DEGREES OF RANDOM SETS

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An explicit recursion-theoretic definition of a random sequence or random set of natural numbers was given by Martin-Löf in 1966. Other approaches leading to the notions of *n-randomness* and *weak n-randomness* have been presented by Solovay, Chaitin, and Kurtz. We investigate the properties of n-random and weakly n-random sequences with an emphasis on the structure of their Turing degrees.

After an introduction and summary, in Chapter II we present several equivalent definitions of n-randomness and weak n-randomness including a new definition in terms of a forcing relation analogous to the characterization of n-generic sequences in terms of Cohen forcing. We also prove that, as conjectured by Kurtz, weak n-randomness is indeed strictly weaker than n-randomness.

Chapter III is concerned with *intrinsic* properties of n-random sequences. The main results are that an $(n+1)$-random sequence A satisfies the condition $A^{(n)} \equiv_T A \oplus 0^{(n)}$ (strengthening a result due originally to Sacks) and that n-random sequences satisfy a number of strong *independence* properties, e.g., if $A \oplus B$ is n-random then A is n-random relative to B. It follows that any countable distributive lattice can be embedded in the 2-random degrees. We also prove that, surprisingly, this independence property fails for weak n-randomness.

In Chapter IV we consider a number of known measure-theoretic results of the form "almost every degree has property P", and use the hierarchy of n-randomness to analyze "how much" randomness is needed for a given property to hold. We obtain fairly sharp results for most of the known properties. For example, Kurtz showed that a.e. degree has a 1-generic predecessor and is relatively r.e. We analyze both proofs to show that 2-randomness is sufficient. That 1-randomness is not enough follows from a new "basis" theorem: every nonempty Π^0_1 -class contains a member with no relatively r.e. predecessor.

The notion of "almost every" degree and the explicit definitions of randomness we use depend on the measure employed. We conclude by proving a series of results concerning the invariance of the n-random degrees with respect to changes in this measure.

Biographical Sketch

Steven M. Kautz was born in San Francisco, California in 1954 and grew up in the formerly pleasant region near there now known as Silicon Valley. Although he began high school with honest intentions, he spent the second half of his sophomore year in Mexico, helping to build a technical school in the small town of San Vicente and reviving its economy by consuming a considerable quantity of excellent Mexican beer. Having “seen Paree”, he could not be persuaded to return to his old high school, and subsequently participated in forming the Tree School, a so-called alternative high school. The Tree School was kind enough to award Mr. Kautz a diploma the following year, despite the fact that he had spent most of his time hitchhiking repeatedly from coast to coast. Fortunately, during this time he was able to master the operation of the electric saw and other significant power tools. It is always important for a young person to learn a useful trade.

On one of these coast-to-coast excursions Mr. Kautz became involved with a small religious commune (it will not be dignified with a name; let it suffice to say that both the religion and the commune are appropriately modified by the adjective “small”). It was here that he met his remarkable wife Carol in 1973; inexplicably, they began to have children at the alarming rate:

$$\frac{4 \text{ children}}{5 \text{ years}}.$$

Mr. Kautz continued to support his family as a carpenter and building contractor for several more years, but being an individual with a naturally short attention span, he became increasingly irritated and bored with his work. It was thus that he started college at the age of twenty-seven with no particular goal in mind other than to find a profession not involving the use of power tools.

He attended Sacramento City College for two years, and completed a B.A. in Mathematics at California State University, Sacramento, two years later. The careful reader will have observed that indeed, the profession of mathematician can be successfully pursued without any power tools whatsoever, which made it an ideal choice. Mr. Kautz was then fortunate to be accepted as a graduate student at Cornell University, where he received an M.S. in Computer Science in May 1990, and is at this very moment completing the requirements for a Ph.D. in Mathematics.

Mr. Kautz is currently employed as an Assistant Professor at Randolph-Macon Woman’s College in Lynchburg, Virginia. It is noted with dismay that Mr. Kautz still does not possess a proper high school diploma.

To Carol

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I thank my parents, Nancy and Bill, for more kinds of support than can be named, and my appreciation goes out to my children, Joi, Judy, Bessa, and Paul—generally acknowledged to be four of the nicest people on the face of the earth—who have given me so much more than they realize during these years. Finally and above all I thank my wife Carol, whose courage, strength, and love made the entire process possible, from beginning to end. This thesis is dedicated to her with the hope that I prove worthy of the sacrifices she has made.

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Chapter I

Introduction

I.1 Randomness and Recursion Theory

The first explicit mathematical definition of randomness to utilize a formal notion of computation was proposed by Church in 1940. Since that time other recursion-theoretic definitions have been given by Kolmogorov, Martin-Löf, Solovay, Chaitin, and Kurtz, among others. In much of this previous work it is tacitly assumed that the subject of interest is the nature of randomness, and recursion theory is regarded merely as a tool for understanding and describing it. The theme of our present work is essentially a complementary view, namely, that *the subject of interest is really recursion theory, and the notion of randomness is a potentially useful tool, analogous to the well-established notion of* genericity, *for understanding the structure of the Turing degrees.* The same point of view can be seen in the work of Kučera on randomness (see [17], for example), which we closely follow in spirit, but can also be traced back to early measure-theoretic results of Spector and Sacks (see Theorem II.5.2). We will also cite Kurtz [15] and van Lambalgen [36] frequently.

Whether the nature of randomness can actually be characterized in terms of computation is at best a contentious question. Computation certainly has a valid *descriptive* role in the study of randomness. The idea of the "unpredictability" of random sequences is a significant part of the common intuition about randomness, and in view of Church's thesis we have no reasonable interpretation of "unpredictable" other than "not predictable by an algorithm" (see Section II.2). From here it is a short step to "not predictable by a *feasible* algorithm", and one useful feature of the recursion-theoretic definitions of Martin-Löf and Kolmogorov is the ease with which they specialize to resource-bounded computations (Hartmanis [8], Ko [13], Lutz [20]). In the setting of computer science, computational definitions of randomness seem to capture all of its relevant properties (see Chaitin [2], for example) and are usually much stronger than necessary.

There is little reason to believe, however, that one can recursion-theoretically characterize precisely the properties of randomness needed in a more general mathematical setting. In his recent work on the foundations of probability theory, van Lambalgen proposes abandoning the attempt to use an explicit definition of randomness and instead provides an *axiomatization* of randomness. A definition such as Martin-Löf's then provides a model for all or part of the axioms, but it is by no means a canonical choice. The motivated reader is referred to van Lambalgen's excellent and thorough investigation [36].

The preceding remarks can be simplified into two extreme points of view: to a computer scientist, a computational definition of randomness characterizes much more than necessary, while to a probabilist it does not describe nearly enough. We suggest instead that it is from the vantage point of recursion theory that the intrinsic connections between randomness and computation are most evident and meaningful; this is at least plausible, since it is only in recursion theory that one has both the continuum 2^ω to work with, so that notions of probability and measure naturally apply, *and* the computational or degree structure on 2^ω. The remainder of this section is devoted to an informal exposition of two kinds of connections which we feel to be particularly significant.

First, we are interested in characterizing "typical" degrees. The indications of recent and current research are that the structure $\mathcal{D}$ (the Turing degrees with the usual ordering $\leq_T$) is structurally and logically as complex as it could possibly be; $\mathcal{D}$ is known to be highly nonhomogeneous and has very few automorphisms, probably none, and thus includes all manner of local, pathological features. It is then meaningful to ask what the typical properties are, that is, what can be said about "most" degrees. A number of results of the form "almost every degree has property P" have been proved, e.g., by Sacks, Stillwell, Paris, and Kurtz; many of these are discussed in Chapter IV.

Here an explicit definition of randomness is exactly what we want, because once we know a property holds for most degrees, we would like to get our hands on *particular* degrees with the property in question, determine where they live in $\mathcal{D}$ (e.g., locate them within the arithmetical hierarchy), and study what other properties they might possess. An excellent example of the application of this approach is Kučera's use of results on random sets and fixed-point-free degrees to settle an open question about the generalized Arslanov completeness criterion (Kučera [18]).

Another reason that an explicit definition of randomness is necessary is in order to talk about structural properties of random degrees in relation to one another. It is known that the first-order theory of $\mathcal{D}$ is as complicated as second order arithmetic. By contrast, Stillwell [34] has shown that the "a.e. theory" of $\mathcal{D}$, that is, the theory of $\mathcal{D}$ in a language containing only quantifiers of the form "for all except a set of measure 0", is *decidable*. This result is tantalizing in that it suggests that the structure of the random degrees should be simpler than that of $\mathcal{D}$; e.g., are the degrees, less a nullset, homogeneous? Decidable? (We hasten to mention that final answers to these questions seem to be a long way off; our early conjectures are a guarded "no" to both.) Stillwell's result is also infuriating in that it offers no clues towards the resolution of the questions it raises, since it says nothing about any *particular* subclass of the degrees. For example, some common ingredients of structural complexity proofs are results on embeddability of various kinds of partially ordered structures into $\mathcal{D}$ or its substructures. Notice there is no way to talk about embedding a given structure into a class of degrees using statements about "almost every degree". We can only hope to do so by explicitly defining a class of *random* degrees, and investigating structural properties of this class.

A second connection between randomness and recursion theory is seen in the analogy between random sets and *generic* sets, those constructed by finite extension arguments. The analogy can be exploited at two different levels.

On the one hand, generic sets can be defined in terms of a *forcing* relation where the forcing conditions are finite strings, i.e., clopen subsets of 2^ω (Cohen forcing). Generic sets are then defined to be exactly those which force every arithmetical sentence or its negation. If we instead take closed arithmetical classes of positive measure as the forcing conditions—this is sometimes called Solovay forcing—the corresponding definition essentially characterizes the random sets defined in Section II.3.

On the other hand, what all this really means is that while random sets are "typical" members of 2^ω in the sense of measure, generic sets are "typical" in the sense of Baire category. Thus while a random set (that is, an ω-random set as in Definition II.1.2) is in every measure one arithmetical class, a generic set is in every *dense open* arithmetical class (equivalently, in every *comeager* arithmetical class). Generic sets can be seen to be typical in a particularly appealing way, namely, they have every property that can be proved to exist by a finite extension argument. Random sets are less familiar, and the constructions are more difficult as they involve more complex classes than just basic intervals (extensions of finite strings), yet the notion of "typicalness" in the sense of measure is intuitively the more natural one. We suggest that random sets are worth investigating because of the analogy with the more familiar generic sets, even if for no other reason.

I.2 Summary of Results

In the first four sections of Chapter II we present definitions of randomness via four entirely different approaches, and then in Section II.5 give sharp results on the extent to which the definitions coincide. Section II.1 is devoted to the notion of *effective approximations in measure* used by Martin-Löf and Solovay: A set $A \in 2^\omega$ is Σ_n^C -*approximable* if there is a recursive sequence of Σ_n^C -classes[1] $\{\mathcal{S}_i\}_{i\in\omega}$ with $\mu(\mathcal{S}_i) \leq 2^{-i}$ and $A \in \bigcap_i \mathcal{S}_i$. A is *n-random relative to C*, or C-n-random, if A is not Σ_n^C -approximable. We prove a basic lemma (Lemma II.1.5) to the effect that an approx-

[1]Notation is defined in Section I.3.

imation in measure can always be replaced by an approximation using *open* classes of the same arithmetical complexity, which shows, for example, that n-randomness is the same as 1-randomness relative to $0^{(n-1)}$. In Section II.2 we examine Church's original attempt to apply a notion of computation to the problem of defining randomness, and then introduce notions of program size complexity due to Kolmogorov and Chaitin; the latter yeilds an alternate characterization of n-randomness. Section II.3 contains Kurtz' direct measure-theoretic approach: $A \in 2^\omega$ is *weakly n-random relative to C* if A is in every Σ_n^C -class with measure one. In Section II.4 we explore the analogy between *randomness* and *genericity*, and give a characterization of weak n-randomness in terms of a forcing relation: while the usual generic and n-generic sets are literally generic with respect to Cohen forcing in arithmetic, random sets are the "generic" objects with respect to Solovay forcing in arithmetic, i.e., forcing with closed sets of positive measure rather than with finite initial segments.

Kurtz showed that

$$C\text{-weakly } (n+1)\text{-random} \Rightarrow C\text{-}n\text{-random} \Rightarrow C\text{-weakly } n\text{-random}$$

and that the first implication is not reversible; we give the brief proofs in Section II.5. Kurtz also noticed that there are weakly 1-random sets which are not 1-random, and conjectured that for all n there are weakly n-random sets which are not n-random. It turns out that the $n = 1$ case is anomalous, and the argument does not generalize. Nonetheless, the conjecture is true; in Theorem II.5.5 we give a direct proof.

Theorem II.5.5 *For each $n \geq 1$ there is weakly n-random set which is not n-random.*

The proof combines the technique of forcing with closed sets of positive measure (to make A avoid every Π_n^0 -nullset) with a finite injury argument (to enumerate a Σ_n^0 -approximation of A). An alternate proof is given in Section III.4, where we show that a weakly n-random set may fail to satisfy some of the strong independence properties of n-random sets.

Much of the material in Chapter II consists of generalizations and improvements of known facts; substantially new results include the proof just described above and the characterization of randomness in terms of a forcing relation in Section II.4. However, the detailed presentation we give is justified, since a coherent synthesis of the various approaches and the relevant lemmas connecting them has not appeared in print.

In Chapter III we prove a number of results on specific properties of n-random sets. Some of these are motivated by the global questions discussed in Chapter IV concerning properties which hold for almost every degree, and it is there that we will put the results of Chapter III to good use in analyzing "how much" randomness is needed to guarantee that such properties hold.

We start in Section III.1 with an exposition of some simple facts about 1-random and n-random sets. We first examine two ways to express the intuitive idea that no part of a random sequence should contain any "information" about any other part. One view, which is seen in the Church-von Mises approach to defining randomness

(Section II.2), is that any recursive section of an n-random sequence should itself be n-random. For example, we have:

Theorem III.1.1(ii) *If A is n-random, then for each $i \in \omega$ the column $A^{[i]}$ is n-random.*

More generally, if A is n-random then the subsequence of A determined by any Δ_n procedure f, which we denote $A/\hat{f}(A)$ (see Definition II.2.1), is itself n-random; this is formulated precisely in Theorem III.1.2. Another view is that, literally, disjoint sections of a random sequence should be Turing-incomparable. For example, the columns of an n-random sequence A are *recursively independent*:

Theorem III.1.4(ii) *If A is n-random, then for each $i \in \omega$,*

$$A^{[i]} \not\leq_T \bigoplus_{j \neq i} A^{[j]}.$$

It will turn out that the properties expressed in both the preceding theorems are aspects of a somewhat deeper result, Theorem III.3.9, which is discussed in Section III.3. Section III.1 concludes with a quick proof that 1-random sets satisfy the law of large numbers.

Section III.2 contains some new results on the jump classes of n-random sets. The most important of these is Theorem III.2.1, which strengthens a result of Sacks' that the class $\{A : A' \equiv_T A \oplus 0'\}$ has measure one.

Theorem III.2.1 *Let $A \in 2^\omega$ and $n \geq 1$. If A is n-random, then $A^{(n-1)} \equiv_T A \oplus 0^{(n-1)}$.*

The above result is the key to generalizing results on 1-randomness to larger n. As a first application, we notice that Kučera's proof that every degree above $0'$ contains a 1-random set can be generalized in the following form.

Theorem III.2.2 *Let $n \geq 1$. For every $B \geq_T 0^{(n)}$ there is an n-random set A with $A^{(n-1)} \equiv_T B$.*

It will also follow that the set A constructed above will be n-random but not $(n+1)$-random.

Theorem III.2.3 *Let $n \geq 1$. The class $\{A : A^{(n-1)} \geq_T 0^{(n)}\}$ has measure zero, and in fact contains no $(n+1)$-random sets.*

Section III.3 is concerned with what can broadly be called *strong independence properties* of n-random sets, in the sense that relative n-randomness models the probabilistic notion of independence. (This can be made precise using van Lambalgen's axiomatization; see [35].) We first show that the results of Section III.1 can be interpreted to yeild more information than is at first apparent. That is, we saw in Theorem III.1.2 that if A is n-random, the subsequence $A/\hat{f}(A)$ determined by a Δ_n procedure f is itself n-random. In fact, the proof actually shows that if f is any function such that A is n-random relative to f, then the subsequence $A/\hat{f}(A)$ is n-random. This means in a strong sense that *almost every* subsequence of an n-random set is

n-random, since we will see as a consequence of Theorem III.3.1 that for an n-random set A,

$$\{\mathbf{deg}(f) : \ A \text{ is } n\text{-random relative to } f\}$$

has measure one.

The results of Section III.1 can also be strengthened in a different way. For example, we saw above that if A is n-random, then by Theorem III.1.1 each column $A^{[i]}$ is n-random, and by Theorem III.1.4 each column $A^{[i]}$ is incomparable to the "rest" of A (the join of the other columns). The result below asserts the independence of the columns in a much stronger sense.

Theorem III.3.7(ii) *If A is n-random, then for each $i \in \omega$, $A^{[i]}$ is n-random relative to $\bigoplus_{j \neq i} A^{[j]}$.*

Then in Theorem III.3.9 we combine the result above with the analysis in the previous paragraph to show that for any f such that A is n-random relative to f, the subsequence $A/\hat{f}(A)$ is n-random relative to $A/\overline{\hat{f}(A)}$; thus almost every subsequence of an n-random set A is "strongly independent" of the rest of A.

Section III.3 concludes with an interesting application of the independence phenomenon. We show that if A and B are relatively 2-random, then A and B form a minimal pair. More generally, we have:

Theorem III.3.14 *If $A \oplus B$ is 2-random relative to C, then $\mathbf{deg}(C)$ is the greatest lower bound of $\mathbf{deg}(A \oplus C)$ and $\mathbf{deg}(B \oplus C)$.*

It then follows using Theorem III.3.7 and standard arguments that any countable distributive lattice can be embedded into the 2-random degrees (or in particular into the columns of any 2-random set).

In Section III.4 we prove a surprising result, namely that the strong independence properties of n-random sets do not necessarily hold for weakly n-random sets. The key lemma is:

Lemma III.4.2 *If B is n-random, there is a set A such that $A \leq_T B \oplus 0^{(n+1)}$ and $A \oplus B$ is weakly n-random.*

Then by Theorem III.2.2, there is an n-random B with $B^{(n-1)} \equiv_T 0^{(n+1)}$, and so $B \oplus 0^{(n+1)} \leq_T B^{(n-1)}$. Thus there is an $A \leq_T B^{(n-1)}$ such that $A \oplus B$ is weakly n-random, but since $\{A\}$ is then a Π^B_n -nullset, we have

Theorem III.4.3 *For each $n \geq 1$ there is a weakly n-random set $A \oplus B$ such that A is not weakly n-random relative to B.*

The above result provides a second proof that there are weakly n-random sets which are not n-random, and provides an explicit example of one of the ways the two notions may differ. Section III.4 concludes with some remarks relating Turing-comparability and independence. Note that if $C >_T 0$, no set $A \geq_T C$ can be weakly 2-random relative to C, since $\{B : \varphi^B_e = C\}$ is a Π^C_2 -nullset for any e. This follows from Sacks' well-known result that the cone above any nonrecursive degree has measure zero (Theorem II.5.2). In Theorem III.4.4 we show by contrast that it is possible to construct $A \geq_T C >_T 0$ such that A is weakly 1-random relative to C.

In Chapter IV we attempt to synthesize the understanding of the n-random/weakly n-random hierarchy developed in Chapter III with a number of known measure-theoretic facts about properties of "almost all" degrees. The idea is that a statement of the form "every n-random degree has property P" is both more informative and more useful than a statement such as "a.e. degree has property P". An example of this sort of analysis is our Theorem III.2.1, where we showed that the condition $A^{(n-1)} \equiv_T A \oplus 0^{(n-1)}$ holds if A is n-random, though it may fail if A is $(n-1)$-random. The repeated usefulness of Theorem III.2.1 provides some evidence of the merit of this approach.

We begin in Section IV.1 by showing that 1-random sets exist satisfying certain *nontypical* properties. We first notice that there are Π^0_1 -classes all of whose members are 1-random, and then apply several *basis theorems*, which state that a nonempty Π^0_1 -class has a member of r.e. degree, of low degree, etc. The bulk of the section is devoted to the proof of a new basis theorem which will be put to use in Section IV.2:

Theorem IV.1.6 *Every nonempty Π^0_1 -class contains a member A such that no $B \leq_T A$ is relatively r.e.*

Section IV.2 begins with a proof of the following effective version of the zero-one law, which is a fundamental tool in our analysis of "how much"randomness is needed for a given property to hold.

Theorem IV.2.2 *A degree-invariant Σ^0_n+1 - or Π^0_n+1 -class contains either all n-random sets or no n-random sets.*

Although the zero-one law is always the starting point, sharper results can usually be obtained by ad hoc methods. For example, referring to item (ii) in the theorem below, the class $\{A \oplus B : \ A, B \text{ form a minimal pair}\}$ can be seen to be a Σ^0_4 -class by counting quantifiers, so by the zero-one law it must contain every 3-random set. But we saw already in Theorem III.3.14 that for any 2-random $A \oplus B$, A and B form a minimal pair.

Most of the known facts are summarized in Theorem IV.2.4:

Theorem IV.2.4

(i) The class $\{A : A \text{ is not minimal}\}$ has measure one (Sacks [31]*), and includes every* 1*-random set.*

(ii) The class $\{A \oplus B : \ A, B \text{ form a minimal pair}\}$ has measure one (Stillwell [34]*); it includes every* 2*-random set but not every* 1*-random set.*

(iii) For each n, the class $\{A : A^{(n-1)} \equiv_T A \oplus 0^{(n-1)}\}$ has measure one (Sacks, Stillwell [34]*); it includes every n-random set but not every* $(n-1)$*-random set.*

(iv) The class $\{A : \mathbf{deg}(A) \text{ is hyperimmune}\}$ has measure one (Martin [21]*); it*

includes every 2*-random set but not every* 1*-random set.*

(v) The class $\{A:\ A$ *has a* 1*-generic predecessor*$\}$ *has measure one (Kurtz* [15]*); it includes every* 2*-random set but not every* 1*-random set.*

(vi) The class $\{A:\ \mathbf{deg}(A)$ *is relatively r.e.*$\}$ *has measure one (Kurtz* [15]*); it includes every* 2*-random set but not every* 1*-random set.*

In Section IV.3 we investigate randomness with respect to computable measures other than Lebesgue measure. Our main objective is to be able to assert, having finally arrived in the previous section at a number of conclusions of the form "every n-random degree has property P", that essentially the same results would have been obtained had we initially defined n-randomness with respect to any computable measure, i.e., that the class of n-random degrees is invariant with respect to changes in the measure. We have fairly sharp results on the extent to which this kind of invariance property holds. A secondary benefit is that we are then able to use other measures whenever convenient to draw conclusions about the n-random degrees; as an application of this approach we give a new proof of a result, due to Demuth, that the nonrecursive tt-predecessors of an n-random set have n-random T-degree (Theorem IV.3.16).

Briefly, a measure μ is *computable* if the measures of basic intervals $\mathrm{Ext}(\sigma)$ can be recursively approximated in a uniform way. The measure μ is said to be *nontrivial* if no countable set of points has measure one, and *nonatomic* if $\mu(\{B\}) = 0$ for every singleton $\{B\}$. For the remainder of this discussion, μ and ν may denote arbitrary computable measures, while the symbol λ will denote Lebesgue measure. Notice that the various definitions of randomness can be interpreted for an arbitrary computable μ as well as for Lebesgue measure; e.g., we say A is *n-random with respect to μ* if for every recursive sequence of $\Sigma_1^{0(n-1)}$ -classes $\{\mathcal{S}_i\}_{i\in\omega}$ with $\mu(\mathcal{S}_i) \leq 2^{-i}$, $A \notin \bigcap_i \mathcal{S}_i$.

Each measure μ induces a natural correspondence between real numbers in $[0,1]$ and sequences in 2^ω. For $a \in [0,1]$ the representation of a with respect to μ is denoted $\mathrm{seq}_\mu(a)$; thus $\mathrm{seq}_\lambda(a)$ is the usual binary expansion of a. Loosely speaking, the correspondence between $[0,1]$ and 2^ω is defined in such a way that μ will act like a uniform measure on $[0,1]$. Precise definitions are given in Section IV.3.

One interesting fact is that randomness may be regarded as an "invariant" of a real number a; we show in Theorem IV.3.14 that given $a \in [0,1]$, and a nonatomic, computable measure ν, the representation $\mathrm{seq}_\nu(a)$ is n-random with respect to ν if and only if for every other nonatomic, computable measure μ, the representation $\mathrm{seq}_\mu(a)$ is n-random w.r.t. μ. In conjunction with the fact, proved in Theorem IV.3.8, that $\mathrm{seq}_\nu(a)$ and $\mathrm{seq}_\mu(a)$ have the same Turing degree, we have the following "degree-invariance" result, which says that we can define randomness with respect to any nonatomic, computable measure, and the class of n-random degrees obtained is always the same.

Corollary IV.3.18 *Let* $A, C \in 2^\omega$*,* $A >_T 0$*, and let* μ, ν *be nonatomic, computable measures. If* A *is* C*-1-random with respect to* ν*, then there is some* $B \equiv_T A$ *such that*

B is C-1-random with respect to μ.

When we consider atomic measures the situation is much more complicated. The idea in Corollary IV.3.18 is that if a nonrecursive A is C-1-random (with respect to ν, say), then A is the representation $\mathrm{seq}_\nu(a)$ of a real number a such that if $\mathrm{seq}_\mu(a)$ is nonrecursive, then it has the same degree as A and is C-1-random w.r.t. μ. But if μ is atomic, it may be that $\mathrm{seq}_\mu(a)$ is recursive; this occurs if $B = \mathrm{seq}_\mu(a)$ is a point for which $\mu(\{B\}) > 0$. But as long as A is at least 2-random, then there is always some other real b of degree $\mathbf{deg}(A)$ (i.e., whose standard representation has the same degree as A) such that $\mathrm{seq}_\mu(b)$ *is* nonrecursive. We then have:

Theorem IV.3.19 *Let $A, C \in 2^\omega$ with $A >_T 0$, and let ν, μ be nontrivial computable measures. Suppose A is C-2-random with respect to ν. Then there is a $B \equiv_T A$ which is C-2-random with respect to μ.*

We next devote some effort to showing that neither Corollary IV.3.18 nor Theorem IV.3.19 can be substantially improved. In particular we show:

Theorem IV.3.21 *There is a nontrivial, computable measure μ such that for any Δ_2 real number a, $B = \mathrm{seq}_\mu(a)$ is recursive, i.e., $\mu(\{B\}) > 0$.*

Corollary IV.3.22 *Let μ be the measure constructed in Theorem IV.3.21; then no nonrecursive Δ_2 set is 1-random with respect to μ.*

The corollary above is obtained by means of a very surprising "duality" between the existence of points A such that $\nu(\{A\}) > 0$ and the existence of intervals $I \subseteq 2^\omega$ such that $\nu(I) = 0$. In general there is no reason to expect that either condition implies the other. What we show in Theorem IV.3.12 is that given a nonrecursive degree $\mathbf{d}$, if for every real a whose standard representation has degree $\mathbf{d}$, $B = \mathrm{seq}_\nu(a)$ is a point with $\nu(\{B\}) > 0$, then every set A of degree $\mathbf{d}$ is contained in an interval I with $\nu(I) = 0$. We then show that whenever $\nu(I) = 0$, every $A \in I$ is Σ^0_1-approximable with respect to ν (Lemma IV.3.15). Applying this argument to Δ_2 degrees $\mathbf{d}$ yeilds the corollary. Since we know there are Δ_2 sets which are 1-random (with respect to λ), the corollary shows that the 1-random degrees may not be exactly the same for all *atomic* computable measures.

I.3 Notation and Conventions

Most of our notation is standard; any undefined terminology can be found in Odifreddi [26] or Soare [32]. Lowercase ω denotes the natural numbers; the central objects of our attention are elements of the *continuum* 2^ω or $\{0,1\}^\omega$. We refer to an element of 2^ω both as an infinite binary *sequence* and as a *set* of natural numbers. We identify sets with their characteristic functions, so that, e.g., $A(n) = 1$ means that $n \in A$ (as a set) or that the $(n+1)$st digit of A (as a sequence) is 1. We generally use e, i, j, k, l, m, n as well as x, y and z for elements of ω and A, B, C, D, S, T, U, V for elements of 2^ω. Subsets of 2^ω will be represented by the script letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}$; we will try to be consistent in referring to subsets of 2^ω as *classes* and elements of 2^ω as sequences or sets.

We use $\overline{A}$ to denote the set complement $\omega - A$, or equivalently the bitwise complement of A as a sequence. $A \upharpoonright n$ is the finite initial segment of A of length n. $A \oplus B$ denotes the set

$$\{2n : n \in A\} \cup \{2n+1 : n \in B\},$$

i.e., the sequence with A on the even bits and B on the odd bits. If B is an infinite set, A/B denotes the subsequence of A defined by

$$(A/B)(n) = A(b_n),$$

where b_n is the $(n+1)$th element of B in order of magnitude (i.e., the location of the $(n+1)$st "1" in B as a sequence). The notation A/B is undefined of B is finite.

Let $\langle \cdot, \cdot \rangle : \omega \times \omega \longrightarrow \omega$ be a fixed recursive bijection. Then $A^{[i]}$, the *ith column of* A, is defined by

$$A^{[i]} = \{n : \langle n, i \rangle \in A\}.$$

Given any countable sequence $\{B_i\}_{i \in \omega}$ we can define a set $\bigoplus_i B_i$ whose ith column is B_i:

$$\bigoplus_{i \in \omega} B_i = \{\langle n, i \rangle : n \in B_i\}.$$

The set of finite binary strings is denoted $2^{<\omega}$; strings are usually represented by σ, τ, ρ, etc. The length of a string σ is denoted $|\sigma|$; $\sigma(n)$ is the $(n+1)$st bit of σ for $n < |\sigma|$. The symbol $\emptyset$ ambiguously denotes an empty set or the string of length 0. We may use $\overline{\sigma}$ for the bitwise complement of the string σ. We write $\sigma \subset \tau$ if σ is an initial segment of τ (not necessarily proper); likewise $\sigma \subset A$ means $\sigma = A \upharpoonright |\sigma|$ for $A \in 2^\omega$. The notation $\sigma \prec \tau$ means that σ precedes τ lexicographically, and $\sigma * \tau$ is the concatenation of σ and τ. When $|\sigma| = |\tau|$, by $\sigma \oplus \tau$ we mean the string π with $\pi(2i) = \sigma(i)$ and $\pi(2i+1) = \tau(i)$ for all $i < |\sigma|$. Likewise σ/τ makes sense as long as $|\sigma| \leq |\tau|$ We assume a canonical identification of $2^{<\omega}$ with ω, so that subsets of ω may always be regarded as sets of strings when required by context.

For σ a string and S a set of strings, let

$$\begin{aligned} \text{Ext}(\sigma) &= \{A \in 2^\omega : \sigma \subset A\} \\ \text{and } \text{Ext}(S) &= \{A \in 2^\omega : (\exists \sigma \in S)\sigma \subset A\}. \end{aligned}$$

We take $\{\text{Ext}(\sigma) : \sigma \in 2^{<\omega}\}$ as the base of a topology on 2^ω; each $\text{Ext}(\sigma)$ is a basic *clopen* set, also called an *interval*. Except where noted otherwise, the symbol μ denotes the measure $\{\frac{1}{2}, \frac{1}{2}\}^\omega$ on 2^ω; generally we identify 2^ω with the interval $[0, 1]$ by associating each real number with its usual binary representation, and picture μ as Lebesgue measure on $[0, 1]$. In any case, we have $\mu(\text{Ext}(\sigma)) = 2^{-|\sigma|}$. It is also useful to note that

$$\mu(\text{Ext}(\sigma \oplus \tau)) = 2^{-|\sigma|} \cdot 2^{-|\tau|}.$$

A property P is said to hold *almost everywhere*, or *for a.e. sequence* A, if the class

$$\{A : A \text{ has property } P\}$$

has measure one. A class with measure zero is called a *nullset*.

We use φ_e to denote the eth partial recursive (p.r.) function, and φ_e^A for the eth p.r. function relative to $A \in 2^\omega$. We write $\varphi_e(x)\downarrow$ if φ_e is defined on x, and $\varphi_e(x)\uparrow$ otherwise; the same holds for the relativized φ_e^A. We often regard φ_e as the p.r. *functional*

$$A \mapsto \phi_e^A,$$

where A is in the domain of φ_e if φ_e^A is *total*, i.e., $\varphi_e^A(x)\downarrow$ for all $x \in \omega$. In such cases it is convenient to assume that φ_e^A is $0,1$-valued (e.g., interpret any nonzero value as 1) so that the range of the functional may be regarded as a subset of 2^ω, though we will not do so consistently. For $s \in \omega$,

$$\varphi_{e,s}(x) = \begin{cases} \varphi_e(x) & \text{if } \varphi_e(x) \text{ converges in} \leq s \text{ steps} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

$\varphi_e^\sigma(x)$ generally abbreviates $\varphi_{e,|\sigma|}^\sigma(x)$; we treat φ_e^σ as a partial function on ω. As usual, $W_e = \mathrm{dom}(\varphi_e)$ is the eth recursively enumerable (r.e.) set, $W_{e,s} = \mathrm{dom}(\varphi_{e,s})$, $W_e^A = \mathrm{dom}(\varphi_e^A)$, and $W_e^\sigma = \mathrm{dom}(\varphi_e^\sigma)$.

