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* Notes on the Existence of Aggregate Production Functions

The general purpose of these cautionary notes is to alert you to certain fundamental difficulties that arise in macroeconomics with regard to aggregation. For concreteness, attention will be focused on production relationships. However, similar issues arise with regard to other forms of aggregation as well, e.g., aggregation over transactors, consumption goods, demand functions and other behavioral relations, and so forth.

The concept of an aggregate production function $Y = F(K, L)$ plays a critical role in AS/AD models and in neoclassical growth and business cycle models. Here Y denotes "aggregate output," K denotes "aggregate capital (services)," L denotes "aggregate labor (services)," and $F{:}R^2_+ \rightarrow R_+$ is a strictly increasing function over R^2_+ (the non-negative orthant of R^2).[1]

In analogy to microeconomic production functions, the aggregate production function is supposed to describe the maximum possible level of aggregate output Y that can be achieved for each given (K, L). The insistence on "maximum possible" implies an *efficient* use of productive inputs, i.e., the relationship $Y{=}F(K, L)$ is meant to describe a production *frontier*.

Consider a macroeconomy that produces only two goods or services, Y^a and Y^b, using two types of capital services (numbered 1 and 2) and two types of labor services (numbered 1 and 2). Suppose the production functions for Y^a and Y^b take the following form:

$$Y^v = f^v(K^v_1, K^v_2, L^v_1, L^v_2) \ , \quad v = a, b \ , \tag{1}$$

where $f^v{:}R^4_+ \rightarrow R_+$ is a strictly increasing function of each of its arguments, $v = a, b$.

Key Definition: A function $F{:}R^2_+ \rightarrow R_+$ is an *aggregate production function* corresponding to the firm-specific production functions (1) if it satisfies the following six conditions:

1. $K = h(K^a_1, K^b_1, K^a_2, K^b_2)$ for some well-defined strictly increasing function $h{:}R^4_+ \rightarrow R_+$ that is independent of current and future prices and wage rates in the economy (i.e., K is a measure of *aggregate capital inputs*);

2. $L = z(L^a_1, L^b_1, L^a_2, L^b_2)$ for some well-defined strictly increasing function $z{:}R^4_+ \rightarrow R_+$ that is independent of current and future prices and wage rates in the economy (i.e., L is a measure of *aggregate labor inputs*);

[1] Production (a flow) should be defined as a function of capital and labor services (flows), not stocks. A common assumption in macroeconomic theory, however, is to assume labor L measures labor service in person-hours (flow) and that K proxies aggregate capital services k assumed to be in fixed proportion to K, i.e., $k = \alpha K$ for some fixed constant α.

3. Y^v satisfies (1), for $v = a, b$.

4. $Y = Q(Y^a, Y^b)$ for some well-defined strictly increasing function $Q: R_+^2 \rightarrow R_+$ that is independent of current and future prices and wage rates in the economy (i.e., Y is a measure of *aggregate produced output*);

5. $F(K, L)$ is a strictly increasing function of K for each $L \geq 0$ and a strictly increasing function of L for each $K \geq 0$;

6. $Y = F(K, L)$ for each $K \geq 0$ and $L \geq 0$.

Note, in particular, that the existence of an aggregate production function implies that aggregate output only depends on the total magnitudes of factors of production and not on the way these factors are distributed across firms or within firms.

Key Aggregation Issue Addressed in These Notes:

> Under what conditions can a collection of production relations such as (1) be combined to form an aggregate production function $Y = F(K, L)$ as defined above? What specific assumptions would be minimally required in order to be able to validly do this? How empirically compelling are these assumptions?

The necessary conditions for the existence of an aggregate production function determined by Nataf [7], reported in Section 5 of Felipe and Fisher [1] (pp. 12-13), can be used to address this issue.[2] If one insists that the labor aggregate L be a "natural" sum of firm-level labor aggregates, Felipe and Fisher claim (top of p. 13) that Nataf's conditions require firm-level production functions to be: (a) linearly additively separable in capital units and labor units; and (b) linear in labor units with the same linear labor coefficient for each firm. As noted on pp. 14-15 of Ref.[1], Nataf's conditions essentially indicate "that aggregate production functions almost never exist."

