Boundary Control of A Linear Thermoelastic Beam

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Abstract

We show in this paper that the motions of a linear thermoelastic beam may be controlled exactly to zero in a finite time by a single boundary control that acts on one end of the beam. The optimal time of controllability depends upon the moment of inertia parameter of the beam and becomes arbitrarily small if this parameter is omitted, as in the Euler-Bernoulli beam theory.

1 Introduction

Consider the following boundary value problem which describes the small vibrations of a homogeneous, isotropic thermoelastic beam:

$$\begin{cases} \ddot{w} - \gamma \ddot{w}_{xx} + w_{xxxx} + \alpha \theta_{xx} = 0, & x \in (0,1), \quad t \geq 0, \\ \dot{\theta} - \theta_{xx} - \alpha \dot{w}_{xx} = 0 \end{cases} \tag{1.1}$$

with initial conditions

$$w(x,0) = w_0(x), \qquad \dot{w}(x,0) = w_1(x), \qquad \theta(x,0) = \theta_0(x) \tag{1.2}$$

and boundary conditions

$$w(0,t) = w_{xx}(0,t) = \theta(0,t) = 0, \quad w(1,t) = 0 \tag{1.3}$$

$$w_{xx}(1,t) + \alpha\theta(1,t) = m(t), \qquad \theta(1,t) = \delta(t) \tag{1.4}$$

where $\gamma \geq 0, \ \alpha > 0$. The notation w_x and $\dot{w}$ refer to differentiation with respect to x and t, respectively. The function w denotes the transverse displacement of the beam and the *thermal moment* θ is proportional to the moment of the temperature through the thickness of the beam. The functions m and δ represent the applied bending moment and thermal moment, respectively, at the right end of the beam. For details on the modeling of (1.1)–(1.4) see Lagnese and Lions, [9].

In this paper we investigate the problem of exact null-controllability of the system (1.1)–(1.4) when the bending moment m or the thermal moment δ at an end is regarded as the control input.

We denote

$$l^2 = \left\{ (c_k)_{k=1}^{\infty} : \ \sum_{k=1}^{\infty} |c_k|^2 < \infty \right\}.$$

For $\beta \in \mathbf{R}$, define

$$S_\beta = \left\{ \sum_{k=1}^{\infty} a_k \sin k\pi x : \ (a_k k^\beta) \in l^2 \right\}. \tag{1.5}$$

S_β beomes a Hilbert space with $\|y\|_{S_\beta} = \|(a_k k^\beta)\|_{l^2}$. It is clear that

$$S_0 = L^2(0,1), \quad S_1 = H_0^1(0,1), \quad S_2 = H^2(0,1) \cap S_1.$$

When $\beta < 0$, S_β is the dual space to $S_{|\beta|}$.

Now we describe the main results of this paper.

The first result concerns regularity of the solutions of the system (1.1)-(1.4). While assuming the initial values are identically zero, the following theorem provides the optimal regularity results of the solution for a given class of boundary inputs.

Theorem 1.1 *Suppose that $w_0 = 0$, $w_1 = 0$ and $\theta_0 = 0$ in (1.2) and that m, $\delta \in L^2(0, \infty)$ in (1.4). Then the following regularity results for (1.1)-(1.4) are valid.*

(i) If $\gamma > 0$, $m = 0$ (the temperature is controlled) then for any $T > 0$

$$(w, \dot{w}, \theta) \in C([0, T], S_2 \times S_1 \times S_{-1/2}), \tag{1.6}$$

(ii) If $\gamma > 0$, $\delta = 0$ (the bending moment is controlled) then for any $T > 0$

$$(w, \dot{w}, \theta) \in C([0, T], S_2 \times S_1 \times S_1), \tag{1.7}$$

(iii) If $\gamma = 0$, $m = 0$ (the temperature is controlled) then for any $T > 0$

$$(w, \dot{w}, \theta) \in C([0, T], S_{3/2} \times S_{-1/2} \times S_{-1/2}), \tag{1.8}$$

(iv) If $\gamma = 0$, $\delta = 0$ (the bending moment is controlled) then for any $T > 0$

$$(w, \dot{w}, \theta) \in C([0, T], S_{3/2} \times S_{-1/2} \times S_{-1/2}). \tag{1.9}$$

These results are optimal in the sense that the indices α of the function spaces S_α cannot be increased.

Our main controllability results follow.

Theorem 1.2 *Let $\gamma > 0$, $0 < \alpha < 1/\sqrt{2}$ and $T > 2\sqrt{\gamma}$.*

(i) For the control problem (1.1)–(1.4) with $m \equiv 0$, given any $(w_0, w_1, \theta_0) \in S_2 \times S_1 \times S_{-1/2})$ there exists $\delta \in L^2(0, T)$ such that $(w, \dot{w}, \theta)$ satisfies (1.6) and $(w, \dot{w}, \theta)(T) = 0$.

(ii) *For the control problem (1.1)–(1.4) with* $\delta \equiv 0$*, given any* $(w_0, w_1, \theta_0) \in S_2 \times S_1 \times S_1)$ *there exists* $m \in L^2(0,T)$ *such that* $(w, \dot{w}, \theta)$ *satisfies (1.7) and* $(w, \dot{w}, \theta)(T) = 0$.

Theorem 1.3 *Let* $\gamma = 0$, $0 < \alpha < 1/\sqrt{2}$ *and* $T > 0$.

(i) *For the control problem (1.1)–(1.4) with* $m \equiv 0$*, given any* $(w_0, w_1, \theta_0) \in S_{3/2} \times S_{-1/2} \times S_{-1/2})$ *there exists* $\delta \in L^2(0,T)$ *such that* $(w, \dot{w}, \theta)$ *satisfies (1.8) and* $(w, \dot{w}, \theta)(T) = 0$.

(ii) *For the control problem (1.1)–(1.4) with* $\delta \equiv 0$*, given any* $(w_0, w_1, \theta_0) \in S_{3/2} \times S_{-1/2} \times S_{-1/2}$ *there exists* $m \in L^2(0,T)$ *such that* $(w, \dot{w}, \theta)$ *satisfies (1.9) and* $(w, \dot{w}, \theta)(T) = 0$.

Note that it is only necessary to utilize one control, either m or δ to obtain exact null-controllability. This is due to a sufficiently strong coupling between the thermal and mechanical components of the solutions. When $\gamma > 0$ the control time T must be larger than $2\sqrt{\gamma}$, while in the case $\gamma = 0$ we obtain null controllability in any time $T > 0$. This is related to the fact that the system (1.1)-(1.4) has infinite propagation speed when $\gamma = 0$.

To prove Theorems 1.2 and 1.3 we use a moment problem approach. In the case $\gamma > 0$ we decompose the dynamics into a part that is parabolic and a part that is hyperbolic. The control problem can then be reduced to a coupled moment problem for which the results of [5] can be applied. In the case $\gamma = 0$ all the eigenvalues lie in a sector of the negative real axis and hence results of [3] concerning parabolic moment problems can be directly applied.

Past literature (in addition to those mentioned) on the topic of controllability of thermoelastic systems includes Lagnese [8], where the problem of controlling only the mechanical portion of the system (i.e., *partial* controllability) is considered, and Zuazua [18] and de Teresa and Zuazua [14], where exact controllability of the mechanical portion of the state together with approximate controllability of the thermal portion is considered using controls supported in a neighborhood of the boundary.

When either the approach in [8] or [14], [18] is applied to the one-dimensional problem considered here, the results obtained are weaker than in Theorems 1.2 or 1.3, although those approaches are not restricted to 1-dimensional problems. Our approach here, although very specialized, is valuable in that we obtain exact controllability of both the mechanical and thermal components of the solution while utilizing only one (scalar) control. Furthermore our results are optimal both in the spaces obtained and the controllability time.

This paper is organized as follows. In Section 2, we discuss the semigroup formulation of the control problem. In Section 3 we discuss spectral properties of the semigroup generator. In Section 4 we review the Carleson measure criterion and then utilize this criterion to obtain the optimal regularity results. The proof of Theorem 1.2 is given in Section 5.

