Some Sums of Some Significance

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May 31, 1994

1 Introduction

After the first year of calculus, any of our students can tell us[1] that

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 2$$

or that

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e,$$

but practically any other infinite sums, such as the closely related

$$\sum_{k=0}^{\infty} \frac{k^n}{2^k} \tag{1}$$

or

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} \tag{2}$$

are regarded as yes-or-no questions, as in: "Yes; by the Ratio Test". It turns out that exact values for these sums are very easy to obtain; the purpose of this note is to give simple derivations of recurrence relations that can be used to compute (1) and (2) and to investigate some of the history and applications of the resulting values.

[1] Well, perhaps this is wishful thinking.

For example, we will show that for each n, the number $\frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}$ is equal to the nth *Bell number* B_n; among other things B_n represents the total number of ways to distribute n distinct objects into identical boxes or to partition an n-element set into nonempty subsets. The value of the sum $\frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k}$, which we denote by P_n, corresponds to the same situation where in addition the boxes or subsets are themselves distinguishable. In Section 3 we show how the sums (1) and (2) can be obtained from the combinatorial applications just described.

The values B_n and P_n also have an interpretation in discrete probability theory which we mention in Section 4. In Section 5 we briefly describe the idea of a *generating function* for a sequence and show how to derive generating functions for the sequences $\{B_n\}$ and $\{P_n\}$.

Both these sequences have a long history, and very little of what follows is new beyond the simple presentation in Section 2 and the recurrence (9). Our interest was piqued when we stumbled by accident into what are essentially the derivations given in Section 2. A source that turned out to be particularly useful for locating prior work was N.J.A. Sloane's *A Handbook Of Integer Sequences* [10]. We have given in most cases the earliest references we know of; those aware of prior or better sources are encouraged to contact the authors.

2 Recurrences

We begin with sums of the form (1). When $n = 1$, the nonzero terms of the sum $\sum_{k=0}^{\infty}\frac{k}{2^k}$ can be arranged in the triangular matrix shown in Figure 1. (Note that when $n \geq 1$, it is immaterial whether the sum starts at $k = 0$ or at $k = 1$.) For each $k \geq 1$,

$$\begin{array}{ccccccccc}
\frac{1}{2^1} & + & \frac{1}{2^2} & + & \frac{1}{2^3} & + & \frac{1}{2^4} & + & \cdots \\
 & + & \frac{1}{2^2} & + & \frac{1}{2^3} & + & \frac{1}{2^4} & + & \cdots \\
 & & & + & \frac{1}{2^3} & + & \frac{1}{2^4} & + & \cdots \\
 & & & & & + & \frac{1}{2^4} & + & \cdots \\
 & & & & & & & \ddots &
\end{array}$$

Figure 1

the sum of the kth row is $\frac{1}{2^{k-1}}$, and so

$$\sum_{k=0}^{\infty}\frac{k}{2^k} = \sum_{k=1}^{\infty}\frac{1}{2^{k-1}} = \sum_{k=0}^{\infty}\frac{1}{2^k} = 2.$$

We can arrange the terms of the sum $\sum_{k=0}^{\infty} \frac{k^2}{2^k}$ in similar way, as shown in Figure 2. For each $k \geq 0$ there are $(k+1)^2 - k^2$ rows which start with the term $\frac{1}{2^{k+1}}$ (and which

$$
\begin{array}{ccccccc}
\frac{1}{2^1} & + & \frac{1}{2^2} & + & \frac{1}{2^3} & + & \cdots \\
 & + & \frac{1}{2^2} & + & \frac{1}{2^3} & + & \cdots \\
 & + & \frac{1}{2^2} & + & \frac{1}{2^3} & + & \cdots \\
 & + & \frac{1}{2^2} & + & \frac{1}{2^3} & + & \cdots \\
 & & & + & \frac{1}{2^3} & + & \cdots \\
 & & & + & \frac{1}{2^3} & + & \cdots \\
 & & & + & \frac{1}{2^3} & + & \cdots \\
 & & & + & \frac{1}{2^3} & + & \cdots \\
 & & & + & \frac{1}{2^3} & + & \cdots \\
 & & & & & \ddots &
\end{array}
$$

