1.7 multiple eigenval's Let A be an $n \times n$ matrix, λ an eigenvalue of multiplicity $m \le n$

Def: Generalized eigenvector: Any nonzero sol. of

$$(A-\lambda I)^k = 0 \qquad k = 1, 2, \dots m$$

Def: N is nilpotent order k iff $N^{k-1} \ne 0$ but $N^k = 0$

Thm Let A be a real $n \times n$ matrix with eigenval's $\lambda_1, \lambda_2, \dots \lambda_n$ (repeated to multiplicity).

Then $\exists$ basis $\{V_1, V_2, \dots, V_n\}$ of generalized ~~eigenvectors~~ eigenvectors.

Moreover, $P = [V_1 \cdots V_n]$ is invertible and

$$A = S + N \quad \text{where} \quad S \text{ is diagonalizable:}$$

$$P^{-1} S P = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

and N is nilpotent order $k \le n$, and

$$SN = NS$$

Cor $\dot{x} = Ax \quad x(0) = x_0$ with A as above, has sol:

$$x(t) = P e^{\Lambda t} P^{-1} \left[I + Nt + \cdots + \frac{N^{k-1} t^{k-1}}{(k-1)!} \right] x_0$$

ex 1 $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad (3-\lambda)(1-\lambda) + 1 = 0 \Rightarrow \lambda^2 - 4\lambda + 4 = 0$

$\lambda = 2, 2.$

$$(A-\lambda I)V = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} V = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow V = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

also note $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow (A-\lambda I)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

and $(A-\lambda I)^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (A-\lambda I)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{0}$

$$\Rightarrow P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

Note that $\Lambda = \begin{bmatrix} 2 & \\ & 2 \end{bmatrix} \Rightarrow S = P \Lambda P^{-1} = \begin{bmatrix} 2 & \\ & 2 \end{bmatrix}$

$$N = A - S = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow N^2 = 0$$

Thus $e^{Nt} = I + Nt = \begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}$

$$\Rightarrow x(t) = \underbrace{P e^{\Lambda t} P^{-1}}_{\begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}} \left[I + Nt \right] x_0$$

$$= e^{2t} \begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix} x_0$$

<u>ex 3</u> $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ $\lambda = 1, 2, 2$

eigenvectors $V_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $\underbrace{V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{1-dim eigenspace}}$

$\underbrace{A - 2I = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}}_{\text{rank 2}} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ generates eigenspace (1-d eigenspace)

$\therefore$ need to find 1 (more) generalized e.vect.

Note $(A - 2I)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\therefore \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ works

$$\Rightarrow P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$S = P \Lambda P^{-1} = P \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$N = A - S = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow N^2 = 0$$

$$X(t) = P e^{\Lambda t} P^{-1} [I + Nt] x_0$$

– full sol. in book.

Multiple complex eigenvalues $\quad \lambda_j = a_j + ib_j \,,\ \bar{\lambda}_j = a_j - ib_j$
– see book – $\quad j = 1, \dots n$

Main thm (Cor. 2)

$$X(t) = P \operatorname{diag}\left(e^{a_j t} \begin{bmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix} \right) P^{-1} \left(I + \dots + \frac{N^k t^k}{k!} \right) x_0$$

main point: sol' involves products & sums of (polynomials degree $\leq n$), sinusoids, exponentials where arguments of sinusoids, exponentials are determined from eigenvalues

Jordan Forms

Real eigenvalues : λ_j – eigenvalues for $j = 1, \dots k$
Gen. eigenvect's $V_1, V_2, \dots V_k$ basis for eigenspaces of real eigenval's

$$P = [V_1, \dots V_k]$$

$$P^{-1} A P = B = \operatorname{diag}(B_1, \dots, B_r)$$

each B_i of the form $\begin{bmatrix} \lambda & 1 & & \\ 0 & \lambda & \ddots & \\ & & \ddots & 1 \\ & & 0 & \lambda \end{bmatrix}$ ← Jordan Block