If φ_e^A is total, so that $\varphi_e^A = B$ for some set B, we write $B \leq_T A$; if $B \leq_T A$ and $A \leq_T B$, we write $A \equiv_T B$. $A <_T B$ means that $A \leq_T B$ but $B \not\leq_T A$. The equivalence class $\mathbf{deg}(A) = \{B \in 2^\omega : A \equiv_T B\}$ is called the *degree* of A; $\mathcal{D}$ denotes the collection $\{\mathbf{deg}(A) : A \in 2^\omega\}$, the *Turing degrees* or *degrees of unsolvability*. Degrees are denoted by boldface letters $\mathbf{a},\mathbf{b},\mathbf{c},\ldots$ The relation $\leq_T$ induces a well-defined partial order on $\mathcal{D}$ (simply denoted $\leq$) and the operation $\oplus$ induces a well-defined least upper bound operation $\cup$ on $\mathcal{D}$. The *jump* of a set A, denoted A', is the set

$$\{x : \varphi_x^A(x)\downarrow\},$$

and $A^{(n)}$ represents the nth iterate of the jump of A. For functions $f : \omega \longrightarrow \omega$, by $\mathbf{deg}(f)$ we mean the degree of the graph of f, $\{\langle x, y\rangle : f(x) = y\}$. By the measure of a collection $\mathcal{C}$ of degrees, we mean the measure of $\bigcup\mathcal{C}$; a property is said to hold for *almost every degree* if it holds for a collection of degrees with measure one in this sense.

Most of the classes we encounter will be *arithmetical*, i.e., members of an effective Borel hierarchy, possibly relative to some oracle. For this reason issues of measurability arise only very rarely. Arithmetical classes are defined as follows: A class of the form $\mathrm{Ext}(W_e)$ is called a Σ_1^0 - *class*; we sometimes refer to e as an index of the class. A Π_1^0 -class is the complement of a Σ_1^0 -class. In general a Π_n^0 -class is the complement of a Σ_n^0 -class, and a $\Sigma_n^0{+}1$ -class is of the form $\bigcup_i \mathcal{T}_i$, where $\{\mathcal{T}_i\}_{i\in\omega}$ is a uniform sequence of Π_n^0 -classes; likewise a $\Pi_n^0{+}1$ -class is of the form $\bigcap_i \mathcal{T}_i$, where the $\mathcal{T}_i$ are Σ_n^0 -classes. Here, as elsewhere, a *uniform* (or *recursive*) sequence $\{\mathcal{T}_i\}_{i\in\omega}$ is one for which there is a recursive function f such that $f(i)$ is an index for $\mathcal{T}_i$; an index for the function f may be called an index of the sequence. For each n we assume there is a canonical assignment of indices to Σ^0_n -classes (respectively, Π^0_n -classes) with the property that given an index for a Σ^0_n+1 -class (Π^0_n+1 -class) $\mathcal{S}$, we have a uniform way to obtain an index for the sequence $\{\mathcal{T}_i\}_{i\in\omega}$ of Π^0_n -classes (Σ^0_n -classes) such that $\mathcal{S} = \bigcup_i \mathcal{T}_i$ (respectively, $\mathcal{S} = \bigcap_i \mathcal{T}_i$). A class is called *arithmetical* if it is Σ^0_n for some n. A Σ^0_0 - or Π^0_0 -class is the extension of a *finite* set of strings, i.e., a clopen set.

It is also convenient to note that arithmetical classes can be defined in terms of quantifier complexity. Let $\mathcal{L}^*$ be the language of arithmetic with a set constant X and a membership symbol $\in$. Then for a sentence ϕ of $\mathcal{L}^*$ and $A \in 2^\omega$, $A \models \phi$ means that ϕ is true in the standard model when X is interpreted as A. A Σ^0_n -class is of the form $\{A : A \models \phi\}$, where ϕ is a Σ^0_n sentence of $\mathcal{L}^*$; likewise for Π^0_n -classes. Since notions of computation can be expressed in a simple way in $\mathcal{L}^*$ we can, for example, represent a Σ^0_n -class in the form

$$\{A : (\exists x_1)(\forall x_2)\dots(\exists x_n)[\varphi^A_e(x_1,\dots,x_n)\downarrow]\}$$

if n is odd, and in the form

$$\{A : (\exists x_1)(\forall x_2)\dots(\forall x_n)[\varphi^A_e(x_1,\dots,x_n)\uparrow]\}$$

if n is even. See Rogers [30] for details.

The definitions of arithmetical classes can all be relativized; e.g., a Σ^C_1 -class is of the form $\mathrm{Ext}(W^C_e)$, etc. Note, for example, that a $\Sigma^{0^{(n-1)}}_1$ -class is an *open* Σ^0_n -class, and a $\Pi^{0^{(n-1)}}_1$ -class is a *closed* Π^0_n -class.

We mention just a few more useful facts: A Σ^0_1 -class can always be represented as $\mathrm{Ext}(S)$ for S a *recursive* set of strings, and a Π^0_1 -class can be represented as the set $[T]$ of infinite paths through a recursive tree T. (Here a *tree* is a set of strings closed under initial segments.) It is often useful to note that given (the index of) a Σ^0_1 -class $\mathrm{Ext}(S)$, there is a uniform way to obtain the index of a set of strings T such that $\mathrm{Ext}(T) = \mathrm{Ext}(S)$ and such that the strings in T are all *disjoint*; this enables us to compute the measure of $Ext(T)$ as

$$\begin{aligned}\mu(\mathrm{Ext}(T)) &= \sum_{\sigma\in T}\mu(\mathrm{Ext}(\sigma))\\ &= \sum_{\sigma\in T}2^{-|\sigma|}.\end{aligned}$$

Chapter II

Definitions of Randomness

II.1 Effective Approximations in Measure

In our view, the most useful and intuitively appealing definition of randomness is the one due originally to Martin-Löf. The idea is to characterize a random sequence by describing the properties of "nonrandomness" which it must *avoid*. For example, we might imagine examining successively longer initial segments σ of a sequence $A \in 2^\omega$ and discovering that

$$\frac{\#\text{ of 0's in }\sigma}{|\sigma|} \geq \frac{3}{4}. \tag{II.1}$$

We would then begin to suspect that A is not a sequence we'd normally think of as "random". A "test" for this particular nonrandomness property can be viewed as a recursive enumeration of strings σ for which (II.1) holds; if A has arbitrarily long initial segments satisfying (II.1), we reject A as nonrandom. Now among all possible enumerations of strings, how do we distinguish those which describe a "nonrandomness" property in some sense? Since ultimately we expect the nonrandom sequences to form a class with measure zero, we can require that as we enumerate longer initial segments in the "test", the total measure of their extensions should become arbitrarily small. The mathematical content of this discussion is made precise in following definition.

Definition II.1.1 *A* Martin-Löf test *is a recursive sequence of* Σ^0_1 *- classes* $\{\mathcal{S}_i\}_{i\in\omega}$ *with* $\mu(\mathcal{S}_i) \leq 2^{-i}$. *A sequence* $A \in 2^\omega$ *is* 1-random *if for every Martin-Löf test* $\{\mathcal{S}_i\}_{i\in\omega}$, $A \notin \bigcap_i \mathcal{S}_i$.

The requirement that a test consist of recursively enumerable sets of strings is a natural starting point but is admittedly somewhat arbitrary. A generalized form, where Σ^0_n -classes replace Σ^0_1 -classes, first appeared in Kurtz [15]. We will also find it useful to define randomness relative to an oracle.

Definition II.1.2 *Let $A, C \in 2^\omega$. A is* Σ_n^C -approximable, *or* approximable in Σ_n^C - measure, *if there is a recursive sequence of Σ_n^C -classes $\{\mathcal{S}_i\}_{i\in\omega}$ with $\mu(\mathcal{S}_i) \leq 2^{-i}$ and $A \in \bigcap_i \mathcal{S}_i$. Then A is* C-n-random, *or* n-random relative to C, *if it is not Σ_n^C -approximable. If $C \equiv_T 0$ we simply say A is* n-random. *We also say A is* random *(or for emphasis,* ω-random*) if A is n-random for all n.*

We will see shortly that 2-randomness is the same as 1-randomness relative to $0'$, or more generally, n-random relative to $C^{(m)}$ is the same as $(m+n)$-random relative to C. The key point will be that any approximation in measure can be replaced by an approximation by *open* classes of the same arithmetic complexity.

Approximation Lemmas

We begin by stating two lemmas which will be used repeatedly throughout the sequel. The first asserts that the measure of a Σ_n^C -class is a real number recursive in $C^{(n)}$.

Lemma II.1.3 *The predicate "$\mu(\mathcal{S}) > \epsilon$" is uniformly Σ_n^C , where $\mathcal{S}$ is a Σ_n^C -class and ϵ is a rational. Likewise "$\mu(\mathcal{S}) < \epsilon$" is uniformly Σ_n^C when $\mathcal{S}$ is a Π_n^C -class.*

Proof. See Kurtz [15]. □

The next result extends Kurtz' Lemma 2.2a ([15, p.21]) in several important ways.

Lemma II.1.4 *(i) For $\mathcal{S}$ a Σ_n^C -class and $\epsilon > 0$ a rational, we can uniformly and recursively obtain the index of a $\Sigma_1^{C^{(n-1)}}$ -class (an open Σ_n^C -class) $\mathcal{U} \supseteq \mathcal{S}$ with $\mu(\mathcal{U}) - \mu(\mathcal{S}) \leq \epsilon$.*

(ii) For $\mathcal{T}$ a Π_n^C -class and $\epsilon > 0$ a rational, we can uniformly and recursively obtain the index of a $\Pi_1^{C^{(n-1)}}$ -class (a closed Π_n^C -class) $\mathcal{V} \subseteq \mathcal{T}$ with $\mu(\mathcal{T}) - \mu(\mathcal{V}) \leq \epsilon$.

(iii) For $\mathcal{S}$ a Σ_n^C -class and $\epsilon > 0$ a rational, we can uniformly in $C^{(n)}$ obtain a closed Π_n^C-1 -class $\mathcal{V} \subseteq \mathcal{S}$ with $\mu(\mathcal{S}) - \mu(\mathcal{V}) \leq \epsilon$. (If $n \geq 2$, $\mathcal{V}$ will be a $\Pi_1^{C^{(n-2)}}$ -class.) Moreover, if $\mu(\mathcal{S})$ is a real recursive in $C^{(n-1)}$, the index for $\mathcal{V}$ can be found recursively in $C^{(n-1)}$.

(iv) For $\mathcal{T}$ a Π_n^C -class and $\epsilon > 0$ a rational, we can uniformly in $C^{(n)}$ obtain an open Σ_n^C-1 -class $\mathcal{U} \supseteq \mathcal{T}$ with $\mu(\mathcal{U}) - \mu(\mathcal{T}) \leq \epsilon$. (If $n \geq 2$, $\mathcal{U}$ will be a $\Sigma_1^{C^{(n-2)}}$ -class.) Moreover, if $\mu(\mathcal{T})$ is a real recursive in $C^{(n-1)}$, the index for $\mathcal{U}$ can be found recursively in $C^{(n-1)}$.

Proof. The proof is by induction:

Base step: Let $\mathcal{S}$ be a Σ_1^C -class. For (i) we can simply take $\mathcal{U} = \mathcal{S}$. For (iii), let e be an index such that $\mathcal{S} = \text{Ext}(W_e^C)$. Recursively in C' we can find a rational number q such that $\mu(\mathcal{S}) > q > \mu(\mathcal{S}) - \epsilon$, and then recursively in C find the least s such that $\mu(\text{Ext}(W_{e,s}^C)) \geq q$. The set $W_{e,s}^C$ is a finite set of strings recursive in C, so $\mathcal{V} = \text{Ext}(W_{e,s}^C)$ is a Π_0^C -class. Note that the procedure is uniform in ϵ and an index for $\mathcal{S}$; moreover, the C' oracle is only needed to determine q, so if $\mu(\mathcal{S})$ can be computed from C, then an index for $\mathcal{V}$ can be obtained from C. Given a Π_1^C -class $\mathcal{T}$, we can apply the same argument to the complement of $\mathcal{T}$ to obtain (ii) and (iv).

Induction step: (i) Let $\mathcal{S}$ be a $\Sigma_n^C{+}1$ -class. Recall from Section I.3 that there is a uniform way to express $\mathcal{S}$ as the union of a recursive sequence of Π_n^C -classes $\{\mathcal{T}_i\}_{i\in\omega}$. By the induction hypothesis, for each i we can find, recursively in $C^{(n)}$, a $\Sigma_1^{C^{(n-1)}}$ -class $\mathcal{U}_i \supseteq \mathcal{T}_i$ such that

$$\mu(\mathcal{U}_i) - \mu(\mathcal{T}_i) \leq \frac{\epsilon}{2^{i+1}}.$$

Let $\mathcal{U} = \bigcup_i \mathcal{U}_i$. Note first that $\mathcal{S} \subseteq \mathcal{U}$ and that $\mathcal{U} - \mathcal{S} = \bigcup_i(\mathcal{U}_i - \mathcal{T}_i)$, so

$$\begin{aligned} \mu(\mathcal{U}) - \mu(\mathcal{S}) &= \mu(\mathcal{U} - \mathcal{S}) \\ &= \mu(\bigcup_i(\mathcal{U}_i - \mathcal{T}_i)) \\ &\leq \sum_i \mu(\mathcal{U}_i - \mathcal{T}_i) \\ &\leq \sum_i \frac{\epsilon}{2^i + 1} \\ &\leq \epsilon. \end{aligned}$$

Notice that $\mathcal{U}$ is a union of $\Sigma_1^{C^{(n-1)}}$ -classes whose indices are uniformly computable relative to $C^{(n)}$, so in fact $\mathcal{U}$ can be regarded as a $\Sigma_1^{C^{(n)}}$ -class whose index encodes the procedure described above. Since the above procedure is uniform in ϵ and an index for $\mathcal{S}$, there is a uniform way to compute the index for $\mathcal{U}$ from ϵ and $\mathcal{S}$.

(iii) Let $\mathcal{S}$ be a $\Sigma_n^C{+}1$ -class; again $\mathcal{S} = \bigcup_i \mathcal{T}_i$, where the $\mathcal{T}_i$ are Π_n^C -classes. First find a rational q such that

$$\mu(\mathcal{S}) > q > \mu(\mathcal{S}) - \frac{\epsilon}{2};$$

in general this requires $C^{(n+1)}$. Then recursively in $C^{(n)}$ we can find a $j \in \omega$ such that

$$\mu\left(\bigcup_{i\leq j} \mathcal{T}_i\right) \geq q.$$

Since $\bigcup_{i\leq j} \mathcal{T}_i$ is itself a Π_n^C -class, by the induction hypothesis we can uniformly obtain the index of a $\Pi_1^{C^{(n-1)}}$ -class $\mathcal{V} \subseteq \bigcup_{i\leq j} \mathcal{T}_i$ with measure within $\frac{\epsilon}{2}$. Thus $\mu(\mathcal{S}) - \mu(\mathcal{V}) \leq \epsilon$. Notice that the procedure for obtaining $\mathcal{V}$ from $\mathcal{S}$ is still uniform; note also that the only place a $C^{(n+1)}$ oracle is used is to find the number q approximating the measure of $\mathcal{S}$; thus if $\mu(\mathcal{S})$ is computable from $C^{(n)}$, the index for $\mathcal{V}$ can be found recursively in $C^{(n)}$.

For (ii) and (iv), given a Π^C_n+1 -class $\mathcal{T}$ we apply the arguments in (i) and (iii), respectively, to the complement of $\mathcal{T}$. $\square$

The characterization in terms of approximation by *open* sets now follows easily from Lemma II.1.4.

Lemma II.1.5 *Let $A, C \in 2^\omega$, $n \geq 1$, and $m \geq 0$. Then A is $\Sigma^{C^{(m)}}_n$ -approximable $\iff$ A is Σ^C_m+n -approximable.*

Proof. ($\Rightarrow$) Immediate, since any $\Sigma^{C^{(m)}}_n$ -class is a Σ^C_m+n -class.

($\Leftarrow$) We show that a Σ^C_m+1 - approximation can be replaced by a $\Sigma^{C^{(m)}}_1$ - approximation; the result will then follow by induction on n. Let $C \in 2^\omega$ and $m \geq 1$ be arbitrary. Suppose $\{\mathcal{S}_i\}_{i\in\omega}$ is a Σ^C_m+1 -approximation. By Lemma II.1.4(i) we can uniformly find for each i a $\Sigma^{C^{(m)}}_1$ -class $\mathcal{U}_i \supseteq \mathcal{S}_i$ with $\mu(\mathcal{U}_i) - \mu(\mathcal{S}_i) \leq 2^{-i}$. Thus $\mu(\mathcal{U}_{i+1}) \leq 2^{-i}$, so $\{\mathcal{U}_{i+1}\}_{i\in\omega}$ is a $\Sigma^{C^{(m)}}_1$ -approximation and $\bigcap_i \mathcal{S}_i \subseteq \bigcap_i \mathcal{U}_{i+1}$. $\square$

In particular, n-randomness is the same as $0^{(n-1)}$-1-randomness. The usefulness of this fact is that known results for 1-random sets can often be generalized to n-random sets simply by relativizing the proofs. Two important examples comprise the remainder of this section.

The Universal Martin-Löf Test

We begin by isolating a key feature of (relativized) 1-randomness, namely, that there is a uniform list of all possible Σ^C_1 -approximations. A recursive sequence $\{\mathcal{S}_i\}_{i\in\omega}$ of Σ^C_1 -classes is associated with a recursive function f such that $\mathcal{S}_i = \mathrm{Ext}(W^C_{f(i)})$. If $f = \varphi_e$, we may call e the *index* of the sequence. Now any $e \in \omega$ can be regarded as the index of a sequence of classes whether or not φ_e is total, since there is a recursive function g such that

$$W^C_{g(e,i)} = \begin{cases} W^C_{\varphi_e(i)} & \text{if } \varphi_e(i)\downarrow \\ \emptyset & \text{otherwise.} \end{cases}$$

Lemma II.1.6 *Let $C \in 2^\omega$. There is a recursive list of the indices of all Σ^C_1 - approximations.*

Proof. Although there is no effective way to tell, given an index e, whether the associated sequence $\{\mathcal{S}_i\}_{i\in\omega}$ is a Σ^C_1 - approximation (satisfies $\mu(\mathcal{S}_i) \leq 2^{-i}$), since we have a recursive upper bound on the measure of each class $\mathcal{S}_i$ we can artificially restrict the measure of every Σ_1^C -class we encounter without affecting the legitimate Σ_1^C -approximations. Let h be the recursive function such that

$$W_{h(x,i)}^C = \begin{cases} W_{x,s}^C & \text{if } s = \max\{t : \mu(\mathrm{Ext}(W_{x,t}^C)) \leq 2^{-i}\} \\ W_x^C & \text{if no such } s \text{ exists} \end{cases}$$

That is, we enumerate W_x^C as long as the measure remains below 2^{-i}. Then for any $e \in \omega$ the sequence given by $\mathcal{S}_i = W_{h(g(e,i),i)}^C$ is a Σ_1^C -approximation; moreover, given a Σ_1^C -approximation $\{\mathcal{U}_i\}_{i\in\omega}$ with index e, we have $\mathcal{U}_i = \mathrm{Ext}(W_{h(g(e,i),i)}^C)$, so every Σ_1^C -approximation does appear on the list. □

Theorem II.1.7 (Martin-Löf [22]) *For any $C \in 2^\omega$ and any $n \geq 1$ there exists a universal Σ_n^C -approximation. That is, there is a recursive sequence of $\Sigma_1^{C^{(n-1)}}$ -classes $\{\mathcal{U}_i\}_{i\in\omega}$, with $\mu(\mathcal{U}_i) \leq 2^{-i}$, such that every Σ_n^C -approximable set is in $\bigcap_i \mathcal{U}_i$.*

Proof. Since we can effectively list all Σ_1^C -approximations, we just enumerate in each $\mathcal{U}_i$ a very small class from the "tail" of each Σ_1^C -approximation. Specifically, if $\{\mathcal{S}_i\}_{i\in\omega}$ is the eth Σ_1^C -approximation, we enumerate $\mathcal{S}_{e+i+1}$ into $\mathcal{U}_i$. Then $\mu(\mathcal{U}_i) \leq \sum_e 2^{-(e+i+1)} = 2^{-i}$, and any Σ_1^C -approximable set is in each class $\mathcal{U}_i$. □

A Characterization due to Solovay

The recursive bound $\mu(\mathcal{S}_i) \leq 2^{-i}$ in the definition of a Σ_1^0 -approximation seems to be used in an essential way in Theorem II.1.7, so it is somewhat surprising that it can be eliminated from the definition of n-randomness. It also turns out that the condition "$A \in \bigcap_i \mathcal{S}_i$" is stronger than necessary.

Theorem II.1.8 (Solovay) *Let A, $C \in 2^\omega$ and $n \geq 1$. A is C-n-random $\Longleftrightarrow$ for every recursive sequence of Σ_n^C -classes $\{\mathcal{S}_i\}_{i\in\omega}$ with $\sum_i \mu(\mathcal{S}_i) < \infty$, A is in only finitely many $\mathcal{S}_i$.*

Proof. ($\Leftarrow$) Immediate.

($\Rightarrow$) Assume $\{\mathcal{S}_i\}_{i\in\omega}$ is a recursive sequence of Σ_n^C -classes with $\sum_i \mu(\mathcal{S}_i) < \infty$, and that A is in infinitely many $\mathcal{S}_i$; we show that A is Σ_n^C - approximable. By Lemma II.1.5 we can assume the $\mathcal{S}_i$ are open $\Sigma_1^{C^{(n-1)}}$ -classes. We may also assume, by taking the "tail" of the sequence, that $\sum_i \mu(\mathcal{S}_i) \leq 1$. For each i let S_i be a set of strings r.e. in $C^{(n-1)}$ such that $\mathcal{S}_i = \mathrm{Ext}(S_i)$. For each j let T_j denote a set of strings r.e. in $C^{(n-1)}$, defined by: enumerate a string σ into T_j just if at least 2^j different sets S_i include a string $\tau \subset \sigma$. Let $\mathcal{T}_j = \mathrm{Ext}(T_j)$. We make two claims.

(i) For all j, $A \in \mathcal{T}_j$: Since A is in infinitely many $\mathcal{S}_i$, for any j there will be a stage at which 2^j of the sets S_i include some initial segment of A; the longest of these will be enumerated in T_j.

(ii) $\mu(\mathcal{T}_j) \leq 2^{-j}$: For each σ enumerated in T_j, σ or some initial segment is in 2^j of the sets S_i. Hence
$$2^j \cdot \mu(\mathcal{T}_j) \leq \sum_i \mu(\mathcal{S}_i) \leq 1.$$

□

II.2 Program Size Complexity Measures

The Church-von Mises Definition

A significant feature of most of our intuitive conceptions of randomness is the notion of *unpredictability*; i.e., knowing the first n bits of a random sequence should not be of any help in predicting the $(n+1)$st bit. That is, consider a game consisting of successive tosses of a fair coin; before each toss we decide whether or not to place a bet on the outcome, according to some strategy based on the results of the previous tosses. Let A denote the sequence of outcomes (writing down a 1 for heads and a 0 for tails, say); if the strategy requires that eventually we bet infinitely often, the sequence of outcomes on which we placed a bet form a subsequence of A. Intuitively the randomness of A means that no such betting strategy can give us any advantage, i.e., yeild a subsequence of A with a different distribution.

More precisely, a "strategy" is a function $f : 2^{<\omega} \longrightarrow \{0,1\}$, the idea being that if σ is an initial sequence of outcomes, $f(\sigma) = 1$ means that we would place a bet on the next outcome. The function f determines a *place selection* on sequences as defined below (the definition here is adapted from van Lambalgen [36]). Recall that if $A, B \in 2^\omega$ and B is infinite, the notation A/B denotes the subsequence of A such that $A/B(n) = A(b_n)$, where b_n is the $(n+1)$st element of B in increasing order.

Definition II.2.1 *Let $f : 2^{<\omega} \longrightarrow \{0,1\}$. Let $\hat{f}$ be defined on strings by*
$$\begin{aligned} \hat{f}(\emptyset) &= \emptyset \\ \hat{f}(\sigma * i) &= \hat{f}(\sigma) * f(\sigma) \end{aligned}$$
and extended to sets $A \in 2^\omega$ by
$$\hat{f}(A) = \bigcup_n \hat{f}(A \restriction n).$$
Formally, the place selection determined by f *is the partial function $2^\omega \longrightarrow 2^\omega$ defined by*
$$A \mapsto A/\hat{f}(A)$$
whenever $\hat{f}(A)$ is infinite (i.e., as a sequence it contains infinitely many ones). We may also refer to f itself as a place selection. If f is constant on strings of each length n, then for some fixed set B, $\hat{f}(A) = B$ for every A; we refer to f as a constant *place selection.*

The original definition of randomness due to von Mises was in part an attempt to capture the intuitive idea above; he defined a sequence $A \in 2^\omega$ to be random if the limiting relative frequency of zeros and ones exists, i.e.

$$\lim_{n\to\infty} \frac{\#\text{ of 0's in } A\restriction n}{n} = p$$

for some p, and for any "admissible" betting strategy f, the subsequence $A/\hat{f}(A)$ has the same limiting relative frequency.

The apparent ambiguity of the notion of an "admissible" place selection has been problematic and controversial; e.g., platonistically for any A there certainly exists a function f such that $A/\hat{f}(A)$ is a sequence of all 1's. The difficulty suggests a natural role for a notion of computation in defining randomness; we invoke Church's thesis to conclude that intuitively what we mean by a "betting strategy" is an *algorithm* $\varphi : 2^{<\omega} \to \{0,1\}$. This yeilds precisely the definition of randomness proposed by Church in [3]. (We emphasize, however, that invoking Church's thesis is by no means the only way, or necessarily the correct way, to resolve the controversy. See van Lambalgen's [36] for a much deeper analysis of von Mises' original conception.)

It turns out that although Church's definition captures the intuitive property described above, this property alone is too weak to characterize a satisfactory notion of randomness, in the sense that there are Church-random sequences that fail to satisfy known probabilistic laws (see the discussion of Ville's theorem in [36]). Further evidence of weakness can be seen in the fact that although there are only countably many recursive place selections, given an $A \in 2^\omega$ with limiting frequency

$$\lim_{n\to\infty} \frac{\#\text{ of 0's in } A\restriction n}{n} = p,$$

there are uncountably many subsequences of A with the same limiting frequency. In fact, the class of $B \in 2^\omega$ such that A/B has the same limiting frequency has measure one. (This follows from a theorem attributed to Steinhaus in [36, p. 89]; we will see a proof of a similar result in Section III.3.)

Another way in which the Church definition can be seen to be "weak" is in the uniformity required of the "betting strategy": for A to be Church-random, we require that there is no single algorithm which can uniformly compute $A\restriction n$ from $A\restriction(n-1)$ with any significant success. What if instead we insist that *no* algorithm can *ever* compute $A\restriction n$ from *any* shorter string? With minor modifications, this description can be turned into a definition characterizing the 1-random sequences of the previous section.

Kolmogorov and Chaitin Complexity

What we have just described is, in effect, a complexity measure based on "program size" originating with the following definition due to Kolmogorov [14]. We call a partial recursive function $\psi : 2^{<\omega} \longrightarrow 2^{<\omega}$ *universal* if it interprets its input as a pair $\langle e, \sigma \rangle$ in some canonical way and simulates $\varphi_e(\sigma)$.

Definition II.2.2 *Let $\psi : 2^{<\omega} \to 2^{<\omega}$ be a universal p.r. function. For $\sigma \in 2^{<\omega}$, the* Kolmogorov complexity of *σ is defined by*

$$K(\sigma) = \min\{|\tau| : \psi(\tau) = \sigma\}.$$

Obviously the definition depends on the exact choice of the universal function ψ, but since one universal function can simulate another, the complexity measures resulting from any two universal functions differ by at most a fixed constant on all inputs.

Usually one then defines a string σ to be "random" if $K(\sigma)$ is approximately equal to $|\sigma|$. The obvious next step is to try to define an infinite sequence A to be random if all its initial segments have high Kolmogorov complexity, i.e., for some constant c,

$$(\forall n)[K(A\restriction n) \geq n - c]. \tag{II.2}$$

Hovever, *no* sequence A satisfies this condition; but surprisingly, making a deceptively subtle change in the definition of K yeilds a meaningful definition of randomness.

It is shown in Chaitin [1] that there is a universal p.r. function ψ with a *prefix-free* domain, that is, if $\psi(\sigma)\downarrow$, then $\psi(\tau)\uparrow$ for any τ which is a proper initial segment or extension of σ. (The intuitive interpretation is that σ is a program delimited by an endmarker or keyword; no proper initial segment or extension of a valid program can be a valid program, since the endmarker won't be in the right place.)

Definition II.2.3 (Chaitin) *Let $\psi : 2^{<\omega} \longrightarrow 2^{<\omega}$ be a universal p.r. function with prefix-free domain. Define*

$$I(\sigma) = \min\{|\tau| : \psi(\tau) = \sigma\}.$$

The quantity $I(\sigma)$ is variously called the *Chaitin complexity* of σ or the *algorithmic information content* of σ. The following theorem, due to Solovay, shows not only that the sequences satisfying the analog of (II.2) for I form a class of measure one, but also that they are precisely the 1-random sequences defined in the previous section. The relativization of I to $0^{(n-1)}$ also yeilds a characterization of n-randomness. The proof can be found in [1].

Theorem II.2.4 *$A \in 2^\omega$ is 1-random iff* $(\exists c)(\forall n)[I(A\restriction n) \geq n - c]$.

II.3 Measure-theoretic Approach

In modern formulations of probability theory, random sequences are not explicitly defined. Rather, one would say that a property P holds of a random sequence if the class

$$\{A \in 2^\omega : A \text{ has property } P\}$$

has measure one. This does not constitute a *definition* of randomness, since no sequence can be a member of every class with measure one. A natural remedy is to restrict the kinds of properties or classes considered. From a recursion theorist's point of view the obvious first thing to try is to consider just the *arithmetical* classes of measure one.[1] The definition below first appeared in Kurtz ([15]).

Definition II.3.1 *Let $A, C \in 2^\omega$. A is* C-weakly n-random *if A is a member of every Σ_n^C -class with measure one.*

We will see in Section II.5 that, as the terminology implies,

$$C\text{-weakly } (n+1)\text{-random} \Rightarrow C\text{-}n\text{-random} \Rightarrow C\text{-weakly } n\text{-random}$$

and that neither implication is reversible. At any rate this shows that the two approaches coincide at the ω-level.

Corollary II.3.2 *$A \in 2^\omega$ is ω-random iff A is in every arithmetical class with measure one.*

The analog of Lemma II.1.5 holds for weak n-randomness only if $n \geq 2$, that is, $C^{(m)}$-weakly n-random is the same as C-weakly $(m+n)$-random if $n \geq 2$. To prove this we use the effective analog of the fact that a measurable set can be expressed as the union of an F_σ set (union of closed sets) and a set of measure 0.

Lemma II.3.3 *Let $n \geq 2$ and $C \in 2^\omega$.*

(i) For any Σ_n^C -class $\mathcal{S}$ we can uniformly and recursively obtain the index of a $\Sigma_2^{C^{(n-2)}}$ -class (i.e., an F_σ Σ_n^C -class) $\mathcal{V} \subseteq \mathcal{S}$ with $\mu(\mathcal{V}) = \mu(\mathcal{S})$.

(ii) For any Π_n^C -class $\mathcal{T}$ we can uniformly and recursively obtain the index of a $\Pi_2^{C^{(n-2)}}$ -class (i.e., a G_δ Π_n^C -class) $\mathcal{U} \supseteq \mathcal{T}$ with $\mu(\mathcal{U}) = \mu(\mathcal{T})$.

Proof. (i) Let $\mathcal{S}$ be a Σ_n^C -class, so $\mathcal{S} = \bigcup_i \mathcal{T}_i$, where the $\mathcal{T}_i$ are Π_n^C-1 . By Lemma II.1.4 we can uniformly find, for each i and j, a $\Pi_1^{C^{(n-2)}}$ -class $\mathcal{V}_{i,j} \subseteq \mathcal{T}_i$ with $\mu(\mathcal{T}_i) - \mu(\mathcal{V}_{i,j}) \leq 2^{-j}$. Let

$$\mathcal{V} = \bigcup_{i,j} \mathcal{V}_{i,j}.$$

Clearly $\mathcal{V} \subseteq \mathcal{S}$ and $\mu(\mathcal{V}) = \mu(\mathcal{S})$. The proof for (ii) is similar. □

[1] Martin-Löf suggests in [23] that the "right" choice is to consider only the hyperarithmetical classes of measure one.

Theorem II.3.4 *Let $A, C \in 2^\omega$ and $n \geq 2$. A is C-weakly n-random $\iff$ A is a member of every $\Sigma_2^{C^{(n-2)}}$ -class with measure one. It follows that A is $C^{(m)}$-weakly n-random if and only if A is C-weakly $(m+n)$-random.*

Proof. ($\Rightarrow$) Immediate.

($\Leftarrow$) Suppose A is a member of every $\Sigma_2^{C^{(n-2)}}$ - class with measure one; let $\mathcal{S}$ be a Σ_n^C -class with measure one. There is a $\Sigma_2^{C^{(n-2)}}$ -class $\mathcal{V} \subseteq \mathcal{S}$ with measure one, so $A \in \mathcal{V}$, and hence $A \in \mathcal{S}$. $\Box$

We will see in Section II.5 that there are always C'-weakly 1-random sets which are not C-weakly 2-random, so the theorem generally fails in the case $n = 1$.