Figure 2

therefore have the sum $\frac{1}{2^k}$). Thus

$$\sum_{k=0}^{\infty} \frac{k^2}{2^k} = \sum_{k=0}^{\infty} \frac{(k+1)^2 - k^2}{2^k} = \sum_{k=0}^{\infty} \frac{2k+1}{2^k} = 2\sum_{k=0}^{\infty} \frac{k}{2^k} + \sum_{k=0}^{\infty} \frac{1}{2^k} = 2(2) + 2 = 6.$$

Similar reasoning suggests that for any $n \geq 1$,

$$\sum_{k=0}^{\infty} \frac{k^n}{2^k} = \sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{2^k},$$

a fact which turns out to be easy to check algebraically since

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(k+1)^n}{2^k} &= 2\sum_{k=0}^{\infty} \frac{(k+1)^n}{2^{k+1}} \\
&= 2\sum_{k=1}^{\infty} \frac{k^n}{2^k} \\
&= 2\sum_{k=0}^{\infty} \frac{k^n}{2^k}.
\end{aligned}$$

If we define

$$S_n = \sum_{k=0}^{\infty} \frac{k^n}{2^k}, \tag{3}$$

then $S_0 = 2$ and using the binomial theorem, for $n \geq 1$,

$$\begin{aligned} S_n &= \sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{2^k} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} \binom{n}{j} \frac{k^j}{2^k} \\ &= \sum_{j=0}^{n-1} \binom{n}{j} S_j. \end{aligned} \tag{4}$$

Values of S_n can be easily obtained using (4); the first few values are shown in Figure 3.

n	S_n
0	2
1	2
2	6
3	26
4	150
5	1,082
6	9,366
7	94,586
8	1,091,670
9	14,174,522
10	204,495,126

Figure 3

There is nothing special about the number "2" appearing in (3); the reader is invited to verify that for any $r > 1$,

$$\sum_{k=0}^{\infty} \frac{k^n}{r^k} = \frac{1}{r-1} \sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{r^k} = \frac{1}{r-1} \sum_{j=0}^{n-1} \binom{n}{j} \left[\sum_{k=0}^{\infty} \frac{k^j}{r^k} \right].$$

This provides a straightforward way to compute the exact value of $\sum_{k=0}^{\infty} \frac{p(k)}{r^k}$ recursively for any polynomial p.

The recurrence (4) appeared without reference to the sum (1) in [2], an 1859 paper by Arthur Cayley (see Section 3). Both (1) and (4) appear in [6] in connection with the combinatorial applications discussed in Section 3.

We now turn our attention to sums of the form (2). First of all, it is easy to see that

$$\sum_{k=0}^{\infty} \frac{k}{k!} = \sum_{k=1}^{\infty} \frac{k}{k!} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} = \sum_{k=0}^{\infty} \frac{1}{k!} = e. \tag{5}$$

Performing the same sequence of steps for an arbitrary $n \geq 1$, we discover that

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} = \sum_{k=1}^{\infty} \frac{k^n}{k!} = \sum_{k=1}^{\infty} \frac{k^{n-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{(k+1)^{n-1}}{k!}.$$

Let

$$M_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}. \tag{6}$$

Then $M_0 = e$ and using the binomial theorem, for $n \geq 1$,

$$\begin{aligned} M_n &= \sum_{k=0}^{\infty} \frac{(k+1)^{n-1}}{k!} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{k^j}{k!} \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} M_j. \end{aligned} \tag{7}$$

The first few values of M_n are given in Figure 4. Again the relation above provides

n	M_n
0	e
1	$2e$
2	$5e$
3	$15e$
4	$52e$
5	$203e$
6	$877e$
7	$4{,}140e$
8	$21{,}147e$
9	$115{,}975e$
10	$678{,}570e$

Figure 4

an easy way to compute exact values for sums of the form $\sum \frac{p(k)}{k!}$ for any polynomial p.

The sums (2) were investigated in 1877 by Dobinski, who gave a demonstration of the similar relationship

$$M_n = M_{n-1} + \sum_{j=0}^{n-2} \binom{n-2}{j} M_{j+1}.$$

Further references concerning the recurrence (7) are given in Section 3.

