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Metal-insulator transition in random superconducting networks

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The nature of the eigenstates and the effects on the superconducting-to-normal phase boundary in a two-dimensional random superconducting network are examined by finite-size scaling transfer-matrix calculations within the mean-field Ginzburg-Landau theory of second-order phase transitions. Results for a site-diluted square lattice are presented and a rich structure in the mobility-edge trajectory is obtained. The critical exponent for the slope of the critical field on $(p-p_c)$ is calculated and compared with previous estimates.

In recent years, considerable attention has been devoted to the study of flux quantization effects, which have theoretically predicted[1-3] and experimentally observed,[4-7] in arrays of superconducting honeycomb,[4] square,[5] self-similar,[6] and quasicrystal[7] networks. The main feature of this quantization effect is the oscillation of the superconducting transition temperature $T_c(H)$ as a function of the external magnetic field H. For example, in a perfect periodic network, $T_c(H)$ exhibits singularities at rational values of the reduced flux ϕ/ϕ_0, where $\phi = HL^2$ is the magnetic flux through the unit cell of the superconducting network and $\phi_0 = hc/2e = 2.07\times10^{-7}\text{G cm}^2$ is the superconducting flux quantum. It was recognized[5] several years ago that such a superconducting network is an ideal experimental system for studying the effects of frustration in a well-characterized system. In this system, the reduced flux is the frustration parameter and can be changed continuously. The mean-field Ginzburg-Landau theory of second-order phase transitions, as developed by de Gennes[1] and Alexander,[2] provides a very accurate description of these phenomena. Most of the research effort so far has been concentrated on understanding the superconducting-to-normal phase boundary in various periodic networks. Only very recently some attempts have been made to experimentally[8,9] and theoretically[9,10] calculate the superconducting phase boundary for random superconducting networks. The expected dimensionality crossover between the homogeneous and fractal regimes was not observed[8] in the random square-lattice percolation networks. Furthermore, the interesting problem of the superconducting phase below T_c has not been addressed[11] beyond the mean-field approximation.

In this Rapid Communication, we study the superconducting-to-normal (SN) phase boundary for a random percolating superconducting network. As expected, the structure of the phase boundary curve $T_c(H)$ is washed out, due to the randomness of the percolation networks. By examining the nature of the eigenstates for different values of the concentration of the sites present, p, as a function of ϕ/ϕ_0, we are able to obtain bounds for the SN phase boundary. A very interesting mobility-edge trajectory is found for the metal-to-insulator transition, which may be observable experimentally.

To examine the problem of the superconducting networks theoretically, we study the linearized Ginzburg-Landau (GL) equations appropriate near the phase boundary. For a network with current conservation at the nodes or lattice points,[1-3,10]

$$-\Delta_i\sum_j \cot(\theta_{ij})+\sum_j \Delta_j e^{i\gamma_{ij}}/\sin(\theta_{ij})=0\,, \tag{1}$$

where Δ_i is the order parameter at node or site i, $\theta_{ij}=l_{ij}/\xi$ is the distance between nodes divided by the coherence length, and $\gamma_{ij}=(2\pi/\phi_0)\int_i^j \mathbf{A}\cdot d\mathbf{l}$ is the circulation of the vector potential $\mathbf{A}$ along the link ij. Equation (1) is, in general, difficult to solve but becomes more tractable for simple geometries. For regular networks which have links of equal length, i.e., $l_{ij}=L$, the $\sin(\theta_{ij})$ term can be removed from the sum in Eq. (1), and the problem reduces to that of an electron on a similar lattice in a magnetic field.[1-3,12]