Weak n-randomness can also be characterized in terms of a kind of effective approximation in measure. Note that a set A is weakly n-random just if A avoids every Π_n^0 -nullset $\mathcal{T}$. Now a Π_n^0 -nullset $\mathcal{T}$ is of the form $\bigcap_i \mathcal{U}_i$, where $\{\mathcal{U}_i\}_{i\in\omega}$ is a uniform sequence of Σ_n^0-1 -classes. By replacing each $\mathcal{U}_i$ by $\bigcap_{j\leq i} \mathcal{U}_j$ if necessary we can assume that $\mathcal{U}_i \supseteq \mathcal{U}_{i+1}$ and $\lim_{i\to\infty} \mu(\mathcal{U}_i) = 0$. Using a $0^{(n-1)}$ oracle we can uniformly find for each j an integer $i(j)$ such that $\mu(\mathcal{U}_{i(j)}) \leq 2^{-i}$. Thus if A fails to be weakly n-random, then A is in some Π_n^0 -nullset $\bigcap_i \mathcal{U}_i$, so we can find a $0^{(n-1)}$-recursive sequence of Σ_n^0-1 -classes $\{\mathcal{S}_j\}_{j\in\omega}$ with $\mu(\mathcal{S}_j) \leq 2^{-j}$ and $A \in \bigcap_j \mathcal{S}_j$. Conversely, if such a sequence exists, then A is not weakly n-random, since $\bigcap_j \mathcal{S}_j$ is a Π_n^0 - nullset. Replacing 0 by an arbitrary oracle C, we have proved:

Theorem II.3.5 *Let A, $C \in 2^\omega$. A is C-weakly n-random $\iff$ for every $C^{(n-1)}$-recursive sequence of Σ_n^C-1 -classes $\{\mathcal{S}_i\}_{i\in\omega}$ with $\mu(\mathcal{S}_i) \leq 2^{-i}$, $A \notin \bigcap_i \mathcal{S}_i$.*

The next section provides a characterization of weakly n-random sets in a completely different way.

II.4 Randomness and Genericity

In this section we explore the analogy between random and generic sets, or more generally, between measure and category, and provide a characterization of randomness in terms of an arithmetical forcing relation. This analogical relationship is in some sense a compelling reason for investigating randomness in recursion theory, since generic sets are those which arise in constructions by finite extensions and hence have played a fundamental role in recursion-theoretic arguments for a number of years and are fairly well understood. The underlying idea is that while random sets are "typical" elements of 2^ω in a measure-theoretic sense, generic sets are "typical" in the sense of Baire category. We begin with an illustration of the relationship between finite extension constructions, forcing, and category. The connection between finite extension constructions and category was first observed by Myhill [25].

Let $\mathcal{L}^*$ be the usual language of first order arithmetic with an additional set constant X and a membership symbol $\in$. For ϕ a sentence of $\mathcal{L}^*$ and $A \in 2^\omega$, we write $A \models \phi$ if ϕ is true in the standard model when X is interpreted as the set A.

Consider a typical finite extension argument. We generally begin with a list of "requirements" $\phi_0, \phi_1, \ldots$, i.e., sentences of $\mathcal{L}^*$ describing properties we wish the set A being constructed to possess. We construct initial segments of A in stages, $\sigma_0 \subset \sigma_1 \subset \ldots$, and let $A = \bigcup \sigma_i$. The idea is that at stage s, having already constructed σ_{s-1}, we would like to "force" ϕ_s to be satisfied by finding an extension $\sigma_s \supset \sigma_{s-1}$ such that the sentence ϕ_s holds for *every* A extending σ_s. We can define:

Definition II.4.1 *Let $\sigma \in 2^{<\omega}$ and let ϕ be a sentence of $\mathcal{L}^*$. We say σ* forces ϕ, *written $\sigma \Vdash \phi$, if $A \models \phi$ for every $A \in \mathrm{Ext}(\sigma)$.*

The content of a finite extension proof to show that given an initial segment (or interval) σ_{s-1}, an extension (subinterval) σ_s forcing the next requirement always exists. That is, we show that for each requirement ϕ the class

$$\mathrm{Ext}\{\sigma : \sigma \Vdash \phi\} \tag{II.3}$$

is *dense*. Then the existence of the set A follows from the Baire category theorem: A countable intersection of dense open classes is nonempty.

There are only countably many arithmetical sentences ϕ, and hence only countably many dense classes of the form (II.3), and so there exist sets A meeting *all* the dense requirements ϕ. Such sets are called *generic*; a generic set is "typical" in that it has every property that can be produced by a finite extension argument. It is shown in Jockusch [9] that if the sentence ϕ is Σ_n, the class $\mathrm{Ext}(\{\sigma : \sigma \Vdash \phi\})$ is a Σ^0_n -class; this suggests the following definition, due to Kurtz:

Definition II.4.2 *Let $A \in 2^\omega$.*

(i) A is generic *if A is a member of every dense open arithmetical class.*

(ii) A is weakly n-generic *if A is a member of every dense open Σ^0_n -class.*

The usual definition of genericity is given in terms of a forcing relation. The definition below is from Jockusch [9].

Definition II.4.3 *Let $A \in 2^\omega$ and let ϕ be a sentence of $\mathcal{L}^*$.*

(i) $A \Vdash \phi$ ("A forces ϕ") if for some $\sigma \subset A$, $\sigma \Vdash \phi$.

(ii) A is generic *if for every arithmetical sentence ϕ, either $A \Vdash \phi$ or $A \Vdash \neg\phi$.*

(iii) A is n-generic *if for every Σ_n sentence ϕ, either $A \Vdash \phi$ or $A \Vdash \neg\phi$.*

Kurtz showed that

$$\text{weakly } (n+1)\text{-generic} \Rightarrow n\text{-generic} \Rightarrow \text{weakly } n\text{-generic},$$

which justifies the use of the word "generic" in Definition II.4.2(i).

The method of forcing with finite initial segments was invented by Cohen [4] in 1963 to construct a model of set theory in which the continuum hypothesis is false. The method was applied to arithmetic in 1965 by Feferman [5], and the original constructions of Kleene and Post were recast as forcing arguments. It is known that for both set theory and for arithmetic there are kinds of requirements which can't be forced using only finite information, e.g., producing a set of minimal degree. In 1970 Solovay [33] introduced a method of forcing with closed sets of positive measure to produce a model of set theory in which all sets of reals are Lebesgue measurable; it is essentially this method, specialized to arithmetic, that is used to construct n-random sets. Rather than a sequence of strings $\sigma_0 \subset \sigma_1 \subset \ldots$ (that is, a sequence of basic clopen sets $\mathrm{Ext}(\sigma_0) \supseteq \mathrm{Ext}(\sigma_1) \supseteq \ldots$), we construct a sequence of closed sets $\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \ldots$ where the $\mathcal{C}_i$ may now be arithmetically more complex than basic intervals ($\mathcal{C}_i$ is usually a Π^0_n -class for some n). As usual, each $\mathcal{C}_i$ is constructed to force some requirement ϕ to hold. "$\mathcal{C} \Vdash \phi$" can be interpreted to mean that $A \models \phi$ for every $A \in \mathcal{C}$.

We now define a notion of genericity with respect to this forcing relation which exactly coincides with weak n-randomness. The presentation follows the uniform treatment of Cohen forcing and Sacks forcing given in Odifreddi [27]. Let $\mathbb{P}_n$ denote the collection of all Π^0_n -classes of positive measure, ordered by inclusion.

Definition II.4.4 *(i) Let $\mathcal{T} \subseteq 2^\omega$. We say $\mathcal{T} \Vdash_{\overline{n}} \phi$ if $\mathcal{T} \in \mathbb{P}_n$ and for all $A \in \mathcal{T}$, $A \models \phi$.*

(ii) For $A \in 2^\omega$, $A \Vdash_{\overline{n}} \phi$ iff there exists a $\mathcal{T} \in \mathbb{P}_n$ such that $A \in \mathcal{T}$ and $\mathcal{T} \Vdash_{\overline{n}} \phi$.

(iii) A is Solovay n-generic *if for every Σ_n sentence ϕ, either $A \Vdash_{\overline{n}} \phi$ or $A \Vdash_{\overline{n}} \neg\phi$.*

We first verify some basic properties of the relation $\Vdash_{\overline{n}}$.

Lemma II.4.5 *(i)* Monotonicity: $\mathcal{T} \Vdash_{\overline{n}} \phi \Rightarrow (\forall \mathcal{S} \subseteq \mathcal{T})[\mathcal{S} \in \mathbb{P}_n \Rightarrow \mathcal{S} \Vdash_{\overline{n}} \phi]$.

(ii) Consistency: *It is not the case that both $\mathcal{T} \Vdash_{\overline{n}} \phi$ and $\mathcal{T} \Vdash_{\overline{n}} \neg\phi$.*

(iii) Quasi-completeness: *For every Σ_n sentence ϕ and every $\mathcal{T} \in \mathbb{P}_n$, there is a $\mathcal{S} \subseteq \mathcal{T}$ in $\mathbb{P}_n$ such that either $\mathcal{S} \Vdash_{\overline{n}} \phi$ or $\mathcal{S} \Vdash_{\overline{n}} \neg\phi$.*

(iv) "Forcing = truth": *A is Solovay n-generic if and only if for every Σ_n or Π_n sentence ϕ,*

$$A \Vdash_{\overline{n}} \phi \Longleftrightarrow A \models \phi. \tag{II.4}$$

Proof. (i) and (ii) are immediate. For (iii), let $\{\mathcal{U}_i\}_{i\in\omega}$ be the universal $\Sigma_1^{0^{(n-1)}}$ -approximation, and let $\mathcal{P}_i$ be the complement of $\mathcal{U}_i$. Since $\mu(\mathcal{P}_i) \to 1$, there is an i such that $\mu(\mathcal{P}_i) \geq 1 - \frac{1}{2}\mu(\mathcal{T})$. Then $\mu(\mathcal{T} \cap \mathcal{P}_i) > 0$, so $\mathcal{S} = \mathcal{T} \cap \mathcal{P}_i$ is in $\mathbb{P}_n$, and all its members are n-random. Now if $A \models \phi$ for all $A \in \mathcal{S}$, then $\mathcal{S} \Vdash_{\overline{n}} \phi$ as desired. Otherwise, the Π_n^0 -class

$$\mathcal{S}' = \mathcal{S} \cap \{A : A \not\models \phi\}$$

is nonempty. Since $\mathcal{S}$ contains only n-random sets, the class above must have positive measure; evidently $\mathcal{S}' \Vdash_{\overline{n}} \neg\phi$.

To prove (iv), first suppose A is Solovay n-generic. Let ϕ be Σ_n or Π_n. By definition $A \Vdash_{\overline{n}} \phi$ always implies $A \models \phi$, and if it is not the case that $A \Vdash_{\overline{n}} \phi$ then by Solovay-genericity, $A \Vdash_{\overline{n}} \neg\phi$, so $A \models \neg\phi$, i.e., $A \not\models \phi$. Conversely suppose (II.4) holds for every Σ_n or Π_n sentence ϕ. Let ψ be any Σ_n sentence. We know either $A \models \psi$ or $A \models \neg\psi$; so by (II.4) either $A \Vdash_{\overline{n}} \psi$ or $A \Vdash_{\overline{n}} \neg\psi$. Hence A is Solovay n-generic. □

We now turn to the relationship with randomness.

Theorem II.4.6 *A is Solovay n-generic $\Longleftrightarrow$ A is weakly n-random.*

Proof. ($\Rightarrow$) Suppose A is not weakly n-random; then A is in some Π_n^0 -nullset $\mathcal{S} = \{B : B \models \phi\}$, where ϕ is Π_n. Since $A \models \phi$, it can't be the case that $A \Vdash_{\overline{n}} \neg\phi$; but since $\mathcal{S}$ has measure zero, there is no class $\mathcal{T}$ of positive measure such that $\mathcal{T} \Vdash_{\overline{n}} \phi$. Hence A is not Solovay n-generic.

($\Leftarrow$) Suppose A is weakly n-random, and let ϕ be a Σ_n sentence. Let $\mathcal{S} = \{B : B \models \phi\}$. Suppose $A \in \mathcal{S}$; as $\mathcal{S}$ is a union of Π_n^0-1 -classes $\mathcal{T}_i$, A is in some $\mathcal{T}_i$, which must have positive measure since A is weakly n-random, so $\mathcal{T}_i \Vdash_{\overline{n}} \phi$. If $A \notin \mathcal{S}$, then A is in the Π_n^0 -class $\overline{\mathcal{S}} = \{B : B \models \neg\phi\}$, which again must have positive measure, so $\overline{\mathcal{S}} \Vdash_{\overline{n}} \neg\phi$. □

Randomness and genericity are analogous notions, but the corresponding sets and degrees generally don't coincide. For example, a set which is even weakly 1-generic can't be 1-random. To see this, note that for any k the classes

$$\mathrm{Ext}\left\{\sigma : |\sigma| \geq k \text{ and } \frac{\#\text{ of 0's in } \sigma}{|\sigma|} \geq \frac{3}{4}\right\}$$

$$\mathrm{Ext}\left\{\sigma : |\sigma| \geq k \text{ and } \frac{\#\text{ of 1's in } \sigma}{|\sigma|} \geq \frac{3}{4}\right\}$$

are dense Σ^0_1 -classes; hence

$$\lim_{n\to\infty} \frac{\#\text{ of 0's in } A\upharpoonright n}{n} \tag{II.5}$$

fails to exist for any weakly 1-generic set A. But for any 1-random set A the limit (II.5) is equal to $\frac{1}{2}$ (see Theorem III.1.6).

Kurtz [15] shows that the downward closure of the 1-generic degrees has measure zero; an analysis of the proof also reveals that any predecessor of a 1-generic set is Σ^0_1 -approximable. Thus, for all n, the n-random and n-generic degrees are completely disjoint, although by Theorem IV.2.4, every 2-random degree has a 1-generic predecessor. However, Kurtz also shows that a degree contains a weakly 1-generic set iff it contains a hyperimmune set, so in conjunction with Theorem IV.2.4, every 2-random degree contains a weakly 1-generic set. Note also that every weakly 1-generic set is weakly 1-random, since a measure one Σ^0_1 -class is, in particular, a dense Σ^0_1 -class.

II.5 n-randomness vs. weak n-randomness

We first give a proof of the fact, due to Kurtz, that the implications suggested by the terminology are all valid.

Theorem II.5.1 *Let $C \in 2^\omega$. Then*

$$C\textit{-weakly } (n+1)\textit{-random} \Rightarrow C\textit{-}n\textit{-random} \Rightarrow C\textit{-weakly } n\textit{-random.} \tag{II.6}$$

Proof. First note that for any C, a sequence of Σ^C_1 -classes given by a procedure recursive in C may just as easily be expressed as a recursive sequence of Σ^C_1 -classes: suppose $f = \varphi^C_e$ is the C-recursive function with $\mathcal{S}_i = \mathrm{Ext}(W^C_{f(i)})$; then by the s-m-n theorem there is a recursive g such that

$$W^C_{g(i)} = W^C_{\varphi^C_e(i)}.$$

It then follows using Lemma II.1.5 that a set A is Σ^C_n -approximable iff there is a $C^{(n-1)}$-recursive sequence of Σ^C_n -classes $\{\mathcal{S}_i\}_{i\in\omega}$ with $\mu(\mathcal{S}_i) \leq 2^{-i}$ and $A \in \bigcap \mathcal{S}_i$.

For the first implication of (II.6), if A is not C-weakly n-random, then $A \in \bigcap_i \mathcal{S}_i$, where $\{\mathcal{S}_i\}_{i\in\omega}$ is a Σ^C_n -approximation and so $\bigcap_i \mathcal{S}_i$ is a $\Pi^C_n{+}1$ -nullset. For the second implication, if A is not C-weakly n-random, then by Theorem II.3.5 there is a $C^{(n-1)}$-recursive sequence of Σ^C_n-1 -classes $\{\mathcal{S}_i\}_{i\in\omega}$ with $\mu(\mathcal{S}_i) \leq 2^{-i}$ and $A \in \bigcap_i \mathcal{S}_i$. By the remark above, A is Σ^C_n - approximable. □

Kurtz also showed that the first implication of II.6 is not reversible. The easiest way to see this is to use the following fundamental result, due originally to Sacks, that the cone above any nonzero degree has measure zero. A proof of the relativized form below can be found in Stillwell [34].

Theorem II.5.2 *Let $A, C \in 2^\omega$. If*

$$\{B : C \leq_T A \oplus B\}$$

has positive measure, then $C \leq_T A$.

We frequently use Theorem II.5.2 in the following form.

Corollary II.5.3 *Let $C >_T 0$. Then any $A \geq_T C$ is contained in a Π_2^C -nullset.*

Proof. If $C = \varphi_e^A$, then

$$\{D : \varphi_e^D = C\} = \{D : (\forall x)(\exists s)[\varphi_{e,s}^D(x) \downarrow = C(x)]\}$$

is clearly Π_2^C , contains A, and has measure zero by Theorem II.5.2 □

Now it is easy to see that the first implication of II.6 is not reversible.

Theorem II.5.4 *For each $C \in 2^\omega$ and $n \geq 1$ there is a C-n-random set which is not C-weakly $(n+1)$-random.*

Proof. The relativization of the r.e. basis theorem, Theorem IV.1.1, shows that there is always a C-n-random set recursive in $C^{(n)}$. On the other hand, no C-weakly $(n+1)$-random set can have a nonzero Δ_{n+1}^C-definable predecessor (and hence in particular a C-n-random set below $C^{(n)}$ is not C-weakly $(n+1)$-random). Suppose $0 <_T B \leq_T C^{(n)}$. Then the class

$$\begin{aligned} \{D : \varphi_e^D = B\} &= \{D : (\forall x)(\exists s)[\varphi_{e,s}^D(x) \downarrow = B(x)]\} \\ &= \{D : (\forall x)(\forall s)[\varphi_{e,s}^D(x) \downarrow \rightarrow \varphi_{e,s}^D(x) = B(x)]\} \end{aligned}$$

has measure zero by Theorem II.5.2 and is thus a $\Pi_n^C{+1}$ -nullset since the predicate "$B(x) = i$" can be expressed in $\Pi_n^C{+1}$ form. □

As we saw in the remarks at the end of the last section, a C-weakly 1-generic set is always C-weakly 1-random but never 1-random (relative to any oracle). This shows why the $n = 1$ case of Theorem II.3.4 fails, i.e., if a C'- weakly 1-random set were always C-weakly 2-random, it would also be C-1-random. We also see that the second implication of (II.6) is not reversible for $n = 1$. Is the $n = 1$ case anomalous, as it is for Theorem II.3.4, or are there counterexamples for every n? Kurtz conjectured that for each n, there are weakly n-random sets which are not n-random. The remainder of this section will be devoted to a proof of this conjecture.

There does not seem to be a ghost of a chance of generalizing the $n = 1$ argument; weak 1-randomness is really a pathological case which includes many sets we would not be inclined to label as "random" at all. Nonetheless, there are strong indications in what we have seen so far that Kurtz' conjecture is correct. We saw in the proof of Theorem II.5.1 that A is n-random if and only if:

For every $0^{(n-1)}$-recursive sequence of $\Sigma_1^{0^{(n-1)}}$ - classes $\{\mathcal{S}_i\}_{i\in\omega}$ with $\mu(\mathcal{S}_i) \leq 2^{-i}$, $A \notin \bigcap_i \mathcal{S}_i$.

On the other hand, by Theorem II.3.5, A is weakly n-random (for $n \geq 2$) if and only if:

For every $0^{(n-1)}$-recursive sequence of $\Sigma_1^{0^{(n-2)}}$ - classes $\{\mathcal{S}_i\}_{i\in\omega}$ with $\mu(\mathcal{S}_i) \leq 2^{-i}$, $A \notin \bigcap_i \mathcal{S}_i$.

In the second case, the enumeration procedure for an individual $\mathcal{S}_i$ has access to only a finite amount of information from the oracle $0^{(n-1)}$, since only the *index* of $\mathcal{S}_i$ may depend on $0^{(n-1)}$. In the first case, the enumeration of $\mathcal{S}_i$ itself has unlimited access to $0^{(n-1)}$. The distinction seems to be genuine, as is confirmed by the next theorem[2]. The proof below gives a direct construction of a weakly n-random set which is not n-random; it is easily relativized to show that there are always C-weakly n-random sets which are not C-n-random. Later in Section III.4 we will see another proof which also illustrates a significant way in which the two notions differ naturally.

Theorem II.5.5 *Let $n \geq 1$. There is weakly n-random set which is not n-random.*

Proof. Let $n \geq 1$. The idea is to enumerate, recursively in $0^{(n-1)}$, a sequence $\{\mathcal{S}_e\}_{e\in\omega}$ of open subsets of 2^ω with $\mu(\mathcal{S}_e) \leq 2^{-e}$, such that $\bigcap_e \mathcal{S}_e$ contains an element A avoiding every Π_n^0 - class with measure 0. The proof combines a finite injury style argument (to produce the enumeration of the classes $\mathcal{S}_e$) with a forcing construction on closed sets of positive measure (to guarantee the existence of the set A). Each of the classes $\mathcal{S}_e$ will actually be a clopen subset of 2^ω, i.e., the extension of a finite set of strings.

Let $\mathcal{P}_e$ denote the Π_n^0 -class with index e. We can express $\mathcal{P}_e$ in the form $\bigcap_i \mathcal{V}_i$, where $\{\mathcal{V}_i\}_{i\in\omega}$ is a uniform sequence of Σ_n^0-1 -classes. Let $\mathcal{P}_e(s) = \bigcap_{i<s} \mathcal{V}_i$; then $\mathcal{P}_e(s)$ is a Σ_n^0-1 -class, $\mathcal{P}_e(s) \supseteq \mathcal{P}_e(s+1)$, and $\mu(\mathcal{P}_e(s))$ decreases monotonically to $\mu(\mathcal{P}_e)$ as $s \to \infty$.

Let us first illustrate the argument for the simpler case $n = 1$. To guarantee that $\bigcap_e \mathcal{S}_e$ contains a set A which is not a member of any Π_1^0 -class of measure 0, we identify for each e a distinguished string τ_e extending some string in S_e (i.e., $\mathrm{Ext}(\tau_e) \subseteq \mathcal{S}_e$) such that $\tau_e \subset \tau_{e+1}$ and and meeting the requirements

$$R_e\text{: If } \mu(\mathcal{P}_e) = 0 \text{ then } \mathcal{P}_e \cap \mathrm{Ext}(\tau_e) = \emptyset.$$

Then the set $A = \bigcup_e \tau_e$ will be weakly 1-random, but is approximable in Σ_1^0 - measure by the sequence $\{\mathcal{S}_e\}_{e\in\omega}$, and hence is not 1-random. Note that since $\mathcal{P}_e$ is a Π_1^0 -class we can take $\mathcal{P}_e(s)$ to be (the extension of) a finite set of strings.

[2]The result was evidently known in some form to Gaifman and Snir; a version appears (without a proof) in [7].

Let $\mathcal{S}_e(s)$ denote the approximation of $\mathcal{S}_e$ at stage s, and let $S_e(s)$ denote the corresponding approximation of the r.e. set of strings S_e such that $\mathcal{S}_e = \mathrm{Ext}(S_e)$. Let $\Gamma(s)$ denote a "moveable marker" which will indicate in which set $S_e(s)$ we should next enumerate a string. Let $\tau_e(s)$ denote the distinguished string associated with $S_e(s)$ at stage s, and let $t_e(s)$ denote a "threshold" value for $\mathcal{P}_e$, whose purpose is explained below.

Initially at stage 0 we let $S_0(0) = \{\emptyset\}$, $\tau_0(0) = \emptyset$, and we set a threshold of $t_0(0) = \frac{1}{2}$ for $\mathcal{P}_0$. Let $\Gamma(0) = 0$. For all $i > 0$, $S_i(0) = \emptyset$, and $\tau_i(0)$ and $t_i(0)$ are undefined.

Consider a stage $s + 1$; let $e = \Gamma(s)$, so we have already a nonempty $S_e(s)$ and an associated string $\tau_e(s)$. The idea is that if $\mu(\mathcal{P}_{e+1}) = 0$, we should enumerate in S_{e+1} a string $\sigma \supset \tau_e(s)$ such that $\mathcal{P}_{e+1} \cap \mathrm{Ext}(\sigma) = \emptyset$, and then require that all strings enumerated after stage $s+1$ be extensions of σ. We can ensure that $\mu(\mathcal{S}_{e+1}) \leq 2^{-(e+1)}$ by choosing sufficiently long strings σ. However, the construction has to be recursive, and we can't effectively tell whether $\mu(\mathcal{P}_{e+1}) = 0$. What we do instead is "guess" that $\mu(\mathcal{P}_{e+1}) > 0$, and go ahead and enumerate in S_{e+1} some string $\sigma \supset \tau_e(s)$ with length $|\sigma| \geq 2(e + 1)$. Then let $\tau_{e+1}(s + 1) = \sigma$, and define the threshold value for $\mathcal{P}_{e+1}$ to be $t_{e+1}(s + 1) = 2^{-(|\sigma|+1)}$, i.e., half the measure of $\mathrm{Ext}(\sigma)$. Then for all $i \neq e$ just let $\tau_i(s+1) = \tau_i(s)$ and $t_i(s+1) = t_i(s)$, and move the marker, $\Gamma(s+1) = e+1$.

Now suppose at some later stage $r + 1$ we discover that for some $e \leq \Gamma(r)$, $\mu(\mathcal{P}_e(r))$ has decreased below the threshold value $t_e(r)$, a fact which we can determine recursively. We then return to S_e and find a string σ extending $\tau_e(r)$ such that $\mathcal{P}_e(r) \cap \mathrm{Ext}(\sigma) = \emptyset$; this is always possible (and can be done effectively) since $\mathcal{P}_e(r)$ is the extension of a finite set of strings with measure less than half the measure of $\mathrm{Ext}(\tau_e(r))$. Let $\tau_e(r+1) = \sigma$, and set the threshold value $t_e(r+1) = -1$. We say that requirement R_e *acts* at stage $r+1$. For $i \neq e$ let $\tau_i(r+1) = \tau_i(r)$ and $t_i(r+1) = t_i(r)$. Note that for $i > e$, $t_i(r + 1)$ may no longer be an extension of $\tau_e(r + 1)$; we say R_i is *injured* by R_e. We move the marker back down to level e by setting $\Gamma(r + 1) = e$, and proceed to enumerate a new string in S_{e+1} at stage $r + 2$ as described previously.

Note that once a requirement R_e acts, it never acts again unless injured; the maximum number a_e of times of times that R_e can act is thus one more than the number of times that it can be injured by the R_i with $i < e$. Thus

$$\begin{aligned} a_0 &= 1, \text{ and} \\ a_{k+1} &= a_0 + a_1 + \cdots + a_k + 1 = 2a_k \end{aligned}$$

so in general R_e can act at most 2^e times. We may also enumerate a new string in S_e when R_e is injured, so this shows that at most 2^e strings are enumerated in S_e. Since we require each such string to have length $\geq 2e$, $\mu(\mathcal{S}_e) \leq 2^e/2^{2e} = 2^{-e}$.

One can verify inductively that for all e,

(i) $\lim_{s\to\infty} \Gamma(s) = \infty$

(ii) $t_e(s)$ reaches a limiting value t_e as $s \to \infty$

(iii) $\tau_e(s)$ reaches a limiting value τ_e as $s \to \infty$

(iv) Either $t_e > 0$ and $(\forall s)\mu(\mathcal{P}_e(s)) > t_e$, or else $t_e = -1$ and $\mathcal{P}_e \cap \mathrm{Ext}(\tau_e) = \emptyset$.

Then by construction $\mathrm{Ext}(\tau_e) \subseteq \mathcal{S}_e$ and $\tau_e \subset \tau_{e+1}$, so the set $A = \bigcup_e \tau_e$ is in $\bigcap_e \mathcal{S}_e$, and thus is not 1-random. Item (iv) above guarantees that A is weakly 1-random.

Now fix $n > 1$. The additional difficulty encountered in the case $n > 1$ is that since $\mathcal{P}_e(s)$ is a Σ^0_n-1 -class rather than a clopen set, it may not be possible to find any string τ_e with $\mathcal{P}_e(s) \cap \mathrm{Ext}(\tau_e) = \emptyset$, i.e., the complement $\overline{\mathcal{P}_e(s)}$ need not contain any open sets at all. However, we can instead associate with S_e a closed Π^0_n-1 -class $\mathcal{T}_e$ with positive measure, satisfying $\mathcal{T}_e \subseteq \mathcal{S}_e$ and $\mathcal{T}_e \supseteq \mathcal{T}_{e+1}$, and meeting for each e the requirements

$$R_e\text{: If } \mu(\mathcal{P}_e) = 0 \text{ then } \mathcal{P}_e \cap \mathcal{T}_e = \emptyset.$$

The argument for $n = 1$ is just the special case where $\mathcal{T}_e = \mathrm{Ext}(\tau_e)$. Let $\mathcal{T}_e(s)$ denote the closed set associated with S_e at stage s.

The construction is as follows:

Stage 0 Let $S_0(0) = \{\emptyset\}$, $\mathcal{T}_0(0) = 2^\omega$, $t_0(0) = \frac{1}{2}$, and $\Gamma(0) = 0$. For $i > 0$, $S_i(0) = \emptyset$; $\mathcal{T}_i(0)$ and $t_i(0)$ are undefined.

Stage s+1 We check, recursively in $0^{(n-1)}$, whether there is some $i \leq \Gamma(s)$ such that $\mu(\mathcal{P}_i(s)) \leq t_i(s)$.

Case 1 No such i exists. Let $e = \Gamma(s)$. Find a string σ and a rational $\epsilon > 0$ such that $|\sigma| \geq 2(e+1)$ and $\mu(\mathcal{T}_e(s) \cap \mathrm{Ext}(\sigma)) \geq \epsilon$. This is always possible since $\mu(\mathcal{T}_e(s)) > 0$ (formally, we show by induction on s that this always holds for $e \leq \Gamma(s)$). Moreover, $\mathcal{T}_e(s)$ is a Π^0_n-1 -class so we can find σ and ϵ effectively in $0^{(n-1)}$. We enumerate σ into S_{e+1}, and let $\mathcal{T}_{e+1}(s+1) \subseteq \mathcal{T}_e(s) \cap \mathrm{Ext}(\sigma)$ be a closed Π^0_n-1 -class with measure $\geq \frac{\epsilon}{2}$, which we can find effectively in $0^{(n-1)}$ by Lemma II.1.4. Then set the threshold value to $t_{e+1}(s+1) = \frac{\epsilon}{4}$, i.e., less than half the measure of $\mathcal{T}_{e+1}(s+1)$. Finally, for all $i \neq e$ let $\mathcal{T}_i(s+1) = \mathcal{T}_i(s)$ and $t_i(s+1) = t_i(s)$. Let $\Gamma(s+1) = e+1$.

Case 2 Some such i exists; let e denote the least such i. Then R_e acts at stage $s+1$: since $\mu(\mathcal{P}_e(s)) \leq t_e(s) \leq \frac{1}{2}\mu(\mathcal{T}_e(s))$ we can find a rational $\epsilon > 0$ such that $\mu(\mathcal{T}e(s) \cap \overline{\mathcal{P}_e(s)}) \geq \epsilon$. Then using Lemma II.1.4 we can find a closed Π^0_n-1 -class $\mathcal{T}_e(s+1) \subseteq \mathcal{T}_e(s) \cap \overline{\mathcal{P}_e(s)}$ with measure $\geq \frac{\epsilon}{2}$. This is effective in $0^{(n-1)}$ since $\mathcal{T}_e(s) \cap \overline{\mathcal{P}_e(s)}$ is a Π^0_n-1 -class. Let $t_e(s) = -1$. For $i > e$, R_i is injured by R_e; we move the marker down to $\Gamma(s+1) = e$. For $i \neq e$ let $\mathcal{T}_i(s+1) = \mathcal{T}_i(s)$ and $t_i(s+1) = t_i(s)$.

We then verify inductively that for each e there is an s_0 such that:

(i) $(\forall s \geq s_0)\Gamma(s) > e$, i.e., $\lim_{s\to\infty} \Gamma(s) = \infty$.

(ii) $(\forall s \geq s_0)t_e(s) = t_e(s_0)$; denote the limiting value t_e.

(iii) $(\forall s \geq s_0)\mathcal{T}_e(s) = \mathcal{T}_e(s_0)$; denote the limiting value $\mathcal{T}_e$.

(iv) Either $t_e > 0$ and $(\forall s \geq s_0)\mu(\mathcal{P}_e(s)) > t_e$ or else $t_e = -1$ and $\mathcal{T}_e \cap \mathcal{P}_e = \emptyset$.

Evidently (ii) and (iii) follow from (i), since $\Gamma(s)$ gets value i whenever $t_i(s)$ or $\mathcal{T}_i(s)$ is changed.

Given e, let s_0 be the least stage for which (i)–(iv) hold for all $i < e$. In fact it must be the case that $\Gamma(s_0) = e$, $\mathcal{T}_e(s_0)$ and $t_e(s_0)$ are defined, and $t_e(s_0) > 0$. Note that by (i), no R_i with $i < e$ can act after stage s_0, so R_e is never injured after stage s_0. Note also that $\mathcal{T}_e(s_0)$ and $t_e(s_0)$ are changed only if R_e acts; thus one of two things can happen:

Case 1 If for all s, $\mu(\mathcal{P}_e(s)) > t_e(s_0)$, then R_e never acts. Then $\Gamma(r) > e$ for all $r > s_0$, the limiting values are $t_e = t_e(s_0)$ and $\mathcal{T}_e = \mathcal{T}_e(s_0)$, and the first clause of (iv) holds.