2 Semigroup formulation

Let Δ denote ∂_x^2 and $J_\gamma = (I - \gamma\Delta)^{-1}$, which is the inverse of the operator $I - \gamma\Delta$ with Dirichlet boundary condition, and

$$\vec{y} = (\Delta w, \frac{dw}{dt}, \theta)' = (y_1, y_2, y_3)'.$$

Then (1.1) may be written as

$$\begin{aligned} \frac{d}{dt}\vec{y} &= \begin{pmatrix} 0 & \Delta & 0 \\ -J_\gamma\Delta & 0 & -\alpha J_\gamma\Delta \\ 0 & \alpha\Delta & \Delta \end{pmatrix} \vec{y} \\ &:= \tau\vec{y} \end{aligned} \tag{2.1}$$

We denote by $\mathcal{H}$ the complex Hilbert space

$$\mathcal{H} = L^2(0,1) \times \mathcal{V}_\gamma \times L^2(0,1) \quad \text{where} \quad \mathcal{V}_\gamma = \begin{cases} H_0^1(0,1) & if\ \gamma > 0 \\ L^2(0,1) & if\ \gamma = 0 \end{cases} \tag{2.2}$$

equipped with the norm

$$\|y\|^2_{\mathcal{V}_\gamma} := ((I-\gamma\Delta)^{1/2}y, (I-\gamma\Delta)^{1/2}y)_{L^2(0,1)}$$

for any $y \in \mathcal{V}_\gamma$. We define the operator A by

$$A\vec{y} = \tau\vec{y} \tag{2.3}$$

on $\mathcal{D}(A)$, where $\mathcal{D}(A)$ is given by

$$\mathcal{D}(A) = \{\vec{y} \in \mathcal{H} :\; A\vec{y} \in \mathcal{H},\, \vec{y}(0) = \vec{y}(1) = 0\;\}. \tag{2.4}$$

It is easily checked that

$$\mathcal{D}(A) = \begin{cases} S_1 \times S_2 \times S_2 & \text{if } \gamma > 0 \\ S_2 \times S_2 \times S_2 & \text{if } \gamma = 0. \end{cases} \tag{2.5}$$

The adjoint operator of A is easily calculated and is given by

$$A^*\vec{w} = \begin{pmatrix} 0 & -\Delta & 0 \\ J_\gamma\Delta & 0 & \alpha J_\gamma\Delta \\ 0 & -\alpha\Delta & \Delta \end{pmatrix} \vec{w} \tag{2.6}$$

for $\vec{w} \in \mathcal{D}(A^*) = \mathcal{D}(A)$. When $m = \delta = 0$ in (1.4), the initial-boundary value problem (1.1)-(1.4) can be written as the following evolution equation.

$$\dot{\vec{y}}(t) = A\vec{y}(t), \qquad \vec{y}(0) = \vec{y}_0 \tag{2.7}$$

where $\vec{y}_0 = (\Delta w_0, w_1, \theta_0)'$.

Proposition 2.1 *The operator A defined by (2.3)-(2.4) is the generator of a strongly continuous semigroup $W(t)$ of contractions on $\mathcal{H}$. Consequently, for any $\vec{y}_0 \in \mathcal{H}$, (2.7) has a unique solution $\vec{y}(t) \in C([0,\infty);\mathcal{H})$.*

Proof: Obviously, the operator A is densely defined on the space $\mathcal{H}$. The proposition will then follow from the Lumer-Phillips theorem [12] once we show that A and A^* are dissipative.

In fact, for any $\vec{y} \in \mathcal{D}(A) \cap H^2(0,1) \times H^2(0,1) \times H^2(0,1)$, integration by parts leads to

$$< A\vec{y}, \vec{y} >_{\mathcal{H}} = (\Delta y_2, y_1)_{L^2} - (\Delta y_1 + \alpha \Delta y_3, y_2)_{L^2} + (\alpha \Delta y_2 + \Delta y_3, y_3)_{L^2}$$

$$= (\Delta y_2, y_1)_{L^2} - \overline{(\Delta y_2, y_1)}_{L^2} + \alpha \left(\overline{(\Delta y_3, y_2)}_{L^2} - (\Delta y_3, y_2)_{L^2} \right)$$

$$+(\Delta y_3, y_3)_{L^2}.$$

Thus, since $\mathcal{D}(A) \cap H^2(0,1) \times H^2(0,1) \times H^2(0,1)$ is a dense subset of $\mathcal{D}(A)$,

$$\text{Re } < A\vec{y}, \vec{y} >_{\mathcal{H}} = (\Delta y_3, y_3)_{L^2} = -(D_x y_3, D_x y_3) \leq 0,$$

for any $\vec{y} \in \mathcal{D}(A)$. Similarly, one can show that

$$\text{Re } < A^* \vec{w}, \vec{w} >_{\mathcal{H}} = -(\Delta w_3, w_3)_{L^2} \leq 0,$$

for any $\vec{w} \in \mathcal{D}(A^*)$. So both A and A^* are dissipative. The proof is complete. □

Let $\mathcal{H}_1 = \mathcal{D}(A)$ endowed with the graph norm and let $\mathcal{H}_{-1} = (\mathcal{H}_1)^*$, where the duality is with respect to $\mathcal{H}_0 := \mathcal{H}$. Since $\mathcal{D}(A) = \mathcal{D}(A^*)$, $W(\cdot)$ extends continuously (by duality) to the space $\mathcal{H}_{-1}$. It follows that a version of Proposition 2.1 remains valid when $\mathcal{H}$ is replaced by the larger space $\mathcal{H}_{-1}$.

Let $G : \mathbf{R}^2 \to \mathcal{H}$ denote the Green's map associated with (2.3)-(2.4):

$$G(m, \delta)' = \vec{y}; \qquad \tau \vec{y} = 0 \quad \text{in } (0,1),$$

$$y_1(0) = y_2(0) = y_3(0) = 0, \quad y_1(1) + \alpha y_3(1) = m, \; y_2(1) = 0, \; y_3(1) = \delta.$$

One finds that $G(m, \delta)' = x(m - \alpha\delta, 0, \delta)'$.

Hence, if $\vec{y}|_{t=0} = 0$, and $\vec{h} := (m, \delta)' \in (C_0^\infty(0, \infty))^2$ then the (classical) solution $\vec{y}$ to (1.1)-(1.4) at time t coincides with an element of $\mathcal{H}_{-1} (= \mathcal{D}(A^*)^*)$, also denoted by $\vec{y}(t)$, which is given by

$$\vec{y}(t) \quad = \quad \int_0^t (\frac{d}{dt} W(t - \tau)) G\vec{h}(\tau) \, d\tau$$

$$= \int_0^t W(t-\tau)(AG)\vec{h}(\tau)d\tau$$

$$:= \int_0^t W(t-\tau)B\vec{h}(\tau)d\tau,$$

where $B := AG$ is called the boundary operator. The operator B maps $\mathbf{R}^2$ into $\mathcal{H}_{-1}$ continuously and hence is a sum of two continuous linear functionals on $\mathcal{H}_1$. Letting $\vec{u}(t)$ denote $(m(t), \delta(t))'$, integration by parts gives

$$\begin{aligned} < B\vec{u}, \vec{y} > &= < G\vec{u}, A^*\vec{y} > \\ &= m(-D_x y_2(1)) + \delta(D_x y_3(1)), \end{aligned}$$

for any $\vec{y} \in \mathcal{H}_1$. Thus we define b_m and b_δ as elements of $\mathcal{H}_{-1}$ by

$$\begin{cases} < b_m, \vec{y} >= -D_x y_2(1), \quad \forall \vec{y} \in \mathcal{H}_1 \\ < b_\delta, \vec{y} >= D_x y_3(1), \quad \forall \vec{y} \in \mathcal{H}_1 \end{cases} \tag{2.8}$$

so that

$$\vec{y}(t) = \int_0^t W(t-\tau)(b_m m(\tau)) + b_\delta \delta(\tau))d\tau \quad \text{on } \mathcal{H}_{-1}. \tag{2.9}$$

On the other hand, for any m, $\delta \in L^2(0,T)$, (2.9) well defines a function in the space $C([0,T]; \mathcal{H}_{-1})$. In fact, the map $(m, \delta) \to \vec{y}$ as given by (2.9) is bounded when considered as a map $(L^2(0,T))^2 \to C([0,T]; \mathcal{H}_{-1})$, and thus defines a generalized solution for $(m, \delta) \in (L^2(0,T))^2$. It follows that the boundary value problem (1.1)–(1.4) may be represented as

$$\dot{\vec{y}} = A\vec{y} + b_m m(t) + b_\delta \delta(t), \quad \vec{y}(0) = \vec{y}_0 \in \mathcal{H} \tag{2.10}$$

in the sense that the unique mild solution of (2.10) coincides with the solution given by Proposition 2.1 and (2.9). We therefore have the following proposition.

