3.2 Continuity

Def 3.2.1 Let $f: S \to \mathbb{R}$, $c \in S$. f is <u>contin. at c</u>

if $\forall \varepsilon > 0 \ \exists \delta > 0$ such that

if $x \in S$ and $|x-c| < \delta$ then $|f(x) - f(c)| < \varepsilon$

This is very close to saying $\lim\limits_{x \to c} f(x) = f(c)$

– but not exactly the same since for a limit to exist, c needs to be a cluster pt.

<u>example</u> if $f(x) = \begin{cases} 1 & x = 2 \\ 2 & = 3 \end{cases}$

Then $S = \{2, 3\}$. Let $\varepsilon > 0$. Let $\delta = 1/2$

Then if $x \in S$ and $|x - 2| < \delta \Rightarrow x = 2$

here $|f(x) - f(c)| = |f(2) - f(2)| = 0 < \varepsilon$

$\Rightarrow$ f is contin. at 2

In fact, if c is not ~~an~~ ~~accum~~ a cluster point then f is automatically contin. at c.

Prop. 3.2.2 (f, S, c as in Def 3.2.1)

i) If c not a cluster pt $\Rightarrow$ f contin. at c

ii) if c is a clust. pt then f contin at c iff $\lim\limits_{x \to c} f(x) = f(c)$

iii) f is contin. at c iff $\forall$ seq $x_n \to c$ with $x_n \in S$, the seq $f(x_n) \to f(c)$

ex $f(x) = 1/x$ contin (see example 3.2.3)

ex x, 1 are contin.

Prop. 3.2.5 If f, g are contin. at c

then so are $(f+g)$, $f-g$, fg at c

and $\frac{f}{g}$ at c if $g(c) \neq 0$

consequence $x \cdot x = x^2$ contin.

$x \cdot x^2$ contin

$\vdots$

x^n contin

$\Rightarrow$ polynomials contin.

$\Rightarrow$ rational functions are continuous where there is no divide by 0

(rational function $= \frac{p(x)}{q(x)}$ p, q polynom's)

Compositions of contin. functions are contin (Prop. 3.2.7)

(we proved this in class - but see book for pf)

example $\frac{1}{x}$ contin if $x \neq 0$

$x^2 - 1$ contin on $\mathbb{R}$

$\frac{1}{x} \circ (x^2-1) = \frac{1}{x^2-1}$ contin. at nonzero values of x^2-1

i.e at $x \neq \pm 1$

$x^2 - 1 \circ \frac{1}{x} = \frac{1}{x^2} - 1$ contin. $x \neq 0$

$\sin x$ contin. at $x=0$:

pf: use $|\sin x| \le |x|$

let $x_n \to 0$ then $|\sin x_n| \le |x_n|$

$$\Rightarrow \underbrace{-|x_n|}_{\downarrow\, 0} \le \sin x_n \le \underbrace{|x_n|}_{\downarrow\, 0}$$

$\Rightarrow \sin x_n \to 0$ by squeeze lemma

$\cos x$ contin. at 0 ? let $x_n \to 0$. Can assume $|x_n| \le \frac{\pi}{2}$

$\Rightarrow \cos x_n = \sqrt{1-\sin x_n}$

since $\underbrace{\sqrt{x} \text{ contin. at } 1}_{\text{quiz prob}}$ $\sin x$ contin. at 0

$\Rightarrow \sqrt{1-\sin x}$ contin at 0 (composition of contin functions)

$$\Rightarrow \lim_{x\to 0} \sqrt{1-\sin x} = \sqrt{1-\sin 0} = 1$$

$\Rightarrow \cos x$ contin. at 0.

- can then use trig formulas to prove $\sin x$, $\cos x$ contin everywhere

Prop. $\sin x$, $\cos x$ are contin.

Discontin. functions we covered Sec. 3.7.3 pretty much like the book - review that.

Dirichlet function $\quad f(x) = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \end{cases}$

f is discont. at all $c \in \mathbb{R}$:

1) if c is rational then $\overbrace{c + \frac{\sqrt{2}}{n}}^{x_n} \notin Q$

$\lim_{x \to c} f(x) = f(c)$
iff $\forall$ seq $\{x_n\} \to c$
$x_n \neq c$
$f(x_n) \to f(c)$

$f(x_n) = 0 \to 0$
and $c + \frac{1}{n} \in Q$
$f(c + 1/n) = 1 \to 1$

2 different limits $\Rightarrow$ limit ONE

$\Rightarrow$ discontin.