It is also interesting to note that if the nonzero terms of $\sum_{k=0}^{\infty} \frac{k}{k!}$ are expressed in the tabular form shown in Figure 5, the sum of all terms is e, the sum of the first row

$$\begin{array}{ccccccccc}
\frac{1}{1!} & + & \frac{1}{2!} & + & \frac{1}{3!} & + & \frac{1}{4!} & + & \cdots \\
 & + & \frac{1}{2!} & + & \frac{1}{3!} & + & \frac{1}{4!} & + & \cdots \\
 & & & + & \frac{1}{3!} & + & \frac{1}{4!} & + & \cdots \\
 & & & & & + & \frac{1}{4!} & + & \cdots \\
 & & & & & & & \ddots &
\end{array}$$

Figure 5

is $e - 1$, and hence the sum of all the remaining terms is equal to 1, that is, we have the unexpected fact that

$$\sum_{k=0}^{\infty} \frac{k}{(k+1)!} = 1.$$

We conclude this section by presenting a novel approach to computing the sums S_n, leading to a different recurrence than that obtained in (4). Consider an experiment in which a fair coin is tossed repeatedly, i.e., the sample space consists of infinite sequences of H's and T's. At any point during the experiment, the probability of obtaining an H after a run of k T's is $\frac{1}{2^{k+1}}$, so at any point in the sequence the probability of obtaining at least one more H is always

$$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = 1.$$

That is, for any integer $n \geq 0$, with probability one there are at least $n+1$ H's. On the other hand, the probability of obtaining at least $n+1$ H's can be expressed as the disjoint union of the events

$$\mathcal{E}_k = \begin{array}{l}\text{“There are exactly } n \text{ H's among the first } k \\ \text{tosses and the } (k+1)\text{st toss is an H”}\end{array}$$

for $k \geq n$. Since there are $\binom{k}{n}$ initial sequences of k tosses containing exactly n H's,

$$\mathbf{Pr}(\mathcal{E}_k) = \frac{\binom{k}{n}}{2^{k+1}}$$

and so

$$1 = \sum_{k=n}^{\infty} \mathbf{Pr}(\mathcal{E}_k) = \sum_{k=n}^{\infty} \frac{\binom{k}{n}}{2^{k+1}}.$$

Substituting $k+n-1$ for k and simplifying,

$$\begin{aligned} 1 &= \sum_{k=1}^{\infty} \frac{\binom{k+n-1}{n}}{2^{k+n}} \\ &= \sum_{k=1}^{\infty} \frac{(k+n-1)!}{2^{k+n} n!(k-1)!} \\ &= \frac{1}{2^n n!} \sum_{k=1}^{\infty} \frac{k(k+1)(k+2)\cdots(k+n-1)}{2^k}. \end{aligned} \tag{8}$$

Using notation suggested in [5], let $\left[{n \atop j} \right]$ denote the coefficient of k^j in the expansion of the product

$$k(k+1)(k+2)\cdots(k+n-1).$$

Then (8) becomes

$$1 = \frac{1}{2^n n!} \sum_{k=1}^{\infty} \sum_{j=1}^{n} \left[{n \atop j} \right] \frac{k^j}{2^k} = \frac{1}{2^n n!} \sum_{j=1}^{n} \left[{n \atop j} \right] S_j,$$

so we obtain the recurrence

$$S_n = 2^n n! - \sum_{j=1}^{n-1} \left[{n \atop j} \right] S_j. \tag{9}$$

But what are the coefficients $\left[{n \atop j} \right]$? They can easily be found recursively; by definition the expansion of

$$k(k+1)(k+2)\cdots(k+n-2)$$

is equal to

$$\left[{n-1 \atop n-1} \right] k^{n-1} + \left[{n-1 \atop n-2} \right] k^{n-2} + \cdots + \left[{n-1 \atop 1} \right] k,$$

and after multiplying both sides by $(k+n-1)$ and grouping like terms we find that the coefficient $\left[{n \atop j} \right]$ of k^j must be

$$\left[{n \atop j} \right] = (n-1) \left[{n-1 \atop j} \right] + \left[{n-1 \atop j-1} \right]. \tag{10}$$

Note that $\left[{n \atop n} \right] = 1$ and $\left[{n \atop 0} \right] = 0$ for $n \geq 1$; it is also convenient to define $\left[{0 \atop 0} \right] = 1$. Then the relation (10) can be used to generate a table of values of $\left[{n \atop j} \right]$ similar to Pascal's triangle; the first few rows are shown in Figure 6. (The entry in row n,

n \ j	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	1				
3	0	2	3	1			
4	0	6	11	6	1		
5	0	24	50	35	10	1	
6	0	120	274	225	85	15	1

Figure 6: Some values of $\left[{n \atop j} \right]$

column j is $(n-1)$ times the entry directly above it plus the entry above and to the left.) The numbers $\left[{n \atop j} \right]$ are known as *Stirling numbers of the first kind*; in the next section we encounter Stirling numbers again in a different combinatorial guise.