For simple lattices, Eq. (1) can be written as

$$-\Delta_i z_i\cos\left(\frac{L}{\xi}\right)+\sum_{j=1}^{z_i}\Delta_j e^{i\gamma_{ij}}=\lambda\Delta_i\,, \tag{2}$$

where the sum j is over the z_i nearest neighbors of node site i and λ is the eigenvalue, which for the superconducting networks is equal to zero. Notice that Eq. (2) is very similar to the tight-binding Hamiltonian of an electronic problem. However, here we are most interested in the zero-eigenvalue case and the diagonal disorder depends on z_i and $\cos(L/\xi)$, which is taken to be equal to E. In this particular case, the off-diagonal term $\exp(i\gamma_{ij})$ is correlated with the diagonal through the field H. One way in which disorder can be introduced into Eq. (2) is by randomly removing $(1-p)$ of the sites, i.e., a fraction p of

the sites or nodes are present. A link ij is present if only both nodes i and j are present. To make the connection[8] with the superconducting $T_c(H)$, it is usually assumed that $\xi=\xi_0/\sqrt{1-t}$, where $t=T/T_c(0)$ and, therefore, $L/\xi=L\sqrt{(1-t)}/\xi_0$. $T_c(H)$ can then be determined from the largest value of E for which Eq. (2) has a solution for $\lambda=0$. In order to find the eigenvalues and eigenvectors Δ_i for Eq. (2), one can either diagonalize the Hamiltonian for a given H and p, or use the transfer-matrix method and finite-size scaling.[13,14] This latter method is useful, since it can be used to study the nature of the different eigenstates Δ_i, in particular, whether they are localized or extended. Direct diagonalization or a tridiagonalization procedure could be also used to study the nature of eigenstates, but is known not to give very accurate results.

In the transfer-matrix method,[13,14] one considers coupled one-dimensional (1D) systems. Each 1D system is described by a tight-binding Hamiltonian of the same form as Eq. (2). The corresponding sites of the nearest-neighbor 1D system are coupled together by an interchain matrix element $t_{ij}e^{i\gamma_{ij}}$ that depends on the strength of the magnetic field H and concentration p. In particular, we choose a gauge such that A is parallel to the 1D chains and $t_{ij}=1$ if both sites are present, ϵ if one site is missing, and ϵ^2 if both sites are missing. This is clearly seen in Fig. 1, where a random percolating network is shown with the identifications of the different parameters. In the present work, M chains are coupled together into a 2D array with interchain coupling t_{ij} and $z_i=\sum_j t_{ij}$. The additional term t_{ij} is necessary to ensure that the lattice is connected. For the M connected chains of length N, one determines the largest localization length λ_M as $N\rightarrow\infty$. From a plot of λ_M vs M, one can obtain the localization properties of the system.[13,14] In particular, by studying the scaling plots λ_M/M vs M, one obtains a reasonable estimate of the mobility-edge trajectory. Exactly at the mobility edge, we also find $\lambda_M/M=0.6$ in agreement with previous work,[13,14] while for extended and localized states we have that λ_M/M vs M increases or decreases, respectively. For our studies here, we used M equal to 2 through 64 with N up to 10000, and $\epsilon=10^{-4}$. We found that our results are independent of ϵ provided that $\epsilon\lesssim 10^{-3}$.

In addition for a given H and p, we numerically[15] calculate the density of states (DOS) to obtain the positions and widths of bands and gaps. With the DOS known, the finite-size scaling analysis is used to determine the nature of the eigenstates in the bands.

At $p=1$, we have $z_i=4$ and Eq. (2) with $\lambda=0$ can be transformed to the "incommensurate" 1D tight-binding model. Its energy spectrum has been studied in great detail by Hofstadter.[12] The phase boundary of the superconductive perfect square lattice can be easily obtained.[12] In order to check our numerical methods, we have also calculated for $p=1$ the phase boundary $T_c(H)$. As can be seen from Fig. 2, the largest value of E, which satisfies Eq. (2) for $\lambda=0$ and is assumed to be proportional to the critical temperature $T_c(H)$, is plotted vs ϕ/ϕ_0; excellent agreement is found with previous works.[10,12] Along the y axis, we plot $E\equiv\cos(L/\xi)$, which is usually related to $T_c(H)$ through the relation $\xi=\xi_0/\sqrt{1-T/T_c(0)}$, and therefore $L^2\{1-[T_c(H)/T_c(0)]\}/\xi_0^2=[\arccos(E)]^2$. The $T_c(H)$ shows all the interesting oscillations expected at rational values of ϕ/ϕ_0. The phase boundary $T_c(H)$ is numerically obtained with three independent methods which quantitatively agree with each other: (1) by a partial diagonalization of 10000×10000 matrix for a given H, and p to find the largest value of E for which $\lambda=0$, (2) by the numerical calculation of DOS (Ref. 15) from which we can follow the behavior of the band edge of the highest band, and (3) by calculating the localization length with the finite-size scaling transfer-matrix techniques.

