Immigr. model (Glucose in blood stream)

$$\frac{dG}{dt} = -a_0 G + C$$

In hypoglycemia - blood sugar too low -
glucose can be injected into blood stream.
- without treatment, its absorbed, converted to energy
at rate prop. to amt present

Let $G(t)$ = glucose in bloodstream time t gms)

C = ~~amt~~ rate of glucose injection

G_0 = initial am't.

(here, in $\frac{dG}{dt} = f(G) \simeq C - a_0 G$ + order G^2 error)

sol. (1st order linear - apply integration factor $e^{a_0 t}$ to

$$\frac{dG}{dt} + a_0 G = C(t)$$

$$\left(e^{a_0 t} G\right)' = Ce^{a_0 t}$$

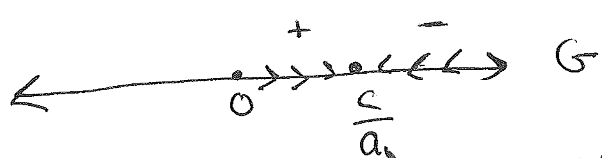
$$\dots \quad G(t) = e^{-a_0 t}\left[G_0 + \int_0^t c(\tau)\, e^{a_0 \tau}\, d\tau\right]$$

constant c analysis $\quad G(t) = e^{-a_0 t}\left[G_0 + \frac{e^{a_0 t} - 1}{a_0} c\right]$

$$= \frac{c}{a_0} + e^{-a_0 t}\left(G_0 - \frac{c}{a_0}\right)$$

$$\lim_{t \to \infty} G(t) = \frac{c}{a_0}$$

phase diagram $\qquad G' = \underbrace{c - a_0 G}_{f(G)}$



$G = \frac{c}{a_0}$ stable equilibrium

2^{-n} $\quad 2^{-M} < \varepsilon \quad \ln(2^{-M}) < \ln \varepsilon$

$-M \ln 2 < \ln \varepsilon$

$M > \frac{\ln \varepsilon}{\ln 2}$

Redo full calculation:

$G' + a_0 G = c$

$(e^{a_0 t} G)' = e^{a_0 t} c$

$e^{a_0 t} G = \int_0^t e^{a_0 s} c(s)\, ds$

$- G(0)$

$e^{a_0 t} G = G_0 + \int_0^t e^{a_0 s} c(s)\, ds$

$G = e^{-a_0 t} G_0 + e^{-a_0 t} \int_0^t e^{a_0 s} c(s)\, ds$

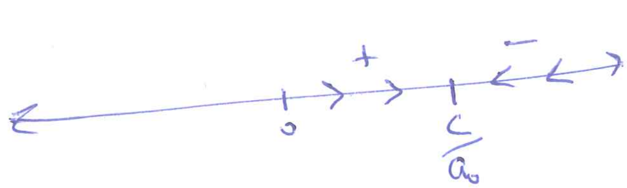
If c constant

$= e^{-a_0 t} G_0 + e^{-a_0 t} c \frac{(e^{a_0 t} - 1)}{a_0}$

$= \frac{c}{a_0} + e^{-a_0 t}\left(G_0 - \frac{c}{a_0}\right)$

$\Rightarrow G(t) \to \frac{c}{a_0}$ exponentially as $t \to \infty$

stability diagram



$G' = c - a_0 G$

$=$

If L is the desired level,

use $c = L a_0$

Suppose a pill or shot is given once per time unit (e.g. day)

$$\frac{dG}{dt} + a_0 G = C(t) = \sum_{n \le t} c_0 \delta(t-n) \qquad (3.2)$$

— each dose is c_0 grams

$$\Rightarrow G(t) = e^{-a_0 t}\left[G_0 + \int_0^t C(\tau) e^{a_0 \tau} d\tau\right]$$

$$\int_0^t C(\tau) e^{a_0\tau} d\tau = \int_0^t e^{a_0\tau} \sum_{n \le \tau} c_0 \delta(\tau - n)\, d\tau$$

