An improved zero-one law for algorithmically random sequences

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Abstract

Results on random oracles typically involve showing that a class $\{X : P(X)\}$ has Lebesgue measure one, i.e., that some property $P(X)$ holds for "almost every X." A potentially more informative approach is to show that $P(X)$ is true for *every* X in some explicitly defined class of random sequences or languages. In this note we consider the *algorithmically random* sequences originally defined by Martin-Löf and their generalizations, the n-random sequences. Our result is an effective form of the classical zero-one law: for each $n \geq 1$, if a class $\{X : P(X)\}$ is closed under finite variation and has arithmetical complexity Σ^0_{n+1} or Π^0_{n+1} (roughly, the property P can be expressed with $n+1$ alternations of quantifiers), then either P holds for every n-random sequence or else holds for none of them. This result has been used by Book and Mayordomo to give new characterizations of complexity classes of the form ALMOST-$\mathcal{R}$, the languages which can be $\leq^{\mathcal{R}}$-reduced to almost every oracle, where $\mathcal{R}$ is a reducibility.

Keywords: Random sequence, algorithmic randomness, n-randomness, zero-one law.

1 Introduction

Results such as the separation of complexity classes relative to random oracles, which have the form "property $P(X)$ holds for a random sequence X," generally rely on the classical zero-one law: if a class $\mathcal{C} = \{X \in \{0,1\}^\infty : P(X)\}$ is closed under finite variation (that is, $A \in \mathcal{C}$ implies that $B \in \mathcal{C}$ for every B which differs from A on only finitely many bits), then either $\mathbf{Pr}(\mathcal{C}) = 0$ or $\mathbf{Pr}(\mathcal{C}) = 1$. Here $\mathbf{Pr}(\mathcal{C})$ can be briefly defined as the probability that a sequence X is in $\mathcal{C}$ when X is generated by successive tosses of a fair coin or, equivalently, as the Lebesgue measure of $\mathcal{C}$. Knowing that $\mathbf{Pr}(\mathcal{C}) = 1$, i.e., that $P(X)$ holds for a "random" sequence or language X, assures that sequences satisfying property P are plentiful but provides no information about any particular X for which $P(X)$ can be presumed to hold. An alternative is to explicitly *define* a class of random sequences and then show that $P(X)$ holds for every X in the class. There are many definitions of randomness, of varying strengths, so in effect one is asking "how much" randomness is required of X to guarantee that $P(X)$ is true.

In this note we consider algorithmically random sequences, as defined by Martin-Löf [15], and their generalizations, the n-random sequences. Our result is an effective form of the zero-one law: if $\mathcal{C}$ is a Σ^0_{n+1} or Π^0_{n+1} class of sequences which is closed under finite variation, then $\mathcal{C}$ either contains every n-random sequence or else contains no n-random sequences. (Terminology is defined in the next section.) Roughly, this may be interpreted to mean that a property P which can be described using $n+1$ alternations of quantifiers is decided in the same way by every n-random sequence. As a simple example consider a property such as $\mathrm{P}^A \neq \mathrm{NP}^A$, shown in [2] to hold for almost every A; since the class $\{A : \mathrm{P}^A \neq \mathrm{NP}^A\}$ can be described arithmetically in Π^0_2 ("$\forall\exists$") form, we immediately have $\mathrm{P}^A \neq \mathrm{NP}^A$ for every algorithmically random (1-random) A. The zero-one law proved here is an improvement of a weaker form appearing in [10], which required the additional ad hoc condition:

$$\text{"if } A \in \mathcal{C} \text{ and } \sigma \text{ is any finite string, then } \sigma A \in \mathcal{C}\text{."} \tag{1}$$

