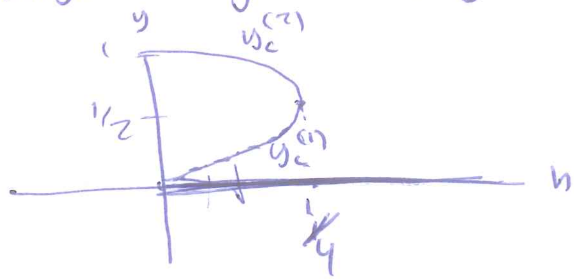
Summary of Fishfarm problem

we analyzed $y' = (1-y)y - h$. In this setting



$0 \le h < 1/4$: If $y_0 > y_c^{(1)}(h) \Longrightarrow$

$y \to y_c^{2}(h)$ as $t \to \infty$

If $y_0 < y_c^{(1)}(h)$

$y \to -\infty$ (model flawed, but population dies out)

~~If $y_0 > y_c$~~

$h = 1/4$: $y_0 \ge 1/2 \to y \to 1/2$ as $t \to \infty$

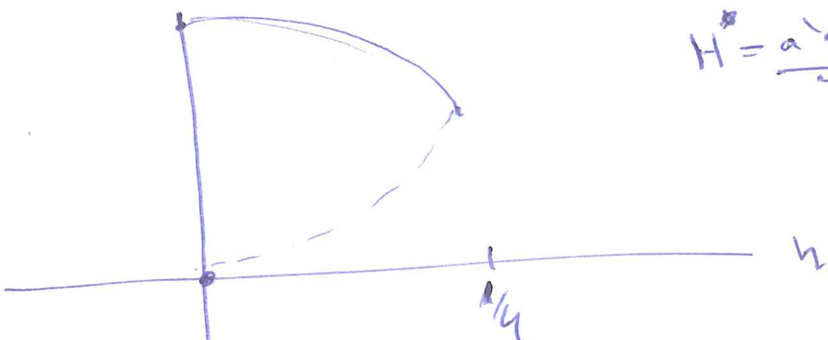
$y < 1/2 \longrightarrow$ population $\to 0$

$h > 1/4$: ~~population~~ $y \to -\infty$

Population pers $\quad y' = ay\left(1 - \frac{y}{Y_c}\right) - H$

$$y_c^{(1)}, y_c^{(2)} = \frac{Y_c}{2}\left(1 \pm \sqrt{1-4h}\right) ; h = \frac{H}{aY_c}$$

$H^* = \frac{aY_c}{4}$: bifurcation point



$H > \frac{aY_c}{4}$: fish $\to 0$

$H < \frac{aY_c}{4}$, $Y_0 > y_c^{(1)} \Longrightarrow y \to y_c^{(2)} = \frac{H}{2a}\left(1 + \sqrt{1 - \frac{4H}{aY_c}}\right)$

For open ocean fishing, const. rate h unrealistic

better: $h \sim$ fish population

but this also not realistic —

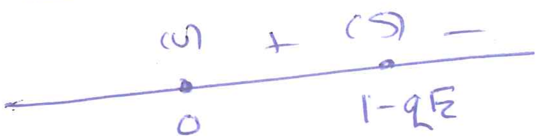
if many boats, more fish

$\therefore$ also prop. to effort E

$\therefore$ assume $h(t) \sim E y(t)$
$$= qEy(t)$$

$$\therefore \; y' = y(1-y) - h$$
$$= y(1-y) - qEy$$
$$= y(1-qE-y) \equiv f_E(y)$$

