BOUNDARY CONTROL OF THERMOELASTIC BEAMS

Scott W. Hansen*, **Bing-Yu Zhang**†
*Department of Mathematics
Iowa State University
Ames, Iowa 50011
shansen@iastate.edu; fax: 01-151-294-5454

†Department of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221*

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Abstract

We show that the motions of a linear thermoelastic beam may be controlled exactly to zero in a finite time by a single boundary control that acts on one end of the beam. The optimal time of controllability depends upon the moment of inertia parameter of the beam and becomes arbitrarily small if this parameter is omitted, as in the Euler-Bernoulli beam theory.

1 Introduction

Consider the following boundary value problem which describes the small vibrations of a homogeneous, isotropic thermoelastic beam:

$$\begin{cases} \ddot{w} - \gamma \ddot{w}_{xx} + w_{xxxx} + \alpha \theta_{xx} = 0, & x \in (0,1), \quad t \geq 0, \\ \dot{\theta} - \theta_{xx} - \alpha \dot{w}_{xx} = 0 & \end{cases} \tag{1.1}$$

with initial conditions

$$w(x,0) = w_0(x),\ \dot{w}(x,0) = w_1(x),\ \theta(x,0) = \theta_0(x) \tag{1.2}$$

and boundary conditions

$$w(0,t) = w_{xx}(0,t) = \theta(0,t) = 0, \quad w(1,t) = 0 \tag{1.3}$$

$$w_{xx}(1,t) + \alpha\theta(1,t) = m(t), \qquad \theta(1,t) = \delta(t) \tag{1.4}$$

where $\gamma \geq 0$, $\alpha > 0$. The notation w_x and $\dot{w}$ refer to differentiation with respect to x and t, respectively. The function w denotes the transverse displacement of the beam and the *thermal moment* θ is proportional to the moment of the temperature through the thickness of the beam. The functions m and δ represent the applied bending moment and thermal moment, respectively, at the right end of the beam. For details on the modeling of (1.1)–(1.4) see Lagnese and Lions, [8].

Our results concern the exact null-controllability of the system (1.1)–(1.4), regarding bending moment m or the thermal moment δ at an end as the control input. For details and proofs of the results described herein, see [5].

We denote

$$l^2 = \left\{ (c_k)_{k=1}^{\infty} : \ \sum_{k=1}^{\infty} |c_k|^2 < \infty \right\}.$$

For $\beta \in \mathbf{R}$, define

$$S_\beta = \left\{ \sum_{k=1}^{\infty} a_k \sin k\pi x : \ (a_k k^\beta) \in l^2 \right\}. \tag{1.5}$$

S_β beomes a Hilbert space with $\|y\|_{S_\beta} = \|(a_k k^\beta)\|_{l^2}$. It is clear that

$$S_0 = L^2(0,1), \quad S_1 = H_0^1(0,1), \quad S_2 = H^2(0,1) \cap S_1.$$

When $\beta < 0$, S_β is the dual space to $S_{|\beta|}$.

The first result concerns regularity of the solutions of the system (1.1)-(1.4).

Theorem 1.1 *Suppose that $w_0 = 0$, $w_1 = 0$ and $\theta_0 = 0$ in (1.2) and that m, $\delta \in L^2(0,\infty)$ in (1.4). Then the following regularity results for (1.1)-(1.4) are valid.*

(i) If $\gamma > 0$, $m = 0$ (the temperature is controlled) then for any $T > 0$

$$(w, \dot{w}, \theta) \in C([0,T], S_2 \times S_1 \times S_{-1/2}), \tag{1.6}$$

(ii) If $\gamma > 0$, $\delta = 0$ (the bending moment is controlled) then for any $T > 0$

$$(w, \dot{w}, \theta) \in C([0,T], S_2 \times S_1) \times S_1, \tag{1.7}$$

(iii) *If $\gamma = 0$, $m = 0$ (the temperature is controlled) then for any $T > 0$*

$$(w, \dot{w}, \theta) \in C([0,T], S_{3/2} \times S_{-1/2} \times S_{-1/2}), \quad (1.8)$$

(iv) *If $\gamma = 0$, $\delta = 0$ (the bending moment is controlled) then for any $T > 0$*

$$(w, \dot{w}, \theta) \in C([0,T], S_{3/2} \times S_{-1/2} \times S_{-1/2}). \quad (1.9)$$

These results are optimal in the sense that the indices α of the function spaces S_α cannot be increased.