Proposition 2.2 *Given $T > 0$, suppose that $w_0 = 0$, $w_1 = 0$ and $\theta_0 = 0$ in (1.2) and that m, $\delta \in L^2(0,T)$. Then the initial-boundary value problem (1.1)-(1.4) has a unique solution (w, θ) for which $(\Delta w, \dot{w}, \theta) \in C([0,T]; \mathcal{H}_{-1})$. In addition, the solution continuously depends on its boundary values in corresponding spaces.*

The regularity obtained in the above proposition is suboptimal and will be improved to the optimal regularity in the Section 4.

3 Spectral analysis of the operator A

For any $\gamma \geq 0$, define

$$m_k = k\pi, \qquad s_k = \sqrt{1+\gamma m_k^2},$$

$$E_{1,k} = \begin{pmatrix} \sin m_k x \\ 0 \\ 0 \end{pmatrix}, \quad E_{2,k} = \frac{1}{s_k}\begin{pmatrix} 0 \\ \sin m_k x \\ 0 \end{pmatrix}, \quad E_{3,k} = \begin{pmatrix} 0 \\ 0 \\ \sin m_k x \end{pmatrix},$$

and

$$\Sigma_k = (E_{1,k}, E_{2,k}, E_{3,k}).$$

Obviously, $\{E_{j,k}\},\ j = 1,2,3;\ k \in \mathbf{N}$ forms a Riesz basis for the space $\mathcal{H}$. In addition, for any $k \in \mathbf{N}$,

$$A\Sigma_k = m_k^2 s_k^{-1} \Sigma_k R_k$$

where

$$R_k = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & \alpha \\ 0 & -\alpha & -s_k \end{pmatrix}. \tag{3.1}$$

Note that if λ is an eigenvalue of R_k with eigenvector $\vec{v}$, then $m_k^2 s_k^{-1}\lambda$ is an eigenvalue of the operator A with eigenfunction $\Sigma_k \vec{v} = \vec{v}\sin m_k$. Thus all the spectral properties A can be determined from those of the matrices R_k.

The characteristic equation of the matrix R_k is

$$\lambda^3 + s_k\lambda^2 + (\alpha^2+1)\lambda + s_k = 0. \tag{3.2}$$

This is the same characteristic equation that occurred in [4, 5] for the case of longitudinal motions in a thermoelastic rod. Referring to those papers, we have the following.

Lemma 3.1 *For each $k \in \mathbf{N}$, the matrix R_k defined in (3.1) has a real eigenvalue μ_k and a non-real complex conjugate pair of eigenvalues σ_k and $\sigma_{-k} = \bar{\sigma}_k$. A corresponding set of eigenvectors $\vec{r}_{\sigma_k}$, $\vec{r}_{\sigma_{-k}}$, $\vec{r}_{\mu_k}$ is given by*

$$(\vec{r}_{\sigma_k}, \vec{r}_{\sigma_{-k}}, \vec{r}_{\mu_k}) = \begin{pmatrix} 1 & 1 & \frac{-\alpha}{1+\mu_k^2} \\ -\sigma_k & -\sigma_{-k} & \frac{\alpha\mu_k}{1+\mu_k^2} \\ \frac{\alpha\sigma_k}{\sigma_k+s_k} & \frac{\alpha\sigma_{-k}}{\sigma_{-k}+s_k} & 1 \end{pmatrix}. \tag{3.3}$$

If $\gamma > 0$ then the eigenvalues of R_k have the following asymptotic form as $k \to \infty$.

$$\begin{cases} \mu_k = -s_k + \alpha^2 s_k^{-1} + O(s_k^{-3}), \\ \sigma_k = -\frac{\alpha^2}{2} s_k^{-1} - i + O(s_k^{-2}) \\ \sigma_{-k} = -\frac{\alpha^2}{2} s_k^{-1} + i + O(s_k^{-2}) \end{cases} \tag{3.4}$$

Note that in the case $\gamma = 0$ the eigenvectors of R_k are independent of k. By this observation and Lemma 3.1 one easily obtains the following.

Proposition 3.1 *Let A be the operator defined by (2.3)-(2.4). The spectrum of A consists of eigenvalues $\{\lambda_{k,j}\}_{k \in \mathbf{N}}$, $j = 1, 2, 3$ where $\lambda_{k,1}$ and $\lambda_{k,2}$ are non-real complex conjugates and $\lambda_{k,3}$ is real. A corresponding set of eigenfunctions $\{\phi_{\lambda_{k,j}}\}$ is given by*

$$(\phi_{\lambda_{k,1}}, \phi_{\lambda_{k,2}}, \phi_{\lambda_{k,3}}) = \begin{pmatrix} 1 & 1 & \frac{-\alpha}{1+\mu_k^2} \\ \frac{-\sigma_k}{s_k} & \frac{-\sigma_{-k}}{s_k} & \frac{\alpha\mu_k}{s_k(1+\mu_k^2)} \\ \frac{\alpha\sigma_k}{\sigma_k+s_k} & \frac{\alpha\sigma_{-k}}{\sigma_{-k}+s_k} & 1 \end{pmatrix} \sin m_k x. \tag{3.5}$$

Moreover,

(i) If $\gamma \neq 0$, then

$$\lambda_{k,1} = -\frac{\alpha^2}{2\gamma} - \frac{i m_k}{\sqrt{\gamma}} + O(m_k^{-1}) \tag{3.6}$$

$$\lambda_{k,3} = -m_k^2 + \frac{\alpha^2}{\gamma} + O(m_k^{-2}) \tag{3.7}$$

as $k \to \infty$.

(ii) *If* $\gamma = 0$, *then*

$$\lambda_{k,1} = \sigma_0 m_k^2,\ \ \lambda_{k,2} = \bar{\sigma}_0 k^2 m_k^2,\ \ \lambda_{k,3} = \mu_0 m_k^2 \tag{3.8}$$

where σ_0, $\bar{\sigma}_0$ *and* μ_0 *are the three eigenvalues of* R_k *in (3.1). (Note that* R_k *is independent of* k *when* $\gamma = 0$*.) The corresponding eigenfunctions are given by*

$$(\phi_{\lambda_{k,1}}, \phi_{\lambda_{k,2}}, \phi_{\lambda_{k,3}}) = (\vec{r}_{\sigma_0} \sin m_k x,\ \ \vec{r}_{\bar{\sigma}_0} \sin m_k x, \vec{r}_{\mu_0} \sin m_k x).$$

(iii) *The spectrum of* A^* *consists of eigenvalues* $\{\bar{\lambda}_{k,j}\}_{k\in N}$, $j = 1, 2, 3$. *The corresponding eigenfunctions* $\psi_{\lambda_{k,j}}$, $j = 1, 2, 3$ *satisfy*

$$\psi_{\lambda_{k,j}}(s_k) = \phi_{\lambda_{k,j}}(-s_k),$$

where s_k *is the parameter appearing in (3.5). In particular, the eigenfunctions of* A^* *have the same asymptotic orders as the eigenfunctions of* A.

A set of vectors $\{f_k\}$ are said to form a *Riesz basis* for the Hilbert space X if there exists a bounded and invertible operator $L : X$ onto X such that $f_k = Le_k$, where $\{e_k\}$ is an orthonormal basis for X. We refer the reader to [17] for details.

Proposition 3.2 *The eigenfunctions of* A (A^*), *as given in Proposition 3.1, form a Riesz basis for the space* $\mathcal{H}$.