(use $\int \delta(\tau-n) f(\tau) d\tau = f(n)$)

$$= \sum_{n \le t} e^{a_0 n}$$

$$\Rightarrow G(t) = e^{-a_0 t}\left[G_0 + c_0 \underbrace{\sum_{n \le t} e^{a_0 n}}_{h(t)}\right]$$

$h(t)$ is a geomet series:

$$1 + r + r^2 + \dots + r^n = S$$

$$\Rightarrow rS + 1 - r^{n+1} = S$$

$$\Rightarrow (r-1)S = r^{n+1} - 1$$

$$S = \frac{r^{n+1} - 1}{r-1}$$

$\therefore$ if M = largest integer $\le t$

$$h(t) = \frac{e^{a_0(M+1)} - 1}{e^{a_0} - 1}$$

If $N \in \mathbb{N}$, $G(N) = e^{-a_0 N}\left[G_0 + c_0 \frac{e^{a_0(N+1)} - 1}{e^{a_0} - 1}\right]$

for $t \in [N, N+1)$ $h(t) = h(N)$

$$\Rightarrow \lim_{t \to (N+1)^-} G(t) = e^{-a_0(N+1)}\left[G_0 + c_0 \frac{e^{a_0(N+1)} - 1}{e^{a_0} - 1}\right]$$

Thus $\lim\limits_{t \to N+1} G(t) = e^{-a_0} \cdot G(N)$

For large N, $G(N) = \underbrace{e^{-a_0 N}}_{\to 0} G_0 + \dfrac{e^{a_0} - e^{-a_0 N}}{e^{a_0} - 1} \cdot C_0$

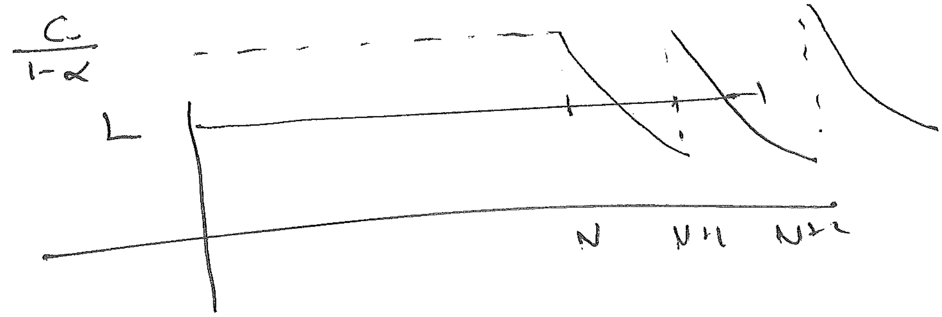
$$\to \frac{e^{a_0}}{e^{a_0} - 1} C_0$$

Suppose without treatment, an initial amount G_0 reduces to $\alpha \cdot G_0$ $(\alpha < 1)$ by end of 1 time unit

$$\Rightarrow G_0 e^{-a_0} = \alpha G_0$$

$$\Rightarrow G(N) \to \frac{\alpha^{-1} C_0}{\alpha^{-1} - 1} = \frac{1}{1-\alpha} C_0$$

