}(b)$ to be the subset of all functions in $\mathbf{K}(B)$ of the form $k:[0, b_1] \rightarrow R$ with $k(0) = k_0$ and $k(b_1) = b_2$; see Fig. 4(b). Consider the problem

$$\max_{\mathbf{k} \in \mathbf{K}(b)} J(\mathbf{k}) \ . \tag{42}$$

Problem (42) is entirely analogous to the basic optimal growth problem (11) with given initial and terminal boundary conditions. Thus, by Theorem 1, it holds as before that a necessary and sufficient condition for $\mathbf{k}$ in $\mathbf{K}(b)$ to solve (42) is that $\mathbf{k}$ solve the Euler-Lagrange differential equations (27) and (28).

For each $b \in B$, suppose there exists a unique function in $\mathbf{K}(b)$ that solves the optimization problem (42). Let this solution be denoted by $\mathbf{k}(b)$. The second stage of the extended optimal growth problem (41) then consists in selecting the optimal terminal point $b \in B$, a finite-dimensional maximization problem. In particular, letting $H:B \rightarrow R$ be defined by $H(b) = J(\mathbf{k}(b))$ for each $b \in B$, this second-stage optimization problem takes the form

$$\max_{b \in B} H(b) \ . \tag{43}$$

By construction, given any point $b^* \in B$ that solves (43), the corresponding trajectory $\mathbf{k}(b^*)$ is a solution for the extended optimal growth problem (41).

The first-order necessary conditions for a point $b^* \equiv (T^*, k_T^*) \in B$ to solve the maximization problem (43) are known as *transversality conditions.* In particular, if B takes the form $\{(T, k_T) \in R_+^2 \mid G(T, k_T) \geq \mathbf{0}\}$, where $G:R_+^2 \rightarrow R^2$ satisfies certain regularity conditions, then problem (43) has the standard form of a nonlinear programming problem with inequality constraints and the transversality conditions are simply the Karush-Kuhn-Tucker conditions for this problem.

For example, suppose the terminal time T is required to satisfy $T = T^*$ for some exogenously given value $T^* < \infty$, and the terminal value $k(T^*)$ is simply restricted to be nonnegative; that is, suppose

$$B = \{(T, k_T) \in R^2 \mid T = T^*,\ k_T \geq 0\} \tag{44}$$

For this special case, it can be shown that the transversality condition for the choice of $k(T^*)$ reduces to $k(T^*) = 0$. This is intuitively sensible. Capital derives its value, in the problem at hand, only through its effects on future consumption, hence planning to hold a positive quantity of capital at the final time T^* would be a waste of resources.

In summary, an optimal solution $\mathbf{k}(b^*)$ for the extended optimal growth problem (41) that requires choice of a terminal point $b = (T, k(T)) \in B$ can in principle be determined by the successive solution of the two finite-dimensional maximization problems (42) and (43). The first-order necessary conditions for problem (42) yield the Euler-Lagrange differential equations (27) and (28) as necessary restrictions on $\mathbf{k}(b^*)$, and the first-order necessary conditions for problem (43) provide additional necessary restrictions on $\mathbf{k}(b^*)$ that are referred to as transversality conditions.

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