Case 2 If for some s, $\mu(\mathcal{P}_e(s)) \leq t_e(s_0)$, let r be the least such s greater than s_0. Then since $t_e(r) = t_e(s_0)$, R_e acts at stage r: by construction $t_e(r+1) = -1$, so R_e can't act again. Thus $\Gamma(s) > e$ for all $s > r+1$, and the limiting values are $t_e = t_e(r+1)$ and $\mathcal{T}_e = \mathcal{T}_e(r+1)$. By construction, $\mathcal{T}_e(r+1) \subseteq \overline{\mathcal{P}_e(r)} \subseteq \overline{\mathcal{P}_e}$, so $\mathcal{T}_e \cap \mathcal{P}_e = \emptyset$; thus the second clause of (iv) holds.

Note that by construction, at any stage s, $\mu(\mathcal{T}_e(s)) > 0$ and $\mathcal{T}_e(s) \supseteq \mathcal{T}_e(s+1)$ for all $e < \Gamma(s)$, so these properties hold for the limiting values $\mathcal{T}_e$ as well. Then $\bigcap_e \mathcal{T}_e$ is nonempty and contains a unique set A. By (iv), each requirement R_e is satisfied, so A is not contained in any Π^0_n -class with measure 0, i.e., A is weakly n-random.

We also have $\mathcal{T}_e \subseteq \mathcal{S}_e$ by construction, so $A \in \bigcap_e \mathcal{S}_e$. The argument described in the $n = 1$ case shows that $\mu(\mathcal{S}_e) \leq 2^{-e}$. The construction at each stage is recursive in $0^{(n-1)}$, so the sequence $\{\mathcal{S}_e\}_{e\in\omega}$ is r.e. in $0^{(n-1)}$ and may thus be viewed as a recursive sequence of Σ^0_n -classes ($\Sigma^{0^{(n-1)}}_1$ -classes, in fact). Thus A is not n-random. □

Chapter III

Properties of n-random Sets

III.1 General Results

In this section we warm up with some fairly simple results showing that 1-random and n-random sequences satisfy a number of properties intuitively associated with randomness. To begin with, there are several ways to capture the intuitive idea that no part of a random sequence should contain any information about any other part. One is expressed by the Church-von Mises definition of Section II.2, that is, that knowing the first n bits of a random sequence should not help predict the $(n+1)$st bit, or equivalently, for any random sequence, a subsequence determined by a recursive place selection should have all the same randomness properties as the original sequence. Theorem III.1.2 below asserts in a precise way that this property holds for n-random sequences. We will first prove a simpler result, Theorem III.1.1, to the effect that recursive sections—even and odd halves, columns, etc.—of an n-random sequence are themselves n-random.

A completely different approach is simply to take a literal interpretation of the statement "no part of a random sequence is computable from any other part"; that is, disjoint recursive sections should be Turing-incomparable. This idea is expressed in Theorem III.1.4, which in turn can be regarded as a special case, for constant place selections, of the more general Theorem III.1.5.

It also turns out that *both* points of view are really just facets of a third, deeper phenomenon which we explore in Section III.3. In fact, the four theorems just mentioned are all consequences of Theorem III.3.9. We have given separate proofs of several of the results in this section since the arguments are quite simple and the underlying ideas more transparent in these easier cases.

We begin with the results expressing the idea that randomness is inherited by subsequences.

Theorem III.1.1 *Let $A, B, C \in 2^\omega$.*

(i) If $A \oplus B$ is C-n-random, then A is C-n-random.

(ii) If A is C-n-random, then for any $i \in \omega$ the column $A^{[i]}$ is C-n-random.

(iii) If A is C-n-random and $B \leq_T C^{(n-1)}$ is infinite, then A/B is C-n-random.

Proof. We give only the proof for (i); the others involve no new ideas. Suppose A is not C-n-random; then A is $\Sigma_1^{C^{(n-1)}}$-approximable by a uniform sequence of the form $\{\text{Ext}(S_i)\}_{i\in\omega}$. We can assume that each set S_i is a disjoint set of strings r.e. in $C^{(n-1)}$. Now fix i; we describe a procedure for enumerating a set T_i: For each σ in S_i, let

$$T_{i,\sigma} = \{\sigma \oplus \tau : |\tau| = |\sigma|\},$$

and let

$$T_i = \bigcup_{\sigma \in S_i} T_{i,\sigma}.$$

Note that $\mu(\text{Ext}(T_{i,\sigma})) = \mu(\text{Ext}(\sigma))$, so

$$\begin{aligned}\mu(\text{Ext}(T_i)) &= \sum_{\sigma\in S_i} \mu(\text{Ext}(T_{i,\sigma})) \\ &= \sum_{\sigma\in S_i} \mu(\text{Ext}(\sigma)) \\ &= \mu(\text{Ext}(S_i)),\end{aligned}$$

since S_i is disjoint. Clearly the procedure for enumerating T_i is uniform and $A \oplus B \in \text{Ext}(T_i)$ (for any B), so $\{\text{Ext}(T_i)\}_{i\in\omega}$ is a $\Sigma_1^{C^{(n-1)}}$-approximation of $A \oplus B$. □

Part (iii) of the above theorem says that certain subsequences of an n-random set—those determined by constant Δ_n place selections—are n-random; this fact is included with the above theorem since the proof is essentially the same as it is for part (i). The general result, which applies to all Δ_n place selections, requires some new ideas.

Theorem III.1.2 *Let $A, C \in 2^\omega$ and $f : 2^{<\omega} \longrightarrow \{0,1\}$. Let $\hat{f}$ be as in Definition II.2.1. Suppose $\hat{f}(A)$ is infinite. If A is C-n-random and $f \leq_T C^{(n-1)}$, then $A/\hat{f}(A)$ is C-n-random.*

We first isolate the key counting argument in the next lemma. Part (ii) will be used in Theorem III.3.9.

Lemma III.1.3 *Let $f : 2^{<\omega} \longrightarrow \{0,1\}$.*

(i) Let σ be any string. The collection

$$S = \{\rho : \rho/\hat{f}(\rho) \supset \sigma\}$$

is r.e. in f and $\mu(\text{Ext}(S)) \leq 2^{-|\sigma|}$.

(ii) Let σ, π, and σ^* be strings such that $\pi/\hat{f}(\pi) = \sigma_0$ and $\sigma^* \subset \sigma$. Let $k = |\sigma| - |\sigma^*|$. Then

$$S_\pi = \{\rho : \rho \supset \pi \ \& \ \rho/\hat{f}(\rho) \supset \sigma\}$$

is r.e. in f and $\mu(\mathrm{Ext}(S_\pi)) \leq 2^{-|\pi|} \cdot 2^{-k}$.

Proof. (i) Let T be any (possibly infinite) disjoint set of strings and $i \in \{0,1\}$ a fixed bit. Define

$$\begin{aligned} T * i &= \{\delta * i : \delta \in T\} \text{ and} \\ T * f &= \{\gamma : f(\gamma) = 1 \ \& \\ &\quad (\exists \delta \in T)[\gamma \supset \delta \ \& \text{ if } \delta \subset \gamma' \subset \gamma \text{ with } \gamma' \neq \gamma, \text{ then} f(\gamma') = 0]\}. \end{aligned}$$

(If we picture f as a betting strategy and δ as an initial sequence of outcomes, then each γ represents a minimal sequence of outcomes after which we'd place another bet.)

Note that $T * i$ and $T * f$ are still disjoint sets of strings, $\mu(T * i) = \frac{1}{2}\mu(T)$, and $\mu(T * f) \leq \mu(T)$. Let σ be the string given in the hypothesis and let $k = |\sigma|$. Now define the set S inductively as follows.

$$\begin{aligned} S_0 &= \emptyset \\ S_{i+1} &= (S_i * f) * \sigma(i) \text{ for } i < k \\ S &= \{\rho : (\exists \delta \in S_k)[\rho \supset \delta]\} \end{aligned}$$

It can now be verified by induction that $\mu(\mathrm{Ext}(S)) \leq 2^{-k}$, $\rho/\hat{f}(\rho) \supset \sigma$ for all $\rho \in S$, and every ρ such that $\rho/\hat{f}(\rho) \supset \sigma$ is in S. Clearly S is r.e. in f.

For the proof of (ii), apply (i) to the function $g(\delta) = f(\pi * \delta)$ and the string σ_1 such that $\sigma = \sigma^* * \sigma_1$. Then if S is the set constructed by the proof of (i), let $S_\pi = \{\pi * \delta : \delta \in S\}$. □

Proof of Theorem III.1.2 It is sufficient to give the proof for the case $n = 1$ (by Lemma II.1.5). Suppose $\{\mathcal{U}_i\}_{i\in\omega}$ is a Σ_1^C -approximation of $A/\hat{f}(A)$; fix i and let U_i be a set of strings r.e. in C such that $\mathcal{U}_i = \mathrm{Ext}(U_i)$. We may assume without loss of generality that U_i is disjoint. Let

$$S(\sigma) = \{\rho : \rho/\hat{f}(\rho) \supset \sigma\}.$$

By Lemma III.1.3(i), $S(\sigma)$ is r.e. in f and $\mu(S(\sigma)) \leq 2^{-|\sigma|}$. Moreover, if $A/\hat{f}(A) \supset \sigma$, then A extends some string in $S(\sigma)$. Let

$$S_i = \bigcup_{\sigma \in U_i} S(\sigma).$$

Then $\mu(\mathrm{Ext}(S_i)) \leq \mu(\mathcal{U}_i) \leq 2^{-i}$ and $A \in \mathrm{Ext}(S_i)$. Note that S_i is r.e. in $f \oplus C$ and hence in C since $f \leq_T C$ by assumption. It follows that $\{\mathrm{Ext}(S_i)\}_{i\in\omega}$ is a Σ_1^C -approximation of A. □

Theorem III.1.2 above, in conjunction with the law of large numbers for 1-random sequences (Theorem III.1.6, below) shows that a 1-random sequence always satisfies Church's definition of randomness (cf. Section II.2). Another consequence is that there is no *direct* way to code arbitrary information into 1-random or n-random sets, since if there were an effective way to recover the coding locations, the coded sequence itself would have to be 1-random. Surreptitious methods have to be devised, such as the technique used by Kučera (Theorem III.2.2).

We next summarize some results on incomparability. The first result below seems to have been folklore; we are indebted to A. Kučera for patiently explaining it to us.

Theorem III.1.4 *Let $A, B, C \in 2^\omega$.*

(i) If $A \oplus B$ is 1-random, then $A \not\leq_T B$.

(ii) If A is 1-random, then for any $i \in \omega$,

$$A^{[i]} \not\leq_T \bigoplus_{j \neq i} A^{[j]}.$$

(iii) If A is C-n-random and $B \leq_T C^{(n-1)}$ is infinite, then

$$A/B \not\leq_T A/\overline{B}.$$

Proof. Again we prove only part (i). Suppose $A = \varphi_e^B$. For $i \in \omega$ we describe a procedure for recursively enumerating a set of strings S_i so that $\{\mathrm{Ext}(S_i)\}_{i\in\omega}$ is a Σ_1^0 -approximation of $A \oplus B$. We search for strings τ such that $\varphi_e^\tau(x)\downarrow$ for all $x < i$, let $\sigma = \varphi_e^\tau \restriction i$, and enumerate $\sigma \oplus \tau$ in S_i. More precisely, let

$$T_{i,s} = \{\tau \in 2^s : (\forall x < i)[\varphi_e^\tau(x)\downarrow]\},$$

and

$$\begin{aligned} S_{i,s} &= \{\sigma \oplus \tau : \sigma = \varphi_e^\tau \restriction i \ \&\ \tau \in T_{i,s}\}, \\ S_i &= \bigcup_s S_{i,s}. \end{aligned}$$

Certainly $A \oplus B \in \mathrm{Ext}(S_i)$, since eventually we will discover a long enough initial segment $\tau \subset B$ for φ_e^τ to converge on all $x < i$; then $\varphi_e^\tau \restriction i = A\restriction i$ and $A\restriction i \oplus \tau$ is enumerated in $S_{i,|\tau|}$. To show that the measure of $\mathrm{Ext}(S_i)$ is less than 2^{-i} it will suffice to show that for each s, $\mu(S_{i,s}) \leq 2^{-i}$, since $\mathrm{Ext}(S_{i,s}) \subseteq \mathrm{Ext}(S_{i,s+1})$. Note that $T_{i,s}$ is a disjoint set of strings, and each $\tau \in T_{i,s}$ corresponds to a unique $\sigma = \varphi_e^\tau \restriction i$. Then

$$\begin{aligned} \mu(S_{i,s}) &= \sum_{\tau \in T_{i,s}} 2^{-|\sigma|} \cdot 2^{-|\tau|} \\ &= 2^{-i} \cdot \sum_{\tau \in T_{i,s}} 2^{-|\tau|} \\ &\leq 2^{-i} \cdot 1 \end{aligned}$$

since $T_{i,s}$ is disjoint. □

As with Theorem III.1.1, part (iii) of the preceding result also holds for general Δ_n place selections, but the proof requires a different technique.

Theorem III.1.5 *Let $A, C \in 2^\omega$ and $f : 2^{<\omega} \longrightarrow \{0,1\}$. If A is C-1-random, then*

$$A/\hat{f}(A) \not\leq_T A/\overline{\hat{f}(A)}.$$

The proof is omitted as it is somewhat complicated and all the relevant ideas are present in the proof of the stronger Theorem III.3.9.

We conclude this section with a short proof that 1-randomness is sufficient for the strong law of large numbers to hold. This result has evidently been known for some time; a proof based on the usual probabilistic arguments can be found in van Lambalgen ([36, p. 69]). We give instead a proof based directly on the definition of 1-randomness, i.e., we explicitly enumerate a Σ^0_1 -approximation.

Theorem III.1.6 *Let $A \in 2^\omega$ be 1-random, and let*

$$l(A\restriction n) = \frac{\#\textit{ of } 0\textit{'s in } A\restriction n}{n}.$$

If A is 1-random, $\lim_{n\to\infty} l(A\restriction n) = \frac{1}{2}$.

Proof. Fix $m \geq 1$. For $k = 1, 2, \ldots$, let

$$T_k = \{\sigma \in 2^{<\omega} :\ |\sigma| = 2mk^3 \text{ and } \sigma \text{ contains at most } (m-1)k^3 \text{ zeros }\}$$

and

$$S_k = \{\sigma \in 2^{<\omega} :\ |\sigma| = 2mk^3 \text{ and } \sigma \text{ contains at most } (m-1)k^3 \text{ ones }\}.$$

If A is in only finitely many of the classes $\mathrm{Ext}(T_k)$ and $\mathrm{Ext}(S_k)$, then

$$\frac{1}{2} - \frac{2}{m+1} \leq l(A\restriction n) \leq \frac{1}{2} + \frac{2}{m+1} \tag{III.1}$$

for all sufficiently large n. That is, suppose there is some j such that A avoids $\mathrm{Ext}(T_k)$ for all $k \geq j$. Then for each $k \geq j$, the initial segment of A of length $2mk^3$ contains at least $(m-1)k^3$ zeros, and so for each n between $2mk^3$ and $2m(k+1)^3$,

$$l(A\restriction n) \geq \frac{(m-1)k^3}{2m(k+1)^3}.$$

Then as long as $k \geq m$, we have

$$\begin{aligned} l(A\upharpoonright n) &\geq \frac{(m-1)k^3}{2m(k+1)^3} \\ &\geq \frac{(m-1)m^3}{2m(m+1)^3} \\ &\geq \frac{1}{2}\left[\frac{(m+1)^3-(4m^2+3m+1)}{(m+1)^3}\right] \text{ (completing the cube)} \\ &\geq \frac{1}{2}-\frac{2}{m+1}. \end{aligned}$$

Likewise the second inequality of (III.1) holds if A is in only finitely many of the classes $\mathrm{Ext}(S_k)$.

Using Solovay's characterization (Theorem II.1.8) it will follow that the 1-random set A is in only finitely many of the classes $\mathrm{Ext}(T_k)$ if we can show that

$$\sum_k \mu(\mathrm{Ext}(T_k)) < \infty;$$

likewise for $\mathrm{Ext}(S_k)$. Since m was arbitrary, it will then follow that the inequalities (III.1) hold for each m, for sufficiently large n; thus $\lim_n l(A\upharpoonright n) = \frac{1}{2}$.

To prove (III.2) we can get by with some fairly crude estimates. The set T_k contains

$$\sum_{i=0}^{(m-1)k^3} \binom{2mk^3}{i}$$

strings of length $2mk^3$. A standard combinatorial inequality (see, e.g., [6, p. 140]) shows that when $s \leq n/2$,

$$\sum_{i=0}^{s} \binom{n}{i} \leq \binom{n}{s} \cdot \left[\frac{n+1-s}{n+1-2s}\right].$$

Taking $n = 2mk^3$ and $s = (m-1)k^3$ in the above, the number of strings in T_k is bounded by

$$\binom{2mk^3}{(m-1)k^3} \cdot \left[\frac{mk^3-k^3+1}{2k^3+1}\right].$$

As long as $k \geq m$, the term in brackets is bounded by $2m$, independent of k. It is also the case that

$$\binom{2mk^3}{(m-1)k^3} \leq \binom{2mk^3}{mk^3}$$

and it is easy to show using Stirling's formula (see Feller, [6, p. 52]) that

$$\binom{2mk^3}{mk^3} \leq \frac{2^{2mk^3}}{\sqrt{k^3}}.$$

Thus the number of strings in T_k is bounded by

$$\frac{2^{2mk^3}}{\sqrt{k^3}} \cdot 2m,$$

so

$$\begin{aligned}\sum_{k \geq m} \mu(\mathrm{Ext}(T_k)) &\leq \sum_{k \geq m} \frac{(2^{2mk^3})/(\sqrt{k^3})}{2^{2mk^3}} \cdot 2m \\ &\leq \sum_{k \geq m} \frac{2m}{\sqrt{k^3}} \\ &< \infty.\end{aligned}$$

The same argument clearly works for S_k, so the proof is complete. □

III.2 Jump Classes

The theorem below is one of the single most useful facts about n-randomness; in connection with Lemma II.1.5, it allows many results for 1-randomness to be relativized to n-random sets. It strengthens a result originally due to Sacks that the class $\{A : A' \equiv_T A \oplus 0'\}$ has measure one (Stillwell [34]).

Theorem III.2.1 *For $n \geq 0$, if A is $(n+1)$-random then $A^{(n)} \equiv_T A \oplus 0^{(n)}$.*

Proof. The theorem is trivial for $n = 0$, so fix $n \geq 1$. Define for each $e \in \omega$ a class

$$\mathcal{B}_e = \{A : e \in A^{(n)}\}.$$

$\mathcal{B}_e$ is a Σ^0_n -class: it can be shown by induction that the predicate "$\sigma \subset A^{(n-1)}$" is Δ^A_n, and

$$\begin{aligned} A \in \mathcal{B}_e &\iff \varphi_e^{A^{(n-1)}}(e)\downarrow \\ &\iff (\exists\sigma)[\sigma \subset A^{(n-1)} \text{ and } \varphi_e^{\sigma}(e)\downarrow]. \end{aligned}$$

Note that an index for the class $\mathcal{B}_e$ can be found uniformly from e. Now by Lemma II.1.4 we can uniformly in $0^{(n)}$ find a $\Sigma_1^{0^{(n-1)}}$ -class $\mathcal{U}_e$ such that $\mathcal{B}_e \subseteq \mathcal{U}_e$ and $\mu(\mathcal{U}_e) - \mu(\mathcal{B}_e) \leq 2^{-(e+1)}$. $\mathcal{U}_e$ is the extension of a set of strings r.e. in $0^{(n-1)}$, say $\mathcal{U}_e = \mathrm{Ext}(W_z^{0^{(n-1)}})$. Let the approximation at stage s be denoted

$$\mathcal{U}_{e,s} = \mathrm{Ext}(W_{z,s}^{0^{(n-1)}}).$$

We can uniformly in $0^{(n)}$ find a stage $s(e)$ such that $\mu(\mathcal{U}_e) - \mu(\mathcal{U}_{e,s(e)}) \leq 2^{-(e+1)}$.

The procedure above for obtaining $\mathcal{U}_{e,s(e)}$ defines, in essence, a partial recursive functional Ψ such that for each e, if B happens to be $0^{(n)}$ then $\Psi^B(e)$ converges and its value is (the index of) a finite set of strings such that $\mathcal{U}_{e,s(e)} = \mathrm{Ext}(\Psi^B(e))$. Now we can define a p.r. functional Φ by

$$\Phi^{A\oplus B}(e) = \begin{cases} 1 & \text{if } \Psi^B(e)\downarrow \text{ and } A \in \mathrm{Ext}(\Psi^B(e)) \\ 0 & \text{if } \Psi^B(e)\downarrow \text{ and } A \notin \mathrm{Ext}(\Psi^B(e)) \\ \uparrow & \text{otherwise.} \end{cases}$$

In particular what we have is that

$$\Phi^{A\oplus 0^{(n)}}(e) = \begin{cases} 1 & \text{if } A \in \mathcal{U}_{e,s(e)} \\ 0 & \text{otherwise.} \end{cases}$$

We now show that the class

$$\{A : A^{(n)}(e) \neq \Phi^{A\oplus 0^{(n)}}(e) \text{ for infinitely many } e\}$$

is $\Sigma^0_n{+}1$ -approximable. We know that $A^{(n)}(e) \neq \Phi^{A\oplus 0^{(n)}}(e)$ if and only if either

$$\begin{aligned} & A \in \mathcal{B}_e \text{ and } A \notin \mathcal{U}_{e,s(e)}, \\ \text{or} \quad & A \notin \mathcal{B}_e \text{ and } A \in \mathcal{U}_{e,s(e)}. \end{aligned}$$

Since $\mu(\mathcal{B}_e - \mathcal{U}_{e,s(e)}) \leq 2^{-(e+1)}$ and $\mu(\mathcal{U}_{e,s(e)} - \mathcal{B}_e) \leq 2^{-(e+1)}$, the class

$$\mathcal{S}_e = (\mathcal{B}_e - \mathcal{U}_{e,s(e)}) \cup (\mathcal{U}_{e,s(e)} - \mathcal{B}_e)$$

has measure at most 2^{-e}. $\mathcal{U}_{e,s(e)}$ is the extension of a finite set of strings recursive in $0^{(n-1)}$, so it is both a Σ^0_n -class and a Π^0_n -class, and so $\mathcal{S}_e$ is a $\Sigma^0_n{+}1$ -class. Thus $\{\mathcal{S}_e\}_{e\in\omega}$ is a $0^{(n)}$-recursive sequence of $\Sigma^0_n{+}1$ -classes, i.e., a $\Sigma^0_n{+}1$ -approximation.

If A is $(n+1)$-random, by Theorem II.1.8 it is in only finitely many of the classes $\mathcal{S}_e$, so $A^{(n)}(e) = \Phi^{A\oplus 0^{(n)}}(e)$ for all but finitely many e; thus $A^{(n)} \leq_T A \oplus 0^{(n)}$. $\square$

In [17] Kučera uses a strategy of coding in recursive trees to prove that every degree above $0'$ contains a 1-random set. Theorem III.2.1 enables us to use the same method to prove the following general form of his result.

Theorem III.2.2 *Let $n \geq 1$. For every $B \geq_T 0^{(n)}$ there is an n-random set A with $A^{(n-1)} \equiv_T B$.*

It follows immediately that the hypothesis that A is $(n{+}1)$-random in Theorem III.2.1 is necessary, since by the result above there are n-random sets A with $A^{(n-1)} \equiv_T 0^{(n)}$ and hence $A^{(n)} >_T 0^{(n)} \equiv_T A \oplus 0^{(n)}$.

We can also show that the n-random set constructed in Theorem III.2.2 cannot be $(n+1)$-random.

Theorem III.2.3 *For $n \geq 1$, the class $\{A : A^{(n-1)} \geq_T 0^{(n)}\}$ has measure zero, and in fact contains no weakly $(n+1)$-random sets.*

Proof. By Theorem II.5.2 we know that

$$\{A : A \oplus 0^{(n-1)} \geq_T 0^{(n)}\}$$

must have measure zero; otherwise $0^{(n)} \leq_T 0^{(n-1)}$. By Theorem III.2.1 every $(n+1)$-random A satisfies $A^{(n-1)} \equiv_T A \oplus 0^{(n-1)}$, so

$$\begin{aligned} & \{A : A^{(n-1)} \geq_T 0^{(n)}\} \cap \{A : A \text{ is } (n+1)\text{-random}\} \\ = \ & \{A : A \oplus 0^{(n-1)} \geq_T 0^{(n)}\} \cap \{A : A \text{ is } (n+1)\text{-random}\} \end{aligned}$$

has measure zero. A quantifier-counting argument shows that the above class is a $\Sigma^0_n{+2}$ -class, and thus is a union of $\Pi^0_n{+1}$ -nullsets, so it can't contain any weakly $(n+1)$-random sets. □

III.3 Strong Independence Properties

Relative Randomness as Independence

Thus far we have used the definition of relative randomness mainly as a technical convenience (e.g., in order to describe n-randomness as 1-randomness relative to $0^{(n)}$) but it can be interpreted in a much more meaningful way. We noticed in Section II.2 that Church's definition of randomness is "weak" because it restricts the admissible place selections to a countable collection, whereas there are many more place selections (a class of measure one, in fact) which in some sense should be counted as "admissible", in that they preserve the desired limiting frequency. Evidently von Mises' original conception of admissibility included a notion of *independence*: the idea, for example, that if A is generated by a sequence of coin tosses, and B is generated by a (physically) independent sequence of coin tosses, then the subsequence A/B should also be random. Independence plays a central role in van Lambalgen's axiomatization of randomness in [35], and there it is also argued that "A is 1-random relative to B" more or less faithfully captures the notion "A is independent of B" as described by the axioms.

With the foregoing in mind let us look again at Theorem III.1.1. What the general proof shows is that

$$\text{If } A/B \text{ is } \Sigma^C_1 \text{ -approximable, then } A \text{ is } \Sigma^{B\oplus C}_1 \text{ - approximable.} \qquad \text{(III.2)}$$

Likewise, what we actually prove in Theorem III.1.2 is

$$\text{If } A/\hat{f}(A) \text{ is } \Sigma^C_1 \text{ -approximable, then } A \text{ is } \Sigma^{f\oplus C}_1 \text{ -approximable.} \qquad \text{(III.3)}$$

Assume that A is n-random and take $C = 0^{(n-1)}$. Not only can we conclude that A/B is n-random whenever B is Δ_n (or that $A/\hat{f}(A)$ is n-random whenever f is Δ_n), but also that A/B is n-random for any B such that A is *n-random relative to* B (likewise, $A/\hat{f}(A)$ is n-random for any f such that A is n-random relative to f). Thus relative randomness does act like "independence" in the intuitive sense discussed above. What is not immediately obvious is that we are actually saying much more than in Theorems III.1.1 or III.1.2: In fact, *almost every* subsequence of an n-random set A is n-random, since the class

$$\{B : A \text{ is } n\text{-random relative to } B\},$$

or equivalently the class

$$\{\mathbf{deg}(f) : A \text{ is } n\text{-random relative to } f\},$$

has measure one. This will be a consequence of the following theorem, in conjunction with Theorem III.2.1.

Theorem III.3.1 *Let $A, C \in 2^\omega$. If $\{B : A$ is $\Sigma_1^{B\oplus C}$ -approximable$\}$ has positive measure, then A is Σ_1^C -approximable.*

The proof of Theorem III.3.1 is a "majority vote" style argument similar to Sacks' original proof of Theorem II.5.2. We first state a simple counting principle used in the proof. It is really just an application of a discrete form of Fubini's theorem.

Lemma III.3.2 *Let X be an n-element set, $m \leq n$, and $\mathcal{F} = \{Y_1, \dots Y_k\}$ a family of subsets of X. If each $Y \in \mathcal{F}$ contains at least m elements, then some $x \in X$ appears in at least $\left\lfloor \frac{m}{n} \cdot k \right\rfloor$ of the sets Y in $\mathcal{F}$. (Here $\lfloor r \rfloor$ denotes the greatest integer $\leq r$.)*

Proof. Consider the sum of the cardinalities

$$s = \sum_{Y \in \mathcal{F}} |Y|.$$

We know $s \geq m \cdot k$ since $|Y| \geq m$ for each $Y \in \mathcal{F}$. Suppose every $x \in X$ occurs in strictly less than $\left\lfloor \frac{m}{n} \cdot k \right\rfloor$ of the sets Y; then the sum s would be less than $n \cdot \left\lfloor \frac{m}{n} \cdot k \right\rfloor$, which is less than $m \cdot k$, a contradiction. Formally we are just reversing the order of summation:

$$s = \sum_{Y \in \mathcal{F}} \left(\sum_{x \in X} Y(x) \right) = \sum_{x \in X} \left(\sum_{Y \in \mathcal{F}} Y(x) \right),$$

where Y is identified with its characteristic function in the sums above. □

We will also need the following standard measure-theoretic result. A proof can be found in [15].

Definition III.3.3 *Let $\mathcal{U}$ be a measurable subset of 2^ω and $\sigma \in 2^{<\omega}$. The* density of $\mathcal{U}$ in $\mathrm{Ext}(\sigma)$ *is the quantity*

$$\frac{\mu(\mathcal{U} \cap \mathrm{Ext}(\sigma))}{\mu(\mathrm{Ext}(\sigma))}.$$

The density of $\mathcal{U}$ in $\mathrm{Ext}(\sigma)$ can intuitively be pictured as the conditional probability

$$\mathbf{Pr}(A \in \mathcal{U} \mid \sigma \subset A).$$

Lemma III.3.4 (Density lemma) *Let $\mathcal{U} \subseteq 2^\omega$ be measurable and $\epsilon < 1$. Then there is a string σ such that the density of $\mathcal{U}$ in $\mathrm{Ext}(\sigma)$ is greater than ϵ.*

Proof of Theorem III.3.1 For simplicity we present the proof in unrelativized form; the relativization to an oracle C is straightforward. Let $A \in 2^\omega$ and suppose

$$\mu\{B : A \text{ is } \Sigma_1^B\text{-approximable }\} > 0.$$

There are only countably many indices for Σ_1^B - approximations, independent of the oracle B, so by countable additivity there is a recursive function f such that the class

$$\{B : A \text{ is } \Sigma_1^B\text{-approximable via } \{\mathrm{Ext}(W^B_{f(i)})\}_{i\in\omega}\} \tag{III.4}$$

has positive measure. By the Density lemma, we can choose a string σ such that the class (III.4) has density greater than $\frac{3}{4}$ in $\mathrm{Ext}(\sigma)$.

Fix i; we give a uniform procedure for recursively enumerating a set of strings S_i such that $\{\mathrm{Ext}(S_i)\}_{i\in\omega}$ is a Σ_1^0-approximation of A. Let $e = f(i)$. For each string ρ of length at least $|\sigma|$ define

$$T_\rho = \{\tau \in 2^{|\rho|} : (\exists \rho' \subset \rho)(\exists \tau' \subset \tau)[\tau' \supset \sigma \ \&\ \rho' \in W_e^{\tau'} \ \&\ \mu(\mathrm{Ext}(W_e^{\tau'})) \leq 2^{-i}]\}.$$

The idea is then to enumerate in S_i those strings ρ such that the density of $\mathrm{Ext}(T_\rho)$ in $\mathrm{Ext}(\sigma)$ exceeds $\frac{1}{2}$; that is, the strings ρ which amass a majority of the "votes". Let

$$S_{i,s} = \{\rho \in 2^s : \frac{\mu(\mathrm{Ext}(T_\rho))}{\mu(\mathrm{Ext}(\sigma))} > \frac{1}{2}\}$$

and

$$S_i = \bigcup_{s \geq |\sigma|} S_{i,s}.$$

Notice that if $\rho' \subset \rho$, then $\mathrm{Ext}(T_{\rho'}) \subseteq \mathrm{Ext}(T_\rho)$, and so for all s,

$$\mathrm{Ext}(S_{i,s}) \subseteq \mathrm{Ext}(S_{i,s+1}). \tag{III.5}$$

There are two claims to verify:

(i) $A \in \text{Ext}(S_i)$, and

(ii) $\mu(\text{Ext}(S_i)) \leq 2 \cdot 2^{-i}$.

(i) Since the density of the class (III.4) in $\text{Ext}(\sigma)$ is greater than $\frac{3}{4}$, it follows that the class

$$\mathcal{U} = \{B : A \in \text{Ext}(W_e^B) \ \&\ \mu(\text{Ext}(W_e^B)) \leq 2^{-i}\}$$

has density greater than $\frac{3}{4}$ in $\text{Ext}(\sigma)$. Since $\mathcal{U}$ is open there exists an initial segment $\rho \subset A$ and a finite, disjoint set of strings $T = \{\tau_1, \ldots, \tau_n\}$ such that $\text{Ext}(T) \subset \mathcal{U}$, $\text{Ext}(T)$ has density greater than $\frac{1}{2}$ in $\text{Ext}(\sigma)$, and for each $\tau \in T$:

- $\tau \supset \sigma$,
- $\mu(\text{Ext}(W_e^\tau)) \leq 2^{-i}$, and
- some initial segment ρ' of ρ is in W_e^τ.

Let s be the larger of $|\rho|$ and $\max\{|\tau| : \tau \in T\}$, and let $\rho^* = A\restriction s$. Then by construction $\text{Ext}(T_{\rho^*})$ contains $\text{Ext}(T)$, so $\text{Ext}(T_{\rho^*})$ has density greater than $\frac{1}{2}$ in $\text{Ext}(\sigma)$, and thus ρ^* is enumerated into $S_{i,s}$.