Complex eigenval's : $\lambda_j = a_j + ib_j \qquad \bar{\lambda}_j = a_j - ib_j$

basis of gen. eigenvectors for complex eigenspace $W_j = U_j \pm i V_j$

$$\Rightarrow P = [V_1\ U_1\ V_2\ U_2 \dots V_k\ U_k]$$

$$P^{-1} A P = \operatorname{diag}(B_1, \dots B_n)$$

B_i of the form $\begin{bmatrix} D & I_2 & & \\ 0 & D & \ddots & \\ \vdots & & \ddots & I_2 \\ 0 & \dots & 0 & D \end{bmatrix} \qquad D = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

Jordan forms for 2x2 blocks

a) $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ and b) $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

a): basis of eigenvectors

b) eigenvector and ~~2~~ gen. eigenvect

For 3x3

$\underbrace{\begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix}}_{\text{basis of eigenvects}}$, $\underbrace{\begin{bmatrix} \lambda & 1 & \\ & \lambda & \\ & & \lambda \end{bmatrix}}_{\text{2 lin. indep eigenvect's + 1 order 2 gen. eigenv.}}$, $\begin{bmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$

basis of eigenvects: $A v_i = \lambda v_i$, $i = 1, 2, 3$

2 lin. indep eigenvect's, 1 ~~order 2~~ order 2 gen. eigenv.:
$(A - \lambda I) v_3 = v_2$,
v_1, v_2 eigenvects
$\{v_2, v_3\}$ Jordan chain

1 eigenvect (gen eigenvect. ord 1)
1 gen eigenvec ord ~~er~~ 2
1 " " order 3

$(A - \lambda I) v_3 = v_2$
$(A - \lambda I) v_2 = v_1$
v_1 : eigenvector
$\{v_1, v_2, v_3\}$ Jordan chain

Ex 3

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix} \qquad \lambda = 2, 2, 2.$$

$$(A - \lambda I) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{\text{rank } 1} \qquad \text{Eigenspace } \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\} \triangleq E$$

rank 1 $\Rightarrow$ 2 lin indep eigenvects

need 1 gen. eigenvect w

so $(A - \lambda I) w = v_1$, where $v_1 \in E$

if $w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in E$ ✓

now pick v_2 so $\{v_1, v_2\}$ basis for E e.g. $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \Rightarrow J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = P^{-1} A P$$

Jordan chain

e^{At} when $A = PJP^{-1}$ (real eigenval's)

$$e^{At} = P e^{Jt} P^{-1} \quad \text{where}$$

$$e^{Jt} = \operatorname{diag}\left(e^{B_j t}\right) \qquad B_j = \begin{bmatrix} \lambda_j & & & \\ & \lambda_j & & \\ & & \ddots & \\ & & & \lambda_j \end{bmatrix} + N$$

N: 1's on superdiagonal

$$e^{B_j t} = e^{\lambda_j t} e^{Nt}$$

e.g.

$$e^{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} t} = I + tN + \frac{t^2}{2} N^2$$

$$= \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} t} = \begin{bmatrix} 1 & t & t^2/2 & t^3/3! & t^4/4! \\ 0 & 1 & t & t^2/2! & t^3/3! \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{N^2} ; \; N^3 = 0$$

1.9

Stable, unstable, center subspaces E^s, E^u, E^c

for $\dot{x} = Ax$

A: $n \times n$ with eigenval's $\lambda_j = \underbrace{a_j \pm i b_j}_{\text{corresp. to } u + iv}$ or $\underbrace{\lambda_k = a_k}_{\text{eigenvect } u}$ if real.

