HW Perko 1.2: 3a   1.3 1a, b   5 ~~a~~bc 7c
1.4 2, 4
5ab
1.5 1a,bc,
3

Thm Let A be $n\times n$, $x_0 \in \mathbb{R}^n$. Then the initial value problem

$$\dot{x} = Ax$$
$$x(0) = x_0$$

has the unique sol.

$$x(t) = e^{At}x_0$$

pf: We have shown $e^{At}x_0$ is a sol. let ~~x(t)~~ ~~be~~ $y(t)$ be another solution, and let

$$z(t) = e^{-At}y(t).$$

$$\Rightarrow z'(t) = -Ae^{-At}y(t) + e^{-At}\dot{y}$$
$$= -Ae^{-At}y(t) + e^{-At}Ay$$
$$= 0 \Rightarrow z = z(0) = y(0)$$
$$\Rightarrow y(t) = e^{At}y(0) \quad \square.$$

1.5 Classification of linear systems in $\mathbb{R}^2$

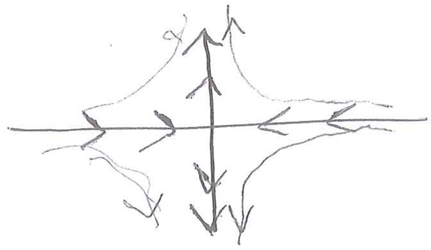
- describe phase portraits of $\dot{x} = Ax$ -

1st Assume A is diagonalizable with

$$B = P^{-1}AP \quad ; \text{ let } x = Py$$

$$\dot{y} = By \qquad B = P^{-1}AP$$

Case I $\quad B=\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}$ $\quad \lambda<0<\mu$



Saddle at $\vec{0}$

(similar if $\mu<0<\lambda$)

Case II $\quad B=\begin{bmatrix}\lambda & \\ & \mu\end{bmatrix}$ $\quad \lambda\le\mu<0$ or $\lambda=\mu<0$ with $B=\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}$

$$\underbrace{\qquad}$$

$$e^{Bt}=e^{\lambda I t}\,e^{\begin{bmatrix}0&1\\0&0\end{bmatrix}t}$$

$$=e\,e^{\lambda t}\left[I+\begin{bmatrix}0&t\\0&0\end{bmatrix}\right]\vec{c}$$

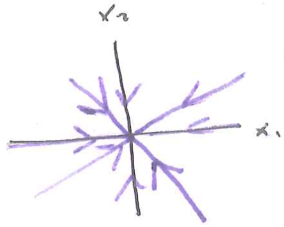
$$=e^{\lambda t}\begin{bmatrix}1&t\\0&1\end{bmatrix}\vec{c}$$

$$=\begin{bmatrix}e^{\lambda t}(c_1+c_2t)\\ e^{\lambda t}c_2\end{bmatrix}$$

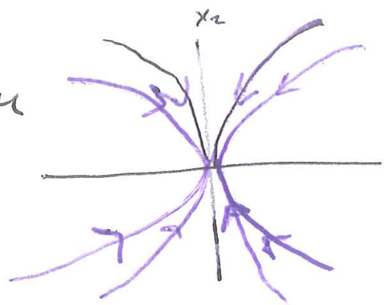
$$=e^{\lambda t}\begin{bmatrix}c_1+c_2t\\ c_2\end{bmatrix}.$$

(proper node)

$\lambda=\mu$



$\lambda<\mu$

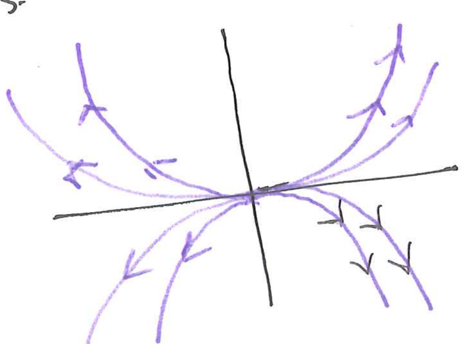


$y=\begin{pmatrix}c_1e^{\lambda t}\\ c_2e^{\mu t}\end{pmatrix}$

"proper" stable nodes

if $0<\lambda\le\mu$, very similar but origin is unstable — "unstable nodes"

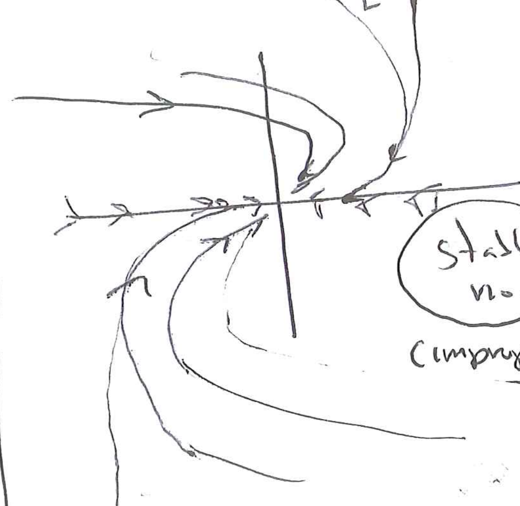
e.g. $0<\lambda<\mu$ $\quad B=\begin{bmatrix}\lambda & \\ & \mu\end{bmatrix}$