Proof: Let M_k denote the matrix of eigenvectors in (3.3). According to Lemma 3.1, R_k has three simple eigenvalues, one real μ_k, and two complex eigenvalues σ_k and $\sigma_{-k} = \bar{\sigma}_k$. Thus $\det M_k \neq 0$, for all k. Their asymptotic forms are provided in (3.4). As a result, we have that

$$\det M_k \to -i \qquad \text{as } k \to \infty$$

and

$$\sup_{k,j,l} |(M_k)_{j,l}| < +\infty.$$

Recall that if $\{g, \lambda\}$ is an eigenpair of R_k, then $\{\Sigma_k g, \lambda m_k^2 s_k^{-1}\}$ is an eigenpair of A. Thus we have $(\phi_{\sigma_k},\ \phi_{\sigma_{-k}},\ \phi_{\mu_k}) = \Sigma_k M_k$. It follows that

$$\begin{pmatrix} \phi_{\sigma_k} \\ \\ \phi_{\sigma_{-k}} \\ \\ \phi_{\mu_k} \end{pmatrix} = M_k^T \begin{pmatrix} E_{k,1} \\ \\ E_{k,2} \\ \\ E_{k,3} \end{pmatrix}.$$

Since $\{E_{k,j}\}$, $j = 1, 2, 3$, $k = 1, 2, \ldots$ forms an orthogonal basis for the space $\mathcal{H}$, applying [1, Proposition 2.3], we know that

$$\{\phi_{\sigma_k}\}_{k=1}^{\infty} \cup \{\phi_{\sigma_{-k}}\}_{k=1}^{\infty} \cup \{\phi_{\mu_k}\}_{k=1}^{\infty}$$

forms a Riesz basis for the space $\mathcal{H}$. □

Remark 3.1 The eigenfunctions $(\phi_{\lambda_{k,j}})$ and adjoint eigenfunctions $(\psi_{\lambda_{k,j}})$ given in Proposition 3.1 are *almost normalized*; that is their norms are bounded above and below by positive constants. It follows that these eigenfunctions satisfy the following biorthogonality relationship:

$$(\phi_{\lambda_{k,j}}, \psi_{\bar{\lambda}_{l,m}})_{\mathcal{H}} = \begin{cases} \nu_{k,j}, & \text{if } (k,j) = (l,m) \\ 0 & \text{otherwise,} \end{cases} \tag{3.9}$$

where $(\nu_{k,j})_{k\in\mathbf{N}, j=1,2,3}$ is bounded above and below by positive constants.

As a result of the above proposition, we can prove have the following result, which shows that the energy in a thermoelastic beam decays at a *uniform exponential* rate.

Proposition 3.3 *The operator A defined in (2.3)-(2.4) is the generator of an exponentially stable semigroup $W(t)$ on the space $\mathcal{H}$ which satisfies*

$$\|W(t)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \le M e^{-\nu t} \qquad \forall t \ge 0$$

for some $M \ge 1$ where

$$-\nu = \sup_{\lambda\in\sigma(A)} \operatorname{Re}\lambda < 0.$$

If in addition, $\gamma = 0$, then W extends to an analytic semigroup in the sector $|\arg(t)| < \theta$, where $\theta = |\arg(-\sigma_0)| > 0$ and σ_0 is given in (3.8).

Proof: First note that Proposition 2.1 implies that $-\nu \leq 0$. If $i\omega$, $\omega \in \mathbf{R}$, is an eigenvalue of A then $\pm i\omega s_k/m_k^2$ are roots of the characteristic polynomial in (3.2). Plugging $\pm i\omega s_k/m_k^2$ into (3.2) leads to two cubic equations that have solutions only if $\alpha = 0$, which is a contradiction. Therefore there are no eigenvalues of A on the imaginary axis. The fact that the eigenvalues are bounded away from the imaginary axis then follows from the asymptotic estimates (3.6)-(3.7). The proof of the exponential decay thus follows if the decay rate of the semigroup $W(\cdot)$ is determined by the spectrum. However this follows from Proposition 3.2 since $W(\cdot)$ can be diagonalized with respect to the Riesz basis of eigenfunctions and consequently is equivalent to a diagonal semigroup in l^2. For such semigroups it is easy to show that that the decay rate is determined by the spectrum. (See [4] for a proof.)

In a similar way, the analyticity for the case $\gamma = 0$ follows from Proposition 3.2 and the fact that the spectrum is contained in the angular sector of the negative real axis $\{\lambda \in \mathbf{C} : \arg(-\lambda) < \theta\}$. The proof is complete. □

Remark 3.2 Actually Proposition 3.3 remains valid for the case of a thermoelastic *plate* on a bounded domain with $\theta = w = \Delta w = 0$ on the boundary. In this case Δ represents the Dirichlet Laplacian on a bounded domain. To adjust the proof simply replace the eigenvalues (m_k) and eigenfunctions $(\sin m_k x)$ by those of the two-dimensional Dirichlet Laplacian.

Many results in this direction have been obtained in recent years. Concerning the case $\gamma = 0$, Kim [7] proved the uniform exponential stability for the case of a thermoelastic plate that is clamped, with $\theta = 0$ on the boundary. More recently Liu and Renardy [11] proved the analyticity, again for a thermoelastic plate, with boundary conditions that include the ones we consider here. In the case $\gamma > 0$, Avalos and Lasiecka [2] recently proved the uniform exponential stability for a thermoelastic plate with both clamped and simply supported boundary conditions, with Newton's law of cooling applied to θ on the boundary. Additional stability and analyticity results that apply to a class of abstract systems (that include the case $\gamma = 0$ with $\theta = w = \Delta w = 0$ on the boundary) are given in Russell [13], Ammar Khodja and Benabdallah [1].

The following result is essential in proving our controllability results.

Proposition 3.4 *All the eigenvalues of A are simple if $0 < \alpha \leq 1/\sqrt{2}$.*

Proof: Let $P_s(\lambda) = \lambda^3 + s\lambda^2 + (\alpha^2+1)\lambda + s$ denote the polynomial in (3.2). Let $\sigma(s)$, $\bar{\sigma}(s)$ and $\mu(s)$ denote the roots of $P_s(\lambda) = 0$, with μ real and σ in the lower half-plane for $s > 0$. Let $(\lambda_1(s), \lambda_2(s), \lambda_3(s)) = M(s)(\sigma(s), \bar{\sigma}(s), \mu(s))$, where $M(s) = (s^2-1)/s\gamma$. Then

$$(\lambda_{1,k}, \lambda_{2,k}, \lambda_{3,k}) = (\lambda_1(s_k), \lambda_2(s_k), \lambda_3(s_k)) \qquad \forall k \in \mathbf{N}.$$

Writing $P_s(\lambda) = 0$ as $(\lambda-\mu)(\lambda-\sigma)(\lambda-\bar{\sigma}) = 0$ leads to the system

$$\begin{cases} s &= -\mu - 2\mathrm{Re}\,\sigma \\ (1+\alpha^2) &= |\sigma|^2 + 2\mu\mathrm{Re}\,\sigma \\ s &= -\mu|\sigma|^2 \end{cases} \tag{3.10}$$

From this we deduce for $s > 0$ that (i) $s+\mu > 0$, (ii) $(1+\alpha^2)|\sigma|^2 > 1$, (iii) $\lim_{s\to\infty} |\sigma| = 1$, (iv) $\lim_{s\to\infty} -\mu/s = 1$ and (v) $\mathrm{Im}\,\sigma \neq 0$. To prove (v) we used the fact that $\alpha < 2$ and a contradiction argument.

By (v) we know that the branches $\{\lambda_1(s_k)\}$, $\{\lambda_2(s_k)\}$ and $\{\lambda_3(s_k)\}$ are distinct. Thus we only need to show that the eigenvalues are distinct within each of these branches. It will therefore suffice to show that the functions $|\lambda_j(s)|$, $j = 1,2,3$, are monotone on $(1,\infty)$. (This is since $s_k = \sqrt{1+\gamma m_k^2} > 1$.)