$E^s = \text{Span}\{$ gen. eigenvect's corresponding to $a_j < 0\}$

$E^u = \text{Span}\{$ " " " " $a_j > 0\}$

$E^c = \text{Span}\{$ " " " " $a_j = 0\}$

ex 1 $\quad A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\qquad \lambda = -2 \pm i, \quad 3$

for $\lambda = 3$: $\quad v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\qquad$ for $\lambda = -2 + i$ $\quad w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$E^u = \text{span} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\qquad E^s = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$

ex 2 $\quad A = \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$ $\qquad \lambda = i$: $\quad E^c = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$

$E^u = \text{span} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

ex 3 $\quad A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\qquad \lambda = 0, 0$ $\quad$ eigenvect. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad$ gen. eigenvect $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$E^c = \mathbb{R}^2$$

sol's $\quad e^{At} = I + t \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$

$x(t) = \begin{bmatrix} x_1(0) \\ t x_1(0) + x_2(0) \end{bmatrix}$ $\qquad$ some bounded sol's, other sol's unbounded

<u>Def 2</u> If all eigenvalues of A have a <u>nonzero real part</u>

then the "flow" $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$

is a <u>hyperbolic flow</u> and

$\dot{x} = Ax$ is a hyperbolic DE

<u>Def 3</u> Subspace E is invariant with respect to the flow e^{At}

if $e^{At} E \subset E$ for $t \in \mathbb{R}$

<u>Lemma</u> Generalized eigenspaces are flow invariant

— see book —

example: suppose v_1, v_2 are eigenvectors of A

$$\Rightarrow e^{At}(\alpha v_1 + \beta v_2) = \alpha e^{At} v_1 + \beta e^{At} v_2 \qquad \text{if } Av_1 = \lambda_1 v_1,\ Av_2 = \lambda_2 v_2$$

$$\Rightarrow \alpha e^{\lambda_1 t} v_1 + \beta e^{\lambda_2 t} v_2$$

$$\in \operatorname{span}(v_1, v_2) \quad \forall t \in \mathbb{R}$$

<u>Thm 7</u> $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$

where E^s, E^u, E^c are flow invariant w.r.t. e^{At}.

<u>Def 4</u> If All eigen val's of A have negative real part

$\Rightarrow$ origin is a <u>sink</u>

positive real parts $\Rightarrow$ origin is a <u>source</u>.

1.9 Thm 2 TFAE:

a) $\forall x_0 \in \mathbb{R}^n$ $(x_0 \neq 0)$ $\{ e^{At} x_0 \to 0$ as $t \to \infty$ and $|e^{At} x_0| \to \infty$ as $t \to -\infty \}$

b) All eigenval's have neg. real part.

c) $\exists$ positive constants a, c, m, M :

$$|e^{At} x_0| \leq M e^{-ct} |x_0| \quad \forall t \geq 0$$

and

$$|e^{At} x_0| \geq m e^{-at} |x_0| \quad \forall t \leq 0$$

~~Idea of pf:~~

~~$a \Rightarrow b$: If $\lambda > 0$ or $\lambda = a + ib$ with $a > 0$~~
~~a jordan block of $J = P^{-1}AP$~~
~~has a solution with form~~
~~(for real case): $e^{\lambda t}\left[I + Nt + \cdots + \frac{N^k t^k}{k!}\right] \to \infty$ as $t \to \infty$~~

~~$\Rightarrow x(t) = Pe^{Jt}c \to \infty$ as $t \to \infty$~~

~~$b \Rightarrow c$: if $\operatorname{Re} \lambda_j < \alpha < -c < 0$~~
~~then any term of the form $\Big|\underbrace{(\text{poly degree } k)}_{p(t)} \underbrace{(\text{sinusoid})}_{s(t)} e^{\lambda t}\Big|$~~
~~$\leq \left| \left(p(t) s(t) e^{-\varepsilon t}\right) e^{-ct} \right|$~~
~~for ε suff small $(c + \varepsilon < \alpha)$~~
~~$\leq M e^{-ct} \quad \forall t \geq 0$~~

~~similarly $|e^{At} x_0| \leq \tilde{M} e^{-ct} |x_0| \quad \forall t \geq 0$~~

~~$c \Rightarrow a$: follows from squeeze lemma~~

next page

Idea of pf $\quad a \Rightarrow b$: Consider the case of real eigenvalues.

Suppose $\lambda = a \geq 0$. $\Rightarrow e^{At}v = e^{\lambda t}v = e^{at}v$

where $Av = \lambda v$. $\therefore$ ~~with~~ if $a \geq 0$,

let $x_0 = v$ then $e^{at}x_0 \not\to 0$ $\rightarrow \leftarrow$

Similar situation occurs when $\lambda = a + ib$ $\quad a \geq 0$.