3 Combinatorial Applications

In this section we show how the numbers S_n and M_n of the previous section arise in a combinatorial context not obviously related to the infinite sums (1) and (2), and we derive the recurrences (4) and (7) in an intuitive way in this context. We must begin, however, with a short digression on *Stirling numbers*.

If we have a set of n distinct objects and j identical boxes, where $0 \leq j \leq n$, the total number of ways of selecting one object to put in each box is the familiar binomial coefficient $\binom{n}{j}$. But suppose instead we want to distribute *all* the objects among the j boxes (leaving none of them empty); that is, we want to partition the n elements into j nonempty subsets. The number of ways of doing this is known as a *Stirling number of the second kind*, and following [5] we use the notation $\left\{ {n \atop j} \right\}$. Values of $\left\{ {n \atop j} \right\}$ can be computed recursively as follows: To partition an n-element set $\{a_1, a_2, \ldots, a_n\}$ into j nonempty subsets, where $1 < j < n$, there are two choices: we can partition $a_1, \ldots, a_{n-1}$ into j nonempty subsets and then select one of the j subsets in which to put a_n, which gives $j\left\{ {n-1 \atop j} \right\}$ possibilities, or else we can keep a_n in a subset by itself and partition $a_1, \ldots, a_{n-1}$ into $j-1$ subsets, for $\left\{ {n-1 \atop j-1} \right\}$

additional possibilities. Thus

$$\left\{ \begin{matrix} n \\ j \end{matrix} \right\} = j \left\{ \begin{matrix} n-1 \\ j \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ j-1 \end{matrix} \right\} \tag{11}$$

for $n > 0$, $1 < j < n$; additionally $\left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1$ and $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = 0$ for $n \geq 1$ (for convenience $\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\}$ is defined to be 1). As we saw in (10), the relation (11) can be used to generate a table of values similar to Pascal's triangle, shown in Figure 7. (The entry in row n, column j, is j times the entry above it plus the entry above and to the left.)

	j = 0	1	2	3	4	5	6
n = 0	1						
1	0	1					
2	0	1	1				
3	0	1	3	1			
4	0	1	7	6	1		
5	0	1	15	25	10	1	
6	0	1	31	90	65	15	1

Figure 7: Some values of $\left\{ \begin{smallmatrix} n \\ j \end{smallmatrix} \right\}$

What, then, are the numbers $\left[\begin{smallmatrix} n \\ j \end{smallmatrix} \right]$, the Stirling numbers of the "first" kind we encountered in Section 2? Though we will not prove this fact here, the value $\left[\begin{smallmatrix} n \\ j \end{smallmatrix} \right]$ represents the number of ways to partition an n-element set into j nonempty *cycles*; i.e., among the $n!$ possible permutations of an n-element set, $\left[\begin{smallmatrix} n \\ j \end{smallmatrix} \right]$ is the number of permutations which can be decomposed into exactly j cycles. It follows, for example, that

$$n! = \sum_{j=0}^{n} \left[\begin{matrix} n \\ j \end{matrix} \right]. \tag{12}$$

Stirling numbers (surprisingly) were described by Stirling [11] and the Stirling numbers of the second kind were apparently rediscovered in [4]. An excellent introduction to both kinds can be found in Chapter 6 of [5].