(ii) By (III.5) it will suffice to show that for any fixed $s \geq |\sigma|$, $\mu(\text{Ext}(S_{i,s})) \leq 2^{-i}$. Note that $S_{i,s}$ is always a (disjoint) collection of strings ρ of length s, and for each $\rho \in S_{i,s}$, T_ρ is also a (disjoint) collection of strings τ of length s, each of which extends σ. Let X denote the set of all strings of length s which extend σ, and let $\mathcal{F} = \{T_\rho : \rho \in S_{i,s}\}$. Now if $T_\rho \in \mathcal{F}$, the density of $\text{Ext}(T_\rho)$ in $\text{Ext}(\sigma)$ exceeds $\frac{1}{2}$, which simply means that T_ρ contains at least half the strings in X. By Lemma III.3.2, there is some string τ in X appearing in at least half the sets T_ρ in $\mathcal{F}$. This means that for at least half the strings $\rho \in S_{i,s}$, there is some $\rho' \subset \rho$ and $\tau' \subset \tau$ with $\rho' \in W_e^{\tau'}$ and $\mu(\text{Ext}(W_e^{\tau'})) \leq 2^{-i}$. Let τ^* be the longest initial segment of τ such that $\mu(\text{Ext}(W_e^{\tau^*})) \leq 2^{-i}$. Then at least half of the strings ρ in $S_{i,s}$ have an initial segment ρ' in $W_e^{\tau^*}$, and so

$$2^{-i} \geq \mu(\text{Ext}(W_e^{\tau^*})) \geq \frac{1}{2} \cdot \mu(\text{Ext}(S_{i,s})),$$

which is the desired conclusion. □

Corollary III.3.5 *Let $A \in 2^\omega$ and $n \geq 1$. If A is n-random, then the class*

$$\{B : A \textit{ is } n\textit{-random relative to } B\}$$

has measure one.

Proof. Suppose $\{B : A \text{ is } \Sigma_n^B\text{-approximable}\}$ has positive measure; since the collection of all n-random sets has measure one (see Theorem II.1.7), the class

$$\{B : B \text{ is } n\text{-random and } A \text{ is } \Sigma_n^B\text{-approximable}\}$$

has positive measure, and hence by Lemma II.1.5 and Theorem III.2.1, the class

$$\{B: \ A \text{ is } \Sigma_1^{B\oplus 0^{(n-1)}}\text{-approximable}\}$$

has positive measure. By Theorem III.3.1, A must then be $\Sigma_1^{0^{(n-1)}}$-approximable, contradicting the assumption that A is n-random. □

More Independence

We can strengthen statements (III.2) and (III.3) still further. Suppose a set $A \oplus B$ is n-random. Theorem III.1.1 asserts that A itself (likewise B itself) is n-random, and Theorem III.1.4 asserts that A is not computable from B. The result below implies further that A is not even Σ_n^B-approximable, i.e., A is n-random *relative* to B.

Lemma III.3.6 *Let $A, B, C \in 2^\omega$. If A is $\Sigma_1^{B\oplus C}$-approximable, then $A \oplus B$ is Σ_1^C-approximable.*

Proof. As the relativization is straightforward we will suppress the oracle C for readability. Suppose A is Σ_1^B-approximable, say by $\{\mathcal{T}_i\}_{i\in\omega}$. Let f be a recursive function giving the indices of the classes $\mathcal{T}_i$, i.e., $\mathcal{T}_i = \operatorname{Ext}(W_{f(i)}^B)$. Fix i and let $e = f(i)$. We describe a uniform procedure for enumerating a set of strings S_i such that $\{\operatorname{Ext}(S_j)\}_{j\in\omega}$ is a Σ_1^0-approximation of $A \oplus B$. Let $S_{i,s}$ be the set of strings

$$\{\sigma\oplus\tau : |\sigma| = |\tau| = s \ \& \ (\exists\sigma'\subset\sigma)(\exists\tau'\subset\tau)[\sigma'\in W_e^{\tau'} \ \& \ \mu(\operatorname{Ext}(W_e^{\tau'})) \le 2^{-i}]\}$$

and $S_i = \bigcup_s S_{i,s}$. Note that S_i is r.e. Certainly $A \oplus B$ is in $\operatorname{Ext}(S_i)$, since some initial segment σ' of A is in W_e^B and hence in W_e^τ for some $\tau \subset B$ with $s = |\tau| \ge |\sigma'|$. Thus $\sigma \oplus \tau$ is enumerated in $S_{i,s}$, where σ is the initial segment of A of length s.

To show that $\mu(\operatorname{Ext}(S_i)) \le 2^{-i}$, since $\operatorname{Ext}(S_{i,s}) \subseteq \operatorname{Ext}(S_{i,s+1})$ it will suffice to show that for each s, $\mu(\operatorname{Ext}(S_{i,s})) \le 2^{-i}$. Fix s and fix a string τ of length s. Let τ^* be the longest initial segment of τ such that $\mu(\operatorname{Ext}(W_e^{\tau^*})) \le 2^{-i}$. Then for every string of the form $\sigma\oplus\tau$ in $S_{i,s}$ there must be some $\sigma' \subset \sigma$ in $W_e^{\tau^*}$, so the measure contributed to $S_{i,s}$ by strings of the form $\sigma\oplus\tau$ cannot exceed

$$\begin{aligned} & \sum_{\sigma'\in W_e^{\tau^*}} 2^{-|\sigma'|}\cdot 2^{-|\tau|} \\ = \ & 2^{-|\tau|}\cdot\mu(\operatorname{Ext}(W_e^{\tau^*})) \\ \le \ & 2^{-s}\cdot 2^{-i} \end{aligned}$$

(where we have tacitly assumed that $W_e^{\tau^*}$ is disjoint). There are 2^s strings τ of length s, so the total measure of $\operatorname{Ext}(S_{i,s})$ is at most 2^{-i}. □

As with the previous results of this type, we summarize some if the different forms it may take.

Theorem III.3.7 *Let $A, B \in 2^\omega$.*

(i) If $A \oplus B$ is n-random, then A is B-n-random and B is A-n-random.

(ii) If A is n-random, then for each $i \in \omega$ the column $A^{[i]}$ is n-random relative to $\bigoplus_{j \neq i} A^{[j]}$.

(iii) If A is n-random relative to B and B is infinite, then A/B is n-random relative to $A/\overline{B}$.

Proof. (i) We apply Lemma III.3.6 and the ubiquitous Theorem III.2.1: If A is Σ_n^B -approximable, then A is $\Sigma_1^{B^{(n-1)}}$ -approximable (by Lemma II.1.5), and hence is $\Sigma_1^{B \oplus 0^{(n-1)}}$ -approximable by Theorem III.2.1 (note that B is n-random by Theorem III.1.1). Thus $A \oplus B$ is $\Sigma_1^{0^{(n-1)}}$ -approximable by Lemma III.3.6, i.e., not n-random. The proofs for (ii) and (iii) are similar. □

The next result then shows that part (iii) above also holds for non-constant place selections.

Lemma III.3.8 *Let $A, C \in 2^\omega$, and $f : 2^{<\omega} \longrightarrow \{0,1\}$. Let $\hat{f}$ be as in Definition II.2.1. Suppose $\hat{f}(A)$ and $\overline{\hat{f}(A)}$ are infinite. If $A/\hat{f}(A)$ is Σ_1^0 -approximable relative to $A/\overline{\hat{f}(A)} \oplus C$, then A is $\Sigma_1^{f \oplus C}$ - approximable.*

Proof. As usual we supress the relativation to C. Let $\{\mathcal{U}_i\}_{i \in \omega}$ be an approximation of $A/\hat{f}(A)$ relative to $A/\overline{\hat{f}(A)}$; fix i, and let e be an index such that

$$\mathcal{U}_i = \mathrm{Ext}(W_e^{A/\overline{\hat{f}(A)}})$$

and such that $W_e^{A/\overline{\hat{f}(A)}}$ is disjoint. We describe a uniform procedure for producing a set of strings S_i which is r.e. in f. Fix an $s \in \omega$ and a string τ with $|\tau| = s$. Let t be the largest integer $\leq s$ such that $\mu(W_{e,t}^\tau) \leq 2^{-i}$. Then for each $\sigma \in W_{e,t}^\tau$, let

$$S(\sigma, \tau) = \{\rho : \rho/\hat{f}(\rho) \supset \sigma \ \&\ \rho/\overline{\hat{f}(\rho)} \supset \tau\}.$$

Then define

$$\begin{aligned} S(\tau) &= \bigcup \{S(\sigma,\tau) : \sigma \in W_{e,t}^\tau\}, \\ S_{i,s} &= \bigcup_{|\tau| = s} S(\tau), \text{ and} \\ S_i &= \bigcup_s S_{i,s}. \end{aligned}$$

It is not difficult to see that S_i is r.e. in f and that $A \in \mathrm{Ext}(S_i)$. We will show below that $\mu(\mathrm{Ext}(S_i)) \leq 2^{-i}$; it then follows that $\{\mathrm{Ext}(S_i)\}_{i \in \omega}$ is a Σ_1^f -approximation of A. The remainder of the proof consists of verifying the following four claims; only the first one is difficult.

(i) $\mu(\text{Ext}(S(\sigma,\tau))) \le 2^{-|\sigma|} \cdot 2^{-|\tau|}$.

(ii) $\mu(\text{Ext}(S(\tau))) \le 2^{-i} \cdot 2^{-|\tau|}$.

(iii) $\mu(\text{Ext}(S_{i,s})) \le 2^{-i}$.

(iv) $\mu(\text{Ext}(S_i)) \le 2^{-i}$.

We can quickly dispense with Claims (ii), (iii), and (iv), assuming that we have (i). For (ii),

$$\mu(\text{Ext}(S(\tau))) \le \sum_{\sigma \in W^{\tau}_{e,t}} 2^{-|\sigma|} \cdot 2^{-|\tau|} \le 2^{-i} \cdot 2^{-|\tau|},$$

since $\mu(\text{Ext}(W^{\tau}_{e,t})) \le 2^{-i}$. For (iii),

$$\mu(\text{Ext}(S_{i,s})) \le \sum_{|\tau|=s} \mu(S(\tau)) = 2^{-i} \cdot \sum_{|\tau|=s} 2^{-|\tau|} = 2^{-i}.$$

Finally, note that $\text{Ext}(S_{i,s}) \subseteq \text{Ext}(S_{i,s+1})$ by construction, so that

$$\mu(\text{Ext}(S_i)) = \lim_{s\to\infty} \mu(\text{Ext}(S_{i,s})) \le 2^{-i},$$

which proves (iv).

To prove (i), let σ and τ be fixed; we first show that there is a unique string π of minimum length such that every $\rho \in S(\sigma,\tau)$ extends π, and either $\pi/\hat{f}(\pi) = \sigma$ or $\pi/\overline{\hat{f}(\pi)} = \tau$.

First let π_0 be a string of maximum length such that every $\rho \in S(\sigma,\tau)$ extends π_0. Let

$$\sigma_0 = \pi_0/\hat{f}(\pi_0) \text{ and } \tau_0 = \pi_0/\overline{\hat{f}(\pi_0)}.$$

Suppose σ_0 and τ_0 are both *proper* initial segments of σ and τ, respectively; let $i_0, j_0 \in \{0,1\}$ with $\sigma_0 * i_0 \subset \sigma$ and $\tau_0 * j_0 \subset \tau$. It also follows that for every $\rho \in S(\sigma,\tau)$, π_0 is a proper initial segment of ρ. There are two cases:

Case 1: $f(\pi_0) = 1$. Let $\rho \in S(\sigma,\tau)$ and let k be the bit such that $\pi_0 * k \subset \rho$. By definition $\hat{f}(\pi_0 * k) = \hat{f}(\pi_0) * 1$, so

$$(\pi_0 * k)/\hat{f}(\pi_0 * k) = \sigma_0 * k \subset \sigma,$$

and so $k = i_0$. This shows that $\pi_0 * i_0 \subset \rho$ for every $\rho \in S(\sigma,\tau)$.

Case 2: $f(\pi_0) = 0$. A similar argument shows that $\pi_0 * j_0 \subset \rho$ for every $\rho \in S(\sigma,\tau)$.

In either case the maximality of π_0 is contradicted. Let us assume then that $\tau_0 \supset \tau$, i.e., $\pi_0/\overline{\hat{f}(\pi_0)} \supset \tau$; the argument for the case $\pi_0/\hat{f}(\pi_0) \supset \sigma$ is symmetric. Now let

π be the least initial segment of π_0 such that $\overline{\pi/\hat{f}(\pi)} = \tau$. A short induction shows that π is unique.

Let $\sigma^* = \pi/\hat{f}(\pi)$. If $\sigma^* \supset \sigma$, then $|\pi| \geq |\sigma| + |\tau|$, so

$$\mu(\mathrm{Ext}(S(\sigma,\tau))) \leq \mu(\mathrm{Ext}(\pi)) \leq 2^{-|\sigma|} \cdot 2^{-|\tau|}$$

as desired. Otherwise let $k = |\sigma| - |\sigma^*|$; note that $|\pi| \geq |\sigma^*| + |\tau|$. Since $\rho/\hat{f}(\rho) \supset \tau$ for any $\rho \supset \pi$, we have

$$S(\sigma,\tau) = \{\rho : \rho \supset \pi \ \& \ \rho/\hat{f}(\rho) \supset \sigma\},$$

so by Lemma III.1.3(ii),

$$\mu(\mathrm{Ext}(S(\sigma,\tau))) \leq 2^{-|\pi|} \cdot 2^{-k} \leq 2^{-|\sigma|} \cdot 2^{-|\tau|}.$$

This concludes the proof of Claim (i), and so the proof of the lemma is complete. □

Theorem III.3.9 *If A is n-random relative to f, then $A/\hat{f}(A)$ is n-random relative to $\overline{A/\hat{f}(A)}$.*

Proof. Similar to the proof of Theorem III.3.7. □

A converse to Lemma III.3.6 is provided by a result of van Lambalgen, relativized using Lemma II.1.5 and the ever-present Theorem III.2.1.

Theorem III.3.10 (van Lambalgen) *If B is n-random and A is B-n-random, then $A \oplus B$ is n-random.*

The corollary below then follows using Theorem III.3.7. A proof can be found in [35].

Corollary III.3.11 *Let A and B be n-random. Then $A \oplus B$ is n-random if and only if A is B-n-random if and only if B is A-n-random.*

Note that Corollary III.3.11 provides an alternate proof of Theorem III.3.1, since it implies that for n-random A,

$$\{B : \ A \text{ is } B\text{-}n\text{-random}\} \supseteq \{B : \ B \text{ is } A\text{-}n\text{-random}\}$$

and the latter has measure one by Theorem II.1.7.

Minimal Pairs and Lattice Embeddings

We saw in Section III.1 that the columns of an n-random set A form a recursively independent collection of n-random sets. This is enough to show, using standard methods, that any countable partial order can be embedded in the n-random degrees: if $\langle P, \leq_P \rangle$ is a countable p.o., where $P = \{p_0, p_1, \ldots\}$ and $\leq_P$ is a recursive relation, the embedding is given by

$$p_i \longmapsto \bigoplus \{A^{[j]} : p_j \leq_P p_i\}. \tag{III.6}$$

Then since we can take $\langle P, \leq_P \rangle$ to be countably universal (see [32, p. 95]), an arbitrary countable p.o. can be embedded as well.

With the "strong" independence properties of Section III.3 we can show in addition that the mapping (III.6) is also a *lattice* embedding if $\langle P, \leq_P \rangle$ is a distributive lattice and if $n \geq 2$. The key idea is the fact that if A and B are relatively 2-random, their greatest lower bound is 0.

Corollary III.3.12 *Let $n \geq 2$. If A and B are relatively n-random, they form a minimal pair. In particular, by Theorem III.3.7, every n-random set is the join of a minimal pair of sets.*

Proof. Suppose A and B are relatively 2-random, and for some $C >_T 0$, $C \leq_T A$ and $C \leq_T B$. Then A is in the Π^C_2 -nullset $\{D : \varphi^D_e = C\}$ for some e (see Corollary II.5.3), so A is not C-2-random, and hence is not B-2-random, contradiction. □

What we would like to show next is that if A, B, and C are, for example, distinct columns of a 2-random set, then $A \oplus C$ and $B \oplus C$ have C as greatest lower bound. That is, in some sense Corollary III.3.12 can be "localized" above the set C. We start with the lemma below, which is based on the following idea: If $C <_T B$, then the set $\{A : B \leq_T A \oplus C\}$ has measure zero by Theorem II.5.2, so any A such that $B \leq_T A \oplus C$ is in a Π^B_2 - nullset, and hence is $\Sigma^{B'}_1$ -approximable. Using a technique similar to that used in the proof of Theorem III.2.1, we can show further that A is $\Sigma^{B \oplus C'}_1$ -approximable:

Lemma III.3.13 *If $C <_T B \leq_T A \oplus C$, then A is $\Sigma^{B \oplus C'}_1$ -approximable.*

Theorem III.3.14 *Let $C \in 2^\omega$. If $A \oplus B$ is 2-random relative to C, then $A \oplus C$ and $B \oplus C$ have C as greatest lower bound.*

Proof. Suppose there is some D such that $D >_T C$, $D \leq_T A \oplus C$, and $D \leq_T B \oplus C$. By the lemma above, A is $\Sigma^{D \oplus C'}_1$ -approximable, and hence is $\Sigma^{B \oplus C'}_1$ - approximable since $D \leq_T B \oplus C$. By Lemma III.3.6, $A \oplus B$ is $\Sigma^{C'}_1$ -approximable, contradicting the fact that $A \oplus B$ is C-2-random. □

The result we need then follows from Theorem III.3.7.

Corollary III.3.15 *Let A be 2-random and $F, G \subseteq \omega$ any recursive sets. Then*

$$\mathbf{deg}\left(\bigoplus_{i \in F \cap G} A^{[i]}\right)$$

is the greatest lower bound of $\mathbf{deg}(\bigoplus_{i \in F} A^{[i]})$ *and* $\mathbf{deg}(\bigoplus_{i \in G} A^{[i]})$.

It now follows by a standard argument that if A is 2-random and $\langle P, \leq_P \rangle$ is a countable distributive lattice, the mapping (III.6) is a lattice embedding of P into the columns of A. Again, since we may take $\langle P, \leq_P \rangle$ to be countably universal, this shows that any countable distributive lattice can be embedded into the columns of A.

III.4 Where Independence Fails

A somewhat surprising fact is the failure of Theorem III.3.7 for weakly n-random sets, that is, there exists a weakly n-random set $A \oplus B$ such that A is not B-weakly n-random. This provides an alternate proof that there are weakly n-random sets which are not n-random. The key ideas are in Lemma III.4.2; we will first prove the following effective version of Fubini's theorem.

Theorem III.4.1 (Effective Fubini's theorem) *If $\mathcal{S}$ is a Σ^0_n or Π^0_n+1 -class with measure one, then for any weakly n-random B the section $\mathcal{S}_B = \{A : A \oplus B \in \mathcal{S}\}$ has measure one.*

Proof. Let $\mathcal{S}$ be a Σ^0_n -class with measure one. For any B, the section $\mathcal{S}_B$ is a Σ^B_n -class, so the class

$$\begin{aligned} \mathcal{S}^* &= \{B : \mathcal{S}_B \text{ has measure one}\} \\ &= \{B : (\forall \epsilon > 0)[\mu(\mathcal{S}_B) > 1 - \epsilon]\} \end{aligned}$$

is a Π^0_n+1 -class, and has measure one by Fubini's theorem (see Oxtoby [28]). Therefore $\mathcal{S}^*$ is an intersection of Σ^0_n -classes, all of which have measure one and therefore contain every weakly n-random set. If $\mathcal{S}$ is a Π^0_n+1 -class, then the class $\mathcal{S}^*$ defined as above is still a Π^0_n+1 -class, so the conclusion follows. $\square$

Lemma III.4.2 *If B is n-random, there is a set A such that $A \leq_T B \oplus 0^{(n+1)}$ and $A \oplus B$ is weakly n-random.*

Proof. Let B be n-random. We construct A as the intersection of a sequence of closed sets $\{\mathcal{T}_i\}_{i\in\omega}$, which in this case will be Π^B_n-1 -classes. Let $\mathcal{P}_e$ denote the eth Π^0_n -class; the idea is that if $\mu(\mathcal{P}_e) = 0$, then $A \oplus B \notin \mathcal{P}_e$. Initially let $\mathcal{T}_0 = 2^\omega$ and $\sigma_0 = \emptyset$. At stage $e+1$ we have a closed Π^B_n-1 -class $\mathcal{T}_e$ of positive measure and an initial segment σ_e such that $\sigma_e \subset A$ for every $A \in \mathcal{T}_e$. At least one of $\mathcal{T}_e \cap \mathrm{Ext}(\sigma_e * 0)$, $\mathcal{T}_e \cap \mathrm{Ext}(\sigma_e * 1)$ must have positive measure; for $i = 0, 1$, using a $B^{(n-1)}$ oracle we search for a rational $\delta(e) > 0$ such that $\mu(\mathcal{T}_e \cap \mathrm{Ext}(\sigma_e * i)) \geq \delta(e)$. When such a $\delta(e)$ is found let $\mathcal{T}'_e = \mathcal{T}_e \cap \mathrm{Ext}(\sigma_e * i)$ and $\sigma_{e+1} = \sigma_e * i$ for the corresponding i.

Now using a $0^{(n+1)}$ oracle determine whether $\mu(\mathcal{P}_e) = 0$. If not, let $\mathcal{T}_{e+1} = \mathcal{T}'_e$. If so, then we know by the effective version of Fubini's theorem that the section $(\mathcal{P}_e)_B = \{A : A \oplus B \in \mathcal{P}_e\}$ has measure zero. By Lemma II.1.4(iv), we can use a $B^{(n-1)}$ oracle to find an open Σ^B_n-1 -class $\mathcal{U}$ such that $(\mathcal{P}_e)_B \subseteq \mathcal{U}$ and $\mu(\mathcal{U}) \leq \frac{1}{2}\delta(e)$. Let $\mathcal{T}_{e+1} = \mathcal{T}'_e - \mathcal{U}$; then $\mathcal{T}_{e+1}$ has measure $\geq \frac{1}{2}\delta(e)$ and is a closed Π^B_n-1 -class avoiding $(\mathcal{P}_e)_B$, that is, for any $A \in \mathcal{T}_{e+1}$, $A \oplus B \notin \mathcal{P}_e$. Evidently $A = \bigcup_i \sigma_i$ has the desired properties; A is recursive in $B^{(n-1)} \oplus 0^{(n+1)}$, but since B is n-random, by Theorem III.2.1 we have $A \leq_T B \oplus 0^{(n+1)}$. □

Theorem III.4.3 *For each $n \geq 1$ there exists a weakly n-random set $A \oplus B$ such that A is contained in a Π^B_n -nullset, i.e., A is not weakly n-random relative to B.*

Proof. By Theorem III.2.2 there is an n-random set B such that $B^{(n-1)} \equiv_T 0^{(n+1)}$. Since then $B \oplus 0^{(n+1)} \leq_T B^{(n-1)}$, Lemma III.4.2 produces a set $A \leq_T B^{(n-1)}$ with $A \oplus B$ weakly n-random. But $A \leq_T B^{(n-1)}$ implies that the singleton set $\{A\}$ is a Π^B_n -nullset. □

The $n = 1$ case of the above theorem produces sets $A \leq_T B$ such that $A \oplus B$ is weakly 1-random, in contrast to the fact that $A \oplus B$ is never 1-random when $A \leq_T B$.

We will see in the proof of Theorem IV.2.4 that using results of Kučera it can be shown that there are 1-random sets $A \oplus B$ such that A and B have a common nonzero predecessor C. Note that A, B must also be 1-random *relative* to C, for if, e.g., A were Σ^C_1 -approximable, then A would also be Σ^B_1 - approximable, contradicting the fact that A and B have to be relatively 1-random. On the other hand, for a 1-random set C no $B \geq_T C$ can be C-1-random; otherwise by Corollary III.3.11 $B \oplus C$ would itself be 1-random, contradicting Theorem III.1.4. We also saw in Corollary II.5.3 that if $C >_T 0$, no $B \geq_T C$ is C-weakly 2-random. Summarizing these facts we can say:

(i) For any $C > 0$, every $B \geq_T C$ is in a Π^C_2 -nullset, i.e., is not C-weakly 2-random.

(ii) If C is 1-random, every $B \geq_T C$ is Σ^C_1 - approximable.

(iii) There exist sets B, C, such that $B >_T C >_T 0$ and B is C-1-random. Note, however, that C cannot be 1-random.

Continuing the above train of thought we can obtain another interesting result relating comparability and relative randomness.

Theorem III.4.4 *There is a weakly* 1*-random set* C *and a set* $B \geq_T C$ *such that* B *is* C*-weakly* 1*-random.*

Proof. By Theorem III.2.2 we can choose a 1-random set $B \equiv_T 0''$. For $e \in \omega$ and $A \in 2^\omega$, let $\mathcal{P}_e^A$ denote the eth Π_1^A -class, i.e., the complement of $\mathrm{Ext}(W_e^A)$. Note that if $D \notin \mathcal{P}_e^A$, the fact is determined by finite information about A and D, as there must be some $\tau \subset D$ and $\sigma \subset A$ such that $\tau \in W_e^\sigma$; it also follows that $D \notin \mathcal{P}_e^C$ for any $C \supset \sigma$. Now for each e, σ define a class

$$\begin{aligned} \mathcal{T}_{e,\sigma} &= \{D : (\forall \tau \subset D)(\forall \sigma' \supset \sigma)[\varphi_e^{\sigma'}(\tau)\uparrow]\} \\ &= \{D : (\forall A \supset \sigma)[D \in \mathcal{P}_e^A]\}. \end{aligned}$$

Note $\mathcal{T}_{e,\sigma}$ is a Π_1^0 -class with index uniformly computable from e and σ. We next show how to construct an $A = \bigcup_i \sigma_i$ by finite approximations, meeting for all e the requirements

R_e: If $\mu(\mathcal{P}_e^A) = 0$ then $B \notin \mathcal{P}_e^A$.

At stage 0 let $\sigma_0 = \emptyset$. At stage $e + 1$ we first determine, recursively in B ($\equiv_T 0''$) whether $\mu(\mathcal{T}_{e,\sigma_e}) = 0$. If so, then $B \notin \mathcal{T}_{e,\sigma_e}$ (as B is 1-random) so there is some $\tau \subset B$ and $\sigma' \supset \sigma_e$ such that $\tau \in W_e^{\sigma'}$; we can find σ' and τ B-recursively, and let $\sigma_{e+1} = \sigma'$. Then $B \notin \mathcal{P}_e^A$ for any A extending σ_{e+1}, and R_e is satisfied.

On the other hand, suppose $\mu(\mathcal{T}_{e,\sigma_e}) > 0$. Then we let $\sigma_{e+1} = \sigma_e$; R_e is satisfied since for $A \supset \sigma_e$, $\mathcal{T}_{e,\sigma_e} \subseteq \mathcal{P}_e^A$, and hence $\mu(\mathcal{P}_e^A) > 0$ also.

Since the construction is by finite extensions, we can construct a 1-generic set $C \leq_T B$ meeting the above requirements (e.g., by forcing the jump at alternate stages). Then C is weakly 1-random and B is C-weakly 1-random. □

Chapter IV

Global Results

IV.1 Basis Theorems

In the previous chapter we examined general properties of all 1-random and n-random sets. In this section we take the opposite approach; we show that 1-random (and to a limited extent, n-random) sets exist satisfying certain *nontypical* properties. These will be useful in Section IV.2 for showing that 1-randomness is "not random enough" to guarantee that certain naturally defined, typical properties hold. The strategy is as follows: Let $\{\mathcal{U}_i\}_{i\in\omega}$ be the universal Σ^0_1 -approximation defined in Theorem II.1.7, and let $\mathcal{P}_i = 2^\omega - \mathcal{U}_i$. Each $\mathcal{P}_i$ is a Π^0_1 -class, all of whose members are 1-random. We can then apply some results known as *basis* theorems, which in general assert that every nonempty Π^0_1 -class has a member with some property P, to conclude that there are 1-random sets with property P. (The class of sets with property P is called a *basis* for Π^0_1 -classes.) Some known facts include the following.

Theorem IV.1.1 (Jockusch [11]) *Every nonempty Π^0_1 -class contains a member of r.e. degree.*

Theorem IV.1.1 can be relativized to show that there are n-random sets of Σ^0_n -degree, and hence recursive in $0^{(n)}$ (but not, however, above $0^{(n-1)}$; see the proof of Theorem II.5.4). It turns out that $0'$ is the *only* r.e. degree containing a 1-random set; this follows from Arslanov's completeness criterion and the fact that a degree containing a 1-random set also contains a fixed-point-free function; see Kučera [17, 18]. There are also 1-random degrees strictly below $0'$, as the next theorem shows.

Theorem IV.1.2 (Jockusch and Soare [12]) *Every nonempty Π^0_1 -class contains a member of low degree, i.e., an $A >_T 0$ such that $A' \equiv_T 0'$.*

The following definition will be needed several times.

Definition IV.1.3 *A set A is* hyperimmune *if for every recursive sequence of disjoint finite sets $F_0, F_1, \ldots$, there is some set F_i in the sequence such that $A \cap F_i = \emptyset$. A degree is hyperimmune if it contains a hyperimmune set. A degree is* hyperimmune-free *if it contains no hyperimmune sets.*

We will also need the characterization provided by the following theorem.

Theorem IV.1.4 (Miller and Martin [24]) *A degree* **a** *contains a hyperimmune set iff there is a function f recursive in* **a** *which is not dominated by any recursive function.*

The next result implies, then, that there are 1-random sets of hyperimmune-free degree. This contrasts with a result due to Martin (Theorem IV.2.4) that almost every degree is hyperimmune. This is our first natural example of a property of random degrees for which 1-randomness is not sufficient.

Theorem IV.1.5 (Jockusch and Soare [12]) *Every nonempty Π^0_1 -class contains a member of hyperimmune-free degree.*

The last result of this section is a new basis theorem which will be used, as in the example above, to show that 1-randomness is not sufficient to guarantee that certain properties hold (cf. Theorem IV.2.4). Recall that a set A is said to be *relatively r.e.* if for some set $B <_T A$, A is r.e. in B.

Theorem IV.1.6 *Every nonempty Π^0_1 -class contains a member A such that no set $B \leq_T A$ is relatively r.e.*

Proof. Let $\mathcal{T} \subseteq 2^\omega$ be a Π^0_1 -class, and let T_0 be a recursive tree with $[T_0] = \mathcal{T}$ ($[T]$ denotes the set of infinite paths through T). We construct a set A in $\mathcal{T}$ by forcing on Π^0_1 -classes, i.e., we construct a sequence of recursive trees $T_0 \supseteq T_1 \supseteq \cdots$ and let $A \in \bigcap_i [T_i]$. A will satisfy for all e, i, j the requirements

$R_{\langle e,i,j\rangle}$: If φ^A_e and φ^A_j are total and if $W_i^{\varphi^A_e} = \varphi^A_j$, then $\varphi^A_j \leq_T \varphi^A_e$.

That is, if $C \leq_T A$ and $B \leq_T A$ and B is r.e. in C, then $B \leq_T C$; thus no $B \leq_T A$ can be r.e. in any strictly smaller degree.

We begin at stage 0 with T_0. At stage $s+1$ assume we have a tree T_s such that for all $\langle e,i,j\rangle < s$, every A on T_s satisfies $R_{\langle e,i,j\rangle}$. Let $s = \langle e,i,j\rangle$.

Step 1 We first determine whether there is some x and an infinite path A on T_s such that $\varphi^A_e(x)$ is undefined. That is, we determine (using a $0''$ oracle) whether

$$(\exists x)(\forall n)(\exists \sigma \in T_s)[|\sigma| = n \ \&\ \varphi^\sigma_e(x)\uparrow]. \tag{IV.1}$$

If so, fix a witness x and let

$$T_{s+1} = T_s \cap \{\sigma : \varphi_e^\sigma(x)\uparrow\}.$$

Then T_{s+1} is infinite and has an infinite path by König's Lemma, and $R_{\langle e,i,j\rangle}$ is vacuously satisfied for all A on T_{s+1}, so we can proceed to stage $s+2$. If (IV.1) fails, then φ_e^A is total for all A on T_s; we go on to step 2.

Step 2 We check whether

$$(\exists x)(\forall n)(\exists \sigma \in T_s)[|\sigma| = n \;\&\; \varphi_j^\sigma(x)\uparrow] \tag{IV.2}$$

and proceed as in Step 1, with φ_j^A in place of φ_e^A.

Step 3 At this point we know that φ_e^A and φ_j^A are total for all A on T_s. We now try to make $W_i^{\varphi_e^A}$ different from φ_j^A. We first determine whether there is some x and some $\sigma \in T_s$ such that $\varphi_j^\sigma(x)\downarrow= 1$, but $\varphi_i^{\varphi_e^A}(x)\uparrow$ for some A on T_s extending σ. Formally we ask whether

$$\begin{aligned}(\exists x)(\exists \sigma \in T_s)\ \ &[\varphi_j^\sigma(x)\downarrow= 1 \;\&\\ &(\forall n)(\exists \tau \in T_s)[|\tau| = n \;\&\; \tau \supset \sigma \;\&\; \varphi_i^{\varphi_e^\tau}(x)\uparrow]].\end{aligned} \tag{IV.3}$$

If so, choose witnesses x and σ, and let

$$T_{s+1} = T_s \cap \{\tau \supset \sigma : \varphi_i^{\varphi_e^\tau}(x)\uparrow\}.$$

Then proceed to stage $s+1$; $R_{\langle e,i,j\rangle}$ is satisfied since for any A on T_{s+1}, $\varphi_j^A(x) = 1$ but $x \notin W_i^{\varphi_e^A}$. Otherwise go on to step 4.