$f(c) = 1$

2) if c not in Q then $f(c) = 0$

$c + 1/n \in$ irrational

$\therefore$ ~~between~~ $\Rightarrow$

$\cancel{c(1+\frac{1}{n})}$ ~~irrational~~

Let $x_n \to c$ where $x_n \in Q$

Popcorn function $\quad f : (0,1) \to \mathbb{R}$ by

$f(x) = \begin{cases} 1/k & \text{if } x = m/k \text{ with } m, k \in \mathbb{N}, \text{ no common factors} \\ 0 & x \text{ irrational} \end{cases}$

If $c \in Q$, by density, there exists a seq $\{y_n\} \in \mathbb{R} \setminus Q$
so $y_n \to c$

$f(y_n) = 0 \to 0 \neq f(c) = 1/k$

If $c \notin Q$, let $\varepsilon > 0$ pick $M : \frac{1}{M} < \varepsilon$.

Then let $S_M = \{ x = \frac{m}{k} : m, k \in \mathbb{N}$ no common factor and $k \leq M \}$

only finitely many such numbers in $(0,1)$

Now let $x_n \to c$. Since $c \notin S_M$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ $x_n \notin S_M \Rightarrow |x_n - 0| < \frac{1}{M} < \varepsilon$

last time : proved that compositions of contin. functions are contin.

ex $f(x) = \sqrt{x} : [0,\infty) \to \mathbb{R}$ contin

pf: let $x_n \to c$ $x_n \geq 0$, $c \geq 0$

then $\lim_{n\to\infty} f(x_n) = \lim \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n} = \sqrt{c}$

$\varepsilon - \delta$: $\lim_{x\to c} \sqrt{x}$ $\quad |\sqrt{x}-\sqrt{c}| = \left|\frac{(\sqrt{x}-\sqrt{c})(\sqrt{x}+\sqrt{c})}{(\sqrt{x}+\sqrt{c})}\right|$

$= \frac{|x-c|}{\sqrt{x}+\sqrt{c}} \leq \frac{|x-c|}{\sqrt{c}} < \frac{\delta}{\sqrt{c}} \leq \varepsilon$

if $\delta = \sqrt{c}\,\varepsilon$

dangly

$\sqrt{x} - 2 = \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(\sqrt{x}+2)}$

$\sqrt{x}$ contin $\sin x$ cont x^2 contin

$\Rightarrow \sqrt{1+\sin^2 x}$ contin.

$\Rightarrow \lim_{x\to 0} \sqrt{1+\sin^2 x} = \sqrt{1+\sin^2 0} = 1$

Dirichlet function $\begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ $f: \mathbb{R}\to\mathbb{R}$

no contin. points $f: [0,1] \to \mathbb{R}$

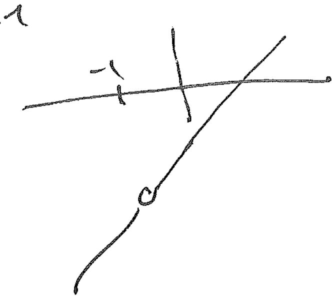
Popcorn function $\begin{cases} \frac{1}{k} & x = \frac{m}{k} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Let $\varepsilon > 0$ let $x_k \to c$ $c \notin \mathbb{Q}$ only finitely many elements $\frac{m}{k}$ with

Removable discontinuity ex $\frac{x^2-1}{x+1} = \begin{cases} x-1 & x \neq -1 \\ \text{undef.} & x = -1 \end{cases}$

f has a rem. discont. at $x = a$ if $\lim_{x\to a} f(x) = L$ exists and the function $\tilde{f}(x) = \begin{cases} f(x) & x \neq a \\ L & x = a \end{cases}$

is contin. at a



3.3 Recall $f: [a,b] \to \mathbb{R}$ is bdd iff $\exists B \in \mathbb{R}$:

$|f(x)| \le B \quad \forall x \in [a,b]$

lemma Let $f: [a,b] \to \mathbb{R}$ contin. Then f is bdd.

pf. If not, $\exists \{x_n\}$: $|f(x_n)| \ge n \quad \forall n$.