Now it is the analog of (12) for $\left\{ \begin{smallmatrix} n \\ j \end{smallmatrix} \right\}$ that we are particularly interested in. Let

$$B_n = \sum_{j=0}^{n} \left\{ \begin{matrix} n \\ j \end{matrix} \right\}. \tag{13}$$

B_n denotes the total number of ways of partitioning an n-element set into nonempty subsets. The numbers B_n have been extensively studied; they are given as examples of *exponential integers* by E.T. Bell in [1], and are elsewhere generally known as *Bell numbers*. The paper [4], dating from 1887, gives the definition of B_n in the form above and also includes an extensive discussion of the properties of Stirling numbers of the second kind. The combinatorial definition above is also given in [3], and the B_n are defined in [12] by the sum (14) below. In [1], Bell uses the theory of finite differences to obtain a formula for B_n. An elegant, non-combinatorial definition is given by Rota in [9] along with an extensive bibliography. As Rota points out, "The properties of these numbers are periodically being rediscovered ...," a fact to which the authors can attest.

It is not difficult to derive a recurrence for B_n. Note $B_0 = 1$ (there is exactly one way of partitioning an empty set). Given $n \geq 1$ elements $a_1, a_2, \ldots, a_n$, we first create a set containing a_n; then for each j, $0 \leq j \leq n-1$, we can choose j additional elements to go in the set with a_n, and we can do so in $\binom{n-1}{j}$ ways. The remaining $n-1-j$ elements can then be partitioned in B_{n-1-j} ways. Summing over all j, the total value is given by

$$\begin{aligned} B_n &= \sum_{j=0}^{n-1} \binom{n-1}{j} B_{n-1-j} \\ &= \sum_{j=0}^{n-1} \binom{n-1}{n-1-j} B_{n-1-j} \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} B_j. \end{aligned}$$

The numbers B_n thus satisfy the same recurrence as the numbers M_n of (6). Since $B_0 = \frac{1}{e} M_0$, it follows that $B_n = \frac{1}{e} M_n$ for all n; that is

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}. \tag{14}$$

The recurrence above can be found in [3] with essentially the derivation given here; the recurrence as well as the connection with the sum (14) appears in [4], [12], and [9]. See also exercises 7.15 and 9.46 in [5].

We next consider the same situation—partitioning an n-element set into j nonempty subsets—where in addition the j subsets are distinguishable, e.g., we regard each subset as a box labelled with a number 1 through j. Again, if we select only j elements from the given n-element set and put one in each box, there are $\binom{n}{j}$ ways to select the elements as before, and for each selection there are $j!$ different ways to permute the objects among the boxes, for a total of $j!\binom{n}{j}$. If we distribute all n elements into the boxes (with no box left empty), there are $\left\{ {n \atop j} \right\}$ ways to partition the elements into j nonempty groups, but again the groups may be permuted among the boxes in $j!$ ways, yeilding

$$j! \left\{ {n \atop j} \right\}$$

possible arrangements. Such a distribution is called a *preferential arrangement* in [6], since it amounts to an arrangement of all the objects into j different ranks or preference groups, namely the boxes 1 through j.

As in the previous case, the value we are actually after is the total number of preferential arrangements of n objects, i.e., the sum

$$P_n = \sum_{j=0}^{n} j! \left\{ {n \atop j} \right\}.$$

To derive a recurrence for P_n, initially $P_0 = 1$; for $n > 0$ there is at least one box, so we may assume one of the boxes is labelled "#1". Then for each possible j, $1 \leq j \leq n$, there are $\binom{n}{j}$ ways to choose j objects to go into box #1, and then P_{n-j} preferential arrangements of the remaining $n - j$ objects, so

$$\begin{aligned} P_n &= \sum_{j=1}^{n} \binom{n}{j} P_{n-j} \\ &= \sum_{j=1}^{n} \binom{n}{n-j} P_{n-j} \\ &= \sum_{j=0}^{n-1} \binom{n}{j} P_j, \end{aligned}$$

which is the same recurrence satisfied by the numbers S_n of (3). Since $P_0 = \frac{1}{2} S_0$, we have

$$P_n = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k^n}{2^k}. \tag{15}$$

The recurrence for P_n is derived in a similar way by Cayley in [2] in the context of the problem of enumerating the number of trees having $n + 1$ terminal nodes. The problem considered by Cayley and its connection to the notion of preferential arrangements is investigated further in [8]. The formula (15) is given in [6] along with a different derivation of the recurrence.