Our main controllability results follow.

Theorem 1.2 *Let $\gamma > 0$, $0 < \alpha < 1/\sqrt{2}$ and $T > 2\sqrt{\gamma}$.*

(i) *For the control problem (1.1)–(1.4) with $m \equiv 0$, given any $(w_0, w_1, \theta_0) \in S_2 \times S_1 \times S_{-1/2}$ there exists $\delta \in L^2(0,T)$ such that $(w, \dot{w}, \theta)$ satisfies (1.6) and $(w, \dot{w}, \theta)(T) = 0$.*

(ii) *For the control problem (1.1)–(1.4) with $\delta \equiv 0$, given any $(w_0, w_1, \theta_0) \in S_2 \times S_1 \times S_1$ there exists $m \in L^2(0,T)$ such that $(w, \dot{w}, \theta)$ satisfies (1.7) and $(w, \dot{w}, \theta)(T) = 0$.*

Theorem 1.3 *Let $\gamma = 0$, $0 < \alpha < 1/\sqrt{2}$ and $T > 0$.*

(i) *For the control problem (1.1)–(1.4) with $m \equiv 0$, given any $(w_0, w_1, \theta_0) \in S_{3/2} \times S_{-1/2} \times S_{-1/2}$ there exists $\delta \in L^2(0,T)$ such that $(w, \dot{w}, \theta)$ satisfies (1.8) and $(w, \dot{w}, \theta)(T) = 0$.*

(ii) *For the control problem (1.1)–(1.4) with $\delta \equiv 0$, given any $(w_0, w_1, \theta_0) \in S_{3/2} \times S_{-1/2} \times S_{-1/2}$ there exists $m \in L^2(0,T)$ such that $(w, \dot{w}, \theta)$ satisfies (1.9) and $(w, \dot{w}, \theta)(T) = 0$.*

Note that it is only necessary to utilize one control, either m or δ to obtain exact null-controllability. This is due to a sufficiently strong coupling between the thermal and mechanical components of the solutions. When $\gamma > 0$ the control time T must be larger than $2\sqrt{\gamma}$, while in the case $\gamma = 0$ we obtain null controllability in any time $T > 0$. This is related to the fact that the system (1.1)-(1.4) has infinite propagation speed when $\gamma = 0$.

To prove Theorems 1.2 and 1.3 we use a moment problem approach. In the case $\gamma > 0$ we decompose the dynamics into a part that is parabolic and a part that is hyperbolic. The control problem can then be reduced to a coupled moment problem for which the results of [4] can be applied. In the case $\gamma = 0$ all the eigenvalues lie in a sector of the negative real axis and hence results of [2] concerning parabolic moment problems can be directly applied.

Past literature (in addition to those mentioned) on the topic of controllability of thermoelastic systems includes Lagnese [7], where the problem of controlling only the mechanical portion of the system (i.e., *partial* controllability) is considered, and Zuazua [12] and de Teresa and Zuazua [9], where exact controllability of the mechanical portion of the state together with approximate controllability of the thermal portion is considered using controls supported in a neighborhood of the boundary.

Although our approach is limited to one dimensional problems, we are able to obtain a much stronger result with the optimal controllability spaces and the optimal control times.

2 Semigroup formulation

Let Δ denote ∂_x^2 and $J_\gamma = (I - \gamma\Delta)^{-1}$, which is the inverse of the operator $I - \gamma\Delta$ with Dirichlet boundary condition, and

$$\vec{y} = (\Delta w, \frac{dw}{dt}, \theta)' = (y_1, y_2, y_3)'.$$

Then (1.1) may be written as

$$\frac{d}{dt}\vec{y} = \begin{pmatrix} 0 & \Delta & 0 \\ -J_\gamma\Delta & 0 & -\alpha J_\gamma \Delta \\ 0 & \alpha\Delta & \Delta \end{pmatrix} \vec{y} \quad (2.1)$$