BW $\Rightarrow \exists$ subseq $x_{n_k} \to y \in [a,b]$

and $|f(x_{n_k})| \ge n_k \ge k \quad \forall k$

$\Rightarrow f(x_{n_k})$ not convergent

yet by continuity since $x_{n_k} \to y$

$\underbrace{f(x_{n_k}) \to f(y)}_{\text{conv.} \Rightarrow \text{bdd}} \Rightarrow \leftarrow$

$f: S \to \mathbb{R}$ achieves an abs. max at $c \in S$ if

$f(x) \le f(c) \quad \forall x \in S$

abs. min if

$f(x) \ge f(c)$

Thm 3.3.2. Let $f: [a,b] \to \mathbb{R}$ contin. Then f achieves an abs. max and abs. min on $[a,b]$

pf abs max: f is bdd. So $\sup f([a,b])$ exists in $\mathbb{R}$

and $\exists y_n \in f([a,b])$: $y_n \to \underbrace{\sup f([a,b])}_{M}$

(prop 2.1.13)

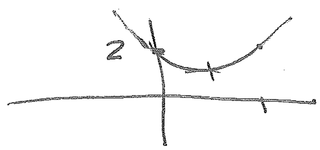
i.e. $\exists x_n \in [a,b]$: $y_n = f(x_n) \to M$

BW $\Rightarrow x_{n_k} \to c \qquad f(x_{n_k}) \to M$

by continuity $f(x_{n_k}) \to f(c)$

$\Rightarrow f(c) = M$

ex $x^2 - x + 2$ on $[-1, 1]$



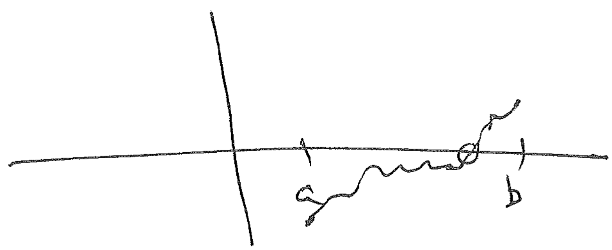
$f(-1)$: abs. max

$f(1/2)$: abs. min

$f(x) = x$ on $(0,1)$

$f(x) = \begin{cases} 1/x & x > 0 \\ 0 & x = 0 \end{cases}$ no max on $[0,1]$

Lemma Let $f: [a,b] \to \mathbb{R}$ contin.
Suppose $f(a) < 0$, $f(b) > 0$.
Then $\exists\, c \in (a,b)$: $f(c) = 0$



pf: Use "bisection method" to make seq's
a_n, b_n such that $f(a_n) \le 0$, $f(b_n) \ge 0$,
$|b_n - a_n| \to 0$

$a_1 = a$ $\quad$ $b_1 = b$ $\quad$ $m_1 = \frac{a_1 + b_1}{2}$

if $f(m_1) > 0$ let $b_2 = m_1$ $\quad$ $a_2 = a_1$ $\quad$ $m_2 = \frac{a_2 + b_2}{2}$

if $f(m_1) < 0$ let $a_2 = m_1$ $\quad$ $b_2 = b_1$

if $f(m_1) = 0$ done $\quad$ repeat

$a_n \nearrow c$ $\quad$ $b_n \searrow d$ $\quad$ $|b_n - a_n| \to 0$
$\Rightarrow c = d$

$f(a_n) < 0$ $\quad$ $f(b_n) > 0$
$\downarrow$ $\quad$ $\downarrow$
$f(c) \le 0$ $\quad$ $f(c) \ge 0$

~~last time~~

<u>Intermediate Value theorem</u> Let $f: [a,b] \to \mathbb{R}$ contin.

Suppose y is between $f(a)$ and $f(b)$. Then there exists c $\in (a,b)$ for which $f(c) = y$.

<u>pf</u>: 2 cases $f(a) < y < f(b)$ or $f(a) > y > f(b)$.

Consider 1st case (pf of 2nd is similar).