Step 4 We determine whether there are x, σ such that $\varphi_j^\sigma(x)\downarrow= 0$ and $\varphi_i^{\varphi_e^A}(x)\downarrow$ for some A on T_s extending σ. That is, we ask whether

$$\begin{aligned}(\exists x)(\exists \sigma \in T_s)\ \ &[\varphi_j^\sigma(x)\downarrow= 0 \;\&\; \varphi_i^{\varphi_e^\sigma}(x)\downarrow\ \&\\ &(\forall n \geq |\sigma|)(\exists \tau \in T_s)[|\tau| = n \;\&\; \tau \supset \sigma]]\end{aligned} \tag{IV.4}$$

If so, choose a witness x and a corresponding σ and let T_{s+1} be the subtree of T_s above σ. $R_{\langle e,i,j\rangle}$ is satisfied since for any A on T_{s+1}, A extends σ, so $\varphi_j^A(x) = 0$ but $x \in W_i^{\varphi_e^A}$. If (IV.4) fails let $T_{s+1} = T_s$; this completes the construction, and we argue below that then $R_{\langle e,i,j\rangle}$ is satisfied because $\varphi_j^A \leq_T \varphi_e^A$ for any A on T_s.

Claim IV.1.7 *If conditions (IV.1), (IV.2), (IV.3), and (IV.4) all fail, then $\varphi_j^A \leq_T \varphi_e^A$ for any A on T_s.*

Proof of Claim. We know that for all A on T_s, φ_j^A and φ_e^A are total and and that for all x and σ,

$$\varphi_j^\sigma(x)\downarrow = 1 \;\Rightarrow\; \varphi_i^{\varphi_e^A}(x)\downarrow \text{ for all } A \text{ on } T_s \text{ extending } \sigma\text{, and} \qquad \text{(IV.5)}$$

$$\varphi_j^\sigma(x)\downarrow = 0 \;\Rightarrow\; \varphi_i^{\varphi_e^A}(x)\uparrow \text{ for all } A \text{ on } T_s \text{ extending } \sigma. \qquad \text{(IV.6)}$$

Suppose D is on T_s, and we are given φ_e^D and wish to compute $\varphi_j^D(x)$ for some x. Since φ_j^A is total for all A on T_s, and T_s is recursive, we can effectively find a cover $S = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ of $[T_s]$ such that every A on T_s extends some $\sigma \in S$ and $\varphi_j^\sigma(x)\downarrow$ for each $\sigma \in S$. If $\varphi_j^\sigma(x) = 0$ for all $\sigma \in S$, we know immediately that $\varphi_j^D(x) = 0$; otherwise let $S' = \{\sigma'_1, \ldots, \sigma'_{n'}\} \subseteq S$ consist of those $\sigma \in S$ for which $\varphi_j^\sigma(x) = 1$. Now by (IV.5), for each $\sigma' \in S'$, $\varphi_i^{\varphi_e^A}(x)\downarrow$ for all A on T_s extending σ', so for each $\sigma'_k \in S'$ we can find a cover $U_k = \{\tau_{k1}, \tau_{k2}, \ldots, \tau_{km_k}\}$ of $[T_s] \bigcap \mathrm{Ext}(\sigma'_k)$ such that $\varphi_i^{\varphi_e^\tau}(x)\downarrow$ for each $\tau \in U_k$, $1 \le k \le n'$. We now look for a string τ in one of the covers U_k such that

$$\varphi_e^\tau(y) = \varphi_e^D(y) \text{ for all } y \text{ in the domain of } \varphi_e^\tau.$$

Suppose we find such a τ; then since $\varphi_i^{\varphi_e^\tau}(x)\downarrow$, it must be the case that $\varphi_i^{\varphi_e^D}(x)\downarrow$ as well. (Note, however, we are not claiming that $\tau \subset D$.) By (IV.6), D must extend one of the $\sigma'_k \in S'$, so $\varphi_j^D(x) = 1$. On the other hand if no such τ is found, then no $\tau \in U_k$, $1 \le k \le n'$, can be an initial segment of D, and hence no $\sigma' \in S'$ can be an initial segment of D. Since S is a cover of $[T_s]$, D must extend some $\sigma \in S$ with $\varphi_j^\sigma(x) = 0$, i.e., $\varphi_j^D(x) = 0$. Since the procedure given above is uniform and recursive in φ_e^D, we can conclude that $\varphi_j^D \le_T \varphi_e^D$. $\square$

IV.2 Properties of Almost All Degrees

In this section we attempt to systematically analyze some properties known to hold for almost every degree and determine how much randomness is necessary for each property to hold. This is essentially what we accomplished in proving Theorem III.2.1, when we showed that $A^{(n-1)} \equiv_T A \oplus 0^{(n-1)}$ if A is n-random, and may fail if A is $(n-1)$-random.

An Effective zero-one Law

It is known by the classical zero-one law (see Oxtoby [28]) that any class which is closed under finite translations, and hence any degree-invariant class, has either measure zero or measure one. One especially useful tool in our analysis is an effective version of the zero-one law, given below. We first prove a lemma generalizing a result due to Kučera.

Lemma IV.2.1 *Let $C \in 2^\omega$ and $n \geq 1$, and let $\mathcal{T}$ be a Π^C_n - or Σ^C_n -class of positive measure. Then $\mathcal{T}$ contains a representative of every C-n-random degree; in particular, if A is any C-n-random set, then for some string σ and $B \in 2^\omega$, $A = \sigma * B$ and $B \in \mathcal{T}$.*

Proof. Let A be any C-n-random set. By Lemma II.1.4 we can assume $\mathcal{T}$ is a $\Pi^{C^{(n-1)}}_1$ -class; let $\mathcal{S} = \overline{\mathcal{T}}$. Let r be a rational number such that $\mu(\mathcal{S}) \leq r < 1$. Now suppose every "tail" of A is in $\mathcal{S}$, that is, suppose for every B such that $A = \sigma * B$, $B \in \mathcal{S}$. Then we can construct a $\Sigma^{C^{(n-1)}}_1$ -approximation $\{\mathcal{S}_i\}_{i\in\omega}$ of A: Let S be a set of strings r.e. in $C^{(n-1)}$ with $\mathcal{S} = \mathrm{Ext}(S)$; without loss of generality we can assume all the strings in S are disjoint. Define

$$\begin{aligned} S_0 &= S \\ S_{i+1} &= \{\sigma * \tau : \sigma \in S_i \;\&\; \tau \in S\} \end{aligned}$$

Clearly each set S_i is r.e. in $C^{(n-1)}$. We first show that A is in each class $\mathrm{Ext}(S_i)$. We know $A \in \mathrm{Ext}(S_0)$ by assumption. Suppose inductively that $A \in \mathrm{Ext}(S_i)$; then $\sigma \subset A$ for some $\sigma \in S_i$. It follows that $A = \sigma * B$ for some B, and since also $B \in \mathcal{S}$ by assumption, there is some string $\tau \in S$ with $\tau \subset B$. Thus $\sigma * \tau \subset A$ and $\sigma * \tau \in S_{i+1}$, so $A \in \mathrm{Ext}(S_{i+1})$. We also have, since we assume all the strings enumerated in S are disjoint,

$$\begin{aligned} \mu(\mathrm{Ext}(S_{i+1})) &\leq \mu(\mathrm{Ext}(S_i)) \cdot \mu(\mathrm{Ext}(S)) \\ &\leq [\mu(\mathrm{Ext}(S))]^{i+1} \\ &\leq r^{i+1} \end{aligned}$$

which suffices to show that A is $\Sigma^{C^{(n-1)}}_1$ -approximable, contradicting the hypothesis that A is C-n-random. Hence it must be the case that for some string σ and some B, $A = \sigma * B$ and $B \notin \mathcal{S}$, i.e., $B \in \mathcal{T}$. □

Theorem IV.2.2 (Effective zero-one law) *Every degree-invariant Σ^0_n+1 -class or Π^0_n+1 -class contains either all n-random sets or no n-random sets.*

Proof. Let $\mathcal{S}$ be a degree-invariant Σ^0_n+1 -class; by the classical zero-one law, $\mathcal{S}$ has either measure zero or measure one. If $\mathcal{S}$ has measure one, it is a union of Π^0_n -classes, at least one of which must have positive measure and hence by the lemma above contains a representative of every n-random degree; then by degree invariance, $\mathcal{S}$ contains every n-random set. On the other hand if $\mathcal{S}$ has measure 0, the complement $\overline{\mathcal{S}}$ is a Π^0_n+1 -class of measure 1; since it is an intersection of Σ^0_n -classes of measure 1 it contains every weakly n-random set, and hence in particular every n-random set. Thus $\mathcal{S}$ contains no n-random sets. For $\mathcal{S}$ a Π^0_n -class we apply the argument to $\overline{\mathcal{S}}$. □

The hypothesis of degree-invariance is clearly stronger than necessary; all that is needed is that the given class $\mathcal{S}$ is closed under finite translations and have the property: if $A \in \mathcal{S}$, then for any string σ, $\sigma * A \in \mathcal{S}$.

An easy consequence of the zero-one law is that the theory of the ordering below a random degree is unique, and the n-quantifier theory below an $(n+2)$-random degree is unique.

Corollary IV.2.3 *(i) Let A, B be ω-random sets and let $\mathbf{a} = \mathbf{deg}(A)$ and $\mathbf{b} = \mathbf{deg}(B)$. Then*

$$\mathrm{Th}(\mathcal{D}(\leq \mathbf{a})) = \mathrm{Th}(\mathcal{D}(\leq \mathbf{b})).$$

(ii) Let $n \geq 1$ and let A, B be n-random. Then

$$\exists_n \cap \mathrm{Th}(\mathcal{D}(\leq \mathbf{a})) = \exists_n \cap \mathrm{Th}(\mathcal{D}(\leq \mathbf{b})).$$

Proof.(Sketch) If ψ is a Σ_n sentence in the language $\{\leq\}$, then for any A,

$$\{A : \mathcal{D}(\leq \mathbf{deg}(A)) \models \psi\}$$

is (at worst) a Σ^0_n+3 -class. (Predecessors of A can be represented by their indices, and the relation $\varphi^A_e \leq_T \varphi^A_i$ requires three quantifiers to express.) □

Summary of Known Properties

We will simply list the known facts in the following theorem.

Theorem IV.2.4

(i) The class $\{A : A$ is not minimal $\}$ has measure one (Sacks [31]), and includes every 1-random set.

(ii) The class $\{A \oplus B :\ A, B$ form a minimal pair $\}$ has measure one (Stillwell [34]); it includes every 2-random set but not every 1-random set.

(iii) For each n, the class $\{A : A^{(n-1)} \equiv_T A \oplus 0^{(n-1)}\}$ has measure one (Sacks, Stillwell [34]); it includes every n-random set but not every $(n-1)$-random set.

(iv) The class $\{A : \mathbf{deg}(A)$ is hyperimmune $\}$ has measure one (Martin [21]); it includes every 2-random set but not every 1-random set.

(v) The class $\{A : A$ has a 1-generic predecessor $\}$ has measure one (Kurtz [15]); it includes every 2-random set but not every 1-random set.

(vi) The class $\{A : \mathbf{deg}(A)$ is relatively r.e.$\}$ has measure one (Kurtz [15]); it includes every 2-random set but not every 1-random set.

Proof.

(i) A 1-random set is never minimal, since by Theorem III.1.4 the columns are recursively independent.

(ii) By Corollary III.3.12, every 2-random set is the join of a minimal pair. On the other hand, Kučera ([16]) has shown that if A and B are 1-random sets $<_T 0'$, then there is a nonzero r.e. set C below A and B. By the Low Basis Theorem (Theorem IV.1.2) there is a 1-random set $A \oplus B <_T 0'$, so A and B are not a minimal pair.

(iii) This is Theorems III.2.1 and III.2.3.

(iv) It is almost immediate from Martin's proof that the class includes every 2-random set. The proof constructs a functional Φ such that the class

$$\{A : \Phi(A) \text{ is total and is not dominated by any recursive function}\}$$

has positive measure (see Theorem IV.1.4). It is a consequence of the construction that whenever $\Phi(A)$ is total, it isn't dominated by any recursive function, and that the Π^0_2 -class $\{A : \Phi(A) \text{ is total}\}$ has positive measure; by Lemma IV.2.1 it contains a representative of every 2-random degree. On the other hand, by Theorem IV.1.5 there are 1-random sets of hyperimmune-free degree.

(v) Kurtz' proof constructs a functional Φ such that the class

$$\{A : \Phi(A) \text{ is total and is 1-generic}\}$$

has positive measure. We can show in addition:

Claim IV.2.5 *Let Φ be the functional constructed in the proof of (v). If $\Phi(A)$ is total and A is weakly 2-random, then $\Phi(A)$ is 1-generic.*

The proof can be found in the appendix. Let B be any 2-random set. Since $\{A : \Phi(A) \text{ is total}\}$ is a Π^0_2 -class with positive measure, by Lemma IV.2.1 it contains a 2-random set C with $C \equiv_T B$. By the claim above, $\Phi(C)$ is 1-generic, and is clearly a predecessor of B as well.

On the other hand, by Theorem IV.1.6 there are 1-random sets with no relatively r.e. predecessor, and hence with no 1-generic predecessor, since a 1-generic set is always relatively r.e. (see [19, p. 81]).

(vi) The original proof in [15] constructs a functional Ξ such that if $\Xi(A)$ is total, then A is r.e. in $\Xi(A)$, and such that

$$\{A : \ \Xi(A) \text{ is total and } \Xi(A) <_T A \ \}$$

has positive measure. With a slight modification of the construction we can also show:

Claim IV.2.6 *There is a functional* Ξ *such that for any* A*, if* $\Xi(A)$ *is total then* A *is r.e. in* $\Xi(A)$*,* Ξ *is total for a class of positive measure, and for any weakly* 2*-random set* A*, if* $\Xi(A)$ *is total then* $\Xi(A) <_T A$*.*

The proof is in the appendix. Then as in (v), since $\{A : \ \Xi(A) \text{ is total } \}$ is a Π^0_2 -class, if B is any 2-random set, there is a 2-random set $C \equiv_T B$ such that $\Xi(C)$ is total. Hence C is r.e. in $\Xi(C)$ and $\Xi(C) <_T C$, and so $\mathbf{deg}(B)$ is relatively r.e. However, by Theorem IV.1.6 there are 1-random degrees which are not relatively r.e. □

We mention two other significant measure-theoretic results which have not yet yeilded to the type of analysis used in Theorem IV.2.4: Paris ([29]) showed that the class

$$\{A : \ A \text{ has no minimal predecessor}\}$$

has measure one, and Kurtz ([15]) used a similar argument to show that the class

$$\{A : \ \text{The 1-generic degrees are downward dense below } A\}$$

has measure one.

IV.3 Degree Invariance

All the definitions and results of the previous sections have been given in terms of Lebesgue measure, i.e., the measure $\{\frac{1}{2}, \frac{1}{2}\}^\omega$ on 2^ω. This is a natural choice for several reasons. Our intuitive view of randomness corresponds well with a uniform distribution on $[0, 1]$; also, the restriction to Lebesgue measure allows the proofs to be presented in the most transparent way possible. Nonetheless, it is worth asking how much generality is lost by restricting our attention to this one measure. The results in this section show that as far as properties of *degrees* are concerned, the answer is "not much". We will show that if $n \geq 2$, the class of n-random degrees is the same regardless of what computable measure is used to define randomness, as long as the measure is *nontrivial* as defined below; moreover the same holds for the 1-random degrees if the measure is also *nonatomic*. This fact will also enable us to use measures other than Lebesgue measure to obtain results on random degrees when it is convenient to do so; an example is the proof of Theorem IV.3.16 below. Throughout this section, the symbol μ may denote an arbitrary measure; we reserve the symbol λ for Lebesgue measure. We will say a real number $a \in [0, 1]$ is *computable* if its usual binary representation is a recursive sequence. We also need to define a computable measure; the definition below is from [36].

Definition IV.3.1 *A measure* $\mu : \mathcal{P}(2^\omega) \longrightarrow [0, 1]$ *is* computable *if the measures of basic intervals can be recursively approximated in a uniform way, i.e., there is a recursive function* $\hat{\mu} : 2^{<\omega} \times \omega \longrightarrow Q$ *such that for any* $\sigma \in 2^\omega$ *and* $i \in \omega$*,*

$$|\hat{\mu}(\sigma, i) - \mu(\text{Ext}(\sigma))| \leq 2^{-i}.$$

A measure μ is atomic *if for some sequence $A \in 2^\omega$, $\mu(\mathrm{Ext}(A\restriction n))$ is bounded away from zero, i.e., $\mu(\{A\}) > 0$. Otherwise μ is* nonatomic *. A sequence A for which $\mu(\{A\}) > 0$ is called an* atom*. A measure μ is* trivial *if all the measure is concentrated on the atoms, i.e.,*

$$\sum_{A \in \mathcal{A}} \mu(\{A\}) = 1,$$

where $\mathcal{A} = \{A : \mu(\{A\}) > 0\}$. Otherwise μ is nontrivial.

Up to this point we have harmlessly identified 2^ω with $[0,1]$, but we should be careful to note that we regard a measure μ as defined on 2^ω, and that here $\mathrm{Ext}(\sigma)$ denotes a subset of 2^ω rather than $[0,1]$, since the correspondence between 2^ω and $[0,1]$ is subject to change when we start looking at different measures.

Representations of Real Numbers

Associated with any measure is a "representation scheme" for real numbers. That is, given a sequence $A \in 2^\omega$, our interpretation of A as a real number depends on the measure of the basic intervals $\sigma \subset A$.

Example. Suppose a sequence A has $01 \subset A$. In the usual representation, the fact that the first digit is 0 tells us that A represents a real number in $[0, \frac{1}{2}]$; this corresponds to the fact that $\lambda(\mathrm{Ext}(0)) = \frac{1}{2}$. Likewise $\lambda(\mathrm{Ext}(00)) = \lambda(\mathrm{Ext}(01)) = \frac{1}{4}$, so the the fact that the second bit is 1 indicates that the number represented by A is in the right half of $[0, \frac{1}{2}]$, or $[\frac{1}{4}, \frac{1}{2}]$. Similarly by looking at the first n bits of A we determine the number the sequence represents up to an accuracy of 2^{-n} by obtaining an interval of length 2^{-n} in which the number must lie. Now suppose we interpret A with respect to the product measure $\mu = \{\frac{2}{3}, \frac{1}{3}\}^\omega$, i.e., a distribution in which zeros are twice as likely as ones. Now since $\mu(\mathrm{Ext}(0)) = \frac{2}{3}$ and $\mu(\mathrm{Ext}(1)) = \frac{1}{3}$, the first digit 0 would indicate that A represents a real in $[0, \frac{2}{3}]$. Likewise $\mu(\mathrm{Ext}(00)) = \frac{4}{9}$ and $\mu(\mathrm{Ext}(01)) = \frac{2}{9}$, so the second bit 1 means the number represented by A is in the rightmost third of $[0, \frac{2}{3}]$, i.e., in $[\frac{4}{9}, \frac{2}{3}]$. The definition below will make the idea precise.

In general a string σ, when interpreted with respect to μ, defines a subinterval of $[0,1]$ which we call $(\sigma)_\mu$. Briefly, to determine $(\sigma)_\mu$ we arrange all strings τ of length $|\sigma| = n$ in lexicographic order,

$$\tau_0 \prec \tau_1 \prec \ldots \prec \tau_{2^{n-1}},$$

and partition $[0,1]$ into 2^n subintervals so that the ith one has length $\mu(\mathrm{Ext}(\tau_i))$. In essence what we are doing is defining a correspondence between 2^ω and $[0,1]$ in a way that makes μ act like a uniform measure on $[0,1]$.

Definition IV.3.2 *Let μ be a measure on 2^ω and $\sigma \in 2^{<\omega}$. The* interval determined by σ with respect to μ, *denoted* $(\sigma)_\mu$, *is the interval* $[l(\sigma), r(\sigma)] \subseteq [0,1]$, *where*

$$\begin{aligned} l(\sigma) &= \sum_{\substack{|\tau|=|\sigma| \\ \tau \prec \sigma}} \mu(\mathrm{Ext}(\tau)) \\ \textit{and } r(\sigma) &= l(\sigma) + \mu(\mathrm{Ext}(\sigma)). \end{aligned}$$

Notice it is always the case that

$$\lambda((\sigma)_\mu) = \mu(\mathrm{Ext}(\sigma)),$$

and since it is possible that $\mu(\mathrm{Ext}(\sigma)) = 0$, $(\sigma)_\mu$ may be an interval of length 0, i.e., consisting of a unique real number.

In practice we will have to work with the approximation $\hat{\mu}$ rather than μ, so we define an approximation $(\sigma)_{\mu,i}$ of $(\sigma)_\mu$.

Definition IV.3.3 *Let μ be a a computable measure, σ a string of length k, and $i \in \omega$. Then $(\sigma)_{\mu,i}$ denotes the interval $[l(\sigma,i), r(\sigma,i)]$, where*

$$\begin{aligned} l(\sigma,i) &= \sum_{\substack{|\tau|=|\sigma| \\ \tau \prec \sigma}} \hat{\mu}(\tau, i+k+1) - 2^{-(i+1)} \\ \textit{and } r(\sigma,i) &= l(\sigma) + \hat{\mu}(\sigma, i+k+1) + 2^{-(i+1)}. \end{aligned}$$

One can verify that $(\sigma)_{\mu,i}$ contains $(\sigma)_\mu$ and differs from it in length by at most 2^{-i}. We can picture $(\sigma)_{\mu,i}$ as a pair of finite strings (i.e., a pair of dyadic rationals). Note also that the endpoints of the intervals $(\sigma)_\mu$ are computable reals since we can approximate them to any predetermined accuracy using the endpoints of the intervals $(\sigma)_{\mu,i}$.

The definitions above provide us with a concise way to define an interpretation of a sequence as a real number. Note that $\bigcap_i (A{\upharpoonright}i)_\mu$ is always nonempty, and if $\mu(\mathrm{Ext}(A{\upharpoonright}i)) \to 0$, i.e., A is not an atom, the intersection contains a unique real number.

Definition IV.3.4 *Let μ be a computable measure and $A \in 2^\omega$. Suppose that*

$$\lim_{i\to\infty} \mu(\mathrm{Ext}(A{\upharpoonright}i)) = 0.$$

Then the real number represented by A with respect to μ, *denoted by* $\mathrm{real}_\mu(A)$, *is the unique member of* $\bigcap_i (A{\upharpoonright}i)_\mu$.

Certainly $\mathrm{real}_\mu(A)$ always exists when μ is nonatomic; we will see in Lemma IV.3.7 that atoms are always recursive, so $\mathrm{real}_\mu(A)$ always exists when A is nonrecursive.

Note also that we could just as well have used the intervals $(\sigma)_{\mu,i}$ in the definition above, since

$$\bigcap_i (A\upharpoonright i)_\mu = \bigcap_i (A\upharpoonright i)_{\mu,i}.$$

On the other hand, given an (abstract) real $a \in [0,1]$, we can define a sequence A which represents a w.r.t. μ.

Definition IV.3.5 *Let μ be a computable measure and $a \in [0,1]$. Suppose for every n there is a unique string σ of length n such that $a \in (\sigma)_\mu$. Then the* sequence representing A with respect to μ, *denoted* $\mathrm{seq}_\mu(A)$, *is the unique $A \in 2^\omega$ such that*

$$\sigma \subset A \Longleftrightarrow a \in (\sigma)_\mu.$$

Note that $\mathrm{seq}_\mu(a)$ fails to exist only if a is an endpoint of some interval $(\sigma)_\mu$. In the case of the usual binary representation $\mathrm{seq}_\lambda(a)$, which is ambiguous for dyadic rationals, it will be convenient to adopt some fixed convention for the ambiguous cases, so that $\mathrm{seq}_\lambda(a)$ is always defined. Note in particular that if a is noncomputable then $\mathrm{seq}_\mu(a)$ always exists.

The next series of results shows that, modulo certain restrictions, any two representations of a given real number have the same Turing degree.

Lemma IV.3.6 *Let μ be a computable measure and $a \in [0,1]$. If $A = \mathrm{seq}_\mu(a)$ is defined, then*

(i) $\mathrm{seq}_\mu(a) \leq_T \mathrm{seq}_\lambda(a)$, *and*

(ii) if $\lim_{n\to\infty} \mu(\mathrm{Ext}(A\upharpoonright n)) = 0$, *then* $\mathrm{seq}_\lambda(a) \leq_T \mathrm{seq}_\mu(a)$.

Proof. (i) To determine $\mathrm{seq}_\mu(a)$ we need to find, for each n, a string σ_n of length n which is an initial segment of $\mathrm{seq}_\mu(a)$, i.e., such that $a \in (\sigma_n)_\mu$. Let $\sigma_0 = \emptyset$, and assume inductively that we have some $\sigma_n \subset \mathrm{seq}_\mu(a)$. We need to determine whether $a \in (\sigma_n{*}0)_\mu$ or $a \in (\sigma_n{*}1)_\mu$. For $i = 0,1$ we compute the approximations $(\sigma_n{*}i)_{\mu,k}$ for $k = 0,1,2,\ldots$, until a stage j is reached such that $a \in (\sigma_n * i)_{\mu,j}$ but $a \notin (\sigma_n * 1 - i)_{\mu,j}$. Such a stage must exist since otherwise a would have to be the common endpoint of the two intervals, and then $\mathrm{seq}_\mu(a)$ would not exist. We can recognize when stage j is reached since we have the representation $\mathrm{seq}_\lambda(a)$ as an oracle, and the endpoints of the intervals $(\sigma_n * i)_{\mu,k}$ are assumed to have finite representations. Let $\sigma_{n+1} = \sigma_n * i$ for the appropriate i.

(ii) The argument is similar to part (i). Suppose we have some initial segment $\sigma \subset \mathrm{seq}_\lambda(a)$; we need to determine whether $\sigma * 0 \subset \mathrm{seq}_\lambda(a)$ or $\sigma * 1 \subset \mathrm{seq}_\lambda(a)$. Let r be the rational number whose (standard) representation is $\sigma * 1$; we need to determine whether $a < r$ or $a > r$. The hypothesis implies that $\bigcap_i (A\upharpoonright i)_\mu$ contains a unique real number, namely a, and since $r \neq a$ we can compute the intervals $(A\restriction n)_{\mu,n}$ up to a stage j at which $r \notin (A\restriction j)_{\mu,j}$. We can then effectively determine whether $a > r$ by examining the endpoints of the interval $(A\restriction j)_{\mu,j}$. □

Lemma IV.3.7 *Let $A \in 2^\omega$ be nonrecursive and μ a computable measure. Then*

$$\lim_{n\to\infty} \mu(\mathrm{Ext}(A\restriction n)) = 0. \tag{IV.7}$$

In particular, no nonrecursive set is an atom of μ.

Proof. Suppose (IV.7) does not hold; then the interval

$$I = \bigcap_n (A\restriction n)_\mu$$

has positive measure. Then for any $b \in I$, if $\mathrm{seq}_\mu(b)$ is defined then $\mathrm{seq}_\mu(b) = A$, so

$$\lambda\{b \in [0,1] : \mathrm{seq}_\mu(b) = A\} = \lambda(I) > 0$$

since the computable reals $b \in I$ only occupy measure 0. By Lemma IV.3.6, from the standard representation of any noncomputable $B \in I$ we can effectively compute A, so we have

$$\lambda\{B \in 2^\omega : A \leq_T B\} > 0.$$

Then by Sacks' majority vote argument (Theorem II.5.2), A is recursive, contrary to hypothesis. □

Note that if a is computable, then $\mathrm{seq}_\lambda(a)$ and $\mathrm{seq}_\mu(a)$ are both recursive. If a is noncomputable, it is still possible that $A = \mathrm{seq}_\mu(a)$ is recursive if A is an atom of μ. However, if μ is nonatomic, then $\mu(\mathrm{Ext}(A\restriction n)) \to 0$, so $\mathrm{seq}_\lambda(a) \leq_T \mathrm{seq}\mu(a)$ by Lemma IV.3.6(ii), and hence $\mathrm{seq}_\mu(a)$ is not recursive. The results are summarized in Theorem IV.3.8.

Theorem IV.3.8 *Let μ be a computable measure and $a \in [0,1]$. Suppose that $\mathrm{seq}_\mu(a)$ is defined. Then:*

- *(i)* $\mathrm{seq}_\mu(a) \leq_T \mathrm{seq}_\lambda(a)$.
- *(ii)* $\mathrm{seq}_\lambda(a) \leq_T \mathrm{seq}_\mu(a)$ *if* $\mathrm{seq}_\mu(a)$ *is nonrecursive or if a is computable.*
- *(iii) If μ is nonatomic, then* $\mathrm{seq}_\mu(a) \equiv_T \mathrm{seq}_\lambda(a)$ *whenever* $\mathrm{seq}_\mu(a)$ *is defined.*

There are really two significant ways in which the translation between real numbers and sequences may become confused. One is that we may have $\mu(\mathrm{Ext}(\sigma)) = 0$ for some string σ, in which case the interval $(\sigma)_\mu$ has zero length and contains a single computable real number a. In this case $\mathrm{real}_\mu(B) = a$ for every $B \in \mathrm{Ext}(\sigma)$, though $\mathrm{seq}_\mu(a)$ is undefined. On the other hand, if μ is atomic, $\mu(\{B\}) > 0$ for some $B \in 2^\omega$, in which case B must be recursive (by Lemma IV.3.7) and the interval $\bigcap_i (B{\restriction}i)_\mu$ has positive length. In this case $\mathrm{seq}_\mu(a) = B$ for every $a \in \bigcap_i (B{\restriction}i)_\mu$, but $\mathrm{real}_\mu(B)$ is undefined.

The next lemma provides, in some sense, a converse to both these observations. It will be helpful to first extend the definition of $(\cdot)_\mu$ in two ways.

Definition IV.3.9 *Let μ be a computable measure.*

(i) For any $A \in 2^\omega$, let $(A)_\mu = \bigcap_i (A{\restriction}i)_\mu$.

(ii) For any subinterval I of 2^ω, let $(I)_\mu = \bigcup\{(\sigma)_\mu : \mathrm{Ext}(\sigma) \subseteq I\}$.

Lemma IV.3.10 *Let μ be a computable measure.*

(i) Suppose a is a noncomputable real number and $A = \mathrm{seq}_\mu(a)$ is recursive. Then $\mu(\{A\}) > 0$.

(ii) Suppose A is nonrecursive and $a = \mathrm{real}_\mu(A)$ is computable. Then there is an interval $I \subseteq 2^\omega$, $\lambda(I) > 0$, such that $A \in I$, $\mu(I) = 0$, and $(I)_\mu = \{a\}$.

Proof. (i) If $\mu(\{A\}) = 0$, then $\mathrm{seq}_\lambda(a) \leq_T \mathrm{seq}_\mu(a)$ by Lemma IV.3.6, contradicting the fact that a is noncomputable.

(ii) If $\mathrm{seq}_\mu(a)$ exists, it must be equal to A; then by Theorem IV.3.8, $\mathrm{seq}_\lambda(a) \equiv_T \mathrm{seq}_\mu(a) = A >_T 0$, contradicting the assumption that a is computable. Thus $\mathrm{seq}_\mu(a)$ does not exist, so for some n there are strings τ_0, τ_1 of length n such that a is in both $(\tau_0)_\mu$ and $(\tau_1)_\mu$, i.e., a is the boundary between the two intervals. By choosing n to be minimal we can assume that $\tau_0 = \tau * 0$ and $\tau_1 = \tau * 1$ for some $\tau \subset A$. Certainly either $\tau_0 \subset A$ or $\tau_1 \subset A$; we will give the argument only for the case $\tau_0 \subset A$ as the other case is symmetric.

Note that if ρ is any string with $\tau_0 \subset \rho \subset A$, $(\rho)_\mu$ is an interval of the form $[b, a]$, i.e., an interval whose right endpoint is a. If ρ is a string extending τ_0 such that A lexicographically precedes ρ, then the interval $(\rho)_\mu$ consists of the single point $\{a\}$. Let $B = \tau_0 * 1^\omega$ and let I consist of all $C \in 2^\omega$ which lie between A and B, inclusive. (Note that $A \neq B$ since A is nonrecursive.) Since $\lim_i \mu(A{\restriction}i) = 0$ and $a = \mathrm{real}_\mu(A)$, it follows that $\mu(I) = 0$ and $(I)_\mu = \{a\}$. $\square$

We can also prove the following "duality" principle which will be needed later.

Theorem IV.3.11 *Let $\mathcal{C} \subseteq 2^\omega$ be a degree-invariant class and μ a computable measure. The following are equivalent:*

(i) For every noncomputable $a \in [0,1]$ *with* $\mathrm{seq}_\lambda(a) \in \mathcal{C}$, $\mathrm{seq}_\mu(a)$ *is recursive.*

(ii) For every nonrecursive $A \in \mathcal{C}$, $\mathrm{real}_\mu(A)$ *is computable.*

Proof. (i) $\Rightarrow$ (ii): Let $A \in \mathcal{C}$ be nonrecursive and $a = \mathrm{real}_\mu(A)$. If a is noncomputable, then $\mathrm{seq}_\mu(a)$ exists and must be equal to A; hence by Theorem IV.3.8, $A = \mathrm{seq}_\mu(a) \equiv_T \mathrm{seq}_\lambda(a)$. This means that $\mathrm{seq}_\lambda(a) \in \mathcal{C}$ since $\mathcal{C}$ is degree-invariant, so $\mathrm{seq}_\mu(a)$ is recursive by (i), a contradiction. Therefore a must be computable.