$b \Rightarrow c$ e.g. consider a term like $|p(t)\, s(t)\, e^{\lambda t}|$

$p(t)$: polynom degree k

$s(t)$ sinusoid

$\lambda = a + ib$

$a < 0$.

If $\mathrm{Re}\,\lambda < -c < 0$ then

$$|p(t)s(t)e^{\lambda t}| \leq |p(t)e^{at}|$$

$$= |p(t)e^{-\varepsilon t}e^{-ct}| \qquad \lambda = \bullet - (\varepsilon + c)$$

$(p(t)e^{-\varepsilon t}$ bdd on $[0,\infty))$

$$\leq M e^{-ct} \quad \forall\, t > 0$$

Since all terms in e^{At} are of this form $\|e^{At}\| \leq \tilde{M} e^{-ct} \quad \forall\, t > 0$

$$\Rightarrow |e^{At}x_0| \leq \tilde{M}e^{-ct}|x_0| \quad \forall\, t > 0$$

Similarly $|e^{At}x_0| \geq m e^{-at}|x_0| \quad \forall\, t < 0$

(here $-a < \mathrm{Re}\,\lambda$)

$c \Rightarrow a$ follows from squeeze lemma

~~Thm 3~~ Note: suppose $\frac{dx}{dt} = Ax$ where A has eigenval's with positive real parts. Then

let $\tau = -t \Rightarrow \frac{dx}{d\tau} = (-A)x$.

and $(-A)$ has eigenval's with all neg. real parts,

$\therefore$ Thm 2 applies to $(-A)$, with time reversed.

or

TFAE (Thm 3)

a) $\forall x_0 \in \mathbb{R}^n - \{0\}$ $\{ e^{At} x_0 \to 0$ as $t \to -\infty$

and

$|e^{At} x_0| \to \infty$ as $t \to \infty \}$

b) all eigenval's have pos. real parts

c) $\exists$ consts pos a, c, m, M s.t.

$$|e^{At} x_0| \le M e^{+ct} |x_0| \quad \forall t \le 0$$

and

$$|e^{At} x| \ge m e^{+at} |x| \quad \forall t \ge 0$$

<u>Corollary</u> If $x_0 \in E^s$ then $e^{At}x_0 \in E^s \quad \forall t \in \mathbb{R}$

and $\lim_{t\to\infty} e^{At}x_0 = 0.$

And if $x_0 \in E^u$ then $e^{At}x_0 \in E^u \quad \forall t \in \mathbb{R}$

and $\lim_{t\to -\infty} e^{At} x_0 = 0$

<u>1.10</u> $\Phi(t)$ ($n\times n$ matrix) is a fundamental matrix if

$$\dot{\Phi} = A\Phi \quad \forall t \in \mathbb{R}$$

e.g. e^{At} is a fund. matrix.

Var. of parameters sol. to $\dot{x} = Ax + b(t)$: $(x(0) = x_0)$

$$x(t) = \Phi(t)\Phi^{-1}(0)x_0 + \int_0^t \Phi(t)\Phi^{-1}(\tau)\, b(\tau)\, d\tau$$

or in case of $\Phi(t) = e^{At}$,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} b(s)\, ds$$

<u>ex</u> $\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ t \end{bmatrix} \qquad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$e^{At} = e^t e^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}t} = e^t\left[I + Nt\right] = e^t\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$x(t) = e^t\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t e^{t-s}\begin{bmatrix} 1 & t-s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ s \end{bmatrix} ds$$

$$= \begin{bmatrix} e^t \\ 0 \end{bmatrix} + \int_0^t e^{t-s}\begin{bmatrix} 1 + (t-s)s \\ s \end{bmatrix} ds$$

$$= \begin{bmatrix} e^t \\ 0 \end{bmatrix} + e^t \int_0^t \begin{bmatrix} e^{-s}(1 + (t-s)s) \\ e^{-s}s \end{bmatrix} ds$$