$$:= \tau\vec{y}$$

We denote by $\mathcal{H}$ the complex Hilbert space

$$\mathcal{H} = L^2(0,1) \times \mathcal{V}_\gamma \times L^2(0,1) \quad (2.2)$$

where

$$\mathcal{V}_\gamma = \begin{cases} H_0^1(0,1) & if\ \gamma > 0 \\ L^2(0,1) & if\ \gamma = 0 \end{cases}$$

equipped with the norm

$$||y||^2_{\mathcal{V}_\gamma} := ((I - \gamma\Delta)^{1/2} y, (I - \gamma\Delta)^{1/2} y)_{L^2(0,1)}$$

for any $y \in \mathcal{V}_\gamma$. We define the operator A by

$$A\vec{y} = \tau\vec{y} \quad (2.3)$$

on $\mathcal{D}(A)$, where $\mathcal{D}(A)$ is given by

$$\mathcal{D}(A) = \{\vec{y} \in \mathcal{H} : A\vec{y} \in \mathcal{H},\ \vec{y}(0) = \vec{y}(1) = 0\}. \quad (2.4)$$

It is easily checked that

$$\mathcal{D}(A) = \begin{cases} S_1 \times S_2 \times S_2 & \text{if } \gamma > 0 \\ S_2 \times S_2 \times S_2 & \text{if } \gamma = 0. \end{cases} \quad (2.5)$$

When $m = \delta = 0$ in (1.4), the initial-boundary value problem (1.1)-(1.4) can be written as the following evolution equation.

$$\dot{\vec{y}}(t) = A\vec{y}(t), \qquad \vec{y}(0) = \vec{y}_0 \quad (2.6)$$

where $\vec{y}_0 = (\Delta w_0, w_1, \theta_0)'$.

The operator A defined by (2.3)-(2.4) is easily seen (by the Lumer-Phillips Theorem) to be the generator of a strongly continuous semigroup $W(t)$ of contractions on $\mathcal{H}$. Consequently, for any $\vec{y}_0 \in \mathcal{H}$, (2.6) has a unique solution $\vec{y}(t) \in C([0,\infty);\mathcal{H})$.

Let $\mathcal{H}_1 = \mathcal{D}(A)$ endowed with the graph norm and let $\mathcal{H}_{-1} = (\mathcal{H}_1)^*$, where the duality is with respect to $\mathcal{H}_0 := \mathcal{H}$. Define b_m and b_δ as elements of $\mathcal{H}_{-1}$ by

$$< b_m, \vec{y} >= -D_x y_2(1), \qquad < b_\delta, \vec{y} >= D_x y_3(1),$$

for all $\vec{y} \in \mathcal{H}_1$. The boundary value problem (1.1)–(1.4) may be represented as

$$\dot{\vec{y}} = A\vec{y} + b_m m(t) + b_\delta \delta(t), \quad \vec{y}(0) = \vec{y}_0 \in \mathcal{H}. \tag{2.7}$$

One can easily obtain the following result.

Proposition 2.1 *Given $T > 0$, suppose that $w_0 = 0$, $w_1 = 0$ and $\theta_0 = 0$ in (1.2) and that $m, \delta \in L^2(0,T)$. Then the initial-boundary value problem (1.1)-(1.4) has a unique solution (w,θ) for which $(\Delta w, \dot{w}, \theta) \in C([0,T];\mathcal{H}_{-1})$. In addition, the solution continuously depends on its boundary values in corresponding spaces.*

However, the regularity obtained in the above proposition is what one obtains from the semigroup theory, considering b_m and b_δ as elements of $\mathcal{H}_{-1}$. An important step in obtaining our controllability results is to obtain the optimal regularity given in Theorem 1.1.

3 Spectral properties

A set of vectors $\{f_k\}$ are said to form a *Riesz basis* for the Hilbert space X if there exists a bounded and invertible operator $L : X$ onto X such that $f_k = Le_k$, where $\{e_k\}$ is an orthonormal basis for X. We refer the reader to [11] for details.

An examination of the spectrum of A leads to the following.

Proposition 3.1 *The eigenfunctions of A (A^*), as given in Proposition 3.1, form a Riesz basis for the space $\mathcal{H}$. Furthermore, the eigenvalues of A all belong to the left half of the complex plane, with the distance to the imaginary axis bounded away from zero. If (in addition) $\gamma = 0$ then $|\arg(-\lambda)| < \theta < \pi/2$ for every eigenvalue λ.*

As a result of the above proposition, we can prove have the following result, which shows that the energy in a thermoelastic beam decays at a *uniform exponential* rate.