(ii) $\Rightarrow$ (i): Let a be noncomputable with $\mathrm{seq}_\lambda(a) \in \mathcal{C}$, and let $A = \mathrm{seq}_\mu(a)$. Suppose A is nonrecursive. Then $A = \mathrm{seq}_\mu(a) \equiv_T \mathrm{seq}_\lambda(a)$ by Theorem IV.3.8, so $A \in \mathcal{C}$. Hence $\mathrm{real}_\mu(A)$ is computable by hypothesis; but $\mathrm{real}_\mu(A) = \mathrm{real}_\mu(\mathrm{seq}_\mu(a)) = a$, contradicting the fact that a is nonrecursive. $\square$

Using Lemma IV.3.10 we can also express the "duality" principle in the following form. The result is somewhat surprising since, for example, given an atomic measure μ there is no reason to suppose that $\mu(I) = 0$ for any interval I of positive length. The theorem asserts that if the image of the atoms of μ under the mapping

$$A \mapsto (A)_\mu$$

contains all the reals of a given degree $\mathbf{d}$, then every sequence of degree $\mathbf{d}$ lies in some interval of positive length to which μ assigns measure zero; moreover, the converse holds as well.

Theorem IV.3.12 *Let* $\mathcal{C} \subseteq 2^\omega$ *be a degree-invariant class and* μ *a computable measure. Then the following are equivalent.*

(i) For every noncomputable a *with* $\mathrm{seq}_\lambda(a) \in \mathcal{C}$, *there is an atom* $B \in 2^\omega$ *such that* $\mu(\{B\}) > 0$ *and* $a \in (B)_\mu$.

(ii) For every nonrecursive $A \in \mathcal{C}$, *there is some interval* $I \subseteq 2^\omega$, $\lambda(I) > 0$, *such that* $A \in I$ *and* $\mu(I) = 0$; *in fact* $(I)_\mu = \{\mathrm{real}_\mu(A)\}$.

Randomness with respect to μ

Definition II.1.2 can be interpreted verbatim for any measure μ.

Definition IV.3.13 *Let* μ *be a computable measure and let* $A, C \in 2^\omega$. A *is* C-1-random with respect to μ *if for every recursive sequence of* Σ_1^C *-classes* $\{\mathcal{S}_i\}_{i\in\omega}$ *with* $\mu(\mathcal{S}_i) \leq 2^{-i}$, $A \notin \bigcap_i \mathcal{S}_i$.

In our work so far we have regarded randomness as a property of *sequences* $A \in 2^\omega$ rather than of reals in $[0,1]$, so that whether a real a is "random" depends on its representation. The result below implies that this dependence is misleading; in a strong sense, randomness can be regarded as an invariant of a real number a.

Theorem IV.3.14 *Let $C \in 2^\omega$, let a be a nonrecursive real number and let μ be a computable measure.*

(i) If $\mathrm{seq}_\lambda(a)$ is C-1-random w.r.t. λ, then $\mathrm{seq}_\mu(a)$ is C-1-random w.r.t. μ.

(ii) If $\mathrm{seq}_\mu(a)$ is nonrecursive and is C-1-random with respect to μ, then $\mathrm{seq}_\lambda(a)$ is C-1-random w.r.t. λ.

Proof. (i) Suppose $\{\mathrm{Ext}(U_i)\}_{i\in\omega}$ is a Σ_1^C - approximation of $\mathrm{seq}_\mu(a)$ with respect to μ. Fix $i \in \omega$; we describe a procedure for enumerating, relative to C, a set of strings S_i. If σ is the kth string enumerated in U_i, we compute the endpoints p, q of the interval

$$[p, q] = (\sigma)_{\mu,k+i+1}.$$

Since p and q are dyadic rationals we can find a finite set of strings $\{\tau_0, \ldots, \tau_n\}$ which exactly cover $[p, q]$; or more precisely, such that

$$[p, q] = \bigcup_{0\le j\le n} (\tau_j)_\lambda.$$

We enumerate $\tau_0, \ldots, \tau_n$ into S_i. Certainly if $\mathrm{seq}_\mu(a)$ extends σ, then $a \in (\sigma)_\mu \subseteq [p, q]$, so $\mathrm{seq}_\lambda(a) \in \mathrm{Ext}(S_i)$. By a standard argument $\lambda(\mathrm{Ext}(S_i)) \le 2 \cdot \mu(\mathrm{Ext}(U_i))$, so $\{\mathrm{Ext}(S_i)\}_{i\in\omega}$ is a Σ_1^C -approximation of $\mathrm{seq}_\lambda(a)$ with respect to λ.

(ii) This part is slightly more complicated than (i). Suppose $\{\mathrm{Ext}(U_i)\}_{i\in\omega}$ is a Σ_1^C -approximation of $\mathrm{seq}_\lambda(a)$ with respect to λ. Fix $i \in \omega$; we describe a uniform procedure for obtaining a set of strings S_i which is r.e. in C. We assume, without loss of generality, that U_i is disjoint. Let σ be the kth string enumerated in U_i, and let p, q be the (rational) endpoints of $(\sigma)_\lambda$. Let $\epsilon = 2^{-(i+k+2)}$, and let

$$I_k = [p - \epsilon, q + \epsilon].$$

Then define

$$\begin{aligned} S_{i,k} &= \{\tau : (\exists n)[(\tau)_{\mu,n} \subseteq I_k]\} \\ \text{and } S_i &= \bigcup_k S_{i,k}. \end{aligned}$$

Clearly $S_{i,k}$ is r.e. given $\sigma \in U_i$, and so S_i is r.e. in C uniformly. Note also that since $\mu(\mathrm{Ext}(\tau)) = \lambda((\tau)_\mu)$, we have

$$\mu(\mathrm{Ext}(S_{i,k})) \le \lambda(I_k)$$

and so

$$\mu(\mathrm{Ext}(S_i)) \le \sum_k \lambda(I_k) \le 2 \cdot 2^{-i}.$$

Hence $\{\mathrm{Ext}(S_i)\}_{i\in\omega}$ is a Σ_1^C - approximation with respect to μ, so we need only show that $\mathrm{seq}_\mu(a) \in \mathrm{Ext}(S_i)$.

The difficulty is that if μ is atomic, it may not be possible to cover $[p, q]$ with intervals of the form $(\tau)_\mu \subseteq I_k$. If μ assigns positive measure to some singleton $\{B\}$, it may be that for every $\tau \subset B$, $(\tau)_\mu$ includes points both inside and outside of I_k; these strings τ will not be enumerated into S_i. However, under the hypothesis that $\mathrm{seq}_\mu(a)$ is nonrecursive, there is some interval $(\tau)_\mu \subseteq I_k$ containing a, so that τ is enumerated in S_i and hence $\mathrm{seq}_\mu(a) \in \mathrm{Ext}(S_i)$. To see this, let $A = \mathrm{seq}_\mu(a)$; note that $a \neq p, q$, and let $\delta = \min\{|a-p|, |a-q|\}$. For each n, $A{\upharpoonright}n$ is the unique string of length n with $a \in (A{\upharpoonright}n)_\mu$. Suppose that for all n, $(A{\upharpoonright}n)_\mu \not\subseteq [p, q]$. Then for all n, either p or q is in $(A{\upharpoonright}n)_\mu$ along with a, so for all n, $(A{\upharpoonright}n)_\mu$ has length greater than δ. But then by Lemma IV.3.7, A is recursive, contrary to hypothesis. This completes the proof. $\square$

While we are mainly interested in the "invariance" results of the next subsection, Theorem IV.3.14 is also useful in that it allows us to use measures other than λ whenever convenient. As an application we give a new proof (and a generalization) of a result due to Demuth that the nonrecursive *tt*-predecessors of a 1-random set are all of 1-random Turing degree. We will first need a lemma which expresses the idea of Lemma IV.3.10 in terms of randomness.

Lemma IV.3.15 *Let μ be a computable measure and $C \in 2^\omega$.*

(i) Suppose a is a noncomputable real number and $A = \mathrm{seq}_\mu(a)$ is recursive. Then A is C-1-random.

(ii) Suppose A is nonrecursive and $a = \mathrm{real}_\mu(A)$ is computable. Then A is Σ^0_1 -approximable.

Proof. (i) By Lemma IV.3.10, $\mu(\{A\}) > 0$, so there is some $\epsilon > 0$ such that for all n, $\mu(A{\upharpoonright}n) \geq \epsilon$. Then any Σ^C_1 -class containing A must have measure at least ϵ, so there is no way to construct a Σ^C_1 -approximation of A.

(ii) Just as in the proof of IV.3.10(ii), if $\mathrm{seq}_\mu(a)$ exists, it has to be equal to A, but then $\mathrm{seq}_\lambda(a) \equiv_T \mathrm{seq}_\mu(a) = A >_T 0$ by Theorem IV.3.8, contradicting the assumption that a is computable. Thus $\mathrm{seq}_\mu(a)$ does not exist, so there are strings τ_0, τ_1 such that a is in both $(\tau_0)_\mu$ and $(\tau_1)_\mu$, and we can assume that $\tau_0 = \tau * 0$ and $\tau_1 = \tau * 1$ for some $\tau \subset A$. Again we will give the argument only for the case $\tau_0 \subset A$ as the case $\tau_1 \subset A$ is symmetric. Now for any $\rho \in 2^{<\omega}$:

- If $\tau_0 \subset \rho \subset A$, then $(\rho)_\mu$ is an interval of the form $[b, a]$, i.e., whose right endpoint is a.
- If $\tau_0 \subset \rho$ and A lexicographically precedes ρ, then $(\rho)_\mu$ is an interval of length zero containing the single point a.

There are two cases. Suppose first that for some ρ such that $\tau_0 \subset \rho \subset A$, $\mu(\mathrm{Ext}(\rho)) = 0$. Then A is trivially Σ^0_1-approximable with respect to μ (by a sequence of Σ^0_1-classes $\mathcal{S}_i = \mathrm{Ext}(\rho)$). Otherwise, for every ρ such that $\tau_0 \subset \rho \subset A$, $\mu(\mathrm{Ext}(\rho)) > 0$. Hence for every such ρ, $(\rho)_\mu$ is an interval of the form $[b, a]$ with b strictly less than a, and for any string $\pi \supset \tau_0$ which precedes ρ lexicographically, $(\pi)_\mu$ has right endpoint $\leq b$. Since we know $A \neq \tau_0 * 0^\omega$, for any $i \in \omega$ there is a string $\pi \supset \tau_0$ such that π precedes A lexicographically and $(\pi)_\mu = [c, b]$, where

$$a - 2^{-i} < b < a.$$

Thus we can approximate A as follows. Given i, we find a string $\pi \supset \tau_0$ and a $k \in \omega$ such that $(\pi)_{\mu,k} = [p, q]$, where the right endpoint q satisfies

$$a - 2^{-i} < q \pm 2^{-k} < a.$$

We know then that π lexicographically precedes A. Let S_i consist of all strings $\rho \supset \tau_0$ for which π lexicographically precedes ρ. Clearly $A \in \mathrm{Ext}(S_i)$, and for any $\rho \in S_i$, $(\rho)_\mu$ is of the form $[b, d]$ with $b > a - 2^{-i}$, so $\mu(\mathrm{Ext}(S_i)) \leq 2^{-i}$. □

Demuth's theorem now follows easily from Theorem IV.3.14 and Lemma IV.3.15.

Theorem IV.3.16 *Let A be C-1-random (with respect to λ). If $B \leq_{tt} A$ and B is nonrecursive, there is a set $D \equiv_T B$ such that D is C-1-random (with respect to λ).*

Proof. $B \leq_{tt} A$ means that $B = \Phi^A$ for some *total* functional Φ (see [26, p. 269]). Consider the computable measure μ defined by

$$\mu(\mathrm{Ext}(\sigma)) = \lambda(\mathrm{Ext}\{\tau : \Phi^\tau \supset \sigma\}).$$

It is almost immediate that if B is Σ^C_1-approximable w.r.t. μ, then A is Σ^C_1-approximable w.r.t. λ, since for any string σ enumerated in an approximation of B we can recursively obtain the preimage $\{\tau : \Phi^\tau \supset \sigma\}$, the measure of which is exactly $\mu(\mathrm{Ext}(\sigma))$, and which includes an initial segment of A if $\sigma \subset B$. Thus since A is C-1-random, B is C-1-random with respect to μ. Since $B >_T 0$, $b = \mathrm{real}_\mu(B)$ is defined, and b is noncomputable by Lemma IV.3.15 so $\mathrm{seq}_\mu(b)$ exists, is equal to B, and is in particular nonrecursive. Then by Theorem IV.3.8, $\mathrm{seq}_\lambda(b) \equiv_T B$, and by Theorem IV.3.14, $\mathrm{seq}_\lambda(b)$ is C-1-random with respect to λ. □

Corollary IV.3.17 *There is a 1-random degree* $\mathbf{a}$ *such that every nonrecursive degree* $\mathbf{b} \leq_T \mathbf{a}$ *is 1-random.*

Proof. By Theorem IV.1.5 there is a 1-random set A of hyperimmune-free degree; let $\mathbf{a} = \mathbf{deg}(A)$. By a result of Martin (see Odifreddi, [26, p. 589]) $B \leq_T A \Rightarrow B \leq_{tt} A$, so the corollary follows from Theorem IV.3.16. □

Invariance properties

At this point we are in a position to give partial answers to the following questions, which ask about degree invariance of randomness in two complementary ways.

Question 1: If a set A is n-random with respect to some computable measure and μ is some other computable measure, is there always a set $B \equiv_T A$ which is n-random with respect to μ?

Question 2: If a set A is n-random with respect to some computable measure, is every set $B \equiv_T A$ also n-random with respect to some computable measure?

If we first restrict our attention to nonatomic measures, both questions can be answered straightforwardly. Regarding the first question, we have the following consequence of Theorems IV.3.8 and IV.3.14.

Corollary IV.3.18 *Let $A, C \in 2^\omega$ with $A >_T 0$. Let ν and μ be computable measures, and assume that μ is nonatomic. If A is C-1-random with respect to ν, then there is a set $B \equiv_T A$ such that B is C-1-random with respect to μ.*

Proof. Assume A is C-1-random with respect to ν; let $a = \mathrm{real}_\nu(A)$. We know a is noncomputable by Lemma IV.3.15, so $\mathrm{seq}_\nu(a)$ and $\mathrm{seq}_\mu(a)$ are defined; $\mathrm{seq}_\nu(a)$ is nonrecursive since $\mathrm{seq}_\nu(a) = A$, and $\mathrm{seq}_\mu(a)$ is nonrecursive by Lemma IV.3.10 and the fact that μ is nonatomic. Then by Theorem IV.3.8,

$$\mathrm{seq}_\nu(a) \equiv_T \mathrm{seq}_\lambda(a) \equiv_T \mathrm{seq}_\mu(a).$$

Let $B = \mathrm{seq}_\mu(a)$; by Theorem IV.3.14, B is C-1-random with respect to μ. $\square$

What the above result means is that if we admit only nonatomic computable measures, *the class of n-random degrees is independent of the measure used to define randomness.* The situation is somewhat more complicated for atomic measures. Consider the situation in Corollary IV.3.18 above; if $A >_T 0$ is C-1-random with respect to a computable ν, and $a = \mathrm{real}_\nu(A)$, then as long as $\mathrm{seq}_\mu(a)$ is nonrecursive it has the same degree as A and is C-1-random with respect to μ. But if μ is atomic, there is no guarantee that $\mathrm{seq}_\mu(a)$ is nonrecursive, nor even that $\mathrm{seq}_\mu(b)$ is nonrecursive for some other b of degree A (i.e., a b for which $\mathrm{seq}_\lambda(b) \equiv_T A$). We can show, however, that as long as A is at least 2-random with respect to ν, and assuming that μ is nontrivial, there will always be some set $B \equiv_T A$ which is 2-random with respect to μ.

Theorem IV.3.19 *Let $A, C \in 2^\omega$ and let ν, μ be nontrivial computable measures. Suppose A is C-2-random with respect to ν. Then there is a $B \equiv_T A$ which is C-2-random with respect to μ.*

Proof. We know by Lemmas IV.3.7 and IV.3.10 that if a real b is noncomputable and $B = \mathrm{seq}_\mu(b)$, then $\mu(\{B\}) > 0$ if and only if B is recursive. Now the assumption of nontriviality means that the total (Lebesgue) measure of the intervals $(B)_\mu$, with $\mu(\{B\}) > 0$, is strictly less than 1. Equivalently, this means that

$$\{a : \ \mathrm{seq}_\mu(a) \text{ exists and is recursive }\}$$

has measure less than 1, and so

$$\{a : \ \mathrm{seq}_\mu(a) \text{ exists and } (\exists\epsilon > 0)(\forall n)[\mu(\mathrm{seq}_\mu(a)\restriction n) \geq \epsilon]\}$$

has measure less than 1. Hence the class

$$\mathcal{C} = \{\mathrm{seq}_\lambda(a) : \ \mathrm{seq}_\mu(a) \text{ exists and } (\forall\epsilon > 0)(\exists n)[\mu(\mathrm{seq}_\mu(a)\restriction n) < \epsilon]\}$$

has positive measure. Let $\{\mathcal{U}_i\}_{i\in\omega}$ be the universal Σ^0_1 -approximation (Theorem II.1.7), and let $\mathcal{P}_i$ denote the complement of $\mathcal{U}_i$. Let k be large enough that $\lambda(\mathcal{P}_k) \geq 1 - \frac{1}{2} \cdot \mu(\mathcal{C})$. Then $\mathcal{P}_k \cap \mathcal{C}$ has positive measure. Now $\mathcal{P}_k$ is a Π^0_1 -class containing only 1-random sets, and if $\mathrm{seq}_\lambda(a)$ is 1-random then a is noncomputable and hence $\mathrm{seq}_\mu(a)$ exists; thus we can express $\mathcal{P}_k \cap \mathcal{C}$ as

$$\mathcal{P}_k \cap \{\mathrm{seq}_\lambda(a) : (\forall\epsilon > 0)(\exists n)[\mu(\mathrm{seq}_\mu(a)\restriction n) < \epsilon]\}$$

which is evidently a Π^0_2 -class, and hence in particular a Π^C_2 -class. By Kučera's lemma (Lemma IV.2.1) it contains a representative of every C-2-random degree; more precisely, for any C-2-random set D (w.r.t. λ) there is a C-2-random set $E \equiv_T D$ in the Π^0_2 -class $\mathcal{P}_k \cap \mathcal{C}$.

Now since A is C-2-random (i.e., C'-1-random) with respect to ν, $A >_T 0$, and λ is nonatomic, by Corollary IV.3.18 there is a set $D \equiv_T A$ which is C-2-random w.r.t λ, and hence a set $E \equiv_T A$ which is C-2-random w.r.t. λ and such that $E \in \mathcal{C}$. Let $b = \mathrm{real}_\lambda(E)$ and $B = \mathrm{seq}_\mu(b)$. Since $E = \mathrm{seq}_\lambda(b) \in \mathcal{C}$, $\mathrm{seq}_\mu(b)$ is nonrecursive, and hence $B \equiv_T E(\equiv_T A)$ by Theorem IV.3.8 and B is C-2-random with respect to μ by Theorem IV.3.14. □

Thus the 2-random degrees are the same for any nontrivial computable measure. Note that the global properties listed in Theorem IV.2.4 are of the form "every 2-random degree has property P", so the same results would be obtained having performed the analysis with respect to any nontrivial computable measure. This justifies the claim that very little generality was lost by our initial restriction to Lebesgue measure. We can also conclude:

Theorem IV.3.20 *Let ν, μ be nontrivial computable measures, and let P be an arithmetical property of degrees. Then ν-a.e. degree has property P if and only if μ-a.e. degree has property P.*

We can also show that neither Theorem IV.3.14 nor Theorem IV.3.19 can be substantially improved. In fact, it is possible to construct a nontrivial, computable measure μ such that no nonrecursive Δ_2 set A is 1-random with respect to μ. Since there are 1-random degrees below $0'$ (e.g., by Theorem IV.1.2), this shows there are sets A which are 1-random w.r.t. λ but such that no $B \equiv_T A$ is 1-random w.r.t. μ. The difficult part is contained in the following result. The proof is rather involved and is deferred to the end of the section.

Theorem IV.3.21 *There is a nontrivial, computable measure μ such that for any Δ_2 set A there is a recursive B with $\mu(\{B\}) > 0$ and $\mathrm{real}_\lambda(A) \in (B)_\mu$.*

Corollary IV.3.22 *Let μ be the measure constructed in Theorem IV.3.21; then no nonrecursive Δ_2 set is 1-random with respect to μ.*

Proof. By the "duality" principle (Theorem IV.3.12), for any nonrecursive Δ_2 set A, $\mathrm{real}_\mu(A)$ is computable. By Lemma IV.3.15, any such A is Σ^0_1 -approximable with respect to μ. $\square$

Returning now to Question 2, there is a straightforward answer, namely "no", if we look only at nonatomic measures. The idea for the following theorem was provided by Stuart Kurtz.

Theorem IV.3.23 *Let μ be a nonatomic computable measure. If A is hyperimmune, then A is Σ^0_1 -approximable with respect to μ.*

Proof. Fix $i \in \omega$. We will describe a uniform procedure for enumerating a set of strings S_i such that $\mu(\mathrm{Ext}(S_i)) \leq 2^{-i}$ and every hyperimmune set is in $\mathrm{Ext}(S_i)$. Let $\hat{\mu}$ denote the recursive approximation of μ given by Definition IV.3.1. We define a sequence of integers $k_0, k_1, \ldots$ by recursion along with a sequence of sets $S_{i,0}, S_{i,1}, \ldots$; then $S_i = \bigcup_m S_{i,m}$. Let

$$k_0 = \text{ least } k \text{ such that } \hat{\mu}(\mathrm{Ext}(0^k), i+2) \leq 2^{-(i+2)},$$

and $S_{i,0} = \{0^{k_0}\}$. Note that the error bound for $\hat{\mu}$ is chosen so that $\mu(\mathrm{Ext}(0^{k_0})) \leq 2^{-(i+1)}$. In general given $S_{i,m}$ and k_m, let

$$\begin{aligned} k^*_m &= \sum_{j=0}^{m} k_j \text{ and} \\ k_{m+1} &= \text{ least } k \text{ such that } \sum_{|\sigma| = k^*_m} \hat{\mu}(\mathrm{Ext}(\sigma * 0^k), i+m+2+k^*_m) \leq 2^{-(i+m+2)}. \end{aligned}$$

Then let

$$S_{i,m+1} = S_{i,m} \cup \{\sigma * 0^{k_{m+1}} : |\sigma| = k^*_m\}.$$

Note that k_m always exists under the assumption that μ is nonatomic. Note also that the error bound $2^{-(i+m+2+k_m^*)}$ is chosen so that the total error over all $2^{k_m^*}$ strings in the sum is at most $2^{-(i+m+2)}$, and so

$$\sum_{|\sigma|=k_m^*} \mu(\mathrm{Ext}(\sigma * 0^{k_{m+1}})) \leq 2^{-(i+m+1)}.$$

Thus

$$\mu(\mathrm{Ext}(S_i)) \leq \sum_m 2^{-(i+m+1)} = 2^{-i}.$$

Now let A be any hyperimmune set. Corresponding to each i is a recursive sequence of disjoint finite sets $F_0, F_1, \ldots$, defined by

$$\begin{aligned} F_0 &= \{x \in \omega : x \leq k_0\} \\ \text{and } F_{m+1} &= \{x \in \omega : k_m^* < x \leq k_{m+1}^*\}. \end{aligned}$$

For some m, $A \cap F_m = \emptyset$, so A extends some string of the form $\sigma * 0^{k_m}$ with $|\sigma| = k_{m-1}^*$ (or $|\sigma| = 0$ is the case $m = 0$). Hence $A \in \mathrm{Ext}(S_i)$. $\Box$

The result above is rather surprising considering the fact that almost every degree contains a hyperimmune set (Theorem IV.2.4) and suggests that randomness is not degree invariant in the sense of Question 2.

We conclude this section with the proof of Theorem IV.3.21.

Proof of Theorem IV.3.21. Let $\{K_s\}_{s\in\omega}$ be a recursive approximation of $K = \{e : \varphi_e(e)\downarrow\}$. For $e, s \in \omega$ let $\sigma_{e,s}$ be the string of length $e+2$ defined by

$$\sigma_{e,s}(x) = \begin{cases} \varphi_{e,s}^{K_s}(x) & \text{if } \varphi_{e,s}^{K_s}(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

for $x < e+2$. Note that if $\varphi_e^K = A$ is total, then there is a stage s such that $\sigma_{e,t} = A\restriction(e+2)$ for all $t \geq s$.

The measure μ will be defined in stages, in such a way that by stage s, $\mu(\mathrm{Ext}(\sigma))$ has at least been defined for all strings σ of length $\leq s$. At any point during the construction we may refer to $\mathrm{dom}(\mu)$, meaning the collection of strings σ for which $\mu(\mathrm{Ext}(\sigma))$ has been defined so far.

At each stage s we will also have a set C_s of pairs (τ, σ), called *commitments*, as well as the set $T_s = \{\tau : (\tau, \sigma) \in C_s \text{ for some } \sigma\ \}$; a $\tau \in T_s$ is said to be *committed* to the corresponding σ. At stage s we have a collection of strings $\sigma_{e,s}$, with $e < s$, each of which is believed to be an initial segment of a corresponding Δ_2 set φ_e^K. It will be the case that the μ-intervals corresponding to the committed strings *cover* each $\sigma_{e,s}$, meaning that

$$(\sigma_{e,s})_\lambda \subseteq \bigcup_{\tau \in T_s} (\tau)_\mu.$$

The idea is that if τ is committed, whenever we enlarge the definition of μ to strings extending τ (say of length $|\tau|+k$), we agree to define $\mu(\text{Ext}(\tau * 0^k)) = \mu(\text{Ext}(\tau))$ and $\mu(\text{Ext}(\tau * \rho)) = 0$ for all ρ of length k with $\rho \neq 0^k$. This means that $(\tau * 0^k)_\mu = (\tau)_\mu$, and if τ remains committed at all later stages of the construction, we will have a set $B = \tau * 0^\omega$ with $\mu(\{B\}) > 0$ (in fact $(B)_\mu = (\tau)_\mu$). Then for any A extending one of the strings $\sigma_{e,s}$, $\text{real}_\lambda(A) \in (B)_\mu$. Our efforts are mainly devoted to ensuring that if A really is a Δ_2 set, so $\varphi_e^K = A$ for some e, then at some stage s after $\sigma_{e,s}$ has stabilized to the value $A\restriction(e+2)$ there will be committed strings τ which cover $\sigma_{e,s}$ and which remain committed at all later stages.

The committed strings T_s will always be disjoint, and the corresponding μ-intervals $(\tau)_\mu$, for $\tau \in T_s$, will *exactly* cover the intervals $(\sigma_{e,s})_\lambda$, $e < s$. This means that the (Lebesgue) measure of all the intervals $(\tau)_\mu$, $\tau \in T_s$, will never exceed the total measure of the intervals $(\sigma_{e,s})_\lambda$, and hence is ultimately bounded by $\frac{1}{2}$. It will then follow that μ is nontrivial. Any committed string $\tau \in T_s$ may cover a portion of many strings $\sigma_{e,s}$, but it is "committed" to just one string $\sigma_{e,s}$, i.e., it appears in exactly one pair (τ, σ) in C_s, and it will also be true that $(\tau)_\mu \subseteq (\sigma)_\lambda$ whenever τ is committed to σ. The heuristic behind this is that τ will be committed to a string $\sigma_{e,s}$ showing the greatest promise of having stabilized, i.e., having remained unchanged for the largest number of stages.

If τ is committed to $\sigma_{e,s}$ and we find that $\sigma_{e,s+1} \neq \sigma_{e,s}$, then the pair $(\tau, \sigma_{e,s})$ is excluded from C_{s+1} and τ is excluded from T_{s+1}; we say τ is *released*. Now it may be that τ partially covered some other string $\sigma_{i,s}$ where $\sigma_{i,s} = \sigma_{i,s+1}$. We may not be able to commit τ to $\sigma_{i,s}$, since τ may be too big, i.e., $(\tau)_\mu \not\subseteq (\sigma_{i,s})_\lambda$. Although τ was committed at previous stages, so for some k we have $\mu(\text{Ext}(\tau * 0^k)) = \mu(\text{Ext}(\tau))$, we can still reapportion the measure of the corresponding interval $(\tau)_\mu$ evenly among the extensions of $\tau * 0^k$, to obtain subintervals of the form $(\tau * 0^k * \rho)_\mu \subset (\tau)_\mu$ which can be committed to $\sigma_{i,s}$.

Formally, the construction is as follows: Initially at stage 0 let $\text{dom}(\mu)$, T_0, and C_0 all be empty. The construction at stage $s+1$ takes place in three substages.

Substage 1 For each $e < s$ and each string $\sigma_{e,s}$ such that $\sigma_{e,s+1} \neq \sigma_{e,s}$, we release all strings committed to $\sigma_{e,s}$: Let

$$C'_s = \{(\tau, \sigma_{i,s}) : \ \sigma_{i,s} = \sigma_{i,s+1} \text{ and } (\tau, \sigma_{i,s}) \in C_s \ \}$$

and

$$T'_s = \{\tau : (\exists\sigma)[(\tau, \sigma) \in C'_s]\}.$$

Substage 2 We refine each uncommitted interval, and ensure that $\text{dom}(\mu)$ includes all strings of length $\leq s$: For each τ in $\text{dom}(\mu)$ of maximal length, if any initial segment τ' of τ is in T'_s, we define

$$\begin{aligned} \mu(\text{Ext}(\tau * 0)) &= \mu(\text{Ext}(\tau')) \\ \text{and } \mu(\text{Ext}(\tau * 1)) &= 0, \end{aligned}$$

and if no initial segment of τ is in T'_s we define

$$\mu(\mathrm{Ext}(\tau * 0)) = \mu(\mathrm{Ext}(\tau * 1)) = \frac{1}{2} \cdot \mu(\mathrm{Ext}(\tau)).$$

Substage 3 We make new commitments as necessary for $\sigma_{e,s+1}$, $e \leq s$. The order in which we make commitments is important; we need to give precedence to those strings which appear to have stabilized. Let

$$l(\sigma_{e,s+1}) = \min\{r : (\forall t)[r \leq t \leq s+1 \rightarrow \sigma_{e,t} = \sigma_{e,s+1}]\}.$$

We will say $\sigma_{e,s+1}$ has *higher priority* than $\sigma_{i,s+1}$ if $l(\sigma_{e,s+1}) < l(\sigma_{i,s+1})$ or $l(\sigma_{e,s+1}) = l(\sigma_{i,s+1})$ and $e < i$.

Then assume the strings $\sigma_{e,s+1}$ are ordered from highest priority to lowest; we then proceed in $s+1$ consecutive steps, always working with the string $\sigma_{e,s+1}$ of highest priority. Initially we let $C_s^0 = C'_s$ and $T_s^0 = T'_s$. Let σ be the jth string in order of priority; at step j we have sets C_s^{j-1} and T_s^{j-1} such that every $\sigma_{e,s+1}$ with higher priority than σ is covered by committed strings in T_s^{j-1}. Let

$$\mathcal{A} = (\sigma)_\lambda - \bigcup\{(\tau)_\mu : \tau \in T_s^{j-1}\},$$

that is, $\mathcal{A}$ consists of the subintervals of $(\sigma)_\lambda$ which are not yet covered by committed strings. Since all the intervals $(\sigma_{i,s+1})_\lambda$ and $(\tau)_\mu$ are dyadic, we can express $\mathcal{A}$ as the extension of a finite set of strings $R = \{\rho_1, \ldots, \rho_k\}$, i.e.,

$$\mathcal{A} = \bigcup_{\rho \in R} (\rho)_\lambda.$$

We then want to exactly cover each string $\rho \in R$ with intervals of the form $(\tau)_\mu$. For every string τ such that $(\tau)_\mu \cap (\rho)_\lambda \neq \emptyset$, and such that τ is maximal in $\mathrm{dom}(\mu)$, we do the following (note we may exclude any string τ for which $\mu(\mathrm{Ext}(\tau)) = 0$; note also that τ is not committed): If $(\tau)_\mu \subseteq (\rho)_\lambda$, then we immediately put (τ, σ) into C_s^j. If $(\tau)_\mu \not\subseteq (\rho)_\lambda$, we need to subdivide $(\tau)_\mu$: in general, having defined $\mu(\mathrm{Ext}(\tau * \pi))$, we let

$$\mu(\mathrm{Ext}(\tau * \pi * 0)) = \mu(\mathrm{Ext}(\tau * \pi * 1)) = \frac{1}{2} \cdot \mu(\mathrm{Ext}(\tau * \rho)).$$

Let m be the least integer such that $(\tau)_\mu \cap (\rho)_\lambda$ can be exactly covered by intervals of the form $(\tau * \pi)_\mu$ with $|\pi| = m$. (Since all the intervals involved are dyadic, such an m will always exist.) We define $\mu(\mathrm{Ext}(\tau * \pi))$ as described above for each π of length m, and leave μ undefined on all longer extensions of τ. Then for each π of length m such that $(\tau * \pi)_\mu \subseteq (\rho)_\lambda$, we put $(\tau * \pi, \sigma)$ into C_s^j and $\tau * \pi$ into T_s^j. After performing the procedure above for each $\rho \in R$, we conclude step j by putting all of C_s^{j-1} into C_s^j and all of T_s^{j-1} into T_s^j.

After step s, we let $C_{s+1} = C_s^s$ and $T_{s+1} = T_s^s$. This completes stage $s+1$ of the construction of μ.