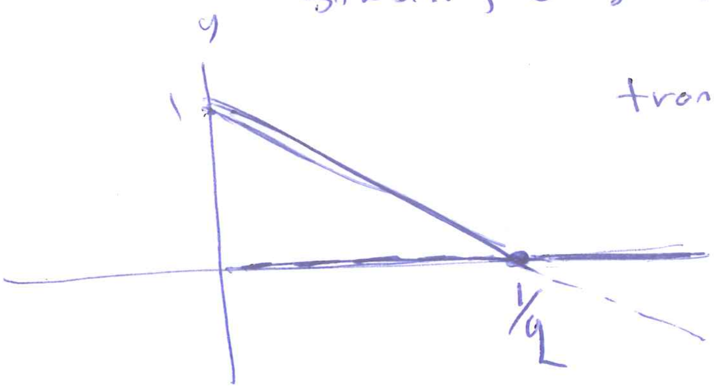
crit. pts $y=0$, $y = 1-qE$



with E the bifurcation parameter

stability changes at $E = 1/q$

transcritical bif. pt



4.3

Perko ch 1

Let A be a $n\times n$ matrix, $x(t)\in\mathbb{R}^n$, $t\in\mathbb{R}$.

Consider $\frac{dx}{dt}=Ax \qquad x(0)=x_0 \qquad (1)$

How to solve (1).

- If A is diagonal $\frac{dx_1}{dt}=a_{11}x_1 \quad \frac{dx_2}{dt}=a_{22}x_2 \cdots \frac{dx_n}{dt}=a_{nn}x_n$ (uncoupled system)

so $x_k(t)=e^{a_{kk}t}x_k(0)$

or $$x(t)=\begin{bmatrix} e^{a_{11}t} & & \\ & e^{a_{22}t} & \\ & & \ddots \\ & & & e^{a_{nn}t}\end{bmatrix}\begin{bmatrix}x_1(0)\\ \vdots \\ x_n(0)\end{bmatrix}=e^{At}x_0$$

☆ insert example

note $$e^{At}=I+At+\frac{A^2t^2}{2!}+\cdots+\frac{A^nt^n}{n!}+$$

$$=\underbrace{\begin{bmatrix}1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1\end{bmatrix}}+\underbrace{\begin{bmatrix}a_{11}t & & \\ & a_{22}t & \\ & & \ddots & \\ & & & a_{nn}t\end{bmatrix}}+\cdots\begin{bmatrix}\frac{a_{11}^nt^n}{n!} & & \\ & \ddots & \\ & & \frac{a_{nn}^nt^n}{n!}\end{bmatrix}$$

$$=\mathrm{diag}(1,1,\dots 1)+\mathrm{diag}(a_{11},a_{22},\dots a_{nn})t+\mathrm{diag}(a_{11}^n\cdots a_{nn}^n)t^n+\cdots$$

$$=\mathrm{diag}\left(1+a_{11}t+\frac{a_{11}^2t^2}{2!}+\cdots\frac{a_{11}^nt^n}{n!}+\cdots,\ \dots,\ 1+a_{nn}t+\cdots\frac{a_{nn}^nt^n}{n!}\right)$$

example $\frac{dx}{dt} = \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}}^{A} x$ sol: $\frac{dx_1}{dt} = 2x_1$, $\frac{dx_2}{dt} = -x_2$

$\Longrightarrow x_1 = c_1 e^{2t} \quad x_2 = c_2 e^{-t} \qquad x = \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{-t} \end{bmatrix} = \underbrace{\begin{bmatrix} e^{2t} & \\ & e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{e^{At} c}$

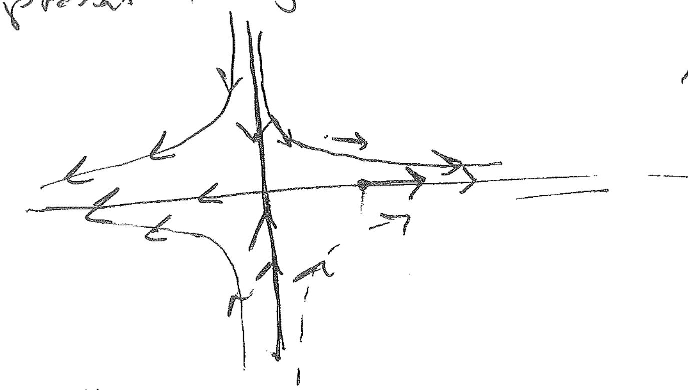
In this case, $f(x) = Ax$

since $f: \mathbb{R}^2 \to \mathbb{R}^2$ f defines a vector field

can represent f by drawing a vector Ax at x

$Ax = \begin{bmatrix} 2x_1 \\ -x_2 \end{bmatrix}$

e.g.,



solutions follow flow of the vector field $\frac{dx}{dt}$ = velocity vector of sol.