asymptotically



pick C_0 to asymptotically keep

$|G(t) - L|$ as close as possible

Equilibrium : need ~~$G_0 + \dots$~~ $G = \frac{C_0}{a_0}$

to have level $\frac{C_0}{a_0}$: if $\frac{C_0}{a_0} \geq g$

~~[illegible]~~ enough sugar is present

$\frac{C_0}{a_0} < g$: need an infusion

Logistic model

$$\frac{dy}{dt} = f(y) \approx f(0) + f'(0)\,y + \frac{f''(0)}{2} y^2$$

if no emmigration (immigration

assume $f(0) = 0$

problem can be written $\frac{dy}{dt} = a_0 y \left(1 - \frac{y}{Y_c}\right)$

$y(0) = y_0$

Sol. similar to

$y' = y(1-y)$: $\frac{dy}{y(1-y)} = dt$

$$\left(\frac{A}{y} + \frac{B}{1-y}\right) dy = dt$$

$$\left(\frac{1}{y} + \frac{1}{1-y}\right) dy = dt$$

$$\ln|y| - \ln|1-y| = t + C$$

$$\ln\left|\frac{y}{1-y}\right| = t + C$$

$$\frac{y}{1-y} = Me^t$$

$$y = Me^t(1-y)$$

$$y + Me^t y = Me^t$$

$$y = \frac{Me^t}{1 + Me^t}$$

$$= \frac{M}{M + e^{-t}}$$

Ch. 3

Consider population with no immigration

- if population grow large, may need another taylor term
  - population can not grow without bound

$$\frac{dy}{dt} = f(y)$$

$$f(0) = 0$$

$$f(y) = f'(0)y + \frac{f''(0)y^2}{2} + \underbrace{R_3}_{\text{cubic remainder term}}$$

$$\simeq a_0 y + C y^2$$

rename $C = -\frac{a_0}{Y_c}$

$$\Rightarrow \quad f(y) = a_0 y\left(1 - \frac{y}{Y_c}\right)$$

or $\frac{dy}{dt} = a_0 y\left(1 - \frac{y}{Y_c}\right)$

of the form: $z' = z(1-z)$

$$\underbrace{\frac{dz}{z(1-z)}} = dt$$

$$\underbrace{\frac{A}{z} + \frac{B}{1-z}} = \frac{A(1-z) + Bz}{z(1-z)} = A + z(B-A)$$

$$\left(\frac{1}{z} + \frac{1}{1-z}\right)$$

$$\ln|z| + - \ln|1-z| = t + c$$

$$\ln\left|\frac{z}{1-z}\right| = t + c$$

$$\left|\frac{z}{1-z}\right| = e^{t} M$$

$$\frac{z}{1-z} = ke^{t}$$

$$z = (1-z)ke^{t}$$

$$z(1+ke^{t}) = ke^{t}$$

$$z = \frac{ke^{t}}{1+ke^{t}}$$

Exact sol. $y(t) = \dfrac{Y_c Y_0}{Y_0 + (Y_c - Y_0)e^{-a_0 t}}$

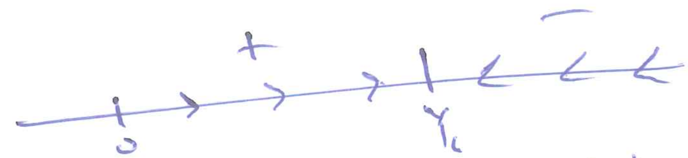
$\longrightarrow Y_c$ as $t \to \infty$

Rmk for $P_0 > 0$, solutions exist on $[0,\infty)$ ($f(y) = a_0 y(1 - y/Y_c) \in C^\infty(\mathbb{R})$)

$y(t) \to Y_c$ as $t \to \infty$ (can't arrive at Y_c in time $T < \infty$ by previous analysis)