First note that since $\sigma(s)$, $\bar{\sigma}(s), \mu(s)$ are *distinct* roots of $P_s = 0$, the implicit function theorem implies that $\sigma(s)$, $\bar{\sigma}(s), \mu(s)$ are analytic functions of s (locally) for each $s > 0$. In partcular, these functions are differentiable for $s > 0$. Since $M(s)$ is also differentiable for $s > 0$ it follows that $|\lambda_j(s)|$, $j = 1,2,3$ are differentiable for all $s > 0$.

Let $r = |\sigma|^2$. Eliminating $\mathrm{Re}\,\sigma$ and μ from (3.10) gives

$$r^3 - (1+\alpha^2)r^2 + s^2r^2 - s^2 = 0.$$

Implicitly solving for $r'(s)$ we obtain

$$r'(s) = \frac{2s(1-r)}{3r^2 - (1+\alpha^2)2r + s^2} =: \frac{N(s)}{D(s)}.$$

Note that by (ii), $N(s)$ is negative for all $s > 0$ and by (ii) and the assumption that $\alpha^2 < 1/2$, $D(s)$ positive for all $s > 0$.

To show that $R(s) := M^2(s)r(s)$ is monotone we calculate

$$R'(s) = M(s)(2M'(s)r(s) + M(s)r'(s)).$$

When $s > 1$ we have $0 < M < s/\gamma$ and $M'(s) = (s^2+1)/s\gamma > s/\gamma$. Therefore for $s > 1$

$$R'(s) > \frac{M(s)s}{\gamma D(s)}(2r(s)D(s) + N(s)).$$

Finally, by using that $s > 1$, $r > 1$ and $\alpha^2 < 1/2$ we obtain

$$2r(s)D(s) + N(s) > 2r^2(3r - 2(1+\alpha^2) > 0,$$

and consequently $R(s)$ is increasing on $(1, \infty)$.

To show that $M(s)\mu(s)$ is monotone, it will be enough to show that $\mu(s)$ is decreasing on $(0, \infty)$ (since M is increasing). We implicitly differentiate the equation $P_s(\lambda) = 0$ to obtain

$$\mu'(s) = \frac{-(\mu^2+1)}{3\mu^2 + 2s\mu + (1+\alpha^2)}.$$

Using (iv) we find that $\lim_{s\to\infty} \mu'(s) = -1$. Furthermore the numerator is always negative. If the denomonator were to change sign, then μ could not be differentiable at that point, which contradicts the separation of the eigenvalues in (v). Thus the denomator is always negative. It follows that μ is decreasing for $s > 0$. The proof is complete. □

For any set $J \subset \mathbf{C}$ we can define an associated spectral projection $P(J) \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ by

$$P(J)\vec{y} = \frac{1}{2\pi i}\int_\Gamma R(\lambda; A)\,\vec{y}\,d\lambda, \quad \forall\,\vec{y} \in \mathcal{H}$$

where $R(\lambda, A)$ is the resolvent operator of A and Γ is an appropriate contour which encloses the eigenvalues in J. In cases where Γ contains infinitely many eigenvalues, convergence for all $\vec{y} \in \mathcal{H}$ is guaranteed by Proposition 3.1. Let us denote

$$\mathcal{P} = P(\mathbf{R}), \qquad \mathcal{Q} = I - P(\mathbf{R})$$

where I denotes the identity operator on $\mathcal{H}$. Let

$$\Lambda = \mathcal{P}\mathcal{H}, \qquad \Sigma = \mathcal{Q}\mathcal{H}.$$

Since the projections are continuous, it follows that $\mathcal{H}$ is the direct sum the spaces Λ and Σ:

$$\mathcal{H} = \Lambda + \Sigma.$$

It is therefore straight-forward to prove (see [5], Proposition 2.3) for details) the following result.

Proposition 3.5 *Let $\gamma > 0$ and let $W(t)$ denote the semigroup generated by the operator A on $\mathcal{H}$. Then for $t \geq 0$,*

$$W(t) = \mathcal{S}(t)\mathcal{P} + \mathcal{G}(t)\mathcal{Q}$$

where $\mathcal{G}(t)$ extends to a strongly continuous group defined for $t \in \mathbf{R}$ and $\mathcal{S}(t)$ extends to an analytic semigroup defined on $\operatorname{Re} t > 0$. The infinitesimal generators of $\mathcal{S}(t)$ and $\mathcal{G}(t)$ are given by the restriction of A, $A|_\Lambda$ and $A|_\Sigma$, respectively.

We have already defined the spaces $\mathcal{H}_1$, $\mathcal{H}_0$ and $\mathcal{H}_{-1}$. Let us introduce a notation for certain interpolation spaces. Since $0 \in \rho(A)$ and $\sigma(-A)$ in $\{\lambda \in C \mid \operatorname{Re}\lambda > 0\}$, for any $a \in [0,1]$, $(-A)^a$ is well-defined by the spectral theorem. Define for such a,

$$\mathcal{H}_a = \mathcal{D}((-A)^a),$$

with graph norm. Since $A : \mathcal{H}_1 \to \mathcal{H}_0$ is an isomorphism it follows that $(-A)^a : \mathcal{H}_a \to \mathcal{H}_0$ is also an isomorphism. Define $\mathcal{H}_{-a}$ to be the completion of $\mathcal{H}$ with respect to the norm

$$\|\vec{y}\|_{\mathcal{H}_{-a}} = \|(-A)^{-a}\vec{y}\|.$$

Since $\mathcal{D}(A) = \mathcal{D}(A^*)$ it follows that $\mathcal{H}_{-a} = \mathcal{H}_a^*$ where the duality is with respect to $\mathcal{H}_0$.

For $a \in \mathbf{R}$, define the Hilbert spaces $\mathcal{H}^a$ by

$$\mathcal{H}^a = \begin{cases} S_a \times S_{a+1} \times S_{2a} & \text{if } \gamma > 0 \\ S_{2a} \times S_{2a} \times S_{2a} & \text{if } \gamma = 0. \end{cases} \tag{3.11}$$

It is easy to see that $\mathcal{H}^a = [\mathcal{H}_0, \mathcal{H}_1]_a$, i.e., is the space obtained by interpolation between $\mathcal{H}_0$ and $\mathcal{H}_1$. (See [10] for details on interpolation.) It can also be shown (due to the Riesz basis property) that $\mathcal{H}_a = [\mathcal{H}_0, \mathcal{H}_1]_a$. Consequently, using duality we have

$$\mathcal{H}_a = \mathcal{H}^a, \qquad \forall\, a \in [-1, 1].$$

It follows from the spectral theorem that the projections $\mathcal{P}$ and $\mathcal{Q}$, and the semigroups W, $\mathcal{S}$ and $\mathcal{G}$ each have unique continuous extensions to the spaces $\mathcal{H}_\alpha$ (for $-1 \leq \alpha < 0$). We define the spaces Λ_α and Σ_α by

$$\Lambda_\alpha = \mathcal{P}\mathcal{H}_\alpha, \qquad \Sigma_\alpha = \mathcal{Q}\mathcal{H}_\alpha$$

for any $\alpha \in \mathbf{R}$. (We will not make a notational distinction between an operator and its possible extensions.) It follows that (see Weiss [16] for theorems of this type)

$$\mathcal{H}_\alpha = \Lambda_\alpha + \Sigma_\alpha, \qquad \forall\, \alpha \in \mathbf{R}.$$

The spaces Λ_α and Σ_α become Hilbert spaces with the norms $\|\cdot\|_{\Lambda_\alpha}$ and $\|\cdot\|_{\Sigma_\alpha}$ inherited from the Hilbert space $\mathcal{H}_\alpha$. Some relationships among the spaces Λ_β, Σ_β, $\mathcal{H}_\beta$ and S_β defined in Section 1 are given in the following lemma.

Lemma 3.2 *If $\gamma > 0$, then for any $\beta \in [-1, 1]$,*

$$\Lambda_\beta \subset S_{2+2\beta} \times S_{2+2\beta} \times S_{2\beta}, \qquad \Sigma_\beta \subset S_\beta \times S_{1+\beta} \times S_{1+\beta}$$

with continuous inclusion.