For a site-diluted square lattice network (i.e., $p_c<p<1$, where $p_c=0.59$ is the site percolation threshold for the square lattice), we find the structure seen in E for $p=1$ washed out even when $p=0.90$ as seen in Fig. 2. Notice, when $p=0.80$, all structure in E has disappeared.[16] Below we will show, after examining the nature of the eigenstates Δ_i which correspond to this particular value of E, that this simple mapping from E to $T_c(H)$ which was discussed above for $p=1$ is not correct for $p<1$ and determination of the phase boundary is actually

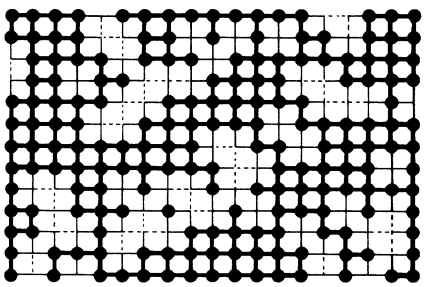

FIG. 1. Random percolating superconducting network with the filled circle representing the fraction p of the sites present equal to 75%. The nearest-neighbor matrix element t_{ij} $=1$ (●—●), ϵ (●—), or ϵ^2 (---) if both, one, or zero sites are present, respectively.

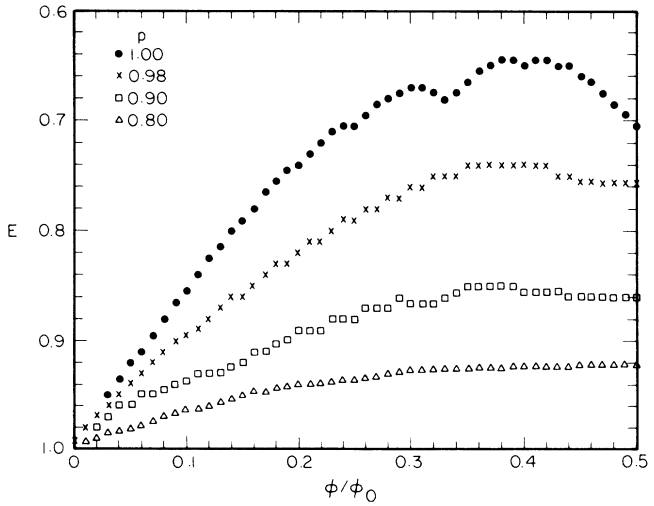

FIG. 2. Phase boundary of the percolating superconducting square network for site concentrations $p=1$, 0.98, 0.90, and 0.80. The y axis represents the largest eigenstate E, while the x axis represents ϕ/ϕ_0, where $\phi=L^2H$ is the magnetic flux through the unit cell and ϕ_0 is the superconducting flux quantum.

more complex.