$= Ax$

phase diagram

The sol. $e^{At}c$ defines a mapping $(\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2)$

by $\underbrace{\Phi(t,v)} = e^{At} v$.

The "dynamical system" defined by the DE

$\circledast$ A linear operator on $\mathbb{R}^n$ is a linear mapping $T:\mathbb{R}^n \to \mathbb{R}^n$.

<u>operator norm</u> $\quad \|T\| = \sup_{|x|\le} |T(x)|$

where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ Euclidean norm

$\|\cdot\|$ can be shown to satisfy norm properties:

a) $\|T\| \ge 0$; $\|T\| = 0$ iff $T = 0$

b) $\|kT\| = |k|\,\|T\|$ $\quad k \in \mathbb{R}$

c) $\|T+S\| \le \|T\| + \|S\|$

$T_n \to T$ iff $\|T_n - T\| \to 0$

properties i) $|T(x)| \le \|T\|\,|x|$

ii) $\|TS\| \le \|T\|\,\|S\|$

iii) $\|T^k\| \le \|T\|^k$

$$e^{At} = \cancel{\sum_{k=}} \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} (At)^k}_{S_n}$$

$\{S_n\}$ Cauchy in $\mathcal{L}(\mathbb{R}^n)$ = Linear operators on $\mathbb{R}^n$ with norm $\|\cdot\|$

So e^{At} exists for any $n \times n$ matrix.

~~Note~~ If $\Phi(t) = e^{At}$, $\quad \Phi(0) = I$

$$e^{A0} = I + 0A + \frac{0^2A^2}{2!} + \cdots = I$$

$$\frac{d}{dt}\Phi(t) = (\text{formally})\ \frac{d}{dt}\sum_{k=0}^{\infty} \frac{(At)^k}{k!} = \sum_{k=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!}$$

$$= A \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = A\,\Phi(t)$$

Thus $\Phi(t)$ is a matrix solution to

$$\dot{X} = AX \qquad X(0) = I \qquad \left(\text{I.e. } \dot{\Phi} = A\Phi \quad \Phi(0) = I\right)$$

multiply by $x_0 \in \mathbb{R}^n$ $\quad$ ~~$y = Xx_0$~~ $\quad y = \Phi(t)x_0$ is a vector

sol. to $\dfrac{dy}{dt} = Ay \quad y(0) = x_0$

Properties $\quad$ 1) e^A is always nonsingular

2) $(e^A)^{-1} = e^{-A}$

(let $t \to -\tau$ $\quad \dfrac{dX}{d\tau} = -AX \quad X(0) = I$

sol.: e^{-At}

$\Rightarrow$ e

3) $e^{A+B} = e^A e^B$ if A, B commute

<u>Prop.</u> If $PTP^{-1} = S$ then

$$e^S = Pe^TP^{-1}$$

<u>pf:</u> $e^S = \sum_{k=0}^{\infty} \frac{(PTP^{-1})^k}{k!} = P\left(\sum_{k=0}^{\infty} \frac{T^k}{k!}\right)P^{-1} = Pe^TP^{-1}$

$PTP^{-1}PTP^{-1} \cdots PTP^{-1}$
$= PT^kP^{-1}$

<u>Prop 2</u> If S, T commute then

$$e^{S+T} = \cancel{e^{S+T}}\, e^S e^T$$

(but ~~or~~ otherwise this is usually false)