As reported in Section 6 (p. 15) of Ref.[1], Fisher [3,5] criticized the analysis of Nataf for trying to do too much, for trying to find conditions for the existence of an aggregate production function under any economic circumstances rather than limiting the analysis to circumstances in which "production has been organized to get the maximum output achievable with the given factors." Fisher argued that imposing the latter desirable condition leads to a different set of aggregation conditions.

The discussion below focuses on the aggregation conditions derived by Fisher [3,5] that are reported in Section 6 of Ref.[1]. Fisher's aggregation conditions are essentially an elaboration of the aggregation analysis of Gorman [6] briefly reported in Section 5 of Ref.[1]. Gorman's analysis takes as given that inputs have been optimally distributed across the individual firms to achieve maximum output.

[2] Note that Leontief's conditions reported in Section 5 of Ref.[1] are only for aggregation *within* a firm.

Fisher's analysis stresses (pp. 15-16) the need to find technical conditions under which *all* needed aggregates can be found simultaneously. This is in accordance with the above Key Definition detailing the need to find technical conditions guaranteeing the simultaneous existence of aggregates Y, K, and L with $Y{=}F(K, L)$, conditional on a given set of micro firm production functions.

The necessary conditions determined by Fisher for the simultaneous existence of aggregates Y, K, and L with $Y{=}F(K, L)$ turn out to be extraordinarily stringent. This is so even though Fisher's analysis is conditioned on the following strong presumptions: (a) labor inputs across firms have been optimally allocated to achieve maximum output (p. 15); (b) capital is firm-specific (i.e., capital service markets do not exist), implying that no optimal allocation problem arises for capital inputs;[3] and (c) firm-level production functions exhibit constant returns to scale.[4]

Roughly stated, Fisher's necessary conditions are as follows: Every firm's capital must be expressable in units of a fixed firm-specific "basket of capital inputs," every firm must employ the same "basket of labor inputs," (differing only by scale), and every firm must produce the same "basket of outputs" (differing only by scale). Hence there can be no specialization across firms either in the employment of labor or in the production of outputs.

For example, consider the firm-specific production functions (1) for firms $v = a, b$. Suppose labor has been optimally allocated across these firms to maximize output, and all capital is firm-specific (no market for capital services). Then Fisher's four necessary conditions for the existence of scalar aggregates Y, K, and L with $Y{=}F(K, L)$, expressed specifically for these two firms, are as follows (cf. Ref[1], pp. 15-21).

- **Condition 1:** First (cf. Ref.[1], top of p. 20), a scalar capital aggregate K^v must exist at the level of each firm v. That is, it must be possible to express the production function of each firm v in the form

  $$Y^v \quad = \quad H^v(K^v, l^v) \quad , \quad v = a, b \; , \tag{2}$$

  where $K^v = h^v(K^v_1, K^v_2)$ is a scalar measure of capital for firm v, and $l^v = (L^v_1, L^v_2)$ is the vector of firm v's labor inputs.

  As explained in Section 5 of Ref.[1], assuming twice-differentiable firm production functions, the necessary and sufficient "Leontieff conditions" for the existence of aggregates within a firm are extremely stringent, requiring that the marginal rate of substitution between each pair of variables in the aggregate be independent of all variables left out

[3] As noted on page 20 of Ref.[1], Fisher (1982) extends the analysis reported in Ref.[1] for firm-specific capital to the case of mobile capital, obtaining essentially the same set of necessary conditions for aggregation.

[4] As noted in footnote 21 of Ref.[1], Fisher was unable to come up with any closed-form characterization for the class of cases in which a capital aggregate K exists across firms in the absence of an assumption of constant returns for the firm-level production functions. On page 20 of Ref.[1] the authors make the broader claim that "if there are nonconstant returns (at the level of individual firms), no aggregate (across firms) will exist in general."

of the aggregate. In particular, in order for a firm-level capital aggregate to exist, the marginal rate of substitution between each pair of distinct capital inputs must be independent of all labor inputs. For example, a *sufficient* condition ensuring that a firm v with a production function (1) satisfies Leontief's conditions is for firm v's production function to have a linearly additively separable form, as follows:

$$Y^v \quad = \quad \theta^v(K_1^v, K_2^v) \ + \ \psi^v(L_1^v, L_2^v) \quad . \tag{3}$$