We briefly mention one application of interest. Book, Lutz, and Wagner [4] show that if $\mathcal{R}$ is a *bounded* reducibility, then for any recursive A, the class $\mathcal{R}^{-1}(A) = \{B : A \leq^{\mathcal{R}} B\}$ is a Σ^0_2-class, i.e., a union of recursively closed sets. (Examples of bounded reducibilities include $\leq^P_m$ and $\leq^P_T$; see [4] for details.), As in [5], let $\mathcal{R}$ be called an *appropriate* reducibility if it is bounded and if $\mathcal{R}^{-1}(A)$ is closed under finite variation for any A. It follows from the effective zero-one law that for appropriate $\mathcal{R}$, if $\mathbf{Pr}(\mathcal{R}^{-1}(A)) = 1$ then $\mathcal{R}^{-1}(A)$ contains every algorithmically random sequence. Book [3] then observed that for an appropriate reducibility $\mathcal{R}$, the class

$$\text{ALMOST-}\mathcal{R} = \{A : \mathbf{Pr}(\mathcal{R}^{-1}(A)) = 1\}$$

can be characterized as exactly the the recursive part of $\{A : A \leq^{\mathcal{R}} B\}$, where B is any algorithmically random sequence. Thus, for example, for any algorithmically random B, the known characterization P = ALMOST-P$_m$ [1] becomes

$$\mathrm{P} = \{A : \; A \leq_m^P B \text{ and } A \text{ is recursive } \}$$

and likewise from [2], BPP is just the recursive part of P^B. The result in [3] is proved for reducibilities satisfying a more restrictive definition of "appropriateness" requiring that $\mathcal{C} = \mathcal{R}^{-1}(A)$ must always satisfy condition (1). The present result shows that (1) is unnecesary. Book and Mayordomo [5] use the general form of the zero-one law described in this note to give further characterizations of ALMOST-$\mathcal{R}$ classes in terms of n-randomness.

2 Preliminaries

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the natural numbers. Let $\{0,1\}^*$ denote the set of finite binary sequences, or *strings*. The concatenation of strings σ and τ is denoted $\sigma\tau$, $|\sigma|$ is the length of σ, and λ is the unique string of length zero. For $\sigma \in \{0,1\}^*$ and $j, k \in \mathbb{N}$ with $0 \leq j \leq k < |\sigma|$, $\sigma[k]$ is the kth bit of σ and $\sigma[j..k]$ is the string consisting of the jth through kth bits of σ (note the leftmost bit of σ is the 0th). The relation $\sigma \sqsubseteq \tau$ means that whenever $\sigma[k]$ is defined, $\tau[k]$ is defined also and $\sigma[k] = \tau[k]$; we say that σ is an *initial segment* or *prefix* of τ, and τ is an *extension* of σ. For an infinite sequence $A \in \{0,1\}^\infty$, the notations $A[k]$, $A[j..k]$, $A[k..\infty]$, σA, and $\sigma \sqsubseteq A$ are defined analogously. Strings σ and τ are *incompatible*, or *disjoint*, if $\sigma \not\sqsubseteq \tau$ and $\tau \not\sqsubseteq \sigma$. A subset $A \subseteq \mathbb{N}$ may be identified with its characteristic sequence $\chi_A \in \{0,1\}^\infty$, defined by $\chi_A[k] = 1 \Longleftrightarrow k \in A$, and we consistently equivocate between the two and write A for χ_A. For $A, B \in \{0,1\}^\infty$, $A \triangle B$ is the bitwise exclusive-OR of A and B or is equivalently the symmetric difference $(A - B) \cup (B - A)$ when A and B are interpreted as sets. $A \subseteq \mathbb{N}$ may be viewed as a subset of $\{0,1\}^*$, i.e., a *language*, by fixing a correspondence between $\{0,1\}^*$ and $\mathbb{N}$. A class $\mathcal{S} \subseteq \{0,1\}^\infty$ is *closed under finite variation* if whenever $A \in \mathcal{S}$ and $B \triangle A$ is finite then $B \in \mathcal{S}$ also. The complement in $\{0,1\}^\infty$ of a class $\mathcal{S}$ is denoted $\mathcal{S}^c$.

We assume the reader is familiar with the basic notions of recursive function theory, such as may be found in the early chapters of [16, 18, 20]. Let φ_e denote the partial recursive (p.r.) function with code or index e and W_e its domain, the eth recursively enumerable (r.e.) set; likewise φ_e^A is the eth p.r. function relative to $A \in \{0,1\}^\infty$ and W_e^A the eth r.e. set relative to A. Given $A \in \{0,1\}^\infty$, the *jump* of A, denoted A', is the set $\{x : \varphi_x^A(x) \text{ is defined } \}$, i.e., the halting set relative to A.

$A^{(n)}$ denotes the nth iterate of the jump of A, and in particular $0', 0'', \ldots, 0^{(n)}, \ldots$ are the iterates of the jump of the empty set. $0'$ is sometimes denoted K.