Let $g(x) = f(x) - y$. Then g is contin. on $[a,b]$

$g(a) < 0$, $g(b) > 0$

$\Rightarrow \exists\ c : g(c) = 0$ by lemma

$\Rightarrow$ $f(c) = y$ $\square$

<u>Prop. 3.3.10</u> Let f be a polynomial of odd degree. Then f has a real root.

<u>pf</u>: $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ $a_n \neq 0$ n odd.

f has a root iff $\frac{f(x)}{a_n}$ has a root so

can assume f is of the form

$$f(x) = \underbrace{b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}}_{h(x)} + x^n$$

If $|x| > 1$ $|h(x)| \leq |b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}|$

$$\leq |b_0| + |b_1||x| + \cdots + |b_{n-1}||x|^{n-1}$$

$$\leq |b_0||x|^{n-1} + |b_1||x|^{n-1} + \cdots + |b_{n-1}|x^{n-1}|$$

$$= \underbrace{(|b_0| + |b_1| + \cdots + |b_{n-1}|)}_{K}|x|^{n-1}$$

if $|x| > K$ $< |x||x|^{n-1} = |x|^n$

Hence if $x > K+1$ $f(x) = x^n + h(x)$

$$\geq |x|^n - |h(x)| > |x|^n - |x|^n = 0$$

$f(x) > 0$

Corollary to IVP, Min-Max value theorem:

If $f: [a,b] \to \mathbb{R}$ contin then either $f([a,b])$ is a single number or a closed, bounded interval $[c,d]$.

— proof of this a HW prob.

3.4 <u>Unif. Continuity</u>

<u>ex</u> $f(x) = \begin{cases} 2x & x \le 0 \\ 17x & x \ge 0 \end{cases}$

ε-δ pf of continuity at $\underline{x=-1}$: Let $\varepsilon > 0$ pick $\delta = \varepsilon/2$

then if $|x - -1| < \delta$, i.e., $|x+1| < \delta$ then

$$|f(x) - f(-1)| = |2x - -2| = 2|x+1| < 2\delta = \varepsilon$$

ε-δ pf at $\underline{x=3}$: Let $\varepsilon > 0$ pick $\delta = \varepsilon/17$

then if $|x - 3| < \delta$ then $|f(x) - f(3)| = |f(x) - 51|$

$$= |17x - 51| = 17|x-3|$$
$$< 17\delta = \varepsilon \quad \square$$

Note that $\delta = \varepsilon/17$ works for $x > 0$ or $x < 0$ — called a <u>uniform delta</u>

Def Let $S \subset \mathbb{R}$ $f: S \to \mathbb{R}$. Suppose $\forall\, \varepsilon > 0$ there exists $\delta > 0$ such that whenever $x, c \in S$ and $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$

then f is <u>uniformly continuous</u>.

Main point: the δ in this def is independent of x or c — it works for all possibilities.

So $f(x)$ in example above is unif. contin.

<u>example</u> $f(x) = x^2$ on $[0,1]$ unif contin.

pf: Let $\varepsilon > 0$, let $\delta = \varepsilon/2$. Then

if $|x-c| < \delta$ with x, c in $[0,1]$,

$$|f(x) - f(c)| = |x^2 - c^2| = |x-c||x+c|$$
$$< \delta |x+c| \le \delta(|x| + |c|)$$
$$\le \delta(1+1)$$
$$= 2\delta = \varepsilon \quad \square$$

But $f(x) = x^2$ on $[0,\infty)$ is not unif contin.

Negation of unif. contin:

$\exists\, \varepsilon > 0$ such that $\forall\, \delta > 0$

there exists x and c with $|x-c| < \delta$ and $x, c \ge 0$

yet $|f(x) - f(c)| \ge \varepsilon$

Take $\varepsilon = 1$, let $\delta > 0$. Then

$$|f(x+\tfrac{\delta}{2}) - f(x)| = (x+\tfrac{\delta}{2})^2 - x^2$$
$$= \delta x + \frac{\delta^2}{4}$$
$$= \delta(x + \tfrac{\delta}{4})$$
$$\ge \delta x > \varepsilon$$

if $x > \frac{\varepsilon}{\delta}$.

since $|(x+\frac{\delta}{2}) - x| = \frac{\delta}{2} < \delta$

this shows f not unif. contin.