We have undertaken a systematic study of the extended or localized nature for all the eigenstates as a function of E. We found that the eigenstate corresponding to the largest value of E is not always extended, as has been implicitly assumed, in determining the phase boundary $T_c(H)$ in Fig. 2. Obviously if the Δ_i are not extended, the solution with the largest eigenvalue cannot correspond to a superconducting state as occurs for $p=1$. For $1>p>p_c$ there exists a mobility edge between localized and extended states. The actual value of $T_c(H)$ must then depend on the solution of the nonlinear Landau-Ginzburg equations and cannot be determined from the linearized equations. Because the nonlinear term can couple different localized eigenstates, the mobility-edge trajectory found here can only be a lower bound of $T_c(H)$.[17] In Fig. 3, we show whether the eigenfunctions are either localized or extended for several values of p and ϕ/ϕ_0 from which one can easily determine the upper and lower limits of $T_c(H)$. The energy at which the first solution appears gives the upper bound, while the energy for the first extended solution gives the lower bound. The $p=1$ results agree very well with those of Ref. 12. For example, for $\phi/\phi_0=\frac{1}{3}$ and $p=1.0$, we have three bands. However, since we only show $E>0$, only the one bands around $E=0.8$ and half of the band around $E=0$ is shown. The eigenstates for $p=1$ are all extended. Notice that as p decreases for a given ϕ/ϕ_0, we first loose some of the gaps that were present at $p=1$ and then localized states begin to appear due to the site dilution. As the disorder increases, we find that all the states become localized. This is the case for $p=0.60$, where for all the magnetic field strengths studied, the eigenstates for all E are localized. However, for intermediate concentrations p, a rich and interesting behavior is obtained. Notice that for $\phi/\phi_0=\frac{1}{3}$ and $E=0.3$, there is no eigenstate at this energy, for $p=1$ case, the state is in a gap, while for $p=0.90$ a localized state exists. It is very interesting that as disorder increases, we recover extended states at $E=0.30$ for $p=0.80$ and $p=0.70$. This is contrary to one's usual intuition from which one expects localized states to appear as disorder increases. This unusual behavior occurs for the disordered superconducting network, because there is a competition between the disorder and the strength of magnetic field in determining the nature of the eigenstates. By diluting the system, we of course introduce more disorder, but the magnetic flux plaquettes rearrange themselves in such a way that an extended eigenstate remains. It would be very interesting to check this rich structure of mobility-edge trajectories experimentally. For a given value of ϕ/ϕ_0 and given p, it might be possible to observe a metal-to-insulator-type transition by changing E which effectively means changing temperature. For our particular superconductivity model, given by Eq. (2), the classical and the quantum percolation thresholds are exactly the same p for $H=0$. This is due to the fact that the site energy $z_i\cos(L/\xi)$ is proportional to the number of nearest neighbors. Therefore, for $H=0$ the $E=\cos(l/\xi)=1$ eigenstate is always extended provided that $p\geq p_c=0.59$.

Another interesting consequence of our results is that we can determine how the slope of the critical field depends on $p-p_c$ $(dH_{c2}/dT)_{T_c}\sim(p-p_c)^{-k}$ for $p>p_c$. For percolation, the linearized GL theory predicts that the critical exponent $k=\nu\theta$, where ν is the correlation length $\xi\sim(p-p_c)^{-\nu}$ exponent and θ is the anomalous diffusion exponent.[8] Taking $\nu=\frac{4}{3}$ and $\theta=0.8$ gives $k=1.06$ in reasonable agreement with the numerical work on linear-

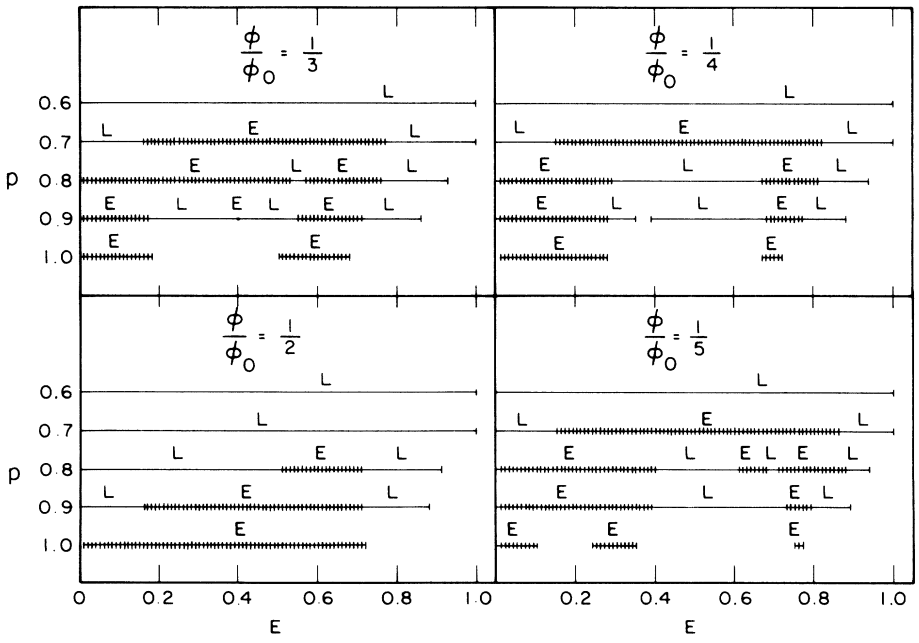

FIG. 3. The concentration dependence of the eigenstates E for different magnetic fields $\phi/\phi_0=\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$. The symbols E and L represent extended and localized states, respectively. For a given p, (——), (HHH), and (blank spaces) represent localized, extended, and no states, respectively.