Proof: We first note that from eigenvalue estimates (3.6)-(3.7) that for $|\beta| \leq 1$,

$$\Lambda_\beta = \left\{ \sum_{k=1}^{\infty} c_k \phi_{\lambda_{k,3}} : \ (c_k k^{2\beta}) \in l^2 \right\}.$$

Thus if $\vec{y} = (y_1, y_2, y_3)' = \sum_{k=1}^{\infty} c_k \phi_{\lambda_{k,3}} \in \Lambda_\beta$, then by (3.5) and the eigenvalues estimates (3.4),

$$y_1 = \sum_{k=1}^{\infty} c_k \cdot O(k^{-2}) \cdot \sin m_k x \ \in S_{2+2\beta},$$

$$y_2 = \sum_{k=1}^{\infty} c_k \cdot O(k^{-2}) \cdot \sin m_k x \ \in S_{2+2\beta},$$

$$y_3 = \sum_{k=1}^{\infty} c_k \cdot O(1) \cdot \sin m_k x \ \in S_{2\beta}.$$

Hence $\Lambda_\beta \subset S_{2+2\beta} \times S_{2+2\beta} \times S_{2\beta}$ and $\|\vec{y}\|_{S_{2+2\beta}\times S_{2+2\beta}\times S_{2\beta}} \leq M\|\vec{y}\|_{\Lambda_\beta}$. The other inclusion is worked out the same way. □

We will also need the following lemma.

Lemma 3.3 *If $\gamma > 0$, then for any $\beta \in [-1/2, 0]$,*

$$\Sigma_0 + \Lambda_\beta = S_0 \times S_1 \times S_{2\beta}.$$

Proof: If $x \in \Sigma_0 + \Lambda_\beta$ it follows from Lemma 3.2 that also $x \in S_0 \times S_1 \times S_{2\beta}$. So suppose that $x = (x_1, x_2, x_3) \in S_0 \times S_1 \times S_{2\beta}$. Obviously we have that $x \in S_\beta \times S_{\beta+1} \times S_{2\beta}$ and consequently

$$x \in \mathcal{H}^\beta = \mathcal{H}_\beta = \Lambda_\beta + \Sigma_\beta.$$

Let $x = \mathcal{P}x + \mathcal{Q}x = y + w$ where $y \in \Lambda_\beta$ and $w \in \Sigma_\beta$. By Lemma 3.2, $y \in S_{2+2\beta} \times S_{2+2\beta} \times S_{2\beta}$ and $w \in S_\beta \times S_{1+\beta} \times S_{1+\beta}$. However, since $w = x - y$, we actually have $w \in S_0 \times S_1 \times S_{1+\beta} \subset \mathcal{H}_0$. Since Q is a projection we have $\mathcal{Q}^2 = \mathcal{Q}$. Consequently $w = \mathcal{Q}w \in \mathcal{Q}\mathcal{H}_0 = \Sigma_0$. Therefore $x \in \Sigma_0 + \Lambda_{2\beta}$. □

4 Well-posedness and Regularity

In this section we determine the optimal regularity of solutions of the initial-boundary value problem (1.1)-(1.4). We begin with a discussion of the Carleson measure criterion.

Consider the control system

$$\dot{x} = \mathcal{A}x + bu(t) \tag{4.1}$$

where $x(t) \in l^2$ is the state, $u \in L^2(0,\infty)$ is the control function. $\mathcal{A}$ is assumed to be diagonal with diagonal elements $\{\nu_k\}$ which satisfy

$$\sup_{k\in N} \operatorname{Re} \nu_k = \omega_0 < 0, \tag{4.2}$$

and $b \in l^2_{-1}$, i.e., is a column vector with components β_k which satisfy

$$\sum_{k=1}^{\infty} \left| \frac{\beta_k}{\nu_k} \right|^2 < \infty.$$

Thus $\mathcal{A}$ generates a strong continuous diagonal semigroup $\mathcal{W}(t)$ on l^2.

For any $h > 0$ and any $\omega \in \mathbf{R}$ let

$$R(h, \omega) = \{z \in \mathbf{C} : \; 0 \leq \; \mathrm{Re}\, z \leq h, \; |\, \mathrm{Im}\, z - \omega| \leq h\}.$$

Definition 4.1 *With $\mathcal{A}$, b and $\mathcal{W}$ as above, b satisfies the Carlson measure criterion for the semigroup $\mathcal{W}$ if there is some $M \geq 0$ such that for any $h > 0$ and any $\omega \in \mathbf{R}$,*

$$\sum_{-\nu_k \in R(h,\omega)} |\beta_k|^2 \leq M \cdot h. \tag{4.3}$$

The Carlson measure criterion is used to determine the admissibility of the input element b in (4.1). The input element b is admissible for $\mathcal{W}$ if for some $t > 0$, the sequence

$$\left(\int_0^t e^{\nu_k (t-s)} \beta_k v(s) ds \right)_{k \in N}$$

lies in l^2 for all $v \in L^2(0, \infty)$. When b is admissible, for any $\tau > 0$, the operator $\Phi_\tau : \; L^2(0, \infty) \to l^2_{-1}$ defined by

$$\Phi_\tau u = \int_0^\tau \mathcal{W}(\tau - s) b u(s) ds \qquad \forall u \in L^2(0, \infty) \tag{4.4}$$

maps continuously into l^2. In this case, for any initial condition $x_0 \in l^2$ and any $u \in L^2(0, \infty)$ a unique solution of (4.1) is given by

$$x(t) = \mathcal{W}(t) x_0 + \Phi_t u, \tag{4.5}$$

with $x \in C([0, \infty), l^2)$.

Theorem 4.1 (Ho and Russell [6], Weiss [15]) *With b, $\mathcal{A}$ and $\mathcal{W}(t)$ as above, b is admissible for $\mathcal{W}(t)$ if and only if b satisfies the Carlson measure criterion for $\mathcal{W}$.*

For $\alpha \in \mathbf{R}$, we denote $l^2_\alpha = \{(c_k) : \; (|\nu_k|^\alpha c_k) \in l^2\}$.

Definition 4.2 *Let $\alpha \in \mathbf{R}$. With b, $\mathcal{A}$ and $\mathcal{W}$ as above, the pair $(b, \mathcal{W})$ is well-posed on l^2_α if for some $\tau > 0$, the operator Φ_τ defined in (4.1) maps continuously into l^2_α.*

If $(b, \mathcal{W})$ is well-posed on l^2_α then we may define solutions of (4.1) by (4.5), and these solutions are continuous in time with values in l^2_α. It follows easily from Theorem 4.1 that $(b, \mathcal{W})$ is well-posed on l^2_α if and only if $(\beta_k|\nu_k|^\alpha)_{k\in N}$ satisfies the Carleson measure criterion for $\mathcal{W}$.

Now we turn to consider the system (2.10). By Proposition 3.5, the projection $\mathcal{P}$ and $\mathcal{Q}$ continuously decompose the solutions in (2.10) by $\vec{y}(t) = \vec{x}(t) + \vec{z}(t)$ where

$$\vec{x}(t) = \int_0^t \mathcal{S}(t-\tau)(\mathcal{P}b_m m(\tau) + \mathcal{P}b_\delta \delta(\tau))d\tau \quad \text{on } \Lambda_{-1}, \tag{4.6}$$

$$\vec{z}(t) = \int_0^t \mathcal{G}(t-\tau)(\mathcal{Q}b_m m(\tau) + \mathcal{Q}b_\delta \delta(\tau))d\tau \quad \text{on } \Sigma_{-1}. \tag{4.7}$$

Note that the Carlson criterion described above for diagonal systems apply to the system (2.10) since A, W, $\mathcal{G}$, $\mathcal{S}$, etc., can be viewed as diagonal operators on l^2 relative to the Riesz basis of eigenfunctions. Likewise an input element b may be identified with a vector in l^2_{-1} whose components are its respective Fourier coefficients. As such, one can then use the Carleson measure criterion to check well-posedness of the pair (b, W) on $\mathcal{H}_\alpha$.

An analysis of the admissibility of the input elements $\mathcal{P}b_j$, $\mathcal{Q}b_j$, $(j = m, \delta)$ will provide the smoothest spaces Λ_α and Σ_β for which $\vec{x}(t)$ and $\vec{z}(t)$ are time-continuous for all $L^2(0,T)$ inputs. This then determines the spaces of maximal regularity for solutions $\vec{y}(t)$ of the system (1.1)-(1.4). We have the following.