Proposition 3.2 *The operator A defined in (2.3)-(2.4) is the generator of an exponentially stable semigroup $W(t)$ on the space $\mathcal{H}$ which satisfies*

$$\|W(t)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \le Me^{-\nu t} \qquad \forall t \ge 0$$

for some $M \ge 1$ where

$$-\nu = \sup_{\lambda\in\sigma(A)} \operatorname{Re}\lambda < 0.$$

If in addition, $\gamma = 0$, then W extends to an analytic semigroup in the sector $|\arg(t)| < \pi/2 - \theta$, where θ is the angle in Proposition 3.1.

Remark 3.2 Actually, Proposition 3.3 remains valid for the case of a thermoelastic *plate* on a bounded domain with $\theta = w = \Delta w = 0$ on the boundary. In this case Δ represents the Dirichlet Laplacian on a bounded domain. Our same proof applies to the case of a thermoelastic plate by simply replacing the eigenvalues (m_k) and eigenfunctions $(\sin m_k x)$ of the one dimensional Dirichlet Laplacian by those of the two-dimensional Dirichlet Laplacian. (Multiple eigenvalues do not matter).

The following result is essential in proving the controllability.

Proposition 3.3 *All the eigenvalues of A are simple and posess a minimum uniform separation if $0 < \alpha \le 1/\sqrt{2}$.*

The proof is similar to a proof in [4] if $\gamma = 0$, however this approach doesn't apply when $\gamma > 0$. Our proof for that case involves an examination of the roots of an associated three dimensional system, which depend upon the mode number k. We replace k by a continuous index s and obtain monotonicity results for these roots that rule out the possibility of repeated roots.

We will refer to the eigenvalues of A as $(\lambda_{k,j})$, $k = 1, 2, \ldots$, $j = 1, 2, 3$. The $j = 3$ branch consists of the sequence of negative eigenvalues and the $j = 1$ and $j = 2$ branches are complex conjugate pairs.

For any set $J \subset \mathbf{C}$ we can define an associated spectral projection $P(J) \in \mathcal{L}(\mathcal{H},\mathcal{H})$ by

$$P(J)\vec{y} = \frac{1}{2\pi i}\int_\Gamma R(\lambda; A)\,\vec{y}\,d\lambda, \quad \forall\,\vec{y} \in \mathcal{H}$$

where $R(\lambda, A)$ is the resolvent operator of A and Γ is an appropriate contour which encloses the eigenvalues in J. Let us denote $\mathcal{P} = P(\mathbf{R})$, $\mathcal{Q} = I - P(\mathbf{R})$ where I denotes the identity operator on $\mathcal{H}$. Let $\Lambda = \mathcal{P}\mathcal{H}$ and $\Sigma = \mathcal{Q}\mathcal{H}$. Since the projections are continuous, it follows that $\mathcal{H}$ is the direct sum the spaces Λ and Σ:

$$\mathcal{H} = \Lambda + \Sigma.$$

It is therefore straight-forward to prove the following result.

Proposition 3.4 *Let $\gamma > 0$ and let $W(t)$ denote the semigroup generated by the operator A on $\mathcal{H}$. Then for $t \ge 0$,*

$$W(t) = \mathcal{S}(t)\mathcal{P} + \mathcal{G}(t)\mathcal{Q}$$

where $\mathcal{G}(t)$ extends to a strongly continuous group defined for $t \in \mathbf{R}$ and $\mathcal{S}(t)$ extends to an analytic semigroup defined on $\operatorname{Re} t > 0$. The infinitesimal generators of $\mathcal{S}(t)$ and $\mathcal{G}(t)$ are given by the restriction of A, $A|_\Lambda$ and $A|_\Sigma$, respectively.

4 Well-posedness and Regularity

Consider the case $\gamma > 0$. According to the previous proposition, the dynamics of our control problem decouple into a hyperbolic part and an analytic part. The optimal regularity of each separate part may be obtained by a variety of methods. (We use the Carleson measure criterion of Ho and Russell [6], Weiss [10] together with interpolation). We obtain the following result.