A string $\sigma \in \{0,1\}^*$ defines a subset $\mathrm{Ext}(\sigma) = \{A \in \{0,1\}^\infty : \sigma \sqsubseteq A\}$ of $\{0,1\}^\infty$, called an *interval*; if S is a subset of $\{0,1\}^*$, $\mathrm{Ext}(S)$ denotes $\bigcup_{\sigma \in S} \mathrm{Ext}(\sigma)$. If $S = W_e$ is an r.e. set of strings, then $\mathrm{Ext}(S)$ is called a Σ_1^0-class; the number e is an *index* of the class. A Σ_1^0-class may be referred to as a *recursively open set*, since it is open in the usual topology on $\{0,1\}^\infty$ in which the intervals $\{\mathrm{Ext}(\sigma) : \sigma \in \{0,1\}^*\}$ are taken as basic open sets. A Π_1^0-class, or a *recursively closed set*, is the complement of a Σ_1^0-class and the same index is associated with it. In general a Π_n^0-class is the complement of a Σ_n^0-class, and a Σ_{n+1}^0-class is of the form $\bigcup_i \mathcal{T}_i$, where $\{\mathcal{T}_i\}$ is a uniform sequence of Π_n^0-classes. Here sequence $\{\mathcal{T}_i\}$ is called *uniform* or *recursive* to mean that there is a recursive function f such that $f(i)$ is an index for $\mathcal{T}_i$; an index for the function f may be called an index of the Σ_{n+1}^0-class. Similarly, a Π_{n+1}^0-class is of the form $\bigcap_i \mathcal{T}_i$, where $\{\mathcal{T}_i\}$ is a uniform sequence of Σ_n^0-classes. A class is called *arithmetical* if it is Σ_n^0 for some n.

It is also convenient to note that arithmetical classes can be defined in terms of quantifier complexity. That is, a class $\mathcal{C}$ is Σ_n^0 if there is a recursive function φ, defined on all inputs for all oracles, such that

$$\mathcal{C} = \{A : (\exists x_1)(\forall x_2) \ldots (Q x_n)[\varphi^A(x_1, \ldots, x_n) = 1]\},$$

where Q is $\exists$ or $\forall$ when n is odd or even, respectively. See [18] or [9] for more detail on the construction of such hierarchies and the equivalence between the two definitions. The definitions of arithmetical classes can all be relativized, e.g., a Σ_1^C-class is of the form $\mathrm{Ext}(W_e^C)$, and so on. It can be shown, in particular, that a $\Sigma_1^{0^{(n-1)}}$-class is an open Σ_n^0-class, and a $\Pi_1^{0^{(n-1)}}$-class is a closed Π_n^0-class.

By a *measure* we simply mean a probability distribution on $\{0,1\}^\infty$, and for our purposes it is sufficient to consider the uniform distribution, i.e., each bit is equally likely to be a zero or a one, also called Lebesgue measure. The measure of a subset $\mathcal{S}$ of $\{0,1\}^\infty$, denoted $\mathbf{Pr}(\mathcal{S})$, can be intuitively interpreted as the probability that a sequence produced by tossing a fair coin is in the set $\mathcal{S}$; in particular the measure of an interval $\mathrm{Ext}(\sigma)$, abbreviated $\mathbf{Pr}(\sigma)$, is just $2^{-|\sigma|}$. For S a set of strings, we abbreviate $\mathbf{Pr}(\mathrm{Ext}(S))$ by $\mathbf{Pr}(S)$; if S is *disjoint*, i.e., all strings in S are pairwise incompatible, then $\mathbf{Pr}(S) = \sum_{\sigma \in S} \mathbf{Pr}(\sigma)$. Standard results of measure theory (see [8]) show that $\mathcal{S}$ is *measurable* (meaning that $\mathbf{Pr}(\mathcal{S})$ is defined) whenever $\mathcal{S}$ is a *Borel* set, i.e., built up from intervals by some finite iteration of countable union and complementation operations; in particular arithmetical classes are Borel sets. A class with measure zero is called a *nullset*. The classical zero-one law (see [17]) states that any measurable class which is closed under finite variation must have measure zero or measure one.