— see book —

<u>Cor:</u> $S, -S$ commute so

$$e^{S+S} = 0 = e^S e^{-S}$$

$$\therefore \left(e^S\right)^{-1} = e^{-S}$$

$$\Longrightarrow \quad \Phi(-t) = \left(\Phi(t)\right)^{-1}$$

Diagonalization

Suppose A has n distinct real eigenvals $\lambda_i \in \mathbb{R}$

$$A\vec{v}_i = \lambda_i \vec{v}_i \qquad \vec{v}_i \in \mathbb{R}^n \setminus \{0\}$$

Let $P = \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}$ ($n \times n$ matrix)

If B is a matrix $(AB)_{col\,j} = A(B_{col\,j})$

so $(AP)_{col\,j} = \lambda_j \vec{v}_j = \vec{v}_j \lambda_j$

$$\Rightarrow AP = \begin{bmatrix} \vec{v}_1\lambda_1 \cdots & \vec{v}_n \lambda_n \end{bmatrix} = P \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{= D \text{ (diagonal)}}$$

so $AP = PD \Rightarrow D = P^{-1}AP$

eigenvals of D: $\lambda_1 \cdots \lambda_n$

eigenvects: $\vec{e}_1 \cdots \vec{e}_n$

To solve $\dot{x} = Ax \qquad x(0) = x_0$

let $y = P^{-1}x \qquad (x = Py)$

$$P\dot{y} = APy \Rightarrow \dot{y} = P^{-1}APy$$
$$= Dy$$

$$\Rightarrow y = e^{Dt}y(0) = e^{Dt}P^{-1}x(0)$$

so $P^{-1}x = e^{Dt}P^{-1}x_0 \qquad x = P\underbrace{e^{Dt}}_{E(t)}P^{-1}x_0$

Then

example $\dot{x}_1 = -x_1 - 3x_2$

$\dot{x}_2 = 2x_2$

$$\frac{dx}{dt} = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix} x$$

$\lambda = -1, 2$

$\lambda = -1$ $\quad (A - \lambda I)v = 0 \quad : \quad \begin{bmatrix} 0 & -3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$c_1 = 1 \quad c_2 = 0$ works

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\lambda = 2$ $\quad \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad E(t) = e^{Dt} = \begin{bmatrix} e^{-t} & \\ & e^{2t} \end{bmatrix}$$

$X(t) = P E(t) P^{-1} \vec{c}$ where $\vec{c} = x(0)$

$$P^{-1}: \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right] \quad P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} e^{-t} & \\ & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}}_{\begin{bmatrix} e^{-t} & e^{-t} \\ 0 & -e^{2t} \end{bmatrix}} \begin{bmatrix} \vec{c} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & e^{-t} - e^{2t} \\ 0 & e^{2t} \end{bmatrix} \vec{c} = e^{At}\vec{c} = \begin{bmatrix} c_1 e^{-t} + c_2(e^{-t} - e^{2t}) \\ c_2 e^{2t} \end{bmatrix}$$

ex $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = aI + b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Since aI, $b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ commute

$$e^A = e^{aI} e^{b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

$$e^{\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = \begin{bmatrix} -b^2 & 0 \\ 0 & -b^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, I$$

$$\begin{bmatrix} 1 - \frac{b^2}{2!} + \frac{b^4}{4!} \cdots & -b + \frac{b^3}{3!} - \frac{b^5}{5!} \\ b - \frac{b^3}{3!} + \frac{b^5}{5!} - & 1 - \frac{b^2}{2!} + \frac{b^4}{4!} \end{bmatrix} = \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$$

$$e^{aI} = e^{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}} = \begin{bmatrix} e^a & \\ & e^a \end{bmatrix} = e^a I$$

$$\Longrightarrow e^A = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$$

ex $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = aI + bN \qquad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

aI commutes with N

$$\Rightarrow e^A = e^{aI} e^{bN} \qquad N^2 = 0$$

$$e^{aI}\left[I + bN + 0 + 0 \cdots\right]$$