Theorem 4.1 *Let $\gamma > 0$ and $\vec{y}_0 = 0$. If $m \equiv 0$ and $\delta \in L^2(0,\infty)$ then the solution to the system (2.7) belongs to $C([0,\infty),\ S_0 \times S_1 \times S_{-1/2})$. If $\delta \equiv 0$ and $m \in L^2(0,\infty)$ then the solution to the system (2.7) belongs to $C([0,\infty),\ S_0 \times S_1 \times S_1)$. Furthermore the indices β in the spaces S_β are the largest possible.*

In the case $\gamma = 0$ the problem is entirely parabolic and we obtain is the following.

Theorem 4.2 *Let $\gamma = 0$ and $\vec{y}_0 = 0$. If $m \equiv 0$ and $\delta \in L^2(0,\infty)$ or if $\delta \equiv 0$ and $m \in L^2(0,\infty)$ then the solution to the system (2.7) belongs to*

$$C([0,\infty),\ \mathcal{H}_{-1/4}) = C([0,\infty),\ \mathcal{S}_{-1/2} \times \mathcal{S}_{-1/2} \times \mathcal{S}_{-1/2}).$$

Furthermore the indices β in the spaces S_β are the largest possible.

Note that the particular control used (m or δ) makes no difference in the regularity.

Theorem 4.1 together with Theorem 4.2 prove Theorem 1.1.

5 Controllability results

Before discussing Theorems 1.2 and 1.3 it will be convenient to review some facts about moment problems.

Consider the moment problem: Find $u \in L^2(0,T)$ such that

$$c_k = \int_0^T e^{s_k t} u(t)\, dt \qquad \forall\, k \in \mathbf{N} \tag{5.1}$$

where (s_k) and (c_k) are given sequences of complex numbers. The *moment space* of (5.1) is the set of sequences (c_k) for which there exist at least one solution u to (5.1).

Let us first recall a result from [2] concerning moment problems of "parabolic type".

Proposition 5.1 *Suppose that there exist positive M, ϵ and $0 \leq \theta < \pi/2$ for which (s_k) satisfies*

(P1) $|\arg(-s_k)| \leq \theta \qquad \forall\ k \in \mathbf{N}$,

(P2) $|s_k - s_j| \geq \epsilon |k^2 - j^2| \quad \forall\, k, j \in \mathbf{N}$,

(P3) $M^{-1} k^2 \leq |s_k| \leq M k^2 \quad \forall\, k \in N$.

Then for any $T > 0$ the moment space to (5.1) contains all sequences (c_k) with the property that for some $p > 0$

$$|c_k| e^{pk} \to 0 \qquad \text{as } k \to \infty. \tag{5.2}$$

Now consider another moment problem: Find $u \in L^2(0,T)$ such that

$$d_k = \int_0^T e^{\omega_k t} u(t)\, dt \qquad \forall\, k \in \mathbf{Z}. \tag{5.3}$$

From [4] (or also see [11] for similar results) we have the following result concerning moment problems of "hyperbolic type".

Proposition 5.2 *Suppose that there exists $\beta \in \mathbf{C}$, $c > 0$ and $(\nu_k)_{k \in \mathbf{Z}} \in l^2$ for which (ω_k) satisfies*

(H1) $\omega_k = \beta + ck\pi i + \nu_k \quad \forall\ k \in \mathbf{Z}$,

(H2) $\omega_k \neq \omega_j$ *unless* $j = k$.

Then if $T \geq 2/c$ the moment space of (5.3) is exactly l^2.

We will also be interested in solving moment problems that have both a parabolic component and a hyperbolic component. In this case, the problem is to find $u \in L^2(0,T)$ which simultaneously solves (5.1) and (5.3). From [4] we have the following result.

Proposition 5.3 *Suppose that $\{(\omega_k)\}_{k \in \mathbf{Z}} \cap \{(s_k)\}_{k \in \mathbf{N}} = \emptyset$ and (s_k) satisfies the hypothesis of Proposition 5.1 and (ω_k) satisfies the hypothesis of Proposition 5.2. Furthermore assume that*

(C1) (c_k) *satisfies the decay condition (5.2),*

(C2) $(d_k) \in l^2$.

Then, for any time $T > 2/c$ there exists $u \in L^2(0,T)$ which simultaneously solves the moment problems (5.1) and (5.3). This is not true if $T \leq 2/c$.