We will need the following more or less standard result of constructive measure theory. A proof (it is a fairly straightforward induction) can be found in [10].

Lemma 2.1 *There is a recursive procedure which, given the index of a Σ^0_n-class $\mathcal{S}$ and a rational $\epsilon > 0$, produces the index of a set U of strings such that U is r.e. relative to $0^{(n-1)}$, $\mathcal{S} \subseteq \mathcal{U} = \mathrm{Ext}(U)$, and $\mathbf{Pr}(\mathcal{U}) - \mathbf{Pr}(\mathcal{S}) \leq \epsilon$.*

Lemma 2.1 is a constructive analog of the fact that any measurable set of real numbers can be approximated from above by an *open* set. (Note that $\mathcal{U}$ is actually a $\Sigma_1^{0^{(n-1)}}$-class.)

The definition of algorithmic randomness below was originally given by Martin-Löf [15] and generalized by Kurtz [11]. (It is shown in [10] that the definition of n-randomness below is equivalent to that originally given in [11].) The definition is quite robust, and equivalent definitions have been given by Levin [13], Schnorr [19], Chaitin [6, 7], and Solovay [21]. See [14] for additional motivation and discussion; see [10] for an investigation of the recursion-theoretic properties of the definition.

Definition 2.2 Let $C \in \{0,1\}^\infty$. A *constructive null cover* or *Martin-Löf test* relative to C is a uniform sequence $\{S_i\}$ of sets of strings, where each S_i is r.e. relative to C and $\mathbf{Pr}(S_i) \leq 2^{-i}$. A sequence $A \in \{0,1\}^\infty$ is 1*-random*, or *algorithmically random*, relative to C if for every constructive null cover $\{S_i\}$ relative to C, $A \notin \bigcap_i \mathrm{Ext}(S_i)$. In particular if A is 1-random relative to $0^{(n-1)}$, we say A is *n-random*.

There is nothing special about the number 2^{-i} in the definition above; we will find it convenient to use r^{-i} for $0 < r < 1$, and indeed Solovay [21] has shown that *any* summable sequence may be substituted for $\{2^{-i}\}$.

Martin-Löf [15] proved the existence of a *universal test*, that is, a constructive null cover $\{U_i\}$ such that for any constructive null cover $\{S_i\}$, $\bigcap_i \mathrm{Ext}(S_i)$ is contained in $\bigcap_i \mathrm{Ext}(U_i)$; thus the complement of $\bigcap_i \mathrm{Ext}(U_i)$ is precisely the class of 1-random sequences. By relativizing his proof to $0^{(n-1)}$ we obtain a universal Σ^0_n test for n-random sequences.

We will also need to refer to the following simple result, proved in [10, 11].

Lemma 2.3 *Every Π^0_n-nullset is contained in a constructive null cover relative to $0^{(n-1)}$.*

3 Main Result

Our main result will be a consequence of the following.

Lemma 3.1 *Let $A, C \in \{0,1\}^\infty$ and let $\mathcal{U}$ be a Σ_1^C-class with $A \in \mathcal{U}$ and $\mathbf{Pr}(\mathcal{U}) < r < 1$. Suppose that for every $B \in \{0,1\}^\infty$ such that $A \triangle B$ is finite, $B \in \mathcal{U}$. Then A is contained in a constructive null cover relative to C.*

Proof. Let U be a set of strings r.e. relative to C for which $\mathcal{U} = \text{Ext}(U)$. We describe a uniform procedure, relative to C, for enumerating sets of strings $S_0 = U, S_1, S_2, \ldots$ so that $\{S_i\}$ is a constructive null cover of A relative to C. Intuitively the idea is as follows: suppose a string σ is enumerated in U; let $k = |\sigma|$. We know that for every string σ' of length k, the sequence $\sigma' A[k..\infty]$ is in $\mathcal{U}$, so there is some string compatible with $\sigma' A[k..\infty]$ enumerated in U. In some cases U may contain a prefix ρ of σ', but this can only occur for less than a fraction r of the strings σ' of length k since the measure of $\mathcal{U}$ is strictly less than r. Thus for all other strings σ' of length k, U must contain an extension of σ' of the form $\sigma'\tau \sqsubseteq \sigma' A[k..\infty]$. The idea is to use the fact that $\tau \sqsubseteq A[k..\infty]$ to approximate A. For example, to enumerate S_1, for each σ enumerated into U we continue to watch the enumeration of U and wait until a string τ can be identified such that for *every* σ' with $|\sigma'| = |\sigma|$ there is some initial segment of $\sigma'\tau$ enumerated in U; then $\sigma\tau$ is enumerated in S_1. It turns out that the measure of S_1 is at most r times the measure of U. To construct $S_2, S_3, \ldots$ we iterate the procedure.