4 Applications in Discrete Probability

Here we briefly mention two instances in which the sums B_n and P_n arise naturally. Consider an experiment in which a fair coin is tossed repeatedly, stopping the first time the coin comes up heads. Let X denote the number of tails obtained before stopping. Then, for example, $\mathbf{Pr}(X = 0) = \frac{1}{2}$ is the probability of getting heads on the first toss, and in general X has the probability distribution

$$\mathbf{Pr}(X = k) = \frac{1}{2^{k+1}}$$

for $k = 0, 1, 2, \ldots$, known as a *geometric* distribution. The *expected value* or *mean* of X, denoted $E(X)$, is then given by

$$\sum_{k=0}^{\infty} k \cdot \mathbf{Pr}(X = k) = \sum_{k=0}^{\infty} \frac{k}{2^{k+1}} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k}{2^k},$$

which is exactly our number P_1. Probabilists also glean additional information about the shape of a distribution from the so-called *higher moments*, i.e., the expected values of X^2, X^3, and so on. Thus the first moment is the mean $E(X)$, and the nth moment turns out to be none other than the familiar sum

$$E(X^n) = \sum_{k=0}^{\infty} k^n \cdot \mathbf{Pr}(X = k) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k^n}{2^k} = P_n.$$

Similarly, suppose Y is a random variable with probability distribution

$$\mathbf{Pr}(Y = k) = \frac{1}{e \cdot k!}$$

for $k = 0, 1, 2, \ldots$. This is an instance of a *Poisson* distribution; if a certain event generally occurs an average of once per time unit, then $\mathbf{Pr}(Y = k)$ can be interpreted as the probability that the event occurs k times within a given time unit (we will not derive this fact here; see [7, pp 183–189]). The nth moment of Y is given by

$$E(Y^n) = \sum_{k=0}^{\infty} k^n \cdot \mathbf{Pr}(Y = k) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$$

which is seen to be the nth Bell number B_n.

5 Generating Functions

A generating function often provides the most concise way to specify the terms of a sequence when a closed-form formula is not available. Suppose f is a function having a power series expansion in some neighborhood of the origin, which (by adjusting the values of the constants as necessary) we express in the form

$$f(x) = \frac{a_0}{0!} + \frac{a_1}{1!} x + \frac{a_2}{2!} x^2 + \frac{a_3}{3!} x^3 \cdots$$

Then we say f is the *exponential generating function* for the sequence $a_0, a_1, a_2, \ldots$, i.e., for each k, a_k is $k!$ times the coefficient of x^k. Notice that the number a_k can be extracted from f by differentiating k times and evaluating at $x = 0$:

$$a_k = \left.\frac{d^k f(x)}{dx^k}\right|_{x=0}.$$

It is not hard to derive generating functions for the sequences $\{B_n\}$ and $\{P_n\}$. For example, setting

$$f(x) = P_0 + P_1\frac{x}{1!} + P_2\frac{x^2}{2!} + \cdots$$

the right-hand side can be simplified to obtain

$$\begin{aligned}
f(x) &= \left(\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\right) + \left(\frac{1}{2}\sum_{k=0}^{\infty}\frac{k}{2^k}\right)\frac{x}{1!} + \left(\frac{1}{2}\sum_{k=0}^{\infty}\frac{k^2}{2^k}\right)\frac{x^2}{2!} + \cdots \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\left(1 + \frac{kx}{1!} + \frac{k^2x^2}{2!} + \cdots\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\left(e^{kx}\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{e^x}{2}\right)^k \\
&= \frac{1}{2}\cdot\frac{1}{1-\frac{e^x}{2}} \\
&= \frac{1}{2-e^x}.
\end{aligned}$$

The cautious reader may be disturbed by our cavalier manipulation of infinite sums above without any mention of a domain of convergence. Note that we are unconcerned with the actual values of x and $f(x)$—all we are interested in are the coefficients—so once the last line is reached we can assume that x is suitably restricted so as to guarantee the absolute convergence of the MacLaurin series for $f(x) = (2-e^x)^{-1}$; then reading the derivation from bottom to top shows legitimately that the nth coefficient is $P_n/n!$.

The derivation above, at least in a general form, can be found in almost any modern probability text. The generating function is also given by Cayley in [2] with a different derivation, and appears implicitly in [6].

We invite the reader to apply the technique above to show that

$$g(x) = e^{e^x-1}$$

is a generating function for the sequence $\{B_n\}$. Other derivations are given in [3], [4], [9], and [12].

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