It is evident that μ is a computable measure. We need to verify the following claims:

(i) μ is nontrivial.

(ii) Let A be any Δ_2 set and $a = \text{real}_\lambda(A)$. Then there is a string τ such that $\mu(\text{Ext}(\tau * 0^\omega)) > 0$ and $a \in (\tau * 0^\omega)_\mu$.

(i) Let $B \in 2^\omega$ be any set such that $\mu(\{B\}) > 0$. We claim that for some initial segment $\tau \subset B$ and some stage s, τ is committed at all stages $t \geq s$. Suppose not: then given any string $\rho \subset B$, there is a stage t at which $\rho \in \text{dom}(\mu)$ and no initial segment of ρ is committed. (By virtue of substage 2, once a string is released, it can't be committed again; only its proper extensions can.) Then by stage $t+1$, substage 2, $\mu(\text{Ext}(\rho * i))$ is defined and has measure equal to $\frac{1}{2}\mu(\text{Ext}(\rho))$, for $i = 0, 1$. Since $\rho \subset B$ was arbitrary, this shows that $\lim_n \mu(\text{Ext}(B \restriction n)) = 0$, a contradiction. So there is a $\tau \subset B$ and a least stage s such that τ is committed at all stages $t \geq s$. It follows that $B = \tau * 0^\omega$ (since otherwise $\mu(\{B\}) = 0$) and that $(B)_\mu = (\tau)_\mu$. By construction the total measure of committed strings at any stage s is bounded by

$$\sum_{e<s} \lambda(\text{Ext}(\sigma_{e,s})) = \sum_{e<s} 2^{-(e+2)} \leq \frac{1}{2}.$$

Hence $\bigcup\{(B)_\mu : \mu(\{B\}) > 0\}$ has measure at most $\frac{1}{2}$.

(ii) Let A be Δ_2 and $a = \text{real}_\lambda(A)$. Choose e such that $A = \varphi_e^K$ and fix a stage s_0 such that $\sigma_{e,t} = A \restriction (e+2)$ for all $t \geq s_0$; let $\sigma_e denote \sigma_{e,s_0}$. By construction, at stage s_0 there is some committed string τ with $a \in (\tau)_\mu$. If τ remains committed at all stages $t \geq s_0$, then $a \in (\tau * 0^\omega)_\mu$ as claimed. Note that once any string is committed to σ_e, it will remain so at all later stages. Notice that at all stages $t \geq s_0$, σ_e has priority over any string $\sigma_{i,t}$ with $i \geq s_0$, and σ_e also has priority over any string $\sigma_{i,t}$ which releases a committed string at a stage $t \geq s_0$. So suppose τ is committed to some other string σ_{j,s_0}, and at a stage $t \geq s_0$, τ is released. By the end of stage t there must be some extension τ' of τ, with $a \in (\tau')_\mu$, which is committed either to σ_e or to some $\sigma_{i,t}$, where $i \neq j$ and $i < s_0$. This can happen at most finitely many times before a stage r is reached at which some $\tau'' \supset \tau$, with $a \in (\tau'')_\mu$, either becomes committed to some $\sigma_{i,r}$ and remains committed at all later stages or else becomes committed to σ_e itself. Since τ'' is committed at all later stages, $(\tau'' * 0^\omega)_\mu = (\tau'')_\mu$, so $\mu(\{\tau'' * 0^\omega\}) > 0$ and $a \in (\tau'' * 0^\omega)_\mu$ as desired. This completes the proof of claim (ii), and hence of the theorem. $\square$

Appendix A

A.1 Proof of Claim IV.2.5

Though we are only giving a short argument concerning the nature of the functional Φ, it will be helpful to at least briefly recall the details of the construction of Φ as given in [15]. An excellent introduction to this type of argument is [15, Chapter III].

The functional Φ will be constructed recursively in stages $\Phi_0, \Phi_1, \ldots$, where at any stage s the domain of Φ_s is a finite tree $D_s \subset 2^{<\omega}$ (recall a *tree* is just a set of strings closed under initial segments), and for $\sigma \in D_s$, $\Phi_s(\sigma)$ is a string. It will always be the case that

- $\Phi_s(\sigma) \subset \Phi_{s+1}(\sigma)$, and
- if $\sigma \subset \tau$, then $\Phi_s(\sigma) \subset \Phi_s(\tau)$.

It is then consistent to let

$$\Phi(A) = \bigcup_s \Phi_s(\sigma_s),$$

where σ_s is the longest initial segment of A appearing in D_s. The maximal elements of D_s (the "leaves" of the tree) will be called the *active* strings at stage s.

The idea is that for a class of sets $A \in 2^\omega$ of positive measure, $\Phi(A)$ should be total and satisfy all the requirements

R_e: For some $\sigma \subset \Phi(A)$, either $\sigma \in W_e$ or else for all $\tau \supset \sigma$, $\tau \notin W_e$.

At each stage various strings in D_s will be marked with colors to indicate their roles in the construction; the colors red$_e$, yellow$_e$, green$_e$, and blue$_e$ may each appear in countable many "hues" $e \in \omega$. The intuitive roles of the colors are more or less as follows:

- Red: construction is stopped above a red$_e$ node ρ; that is, for all $\sigma \supset \rho$, $\Phi(\sigma)$ remains undefined as long as ρ has color red$_e$.

- Yellow: construction proceeds above a yellow_e node "with caution", as the construction may be injured by the action of requirement R_e.

- Green: construction proceeds above a green_e node without interuption from R_e.

- Blue: the blue_e strings are "policemen" which direct the placement of the other colored strings.

The general idea for satisfying a requirement R_e above a given node β (i.e., for sets A extending β) is to set aside a string $\rho \supset \beta$ representing some small fraction of the measure of $\text{Ext}(\beta)$, give ρ color red_e, and "wait" for some τ to appear in W_e which extends $\Phi_s(\rho)$ (i.e., $\Phi_s(\rho) = \Phi_t(\rho)$ for stages $t \geq s$ as long as no extension of $\Phi_s(\rho)$ appears in $W_{e,t}$, and Φ_t remains undefined on proper extensions of ρ). In general we let $\rho = \beta * 0^{e+2}$. Meanwhile, the rest of the extensions $\gamma \supset \beta$ of length $|\rho|$ receive color yellow_e, and the construction proceeds above the yellow_e strings as though R_e were already satisfied. We will have arranged that $\Phi_s(\rho) = \Phi_s(\gamma)$ for the yellow_e strings $\gamma \supset \beta$. If no extension of $\Phi_s(\rho)$ is ever enumerated in W_e, then Φ is undefined on all A extending ρ, but then R_e is satisfied for all A extending a yellow_e string $\gamma \supset \beta$, since it is the case that no extension of $\Phi(\gamma)$ is ever enumerated in W_e. On the other hand, if at some stage t a string τ extending $\Phi_t(\rho)$ is enumerated in $W_{e,s}$, we define $\Phi_{t+1}(\rho) = \tau$ and give ρ color green_e. The construction above the yellow_e strings γ is "injured" by R_e, since we can no longer be sure that R_e is satisfied for A extending γ. We erase all the colors assigned to strings extending γ and re-start our efforts to satisfy requirement R_e, working above each of the active strings in D_{t+1} which extend γ.

Formally, at stage 0 we let $\Phi(\sigma) = \emptyset$ for all strings σ of length ≤ 2; we give $\emptyset$ color blue_0, 00 receives color red_0, and 01, 10, and 11 all receive color yellow_0. The construction at stage $s+1$ takes place in three substages:

Substage 1: For each red_e string ρ in D_s, there are two cases:

Case 1: If no $\tau \supset \Phi_s(\rho)$ is enumerated in $W_{e,s+1}$, we simply let $\Phi_{s+1}(\rho) = \Phi_s(\rho)$.

Case 2: If some $\tau \supset \Phi_s(\rho)$ is enumerated in $W_{e,s+1}$, then R_e *acts* at stage $s+1$: let $\Phi_{s+1}(\rho) = \tau$, and let β be the unique blue_e predecessor of ρ. The string β loses color blue_e, and every proper extension of β loses whatever color it may have had. Then ρ receives color green_e.

Substage 2: For each non-red, active string σ in D_s: Let j be the least $i \in \omega$ such that no $\tau \subset \sigma$ has color yellow_i or green_i. Assign σ the color blue_j.

Substage 3: For each active blue_e string β in D_s, we extend the domain D_s to include the strings of length $|\beta| + e + 2$ extending β. Let $\Phi_{s+1}(\beta * \tau) = \Phi_s(\beta)$ for each string τ of length $e+2$. We assign color red_e to the string $\beta * 0^{e+2}$ and color yellow_e to the strings $\beta * \tau$ where $|\tau| = e+2$ and $\tau \neq 0^{e+2}$.

This completes the construction at stage $s+1$. We say a string σ has *final color* color$_e$ if for some stage s, σ has color$_e$ at every stage $t \geq s$. Define

$$\begin{aligned}
\mathcal{R}_e &= \{A : \text{ some } \sigma \subset A \text{ has final color yellow}_e \text{ or green}_e\}, \\
\mathcal{S}_e &= \{A : \text{ some } \sigma \subset A \text{ has final color red}_e\}, \text{ and} \\
\mathcal{S} &= \bigcup_e \mathcal{S}_e.
\end{aligned}$$

As a notational convenience let $\mathcal{R}_{-1} = 2^\omega$. The following lemmas are then proved to verify that the functional Φ constructed has the desired properties.

Lemma A.1.1 *If $A \in \bigcap_e \mathcal{R}_e$, then $\Phi(A)$ is total and $\Phi(A)$ is 1-generic.*

Proof. Similar to Lemma A.2.1, or see [15]. □

Lemma A.1.2 *(i) $\mu(\mathcal{R}_{e-1} - (\mathcal{R}_e \cup \mathcal{S}_e)) = 0$.*

(ii) $\mathcal{R}_e \subseteq \mathcal{R}_{e-1}$.

(iii) $\mu(\mathcal{S}) + \mu(\bigcap_e \mathcal{R}_e) = 1$.

Proof. Similar to Lemma A.2.2, or see [15]. □

Lemma A.1.3 *(i) $\mu(\mathcal{S}) \leq \frac{1}{2}$.*

(ii) $\mu(\bigcap_e \mathcal{R}_e) \geq \frac{1}{2}$.

Proof. Similar to Lemma A.2.3, or see [15]. □

It is now fairly easy to complete the proof of Claim IV.2.5. By Lemma A.1.2 we have

$$\mu(\mathcal{S}) + \mu(\textstyle\bigcap_e \mathcal{R}_e) = 1.$$

For any fixed e let

$$\begin{aligned}
Y_{e,i} &= \{\tau : \ \tau \text{ receives color yellow}_e \text{ at a stage } t \geq i \ \} \text{ and} \\
\mathcal{Y}_e &= \bigcap_i \mathrm{Ext}(Y_{e,i}).
\end{aligned}$$

Thus A is in $\mathcal{Y}_e$ just if infinitely often, some initial segment of A receives color yellow$_e$. Clearly $\mathcal{Y}_e \cap \mathcal{S} = \emptyset$ and $\mathcal{Y} \cap (\bigcap_e \mathcal{R}_e) = \emptyset$, so $\mathcal{Y}_e$ has measure zero. Evidently $Y_{e,i}$ is r.e., so $\mathcal{Y}_e$ is a Π^0_2 -class.

Let B be a set for which $\Phi(B)$ is total but $B \notin \bigcap_e \mathcal{R}_e$, and fix the least e for which $B \notin \mathcal{R}_e$. We will show that B is in the class $\mathcal{Y}_e$. Choose s_0 so that for each $i < e$, some initial segment of B has received final color yellow$_e$ or green$_e$ by stage

s_0; such a stage exists by the choice of e. Now suppose some initial segment τ of B receives color$_e$ at a stage $t \geq s_0$. Note that by construction, the string σ with final color green$_{e-1}$ or yellow$_{e-1}$ must be a predecessor of τ (this assumes that $e > 0$, but a similar argument applies if $e = 0$ by taking $\sigma = \emptyset$). Note also that τ can only lose color$_e$ by the action of R_e: Action by a requirement R_j removes the color from all strings extending a blue$_j$ string β; for $j > e$, no blue$_j$ string precedes τ, and for $j < e$, any blue$_j$ string preceding τ must also precede σ, so action by R_j removing color$_e$ from τ would also remove the final color from σ.

It is then easy to see that no initial segment τ of B can receive color red$_e$ after stage s_0: since $A \notin \mathcal{S}$, τ must lose color red$_e$ at some later stage, and since this only occurs by the action of R_e, τ then receives color green$_e$. Since it could only lose color green$_e$ by the action of some R_j, $j < e$, τ would then have final color green$_e$, i.e., $B \in \mathcal{R}_e$, a contradiction.

By construction, at any stage $s \geq s_0$, any non-red active string β in D_s without a green$_e$ or yellow$_e$ predecessor receives color blue$_e$, and then the extensions $\beta * \tau$, $|\tau| = e+2$, receive color green$_e$ or yellow$_e$. Since no initial segment of B receives color red$_e$, B extends one of the strings $\beta * \tau$ with color yellow$_e$. This cannot be a final color, so it follows that at infinitely many stages, some initial segment of B receives color yellow$_e$. Thus $B \in \mathcal{Y}_e$.

Since $\mathcal{Y}_e$ is a Π^0_2-nullset, if A is weakly 2-random and $\Phi(A)$ is total, then $A \in \bigcap_e \mathcal{R}_e$ and hence $\Phi(A)$ is 1-generic. □

A.2 Proof of Claim IV.2.6

Kurtz' original proof that a.e. degree is relatively r.e. produces a functional Ξ such that A is r.e. in $\Xi(A)$ whenever $\Xi(A)$ is total, and such that

$$\{A : \ \Xi(A) \text{ is total and } A \not\leq_T \Xi(A)\ \}$$

has measure $\geq \frac{1}{4}$. We need to make a slight modification to the construction. Essentially the only difference between the construction presented in [15] and the construction below is that we require $\Xi(A)$ to be undefined for any A extending a purple string. The meaning of this remark will become clear in context. Although this is a minor difference, it nonetheless changes a number of details in the verification that the construction succeeds, so we will give a complete proof below. However, although self-contained, the description of the construction will be brief; for a first reading we recommend the detailed description in [15, pp. 97–107].

As in the construction of Φ described above in the proof of Claim IV.2.5, Ξ will be constructed in stages; at any stage s the approximation Ξ_s will have domain a finite tree $D_s \subset 2^{<\omega}$. The maximal elements of D_s (the "leaves") will be called *active* strings. As before, certain strings in D_s will be marked with colors to indicate their role in the construction. The roles of red, yellow, green, and blue strings are similar to the previous construction, but the use of the color purple is new.

To ensure that A is r.e. in $\Xi(A)$, if $n \in A$ we will put a pair $\langle n, m\rangle$ in $\Xi(A)$ for some m, and if $n \notin A$ we will make sure no pair $\langle n, m\rangle$ is in $\Xi(A)$. Define a string ξ to be *acceptable* for a string σ if

$$\xi(\langle n, m\rangle) = 1 \Rightarrow \sigma(n) = 1.$$

It will always be the case during the construction that $\Xi_s(\sigma)$ is acceptable for σ.

We will also need to satisfy requirements of the form

R_e: $A \neq \{e\}^{\Xi(A)}$.

We will say that a string θ is *threatening* requirement R_e at stage s if θ extends some yellow$_e$ string ν and there is a string ξ which is acceptable for θ such that

- $|\theta| \leq s$,
- $|\xi| = s$,
- $\xi \supset \Xi_s(\nu)$,
- no initial segment of θ already has color purple$_e$, and
- $\{e\}^{\xi}(k) = \nu(k) = 0$ for some k with $|\beta| < k \leq |\nu|$, where β is the unique blue$_e$ predecessor of ν.

At stage 0 the empty string $\emptyset$ has color blue$_0$, the strings 00, 01, and 10 all have color yellow$_0$, and 11 has color red$_0$. We define $\Xi_0(\sigma) = \emptyset$ for all σ of length 2.

The construction at stage $s+1$ takes place in four substages.

Substage 1: For each $e \leq s$, any string in D_s which is threatening R_e receives color purple$_e$.

Substage 2: For each $e = 0, \dots, s$, we sequentially do the following for each blue$_e$ string β. If the density of the purple$_e$ strings extending β is at least $2^{-(e+3)}$, then we say that R_e *acts*: let ν be the lexicographically least yellow$_e$ string extending β such that the density of the purple$_e$ strings extending ν is at least $2^{-(e+3)}$. Let ρ be the unique red$_e$ string of length $|\nu|$ which extends β. Let $\{\theta_0, \dots, \theta_k\}$ be a list of the purple$_e$ strings extending ν, and let $\{\xi_0, \dots, \xi_k\}$ be a set of strings such that each ξ_i witnesses that θ_i is threatening R_e. Every string in D_s extending β loses its color, *except* those strings with color purple$_j$ for some $j < e$. There are now two cases:

Case 1: The string ρ has a predecessor with color purple$_j$, $j < e$; then we do nothing.

Case 2: The string ρ has no predecessor with color purple$_j$. Then for each of the (formerly) purple$_e$ strings $\theta_i \supset \nu$, let

$$\theta_i'(x) = \begin{cases} \rho(x) & \text{if } x \le |\rho| \\ \theta_i(x) & \text{otherwise.} \end{cases}$$

That is, θ_i' is the translate of θ_i above ρ. We define $\Xi_{s+1}(\theta_i') = \xi_i$ for each $i \le k$. (Note this is consistent since by virtue of substage 4, we will have $\Xi_s(\nu) = \Xi_s(\rho)$.) Finally, each string θ_i' receives color green$_e$.

Substage 3: For each active string σ such that

- σ is not red, and
- σ does not have a predecessor with color purple,

we choose the least e such that no predecessor of σ has color green$_e$ or yellow$_e$. We then define $\Xi_{s+1}(\sigma * 0) = \Xi_{s+1}(\sigma * 1) = \xi$, where ξ is the lexicographically least string such that $\xi \supset \Xi_s(\sigma)$, ξ is acceptable for σ, and such that

$$\sigma(n) = 1 \iff \xi(\langle n, m \rangle) = 1 \text{ for some } m.$$

We give $\sigma * 0$ and $\sigma * 1$ color blue$_e$.

Substage 4: For each e, for each active blue$_e$ string β, we define $\Xi_{s+1}(\beta * \tau) = \Xi_s(\beta)$ for all τ of length $e+2$. We give $\beta * 1^{e+2}$ color red$_e$, and we give color yellow$_e$ to all $\beta * \tau$ with $|\tau| = e+2$ and $\tau \neq 1^{e+2}$. This completes the construction at stage $s+1$.

We say a string σ has *final color* color$_e$ of there is some s such that σ has color$_e$ at all stages $t \ge s$. Define the classes

$$\begin{aligned} \mathcal{R}_e &= \{A : \text{ some } \sigma \subset A \text{ has final color yellow}_e \text{ or green}_e\} \\ \mathcal{S}_e &= \{A : \text{ some } \sigma \subset A \text{ has final color red}_e\} \\ \mathcal{P}_e &= \{A : \text{ some } \sigma \subset A \text{ has final color purple}_e\}. \end{aligned}$$

We will also let

$$\begin{aligned} \mathcal{S} &= \bigcup_e \mathcal{S}_e, \text{ and} \\ \mathcal{P} &= \bigcup_e \mathcal{P}_e. \end{aligned}$$

As a notational convenience let $\mathcal{R}_{-1} = 2^\omega$. We first show

Lemma A.2.1 *If $A \in \bigcap_e \mathcal{R}_e$, then $\Xi(A)$ is total, A is r.e. in $\Xi(A)$, and $A \not\le_T \Xi(A)$.*

Proof. It is clear that if A is in every class $\mathcal{R}_e$, then Ξ is defined for arbitrarily long initial segments of A. It is also immediate from the construction that A is r.e. in $\Xi(A)$. Now suppose that for some e, $A = \{e\}^{\Xi(A)}$. Let σ be the initial segment of A with final color yellow$_e$ or green$_e$, and let s_0 be the stage at which this final color was received. Suppose first that σ is green$_e$. Then by construction, at stage s_0 we defined $\Xi(\sigma) = \xi$ for a string ξ such that $\{e\}^{\xi}(k) = 0$ for some k with $\sigma(k) = 1$; therefore $\{e\}^{\Xi(A)} \neq A$. On the other hand, suppose that σ is yellow$_e$, and let ρ be the associated red$_e$ string. Then for some k we have $\sigma(k) = 0$ and $\rho(k) = 1$, and by assumption $\sigma(k) = \{e\}^{\Xi(A)}(k) = 0$ as well. Let ξ be the shortest initial segment of $\Xi(A)$ such that $\{e\}^{\xi}k \downarrow= 0$. Note that ξ is acceptable for σ, so σ is threatening R_e at any stage $s \geq s_0$ with $s \geq |\xi|$ and $s \geq |\sigma|$. Thus σ receives color purple$_e$ unless some initial segment π of σ already has color purple$_j$ for some $j < e$. Note that if $\pi \subset \sigma$ has color purple$_j$, $j < e$, at any stage $\geq s_0$, then π has final color purple$_j$: if π loses color purple$_j$, then so does every string extending the blue$_j$ string $\beta \subset \pi$ (except the purple$_m$ strings, where $m < j$), so σ would have to lose color yellow$_e$. Hence σ receives color purple$_e$, and this color must be final; but then the construction stops above σ, so Ξ can't be total on any extension of σ. This contradicts the fact that $A \in \bigcap_e \mathcal{R}_e$, and so we can conclude that $A \not\leq_T \Xi(A)$. □

The crucial fact we need is part (iii) of the lemma below.

Lemma A.2.2 *(i)* $\mu(\mathcal{R}_{e-1} - (\mathcal{R}_e \cup \mathcal{S}_e \cup \mathcal{P})) = 0$.

(ii) $\mathcal{R}_e \subseteq \mathcal{R}_{e-1}$.

(iii) $\mu(\mathcal{S} \cup \mathcal{P}) + \mu(\bigcap_e \mathcal{R}_e) = 1$.

Proof. (i) First assume that $e > 0$. The class $\mathcal{R}_{e-1}$ is a disjoint union of basic open intervals $\mathrm{Ext}(\sigma)$ for strings σ with final color green$_{e-1}$ or yellow$_{e-1}$. Suppose $\mu(\mathcal{R}_{e-1} - (\mathcal{R}_e \cup \mathcal{S}_e \cup \mathcal{P})) > 0$; then $\overline{\mathcal{R}_e \cup \mathcal{S}_e \cup \mathcal{P}}$ would have density > 0 in some interval $\mathrm{Ext}(\sigma)$, where σ has final color green$_{e-1}$ or yellow$_{e-1}$. Then by the Density lemma (Lemma III.3.4), there must be a string $\sigma_0 \supset \sigma$ such that $\overline{\mathcal{R}_e \cup \mathcal{S}_e \cup \mathcal{P}}$ has density greater than $1 - 2^{-(2e+5)}$ in $\mathrm{Ext}(\sigma_0)$. Therefore to prove (i) it will suffice to show that for any string σ_0 extending a string σ with final color green$_{e-1}$ or yellow$_{e-1}$, the density of $\mathcal{R}_e \cup \mathcal{S}_e \cup \mathcal{P}$ in $\mathrm{Ext}(\sigma_0)$ is at least $2^{-(2e+5)}$.

Assume σ has final color green$_{e-1}$ or yellow$_{e-1}$ and let $\sigma_0 \supset \sigma$; there is no harm in assuming that σ_0 properly extends σ. Let s_0 be the stage at which σ received color green$_{e-1}$ or yellow$_{e-1}$.

Notice that action by any requirement R_i removes the colors from all extensions of a blue$_i$ string β except the purple$_m$ strings, for $m < i$. In particular this means two things. First, for $j < e$, if any string π comparable to σ ever has purple$_j$ after stage s_0 then purple$_j$ is the *final* color of π, since color purple$_j$ can only be removed by the action of a requirement R_i with $i \leq j$, and the blue$_i$ predecessor of any π comparable to σ is also a predecessor of σ; thus the action of R_i would also remove the final color green$_{e-1}$ or yellow$_{e-1}$ from σ. Second, if any string π extending σ has color$_e$ after stage s_0, π can only lose color$_e$ by the action of requirement R_e. If $i < e$, then action by R_i removing color$_e$ from π would remove the final color from σ; on the other hand if $i > e$, then either color$_e$ = purple$_e$, in which case the action by R_i does not remove the color from π, or else no blue$_i$ string is a predecessor of π, so π is unaffected by the action of R_i.

Suppose some initial segment π of σ has color purple$_j$, (where necessarily $j < e$), at any stage $t \geq s_0$. Then π has final color purple$_j$, and so $\mathcal{P}$ has density 1 in $\mathrm{Ext}(\sigma_0)$, and we are through. Otherwise assume that no initial segment of σ is ever purple after stage s_0. Then some initial segment of σ_0 receives color blue$_e$ (possibly temporarily) at some stage after s_0: Choose $i \in \{0, 1\}$ so that $\sigma * i \subset \sigma_0$, and let $\beta = \sigma * i$; then β receives color blue$_e$ by the end of stage $s_0 + 1$. Hence we can let β_0 be the longest initial segment of σ_0 which ever receives color blue$_e$.

Suppose β_0 has final color blue$_e$. Then every set A extending β_0 passes through a node with final color either red$_e$ or yellow$_e$; hence $\mathcal{R}_e \cup \mathcal{S}_e$ has density 1 in $\mathrm{Ext}(\sigma_0)$. Otherwise β_0 loses color blue$_e$ at some later stage. As we noted above, this can only occur if the density of the purple$_e$ strings above β_0 exceeds $2^{-(e+3)}$, i.e., by the action of requirement R_e.

Now during the stage at which β_0 loses color blue$_e$ we form a disjoint cover $T = \{\pi_0, \ldots, \pi_m\}$ of $\mathrm{Ext}(\beta_0)$ by strings π such that either

- π is green$_e$,
- π has an initial segment which is purple$_j$, for some $j < e$, or
- π is blue$_e$.

If $\pi \in T$ is green$_e$, then its final color is green$_e$ (the color green$_e$ is never removed by the action of R_e), so if $\pi \subset \sigma_0$, $\mathcal{R}_e$ has density 1 in $\mathrm{Ext}(\sigma_0)$. If $\pi \in T$ has a purple$_j$ predecessor π', $j < e$, then the final color of π' is purple$_j$, so if $\pi \subset \sigma_0$, then $\mathcal{P}$ has density 1 in $\mathrm{Ext}(\sigma_0)$.

Otherwise we can assume that σ_0 does not extend any string $\pi \in T$ (note that σ can't extend $\pi \in T$ with color blue$_e$, as this would contradict the choice of β_0). Hence $\mathrm{Ext}(\sigma_0)$ is covered by the strings $\pi \in T$, so it will suffice to show that for each π, the density of $\mathcal{R}_e \cup \mathcal{S}_e \cup \mathcal{P}$ in $\mathrm{Ext}(\pi)$ is at least $2^{-(2e+5)}$. This is clear if π is green$_e$ or has an initial segment which is purple$_j$ for some $j < e$. Then suppose π is blue$_e$. If π has final color blue$_e$ then every set extending π passes through a node with final color red$_e$ or yellow$_e$, so $\mathcal{R}_e \cup \mathcal{S}_e$ has density 1 in $\mathrm{Ext}(\pi)$. If π loses color blue$_e$ at some later stage, it must be because the density of purple$_e$ strings above π exceeds $2^{-(e+3)}$. Then one of two things can happen:

Case 1: There is some purple$_j$ string $\theta \subset \rho$, where $\rho = \pi * 1^{e+2}$ is the red$_e$ string extending π and $j < e$. Then θ has final color purple$_j$, and thus the density of $\mathcal{P}$ in $\mathrm{Ext}(\pi)$ is at least $2^{-(e+2)}$.

Case 2: Each θ' in a collection $\{\theta'_0, \ldots, \theta'_k\}$ of strings extending $\rho = \pi * 1^{e+2}$ receives color green$_e$. Note this must be their final color, and by construction these green$_e$ strings have density at least $2^{-(e+3)}$ in $\mathrm{Ext}(\rho)$, and hence have density at least $2^{-(2e+5)}$ in $\mathrm{Ext}(\pi)$.

The proof for the case $e = 0$, where $\mathcal{R}_{-1} = 2^\omega$, is essentially the same, starting with $\sigma = \emptyset$. This completes the proof of (i).

(ii) Let $A \in \mathcal{R}_e$, and let $\sigma \subset A$ be a string with final color green$_e$ or yellow$_e$. Let $\beta \subset \sigma$ be the associated blue$_e$ string. Certainly if $e = 0$ then $A \in \mathcal{R}_{-1} = 2^\omega$, so assume $e > 0$. By construction, when β received color blue$_e$ there must have been some string $\tau \subset \beta$ with color green$_{e-1}$ or yellow$_{e-1}$. Any action removing the color from τ would also remove the color green$_e$ or yellow$_e$ from σ, so τ has final color green$_{e-1}$ or yellow$_{e-1}$.

(iii) It follows from (i), and the observation that $\overline{\mathcal{S}} \subseteq \overline{\mathcal{S}_e}$, that

$$\mu((\mathcal{R}_{e-1} - \mathcal{R}_e) \cap \overline{(\mathcal{S} \cup \mathcal{P})}) = 0. \tag{A.1}$$

It is also easy to see, using part (ii), that

$$2^\omega = \bigcup_e (\mathcal{R}_{e-1} - \mathcal{R}_e) \cup (\textstyle\bigcap_e \mathcal{R}_e).$$

Since $(\mathcal{S} \cup \mathcal{P}) \cap (\bigcap_e \mathcal{R}_e) = \emptyset$, we can write

$$\begin{aligned} \mathcal{S} \cup \mathcal{P} &= (\mathcal{S} \cup \mathcal{P}) \cap 2^\omega \\ &= (\mathcal{S} \cup \mathcal{P}) \cap \left[\bigcup_e (\mathcal{R}_{e-1} - \mathcal{R}_e) \cup (\textstyle\bigcap_e \mathcal{R}_e) \right] \\ &= (\mathcal{S} \cup \mathcal{P}) \cap \bigcup_e (\mathcal{R}_{e-1} - \mathcal{R}_e). \end{aligned}$$

Then

$$\begin{aligned} \mu\left(\bigcup_e (\mathcal{R}_{e-1} - \mathcal{R}_e) \right) &= \mu\left(\bigcup_e (\mathcal{R}_{e-1} - \mathcal{R}_e) \cap (\mathcal{S} \cup \mathcal{P}) \right) \\ &\quad + \mu\left(\bigcup_e (\mathcal{R}_{e-1} - \mathcal{R}_e) \cap \overline{(\mathcal{S} \cup \mathcal{P})} \right) \\ &= \mu\left(\bigcup_e (\mathcal{R}_{e-1} - \mathcal{R}_e) \cap (\mathcal{S} \cup \mathcal{P}) \right) + 0 \text{ (using A.1)} \\ &= \mu(\mathcal{S} \cup \mathcal{P}). \end{aligned}$$

Thus

$$\begin{aligned} 1 &= \mu(\bigcup_e(\mathcal{R}_{e-1} - \mathcal{R}_e)) + \mu(\bigcap_e \mathcal{R}_e) \\ &= \mu(\mathcal{S} \cup \mathcal{P}) + \mu(\bigcap_e \mathcal{R}_e). \end{aligned}$$

□

The rest of the proof is quite a bit simpler. We next show:

Lemma A.2.3 *(i)* $\mu(\mathcal{S}) \leq \frac{1}{2}$.

(ii) $\mu(\mathcal{P}) \leq \frac{1}{4}$.

(iii) $\mu(\bigcap_e \mathcal{R}_e) \geq \frac{1}{4}$.

Proof. (i) Since $\mathcal{S} = \bigcup_e \mathcal{S}_e$, it is enough to show that for each e, $\mu(\mathcal{S}_e) \leq 2^{-(e+2)}$. By construction, the strings with final color blue$_e$ are disjoint, and the density of red$_e$ strings in any blue$_e$ string β is $\leq 2^{-(e+2)}$. Therefore the total measure of all the strings with final color red$_e$ is bounded by

$$\sum_\beta 2^{-(e+2)} \cdot \mu(\mathrm{Ext}(\beta)) \leq 2^{-(e+2)} \cdot 1,$$

where the sum is taken over all β with final color blue$_e$.

(ii) Follows as in (i), since the density of purple$_e$ strings in any blue$_e$ string β is $\leq 2^{-(e+3)}$.

(iii) Immediate from (i), (ii), and Lemma A.2.2(iii). □

Notice that if $A \in \mathcal{S} \cup \mathcal{P}$, then $\Xi(A)$ is not total, and recall that we showed above that if $A \in \bigcap_e \mathcal{R}_e$, then $\Xi(A) <_T A$. (It is also the case that A is r.e. in $\Xi(A)$ whenever $\Xi(A)$ is total.) Then because of the fact, proved in Lemma A.2.2, that

$$\mu(\mathcal{S} \cup \mathcal{P}) + \mu(\bigcap_e \mathcal{R}_e) = 1,$$

the class

$$\{A : \ \Xi(A) \text{ is total but } A \leq_T \Xi(A)\ \}$$

has measure zero. Let B be any weakly 2-random set. If $\Xi(B)$ is total, then it must be the case that $B \in \bigcap_e \mathcal{R}_e$, since otherwise $B = \{e\}^{\Xi(B)}$ for some e and hence B would be in the Π^0_2 -nullset

$$\{A : \ \Xi(A) \text{ is total and } A = \{e\}^{\Xi(A)}\},$$

contradicting the fact that B is weakly 2-random. This completes the proof of Claim IV.2.6. □

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