Proposition 4.1 *Let $\gamma > 0$. In the above notations,*

(i) $(\mathcal{P}b_\delta, \mathcal{S})$ is well-posed on Λ_α $\forall\, \alpha \leq -1/4$ and $(\mathcal{Q}b_\delta, \mathcal{G})$ is well-posed on Σ_α $\forall\, \alpha \leq 0$.

(ii) $(\mathcal{P}b_m, \mathcal{S})$ is well-posed on Λ_α $\forall\, \alpha \leq 3/4$ and $(\mathcal{Q}b_m, \mathcal{G})$ is well-posed on Σ_α $\forall\, \alpha \leq 0$.

Proof: Let us prove (i). By (2.8), $b_\delta \in \mathcal{H}_{-1}$, and hence $\mathcal{P}b_\delta \in \Lambda_{-1}$. Therefore its series

$$\begin{aligned} \mathcal{P}b_\delta &= \sum_{k\in\mathbf{N}} < b_\delta, \psi_{\lambda_{k,3}} > \phi_{\lambda_{k,3}} \\ &= \sum_{k\in\mathbf{N}} (D_x(\psi_{\lambda_{k,3}})_3(1))\phi_{\lambda_{k,3}} \equiv \sum_{k\in\mathbf{N}} c_k \phi_{\lambda_{k,3}} \end{aligned}$$

converges in Λ_{-1}. The coefficients (c_k) are easily computed using Proposition 3.1 . One finds that there exist positive constants m and M such that

$$mk < |c_k| < Mk \qquad \forall k \in \mathbf{N}. \tag{4.8}$$

For $k \in \mathbf{N}$ let $\beta_k = c_k/|\lambda_{k,3}|^{1/4}$. The semigroup $\mathcal{S}$ can be identified with the diagonal semigroup $\tilde{\mathcal{S}} \equiv \text{diag}\,(e^{\lambda_{1,3}t}, e^{\lambda_{2,3}t}, \cdots)$ relative to the Riesz basis of eigenfunctions. Thus, $(\mathcal{P}b_\delta, \mathcal{S})$ is well-posed in Λ_α for $\alpha \leq -1/4$ if the sequence (β_k) satisfies the Carleson measure criterion for $\tilde{\mathcal{S}}$. Since the eigenvalues $\lambda_{k,3}$ grow quadratically, (4.8) implies that there are constants $m_1 > 0$ and $M_1 > 0$ for which

$$m_1 k < |\beta_k|^2 < M_1 k, \qquad \forall k \in \mathbf{N}.$$

It follows there are positive numbers m_2, m_3, M_2, M_3 for which

$$m_3|\lambda_{n,3}| < m_2 n^2 < \sum_{k=1}^{n} |\beta_k|^2 < M_2 n^2 < M_3|\lambda_{n,3}|, \qquad \forall n \in \mathbf{N}.$$

Thus if $N \in \mathbf{N}$ and $h = |\lambda_{N,3}|$ we have

$$m_3 h \leq \sum_{-\lambda_{k,3}\in R(h,0)} |\beta_k|^2 \leq M_3 h. \tag{4.9}$$

Thus (4.3) holds and $(\mathcal{P}b_\delta, \mathcal{S})$ is well-posed on Λ_α $\forall\, \alpha \leq -1/4$. The first inequality in (4.8) shows that $\alpha = -1/4$ cannot be increased.

To show that $(\mathcal{Q}b_\delta, \mathcal{G})$ is well-posed on Σ_α $\forall\, \alpha \leq 0$, notice that the eigenvalues $(\lambda_{k,1})$ and $(\lambda_{k,2})$ lie in a vertical strip and their imaginary parts possess a uniform asymptotic separation. From this it is easy to show that (4.3) holds if and only the sequence (β_k) in (4.3) is bounded. Using (2.8) and Proposition 3.1, we can show that there exists positive numbers c_0 and C_0 such that

$$c_0 \leq | < b_\delta, \psi_{\lambda_{k,j}} > | \leq C_0, \qquad \forall\, k \in \mathbf{N},\ j = 1, 2. \tag{4.10}$$

Hence $(\mathcal{Q}b_\delta, \mathcal{S})$ is well-posed on $\Sigma_\alpha \ \ \forall\, \alpha \leq 0$ and $\alpha = 0$ is optimal. Thus we have proved (i). The proof of (ii) is similar. □

Combining the above result with Lemma 3.2 yields the following.

Theorem 4.2 *Let $\gamma > 0$ and $\vec{y}_0 = 0$. If $m \equiv 0$ and $\delta \in L^2(0, \infty)$ then the solution to the system (2.10) belongs to $C([0, \infty),\ S_0 \times S_1 \times S_{-1/2})$. If $\delta \equiv 0$ and $m \in L^2(0, \infty)$ then the solution to the system (2.10) belongs to $C([0, \infty),\ S_0 \times S_1 \times S_1)$. Furthermore the indices β in the spaces S_β are the largest possible.*

In the case $\gamma = 0$ the particular control used (m or δ) makes no difference in the regularity. Proceeding in the same manner that Theorem 4.2 was proved (for its parabolic component) we obtain the following.

Theorem 4.3 *Let $\gamma = 0$ and $\vec{y}_0 = 0$. If $m \equiv 0$ and $\delta \in L^2(0, \infty)$ or if $\delta \equiv 0$ and $m \in L^2(0, \infty)$ then the solution to the system (2.10) belongs to*

$$C([0, \infty),\ \mathcal{H}_{-1/4}) = C([0, \infty),\ \mathcal{S}_{-1/2} \times \mathcal{S}_{-1/2} \times \mathcal{S}_{-1/2}).$$

Furthermore the indices β in the spaces S_β are the largest possible.

Theorem 4.2 together with Theorem 4.3 prove Theorem 1.1.

5 Proof of controllability results

Before proving Theorems 1.2 and 1.3 it will be convenient to review some facts about moment problems.

Consider the moment problem: Find $u \in L^2(0, T)$ such that

$$c_k = \int_0^T e^{s_k t} u(t)\, dt \qquad \forall\, k \in \mathbf{N} \tag{5.1}$$

where (s_k) and (c_k) are given sequences of complex numbers. The *moment space* of (5.1) is the set of sequences (c_k) for which there exist at least one solution u to (5.1).

Let us first recall a result from [3] concerning moment problems of "parabolic type".

Proposition 5.1 *Suppose that there exist positive M, ϵ and $0 \leq \theta < \pi/2$ for which (s_k) satisfies*

(P1) $|\arg(-s_k)| \leq \theta \qquad \forall\, k \in \mathbf{N}$,

(P2) $|s_k - s_j| \geq \epsilon |k^2 - j^2| \quad \forall\, k,\, j \in \mathbf{N}$,

(P3) $M^{-1}k^2 \leq |s_k| \leq Mk^2 \quad \forall\, k \in N$.

Then for any $T > 0$ the moment space to (5.1) contains all sequences (c_k) with the property that for some $p > 0$

$$|c_k| e^{pk} \to 0 \qquad \text{as } k \to \infty. \tag{5.2}$$

Now consider another moment problem: Find $u \in L^2(0,T)$ such that

$$d_k = \int_0^T e^{\omega_k t} u(t)\, dt \qquad \forall\, k \in \mathbf{Z}. \tag{5.3}$$

From [5] (or also see [16] for similar results) we have the following result concerning moment problems of "hyperbolic type".

Proposition 5.2 *Suppose that there exists $\beta \in \mathbf{C}$, $c > 0$ and $(\nu_k)_{k \in \mathbf{Z}} \in l^2$ for which (ω_k) satisfies*

(H1) $\omega_k = \beta + ck\pi i + \nu_k \quad \forall\, k \in \mathbf{Z}$,

(H2) $\omega_k \neq \omega_j$ *unless* $j = k$.

Then if $T \geq 2/c$ the moment space of (5.3) is exactly l^2.

We will also be interested in solving moment problems that have both a parabolic component and a hyperbolic component. In this case, the problem is to find $u \in L^2(0,T)$ which simultaneously solves (5.1) and (5.3). From [5] we have the following result.