ized GL theory of Ref. 10 who found $k=0.93$. Experimental values of $k=0.57$ and $k=1.06$ are given in Refs. 18 and 8 respectively, while Ref. 3 gives $k=0.87$. To compare our results with previous work on random percolating networks, we have analyzed the initial part of $\phi/\phi_0 \leq 0.1$ of the phase boundary with the formula $H_{c2}(T)=A(1-t)\phi_0/\xi_0^2$, where $t=T/T_c$. Therefore, $A=(\xi_0^2 T_c/\phi_0)(dH_{c2}/dT)_{T_c}$. In Fig. 4, we plot A as a function of $p-p_c$ on logarithmic axis. Notice that A appears to vary with p for wide ranges of $p-p_c$. In Fig. 3, the triangles represent the largest value of E for which Eq. (2) has a solution for $\lambda=0$, while the filled circles represent the eigenstate E (or temperature T_c) at which the first extended state appears as we reduce E. The extended curve is closer to the experiments.[8] Notice that the exponent k for the extended curve is equal to 0.84 while for the band-edge curve $k=2.50$ which is much higher than the numerical estimations of Ref. 10 which give a value of $k=0.93$. The open circles, are the experimental data for A vs $p-p_c$ obtained by Ref. 8. Notice that the experimental data lie in between the two numerical predictions, which only give bounds to the real $T_c(H)$ suggesting that the agreement for the value of k given in Ref. 8 with the earlier prediction[10] of the linearized GL theory was fortuitous. To determine the actual $T_c(H)$, the nonlinear terms which have been neglected in the present mean-field treatment have to be taken into account.[17] This breakdown of mean-field theory and the importance of nonlinear terms in determining $T_c(H)$ probably also explains the lack of dimensional crossover which was theoretically expected but not observed[8] in percolating superconducting networks.

In conclusion, we showed that the linearized GL theory can not be used to determine the SN phase boundary for disordered systems. It can only be used to bound the phase boundary. In cases of weak disorder, it probably gives a very good estimate of $T_c(H)$, but cannot be trusted for strong disorder. Instead, it is necessary to take account of the nonlinear terms which have been neglected in the linearized theory. We found that the cusplike structures present at $p=1$ for $T_c(H)$ vs H are washed out for $p \leq 0.90$. The extended or localized nature of the eigenstates was also examined. A rich and interesting structure of mobility-edge trajectories was obtained. Finally, the critical exponent of the upper critical field was calculated and found to be $k > 0.84$, consistent with the experimental measurements.[8]

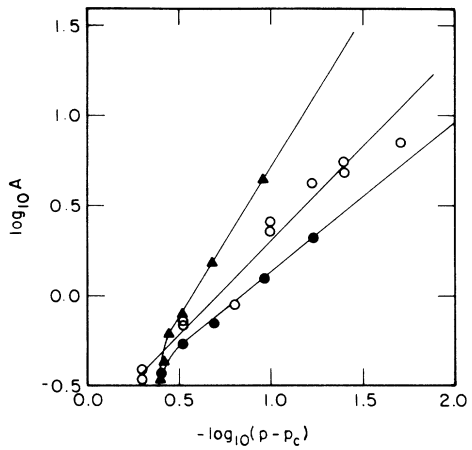

FIG. 4. Log-log plot of parameter $A \sim dH_{c2}/dT$ vs $p-p_c$ for the band edge (▲) and first extended states (●) from our numerical calculations for site percolation. The experimental results (○) of Ref. 8 for bond percolation are also presented.

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[16]This is in disagreement with Ref. 10 where some structure is still seen even at $p=0.70$. We believe that the three independent numerical methods used in our study to find the largest eigenvalue is superior to the tridiagonalization technique used in Ref. 10. Most likely the continued fractions used in 512 iterations (Ref. 10) did not give the correct E. It appears that for $p<1$, Ref. 10 always overestimates E and that is why some structure of E remains even at $p=0.70$.

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