Formally, define $S_0 = U$ and

$$S_{i+1} = \left\{ \sigma\tau : \begin{array}{l} \sigma \in S_i \text{ and for every } \sigma' \text{ with } |\sigma'| = |\sigma|, \\ \text{there is some } \rho \in U \text{ with } \rho \sqsubseteq \sigma'\tau. \end{array} \right\}$$

We first show that $\mathbf{Pr}(S_i) < r^i$. Certainly $\mathbf{Pr}(S_0) < 1$, so it will suffice to show that $\mathbf{Pr}(S_{i+1}) < r\mathbf{Pr}(S_i)$ for all i. Let $i \geq 0$ and fix $\sigma \in S_i$. Then

$$\begin{aligned} \mathbf{Pr}\{\tau : \sigma\tau \in S_{i+1}\} &= \mathbf{Pr}\{\sigma'\tau : |\sigma'| = |\sigma| \text{ and } \sigma\tau \in S_{i+1}\} \\ &\leq \mathbf{Pr}\{\sigma'\tau : |\sigma'| = |\sigma| \text{ and for some } \rho \in U,\ \rho \sqsubseteq \sigma'\tau\} \\ &\leq \mathbf{Pr}(\mathcal{U}) \\ &< r. \end{aligned}$$

Thus for each $\sigma \in S_i$, $\mathbf{Pr}\{\sigma\tau : \sigma\tau \in S_{i+1}\} < r\mathbf{Pr}(\sigma)$, so taking the union over $\sigma \in S_i$ it follows that $\mathbf{Pr}(S_{i+1}) < r\mathbf{Pr}(S_i)$.

Next we show that for all i, $A \in \mathrm{Ext}(S_i)$. Clearly $A \in \mathrm{Ext}(S_0) = \mathcal{U}$. Suppose for an induction that $A \in \mathrm{Ext}(S_i)$; then there is some $\sigma \in S_i$ with $\sigma \sqsubseteq A$. Let $k = |\sigma|$. For each string σ' with $|\sigma'| = k$, since $\sigma' A[k..\infty] \in \mathcal{U}$, there is some string $\rho \in U$ with $\rho \sqsubseteq \sigma' A[k..\infty]$; let $\rho_{\sigma'}$ be the shortest such string. Let ρ be a string in the set $\{\rho_{\sigma'} : |\sigma'| = k\}$ of maximal length; note $|\rho| > k$, since otherwise we would have $\mathbf{Pr}(\mathcal{U}) = 1$. We have $\rho = \sigma'\tau$ for some τ with $\sigma\tau \sqsubseteq A$. By construction, $\sigma\tau$ is enumerated in S_{i+1}. $\square$

Theorem 3.2 *Let $n \geq 1$, and let $\mathcal{S}$ be a Σ^0_{n+1}- or Π^0_{n+1}-class which is closed under finite variation. Then $\mathcal{S}$ either contains all n-random sequences or no n-random sequences.*

Proof. Assume that $\mathcal{S}$ is a Σ^0_{n+1}-class; the other case is obtained by considering the complement of $\mathcal{S}$. Then $\mathcal{S} = \bigcup_i \mathcal{T}_i$, where the $\mathcal{T}_i$ are Π^0_n-classes. If $\mathcal{S}$ has measure zero, then each $\mathcal{T}_i$ has measure zero, so it is sufficient to note that by Lemma 2.3, none of the classes $\mathcal{T}_i$ can contain an n-random sequence. Suppose on the other hand that $\mathbf{Pr}(\mathcal{S}) > 0$; then for some natural number J the class $\mathcal{T} = \bigcup_{i=0}^{J}$ has measure greater than 0 also. $\mathcal{T}$ is still a Π^0_n-class, so its complement $\mathcal{T}^c$ is a Σ^0_n-class with measure strictly less than 1. By Lemma 2.1, there is an open $\Sigma_1^{0^{(n-1)}}$-class $\mathcal{U}$ containing $\mathcal{T}^c$ whose measure is also strictly less than 1. It then follows from Lemma 3.1 that given any n-random sequence A, there must be some B such that $A \triangle B$ is finite and such that B is not in $\mathcal{U}$, i.e., $B \in \mathcal{T} \subseteq \mathcal{S}$. Since $\mathcal{S}$ is closed under finite variation, $A \in \mathcal{S}$ also. $\square$