Proposition 5.3 *Suppose that* $\{(\omega_k)\}_{k\in\mathbf{Z}} \cap \{(s_k)\}_{k\in\mathbf{N}} = \emptyset$ *and* (s_k) *satisfies the hypothesis of Proposition 5.1 and* (ω_k) *satisfies the hypothesis of Proposition 5.2. Furthermore assume that*

(C1) (c_k) *satisfies the decay condition (5.2),*

(C2) $(d_k) \in l^2$.

Then, for any time $T > 2/c$ *there exists* $u \in L^2(0,T)$ *which simultaneously solves the moment problems (5.1) and (5.3). This is not true if* $T \leq 2/c$.

Let us now return to our control problem. Consider

$$\dot{\vec{y}}(t) = A\vec{y}(t) + bu(T-t), \quad 0 < t < T;\ \vec{y}(0) = \vec{y}_0, \tag{5.4}$$

where A is defined in (2.3)-(2.4), $u \in L^2(0,T)$, b represents b_m or b_δ in (2.8) and $\vec{y}_0$ belongs to an appropriate space which we will specify later. If we wish to find a control that drives the initial state $\vec{y}_0$ to 0 in time T, the variation of parameters formula must hold (on an appropriate space):

$$0 = W(T)\vec{y}_0 + \int_0^T W(s)bu(s)\,ds. \tag{5.5}$$

If a u can be found that solves (5.5) then the corresponding control m or δ given by $m(t) = u(T-t)$ or $\delta(t) = u(T-t)$, as the case may be.

First let us consder the case where $\gamma = 0$.

Proof of Theorem 1.3: When (5.5) is integrated against the eigenfunctions of A^* one obtains the moment problem:

$$c_{k,j} = \int_0^T e^{\lambda_{k,j}t}u(t)dt \qquad k \in \mathbf{N},\ j = 1,2,3, \tag{5.6}$$

where $(\lambda_{k,j})$ are the eigenvalues of A and

$$c_{k,j} = \frac{-e^{\lambda_{k,j}T} < \vec{x}_0, \psi_{\bar{\lambda}_{k,j}} >}{< b, \psi_{\bar{\lambda}_{k,j}} >}. \tag{5.7}$$

By (2.8) and Proposition 3.1 we have $| < b, \psi_{\lambda_{k,j}} > | \geq C_1 > 0$, where C_1 is independent of k and j. In particular, the sequence $(c_{k,j})$ is well defined.

Define $s_{1+3k-j} = \lambda_{k,j}$ and $c_{1+3k-j} = c_{k,j}$ for $k \in \mathbf{N}$ and $j = 1, 2, 3$. Then (5.6) is of the same form as the moment problem (5.1). Using the fact that all the eigenvalues are distinct (see Proposition 3.4) and the explicit formula for the eigenvalues (3.8), it is easy to see that (s_k) satisfies the conditions (P1), (P2) and (P3) of Proposition 5.1 (for appropriate constants). Therefore to complete the proof of Theorem 1.3 it will be enough to show that the sequence (c_k) satisfies the decay condition in (5.2).

By hypothesis, $\vec{x}_0 \in (S_{-1/2})^3 = \mathcal{H}_{-1/4}$. It follows (from Proposition 3.1) that

$$| < \vec{x}_0, \psi_{\lambda_{k,j}} > | \leq C_2 |\lambda_{k,j}|^{1/4},$$

for some $C_2 > 0$. Therefore (since $|\lambda_{k,j}| = O(k^2)$) we obtain $| < \vec{x}_0, \psi_{s_k} > | \leq C_3 k^{1/2}$, for some $C_3 > 0$. By Propositions 3.1 and 3.3 there exists $\epsilon_0 > 0$ such that $\mathrm{Re}\, s_k \leq -\epsilon_0 k^2$, for all $k \in \mathbf{N}$. This implies that

$$|c_k| \leq \frac{C_3 k^{1/2} e^{-\epsilon_0 k^2}}{C_1}.$$

It follows that the sequence (c_k) satisfies (5.2). The proof is complete. □

Let us now prove Theorem 1.2.

Proof of Theorem 1.2: Again we seek $u \in L^2(0,T)$ that satisfies (5.5). Using the same decomposition as in (4.6)-(4.7), we must have

$$-\mathcal{S}(T)\vec{x}_0 = \int_0^T \mathcal{S}(\tau)\mathcal{P}bu(\tau)\, d\tau, \tag{5.8}$$

$$-\mathcal{G}(T)\vec{z}_0 = \int_0^T \mathcal{G}(\tau)\mathcal{Q}bu(\tau)\, d\tau, \tag{5.9}$$

where $\vec{x}_0 = \mathcal{P}\vec{y}_0,\ \vec{z}_0 = \mathcal{Q}\vec{y}_0$. Integration of (5.8) and (5.9) against the eigenfunctions of A^* results in the coupled moment problem: Find $u \in L^2(0,T)$ such that

$$c_k = \int_0^T e^{s_k t} u(t)\, dt \qquad \forall\, k \in \mathbf{N} \tag{5.10}$$

$$d_k = \int_0^T e^{\omega_k t} u(t) dt \qquad k \in \mathbf{Z} - \{0\}, \tag{5.11}$$

where

$$s_k = \lambda_{k,3},\ \omega_k = \lambda_{k,1},\ \omega_{-k} = \lambda_{k,2} \qquad \forall\, k \in \mathbf{N} \tag{5.12}$$

and

$$c_k = \frac{-e^{s_k T} < \vec{x}_0, \psi_{s_k} >}{< b, \psi_{s_k} >}, \qquad d_k = \frac{-e^{\omega_k T} < \vec{z}_0, \psi_{\omega_{-k}} >}{< b, \psi_{\omega_{-k}} >}. \tag{5.13}$$

To complete the proof we need to show that for $T > 2\sqrt{\gamma}$ the hypothesis of Proposition 5.3 is satisfied. (Note that Proposition 5.3 remains valid when the equation corresponding to the index $k = 0$ is omitted since there is one less equation that needs to be solved.)

Since $\alpha < 1/\sqrt{2}$, by Proposition 3.4 we know that all the eigenvalues are distinct. Thus obviously we have that $\{(\omega_k)\} \cap \{(s_k)\} = \emptyset$. Furthermore by Proposition 3.4 and the eigenvalue estimates in Proposition 3.1 it is easily verified that (s_k) satisfies (P1), (P2) and (P3) (for appropriate constants) and (ω_k) satisfies (H1) and (H2) $\beta = -\alpha^2/2\gamma$ and $c = 1/\sqrt{\gamma}$. In particular, note that $T > 2/c$. What remains is to prove that (c_k) satisfies the decay condition in (5.2) and that $(d_k) \in l^2$.

If $b = b_\delta$, then $\vec{y}_0$ is given in $S_0 \times S_1 \times S_{-1/2}$. Therefore by Lemma 3.3, $\vec{y}_0 = \vec{z}_0 + \vec{x}_0$ where $\vec{z}_0 \in \Sigma_0$ and $\vec{x}_0 \in \Lambda_{-1/4}$. On the other hand, if $b = b_m$, then $\vec{y}_0$ is given in $S_0 \times S_1 \times S_1$. In this case we have $\vec{z}_0 \in \Sigma_0$ and $\vec{x}_0 \in \Lambda_0 \subset \Lambda_{-1/4}$.

In either case, we have that $\vec{z}_0 \in \Sigma_0$ and $\vec{x}_0 \in \Lambda_{-1/4}$. The proof that (c_k) satisfies the decay condition (5.2) similar to the proof for the case $\gamma = 0$. (See proof of Theorem 1.3.)

To show that $(d_k) \in l^2$, by (5.13) it will be enough to observe that

(i) the sequence $(|e^{\omega_k t}|)$ is bounded above and below by positive constants,

(ii) $(< \vec{z}_0, \psi_{\omega_{-k}} >) \in l^2$,

(iii) the sequence $(| < b, \psi_{\omega_{-k}} > |)$ is bounded below by positive constants.

We have that (i) follows from Proposition 3.1, (ii) follows from the definition of Σ_0 and (iii) follows from (4.10). This completes the proof. □

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