phase diagram

$$y' = a_0 y \left(1 - \frac{y}{Y_c}\right)$$

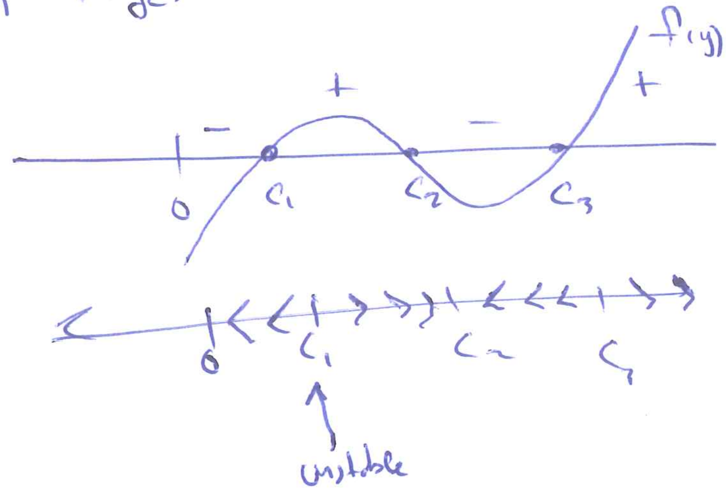


$y = Y_c$ is a stable equilibrium

<u>general population models</u>

suppose $y' = f(y)$

if $f(y_c) = 0$ then $y = y_c$ is an equilibrium sol



phase diagram

3.3 Wan Fish population, Fish harvesting, over fishing

general model $\frac{dy}{dt} = f(t,y)$

could be $f(t,y) = a_0(t)\,y + c_0(t)$
(linear growth)

or $= a_0 y + c_0$
(if autonomous)
a_0, c const.

or could be logistic type

$f(t,y) = \underbrace{a_0 y\left(1 - \frac{y}{Y_c}\right)}$

in most applications there is some form of immigration/emigration

Fish farm with logistic model

$y' = a_0 y\left(1 - \frac{y}{Y_c}\right) - h(t)$

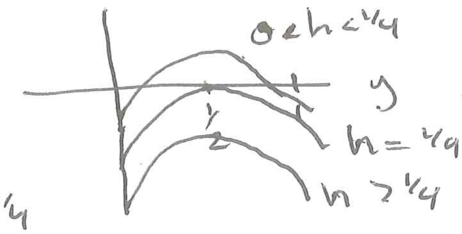
For simplicity, consider

$y' = y(1-y) - h \qquad y(0) = Y_0$

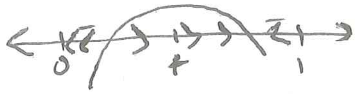
Review Fish farm

$$\underbrace{y' = y(1-y) - h}_{f(y)} \qquad y(0) = Y_0$$

f(y)



stability



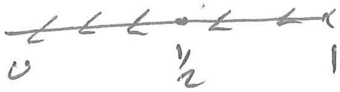
$0<h<1/4$

$$y^c = \frac{1 \pm \sqrt{1-4h}}{2}$$

$$y_1 = \frac{1-\sqrt{1-4h}}{2} \ : \text{unst.}$$

$$y_2 = \frac{1+\sqrt{1-4h}}{2} \ : \text{~~stable~~}$$

$h = 1/4$



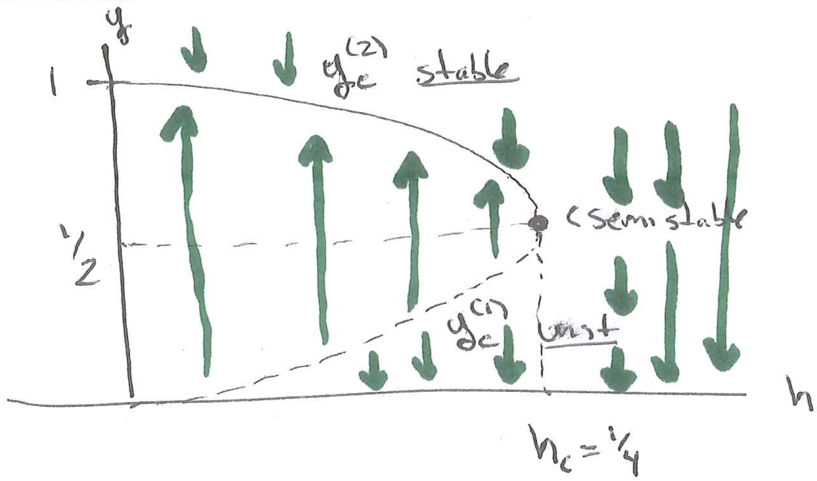
$y = 1/2$ is semistable

$h > 1/4$

y decreases to 0

$\therefore$ if $y \geq 1/2$ $\quad$ h can be up to $1/4$ without depleting fish

Bifurcation diagram



crit. pts:
$h = y(1-y)$

- called a "saddle-node" bifurcation
- $h = 1/4$ is a bifurcation pt
- solid line : asy. stable
- dashed curve : unstable