- **Condition 2:** Second, suppose scalar firm-level aggregates K^v and L^v for capital and labor exist for each firm v,[5] and suppose the production function of each firm v can be expressed in the form

$$Y^v \quad = \quad F^v(K^v, L^v) \quad , \quad v = a, b \quad , \tag{4}$$

where $F^v(\cdot)$ exhibits constant returns to scale. Then (cf. Ref.[1], p. 19 and top of p. 20), in order for the scalar firm-level capital aggregates K^v to be aggregated *across* firms into a scalar capital aggregate K, the production functions $F^v(\cdot)$ in (4) must be expressable in the following form:

$$Y^v \quad = \quad F^v(b^v, L^v) \quad = \quad G(b^v\bar{k}^v, L^v) \quad , \quad v = a, b \ , \tag{5}$$

where b^v is a firm-specific scalar, $\bar{k}^v$ is a fixed vector representing a fixed firm-specific *basket of capital inputs*, and the function $G(\cdot)$ is common across firms $v = a, b$.

- **Condition 3:** Third, suppose condition (5) holds, where the firm-level production functions $F^v(\cdot)$ in (5) exhibit constant returns to scale. Then (cf. Ref.[1], Section 6.2, p. 21), in order for the scalar firm-level labor aggregates L^v to be able to be aggregated *across* firms into a scalar labor aggregate L for any given set of relative wages, the production function $F^v(\cdot)$ for each firm v in (5) must be expressable in the following form:

$$Y^v \quad = \quad F^v(b^v, c^v) \quad = \quad H(b^v\bar{k}^v, c^v\bar{l}) \quad , \quad v = a, b \ , \tag{6}$$

where the nonnegative real number c^v measures the scale of labor usage for firm v, $\bar{l}$ is a fixed vector representing a fixed *basket of labor inputs* common across firms $v = a, b$, and $H(\cdot)$ is common across all firms. Thus, there is a complete absence of specialization in labor across firms.

- **Condition 4:** Suppose that (6) holds, where the firm-level production functions $F^v(\cdot)$ in (6) exhibit constant returns to scale. Then (cf. Ref.[1], Section 6.2, p. 21), in order

[5] If all firm-level production functions are twice differentiable, the Leontieff Conditions reported in Section 5 of Ref.[1] provide necessary and sufficient conditions for this existence. As noted in footnote 25 in Ref.[1], however, Fisher did not establish the necessity of the existence of firm-level output and labor aggregates for the ability to obtain economy-wide aggregates *across* firms; he assumed the existence of these firm-level output and labor aggregates to avoid "strange things" about this across-firm aggregation.

for the firm-level outputs Y^v to be able to be aggregated *across* firms $v = a, b$ into a scalar aggregate output Y for any given set of relative output prices, the firm-level outputs Y^v must be expressable in the form $a^v\bar{Y}$ where a^v is a nonnegative scalar measuring the scale of output for firm v and $\bar{Y}$ is common across both firms. That is, each firm must produce the same output $\bar{Y}$ (differing only by a scale factor) when faced with any given set of relative output prices, implying a complete absence of specialization in production across firms.

Assuming Conditions 1-4 hold, it follows that the production function for each firm v takes the following form:

$$Y^v \quad = \quad a^v\bar{Y} \quad = \quad F^v(b^v, c^v) \quad = \quad H(b^v\bar{k}^v, c^v\bar{l}) \quad , \quad v = a, b \; , \tag{7}$$

where:

- the nonnegative real number a^v measures the scale of output for firm v;
- the nonnegative real number b^v measures the scale of labor usage for firm v;
- the nonnegative real number c^v measures the scale of capital usage for firm v;
- the basket of capital goods $\bar{k}^v$ for firm v is exogenously given, hence it is not affected by current and future relative output prices and relative wage rates;
- the functional form $H(\cdot)$, the labor basket $\bar{l}$, and the output $\bar{Y}$ are common across firms $v = a, b$ and exogenously given, hence they are not affected by current and future relative output prices and relative wage rates

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