4 Remarks

Theorem 3.2 is optimal in the sense that the class of random sequences to which it applies cannot be enlarged. If $\mathcal{R}$ is any class properly containing the n-random sequences, then $\mathcal{R}$ must contain some element A in the universal Σ^0_n test, which is a Π^0_{n+1}-nullset; its complement is thus a Σ^0_{n+1}-class, closed under finite variation, which fails to contain every member of $\mathcal{R}$.

Theorem 3.2 shows that any Π^0_{n+1}-nullset which is closed under finite variation cannot contain an n-random sequence; on the other hand, if a sequence A avoids every Π^0_{n+1}-nullset which is closed under finite variation, then in particular A avoids the universal Σ^0_n test (which is, of course, closed under finite variation). Thus we have another characterization of n-randomness:

Corollary 4.1 *Let $A \in \{0,1\}^\infty$ and $n \geq 1$. Then A is n-random if and only if A avoids every Π^0_{n+1}-nullset which is closed under finite variation.*

Note that the usual definition of n-randomness is also of the form "A avoids every Π^0_{n+1}-nullset of a special type," namely, A avoids the constructive null covers (relative to $0^{(n-1)}$). It is interesting to note that there is a "universal" version of the corollary above, i.e., the union of all Π^0_{n+1}-nullsets which are closed under finite variation is again a Π^0_{n+1}-nullset, since it is precisely the universal Σ^0_n test. The discussion also suggests the following definition, first appearing in [11]: $A \in \{0,1\}^\infty$ is said to be *weakly* $(n+1)$*-random* if A avoids *every* Π^0_{n+1}-nullset. Evidently every weakly $(n+1)$-random sequence is n-random; it is shown in [11, 10] that the converse does not hold.

Lemma 3.1 can also be interpreted in the following way.

Corollary 4.2 *(i) Suppose that $\mathcal{S}$ is a Σ^0_n- or Π^0_{n+1}-class containing an n-random sequence A along with each B for which $A \triangle B$ is finite. Then $\mathcal{S}$ contains every n-random sequence.*

(ii) Let $\mathcal{S}$ be any Σ^0_{n+1}- or Π^0_n-class with positive measure. Then for every n-random sequence A, there is a sequence $B \in \mathcal{S}$ such that $A \triangle B$ is finite.

Proof. (i) Let $\mathcal{S}$ be a Σ^0_n-class. If $\mathbf{Pr}(\mathcal{S}) < r$, there is a $\Sigma_1^{0^{(n-1)}}$-class $\mathcal{U}$ containing $\mathcal{S}$ with $\mathbf{Pr}(\mathcal{U}) < r$, implying by 3.1 that A is not n-random; hence $\mathbf{Pr}(\mathcal{S}) \geq r$ for every $r < 1$, i.e., $\mathbf{Pr}(\mathcal{S}) = 1$. Then $\mathcal{U}$ contains every n-random sequence by Lemma 2.3. The same applies to a Π^0_{n+1}-class, since it can be expressed as an intersection of Σ^0_n-classes. For (ii) apply (i) to the complement of $\mathcal{S}$. □

The above corollary shows that in the zero-one law it is not always strictly necessary to assume that the class $\mathcal{S}$ is closed under finite variation, only that it contains all finite variates of *some* n-random A. Note that (i) does not hold for $\mathcal{S}$ a Σ^0_{n+1}-class, since there exists an n-random sequence A which is not weakly $(n+1)$-random, and hence there is a Π^0_{n+1}-nullset containing A. One consequence of (ii) is that in any Π^0_n- or Σ^0_{n+1}-class with positive measure, there is some representative of every Turing degree (or m-degree) containing an n-random sequence; this fact was observed by A. Kučera [12] using a property similar to (ii).

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