EXERCISE 5: INDIVIDUAL (20 Points Total) **L. Tesfatsion**
DUE: Friday, October 13, 12:10pm **Econ 502/Fall 2017**

****CAUTION: Late assignments will not be accepted – no exceptions.**

Introductory Exercise on the Basic Optimal Growth Model

Basic Reference:

- **[1] (**) "A Simple Illustrative Optimal Growth Model," posted online at Econ 502 Syllabus Section III.D**
  http://www2.econ.iastate.edu/classes/econ502/tesfatsion/OptGrow.pdf

Background Materials from Ref. [1]:

As in [1], let $\rho \geq 0$ be an exogenously given discount rate. Let $g > 0$ denote the exogenously given growth rate of labor, and let $\delta \geq 0$ denote the exogenously given capital depreciation rate. Let $\theta \equiv [g + \delta] > 0$. Finally, let sequences of real-valued per-capita capital and consumption levels over $[0, T]$ be denoted by

$$\mathbf{k} = (k(t) : t \in [0, T]) \; ; \tag{1}$$

$$\mathbf{c} = (c(t) : t \in [0, T]) \; . \tag{2}$$

Consider the following intertemporal utility maximization problem:

$$\max_{\mathbf{c},\mathbf{k}} \int_0^T u(c(t)) e^{-\rho t} dt \tag{3}$$

subject to

$$c(t) = f(k(t)) - \theta k(t) - D_+ k(t) \; , \quad 0 \leq t \leq T \; ; \tag{4}$$

$$k(0) = k_0 \; ; \tag{5}$$

$$k(T) = k_T \; . \tag{6}$$

Let $\mathbf{K}$ denote the collection of all twice differentiable functions $\mathbf{k}$ taking the form $k{:}[0, T] \rightarrow R$ with boundary conditions $k(0) = k_0$ and $k(T) = k_T$. Using equation (3) to substitute out for $c(t)$ in the objective function, one obtains a representation for this objective function as a function only of $\mathbf{k}$, as follows:

$$J(\mathbf{k}) = \int_0^T [u(f(k(t)) - \theta k(t) - Dk(t))] e^{-\rho t} dt \; . \tag{7}$$

Problem (3)-(6) then takes the compact form

$$\max_{\mathbf{k} \in \mathbf{K}} J(\mathbf{k}) \; . \tag{8}$$

THEOREM [20, pp. 10-11]: Let $\rho \geq 0$ and $\theta \equiv [g + \delta] > 0$ be given. Suppose the utility function $u: R_{++} \to R$ is twice continuously differentiable with $u' > 0$ and $u'' < 0$. Suppose the production function $f: R_+ \to R$ is continuous over R_+ and twice continuously differentiable with $f' > 0$ and $f'' < 0$ over R_{++}. Let $\mathbf{K}$ denote the set of all twice differentiable functions $\mathbf{k}$ of the form $k: [0, T] \to R$ with $k(0) = k_0$ and $k(T) = k_T$. Then, in order for a function $\mathbf{k}^*$ in $\mathbf{K}$ to be the unique solution for the optimal growth problem (8), it is necessary and sufficient that $\mathbf{k}^*$ solve the following differential system:

$$Dk^*(t) = f(k^*(t)) - \theta k^*(t) - c^*(t)\ , \quad t \in [0, T]\ ; \tag{9}$$

$$Dc^*(t) = -\frac{u'(c^*(t))}{u''(c^*(t))}[f'(k^*(t)) - \theta - \rho]\ , \quad t \in [0, T]\ . \tag{10}$$

PART A (5 Points): Provide a careful *economic* interpretation for the maximization problem (8).

PART B (10 Points): Reference [1] (Section E, Figure 3) develops a phase diagram portrait for the Euler-Lagrange equations (9) and (10) for generally specified utility and production functions $u(c)$ and $f(k)$. Redo this phase diagram portrait using the following *more precisely* specified utility and production functions:

$$u: R_{++} \to R \quad \text{given by} \quad u(c) = B - \exp(-\beta c),\ B > 1,\ \beta > 0 \tag{11}$$

$$f: R_+ \to R \quad \text{given by} \quad f(k) = k^\alpha\ , \quad 0 < \alpha < 1\ . \tag{12}$$

Be sure to show your work, step by step, justifying carefully each step. Also, be sure your resulting phase diagram portrait is carefully labeled and carefully explained.

PART C (5 Points): What additional dynamic properties (if any) can be deduced from your phase diagram portrait in Part B in comparison to the general phase diagram portrait developed in [1] (Section E, Figure 3)? Explain carefully.