Basic types of bifurcation

Saddle-node

(2nd example) $$z' = \underbrace{\mu - z^2}_{f(z,\mu)}$$

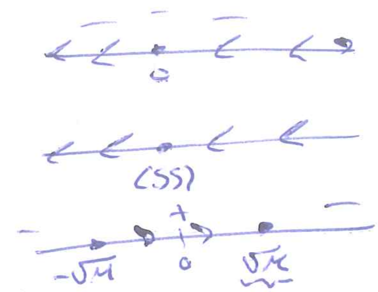
$\Longrightarrow z = \pm\sqrt{\mu}$ $(\mu > 0)$

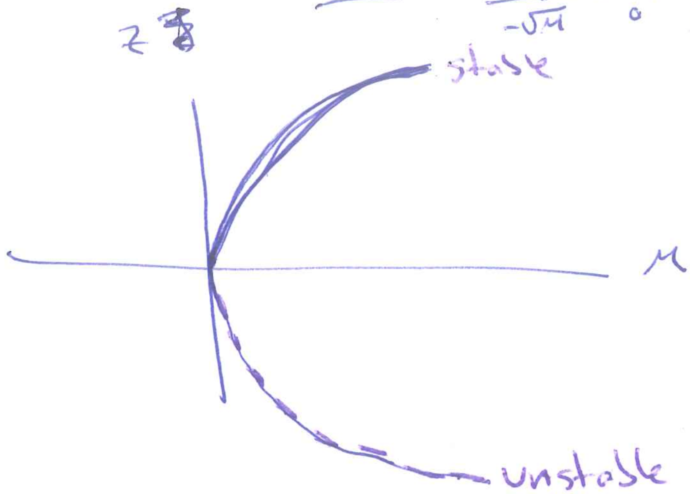
phase diagrams

$\mu < 0$

$\mu = 0$

$\mu > 0$





$f(z,\mu) = 0$
$\Rightarrow \mu = z^2$

$\hookrightarrow$ μ decreases from + to −

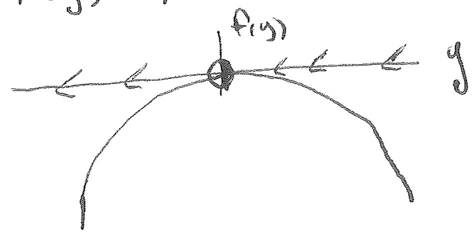
2 crit pts (stable, unst) $\rightarrow$ 1 (SS) crit pt $\rightarrow$ 0

"saddle-node" type

Observation $y' = f(y;\mu)$.

At $\mu = \mu_c$ $y = y_c$ is semistable

and $f(y, \mu_c)$ looks like



note $f(y;\mu_c)$ has a horiz. tangent at $y = y_c$

This is true in general: [Assume f is 2 times diff'ble in y and μ]

$$f(y;\mu) = f(y_c;\mu) + f_y(y_c;\mu)(y-y_c) + \frac{1}{2!} f_{yy}(y_c;\mu)(\mu - y_c)^2$$

$+ \cdots$

if $f(y_c, \mu_c) = 0$ and $f_y(y_c, \mu_c) \neq 0$

then for (y, μ) close enough to

(y_c, μ_c), $f(y;\mu)$ also has a

sign change $\Rightarrow$ μ_c not a bifurcation pt.