$y(t)=\begin{bmatrix}c_1e^{\lambda t}\\ c_2e^{\mu t}\end{bmatrix}$

$\Rightarrow K|x|^{\alpha}=|y|$

as $t\to\infty$ $\sim\begin{bmatrix}c_2te^{\lambda t}\\ 0\end{bmatrix}$



stable node

(improper)

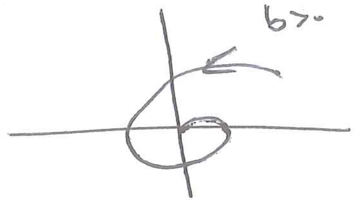
<u>Case III</u> $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$$e^{Bt} = e^{at} \underbrace{\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}}_{\text{rotation through angle } bt}$$

$\boxed{\lambda = a \pm ib}$

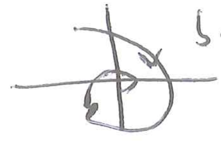
<u>$a<0$</u>

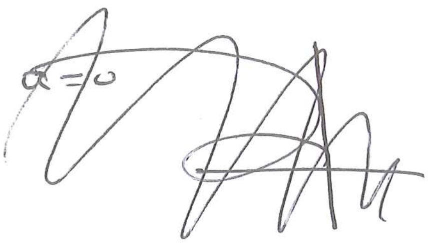
$b>0$



"stable focus"

$b<0$

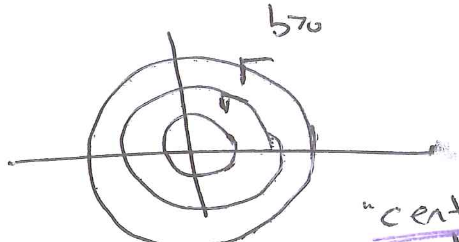




if $a>0$, spirals away from $\vec{0}$
"unstable focus"

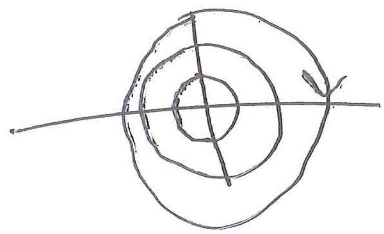
<u>$a=0$</u>

$b>0$



"center" at $\vec{0}$

$b<0$

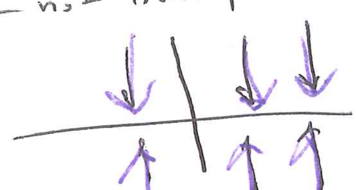


$\boxed{\lambda = \pm ib}$

<u>degenerate equilibria</u> if $\lambda = 0$ & (Degenerate equilibria correspond to one or more 0 eigenvalues)

e.g. $\lambda = \mu = 0$
$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; everywhere is a crit. pt — not isolated — crit. pts: $\mathbb{R}^2$

if $\lambda = 0$,



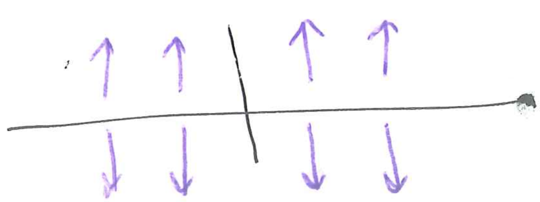
$\mu<0$

crit. pts: x_1 axis

$\mu<0$

$A = \begin{bmatrix} 0 & 0 \\ 0 & \mu \end{bmatrix}$

$x = \begin{bmatrix} c_1 \\ e^{\mu t} c_2 \end{bmatrix}$



$\mu>0$

possibilities (for nondegenerate cases): saddle, node, focus, center

Theorem Let $\delta = \det A$, $\tau = \text{trace } A$ then for $\dot{x} = Ax$,

a) $\delta < 0 \Rightarrow$ saddle

b) $0 < \delta \le \tau^2/4 \Rightarrow$ node

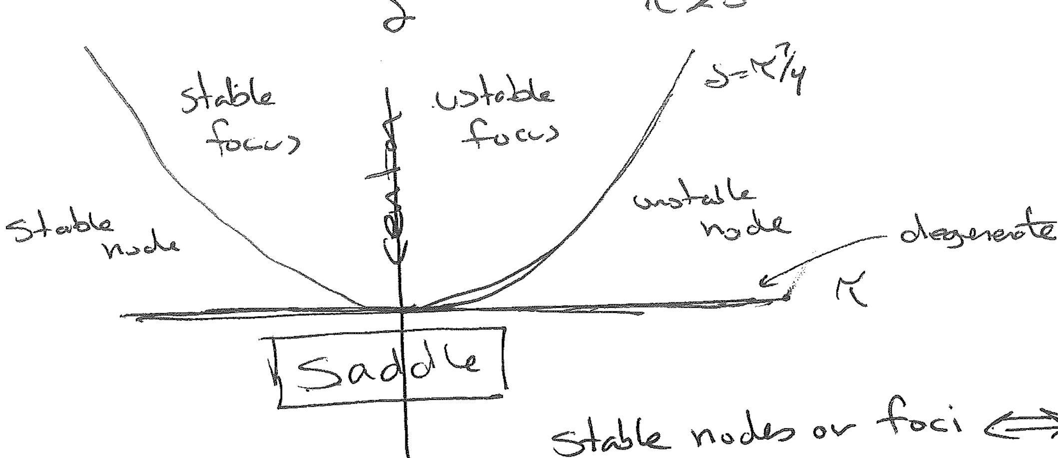
($\tau < 0 \Rightarrow$ stable, $\tau > 0 \Rightarrow$ stable)