Let us now return to our control problem. Consider

$$\dot{\vec{y}}(t) = A\vec{y}(t) + bu(T-t), \quad 0 < t < T;\ \vec{y}(0) = \vec{y}_0, \tag{5.4}$$

where A is defined in (2.3)-(2.4), $u \in L^2(0,T)$, b represents b_m or b_δ. If we wish to find a control that drives the initial state $\vec{y}_0$ to 0 in time T, the variation of parameters formula must hold (on an appropriate space):

$$0 = W(T)\vec{y}_0 + \int_0^T W(s) b u(s)\, ds. \tag{5.5}$$

If a u can be found that solves (5.5) then the corresponding control m or δ is given by $m(t) = u(T-t)$ or $\delta(t) = u(T-t)$, as the case may be.

First let us consder the case where $\gamma = 0$.

When (5.5) is integrated against the eigenfunctions of A^* one obtains the moment problem:

$$c_{k,j} = \int_0^T e^{\lambda_{k,j} t} u(t) dt \qquad k \in \mathbf{N},\ j = 1, 2, 3, \tag{5.6}$$

where $(\lambda_{k,j})$ are the eigenvalues of A,

$$c_{k,j} = \frac{-e^{\lambda_{k,j}T} < \vec{x}_0, \psi_{\bar{\lambda}_{k,j}} >}{< b, \psi_{\bar{\lambda}_{k,j}} >}, \qquad (5.7)$$

and ψ_s is an eigenfunction of A^* corresponding to the eigenvalue s.

We are able to show that the moment problem (5.6) satisfies all the hypothesis of Proposition 5.1 when the initial data is picked in the space of optimal regularity.

Now consider the case $\gamma > 0$. Using the same decomposition as in Proposition 3.4, we must have

$$-\mathcal{S}(T)\vec{x}_0 = \int_0^T \mathcal{S}(\tau)\mathcal{P}bu(\tau)\,d\tau, \qquad (5.8)$$

$$-\mathcal{G}(T)\vec{z}_0 = \int_0^T \mathcal{G}(\tau)\mathcal{Q}bu(\tau)\,d\tau, \qquad (5.9)$$

where $\vec{x}_0 = \mathcal{P}\vec{y}_0$, $\vec{z}_0 = \mathcal{Q}\vec{y}_0$. When (5.8) and (5.9) are integrated against the eigenfunctions of A^* we obtain the following coupled moment problem: Find $u \in L^2(0,T)$ such that

$$c_k = \int_0^T e^{s_k t} u(t)\,dt \qquad \forall\, k \in \mathbf{N} \qquad (5.10)$$

$$d_k = \int_0^T e^{\omega_k t} u(t) dt \qquad k \in \mathbf{Z} - \{0\}, \qquad (5.11)$$

where

$$s_k = \lambda_{k,3},\ \omega_k = \lambda_{k,1},\ \omega_{-k} = \lambda_{k,2} \qquad \forall\, k \in \mathbf{N}$$

and

$$c_k = \frac{-e^{s_k T} < \vec{x}_0, \psi_{s_k} >}{< b, \psi_{s_k} >}, \qquad d_k = \frac{-e^{\omega_k T} < \vec{z}_0, \psi_{\omega_{-k}} >}{< b, \psi_{\omega_{-k}} >}.$$

To complete the proof we show that for $T > 2\sqrt{\gamma}$ the hypothesis of Proposition 5.2 is satisfied.

6 Conclusions

Our result is that the state of a linear thermoelastic beam may be exactly controlled to zero by controlling only one boundary condition at one of the beam. Our general approach is identical to the approach used in [4] to obtain a similar result for the longitudinal vibrations of a one dimensional thermoelastic. However in the present case, the proof of the eigenvalue separation (Proposition 3.3) required more analysis.

Our method applies to more general boundary conditions than those condidered here, however (it seems) , we are restricted to those cases in which all the eigenfunctions can be expressed as sines and cosines. For example, our approach does not apply to the case of a beam that is clamped at one end.

We also mention that Proposition 3.2 concerning the uniform exponential decay for the case $\gamma > 0$ was unknown (for any boundary conditions, even in one dimension) until recently; see Avalos and Lasiecka [1]. Our proof will not apply to the boundary conditions they consider, but is much simpler.

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