Thm 2 $\quad \dfrac{\partial f}{\partial y}(y;\mu_c) = 0$ at $y = y_c$

Rmk can use this to find bifurc. pt:

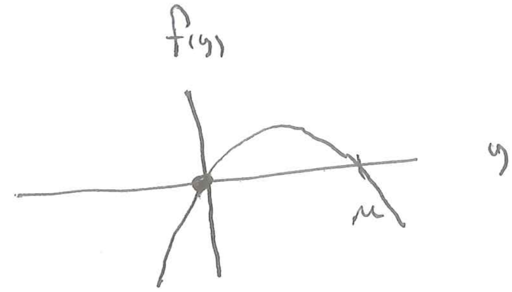
$$f(y_c; h_c) = y_c(1 - y_c) - h_c = 0$$

$$\frac{\partial f}{\partial y}(y_c;\mu_c) = 1 - 2y_c = 0 \qquad y_c = 1/2$$

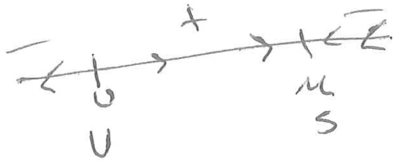
$$h_c = 1/4$$

Transcritical bifurcation

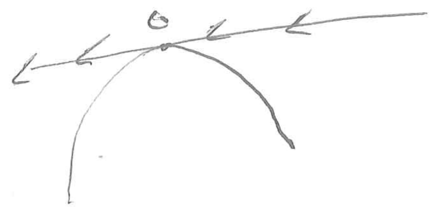
$y' = f(y;\mu) = y(\mu - y)$



$\mu > 0$



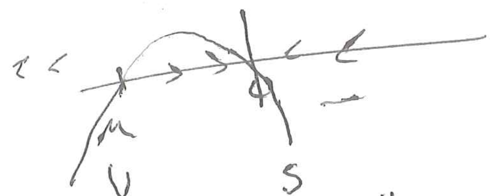
$\mu = 0$



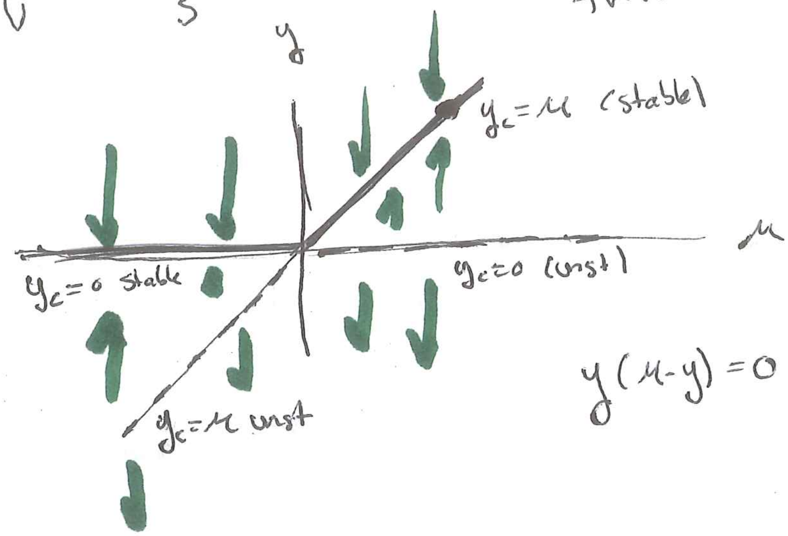
(bif. pt)
$\mu = 0$

stability of root at 0 and μ changes as μ passes through 0

$\mu < 0$



diagram



$y(\mu - y) = 0$

note again that

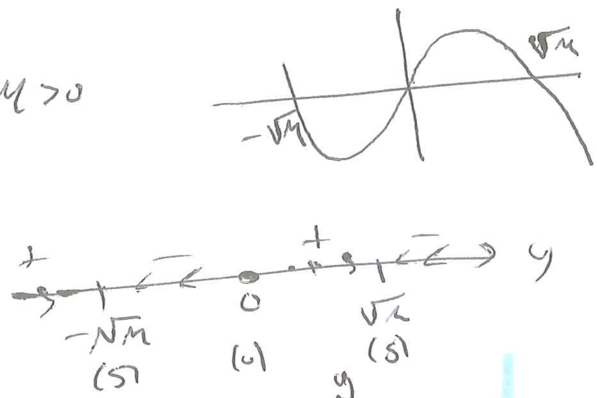
$$f(y;\mu_c) = f(y;0) = -y^2$$