c) $\delta \ge \tau^2/4$, $\tau \ne 0 \Rightarrow$ focus or center

$\tau < 0$ stable

$\tau = 0$ center

$\tau > 0$ unstable



stable nodes or foci $\Longleftrightarrow$ "sinks"

unstable " " " $\Longleftrightarrow$ "sources"

Ex $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$\delta = 3 + 2 = 5$

$\tau = 1 + 3 = 4 \Rightarrow \tau^2/4 = 4$ so

$\delta > \tau^2/4$

$\Rightarrow$ unstable focus

To work out in detail,

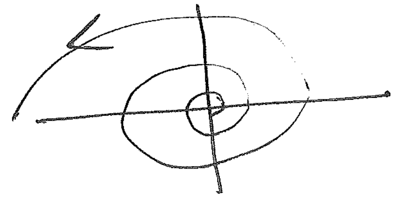
$(1-\lambda)(3-\lambda) + 2 = \lambda^2 - 4\lambda + 3 + 2 = \lambda^2 - 4\lambda + 5$

$= \lambda^2 - 4\lambda + 4 + 1$

$= (\lambda - 2)^2 + 1$ $\quad \lambda = 2 \pm i$

$\Rightarrow B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

$e^{Bt} = e^{2t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$



How to find P   Sec. 1.6

Suppose A is $2n \times 2n$ and has complex conj. pair of eigenvalues

$$\lambda_j = a_j + i b_j$$
$$\bar{\lambda}_j = a_j - i b_j$$

$$W_j = U_j + i V_j \qquad \text{eigenvectors of } \lambda_j,$$
$$\bar{W}_j = U_j - i V_j \qquad \bar{\lambda}_j$$

let $P = \begin{bmatrix} V_1 & U_1 & V_2 & U_2 & \cdots & V_n & U_n \end{bmatrix}$

then $P^{-1}AP$ is the block diagonal matrix with

jth block
$$B_j = \begin{bmatrix} a_j & -b_j \\ +b_j & a_j \end{bmatrix}$$

(or use $Q = \begin{bmatrix} U_1 & V_1 & U_2 & V_2 & \cdots & U_n V_n \end{bmatrix} \Rightarrow B_j = \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}$)

$$\therefore \quad P^{-1}AP = B = \operatorname{diag}\left(\{B_j\}_{j=1}^{n}\right)$$

Ex
$$A = \left[\begin{array}{c|cc} -3 & 0 & 0 \\ \hline 0 & 3 & -2 \\ 0 & 1 & 1 \end{array}\right] \qquad A - \lambda I = \left[\begin{array}{c|cc} -3-\lambda & \multicolumn{2}{c}{\vec{0}} \\ \hline \vec{0} & 3-\lambda & -2 \\ & 1 & 1-\lambda \end{array}\right]$$

$$|A - \lambda I| = (-3-\lambda)\left|(3-\lambda)(1-\lambda) + 2\right|$$
$$= -(3+\lambda)(\lambda^2 - 4\lambda + 5) = \cancel{-(3+\lambda)(\ )}$$
$$\lambda = -3, \ 2 \pm i$$

$\lambda_1 = -3$

$\lambda_2 = 2 + i$

$\cancel{\lambda_3 =}$

$\underline{\lambda = -3}$ $\quad V = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\lambda = 2 + i \qquad \underbrace{\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}}_{\text{eigenvect: } \begin{bmatrix} 1+i \\ 1 \end{bmatrix}} \begin{bmatrix} x \\ 1 \end{bmatrix} \Rightarrow x = 1 + i$$

For 3×3: $W = U + iV = \begin{bmatrix} 0 \\ 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\underbrace{P^{-1}AP}_{B} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

unstable focus

$$e^{Bt} = \begin{bmatrix} e^{-3t} & \\ & e^{B_1 t} \end{bmatrix} = \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}\cos t & -e^{2t}\sin t \\ 0 & e^{2t}\sin t & e^{2t}\cos t \end{bmatrix}$$

– sol's have $y_1(t) \to 0$
$\{y_2, y_3\}$ spirals $\to \infty$

x – system

$x(t) = P e^{Bt} P^{-1}$

– see book

but $x = Py$ takes x_1 axis $\to x_1$ axis
" x_2, x_3 plane $\to x_2, x_3$ plane

x_1 axis : "stable subspace"
$x_2 - x_3$ plane : "unstable subspace"