and $\dfrac{\partial f(0,\mu_c)}{\partial y} = 0$

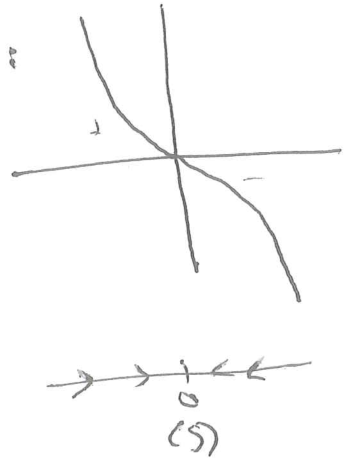
pitchfork bifurcation

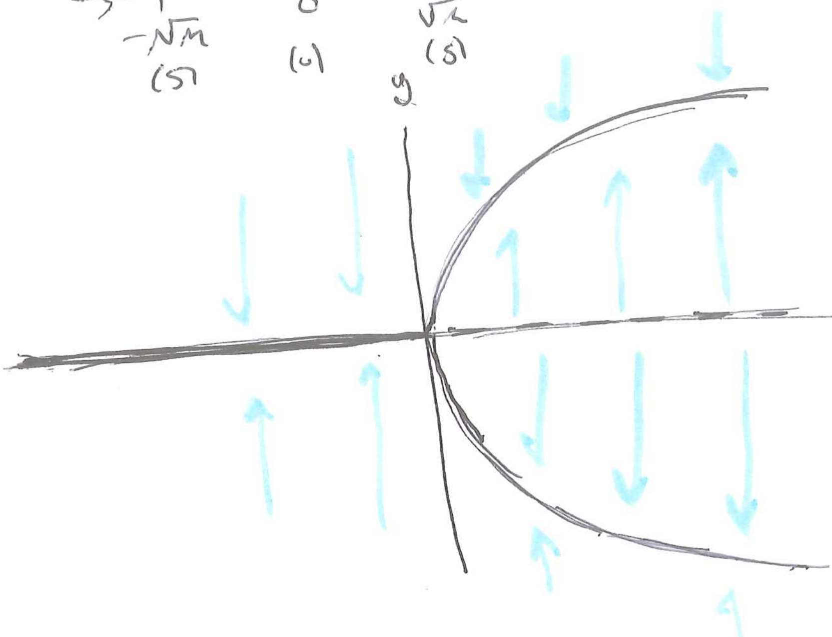
ex. $y' = y(\mu - y^2)$

$\mu > 0$



$\mu < 0$:





$y(\mu - y^2) = 0$

$y = 0$ or $\mu = y^2$

review: saddle node:

$0 \to 1 \to 2$ crit pts

transcritical $2 \to 1 \to 2$

pitchfork $1 \to 3$

1.8 $x^{(k)} = f(x, x^{(1)}, \dots, x^{(k-1)})$

If $\varphi(t)$ is a sol., then show $\varphi(t-t_0)$ also is.

To say φ is a sol means

$$\varphi^{(k)}(t) = f(\varphi(t), \varphi^{(1)}(t), \dots, \varphi^{(k-1)}(t)) \quad \star$$

on some interval I

Let $\psi(t) = \varphi(t-t_0)$. We need to show that ψ satisfies $\star$. By substitution, we have

$$\varphi^{(k)}(t-t_0) = f(\varphi(t-t_0), \varphi^{(1)}(t-t_0), \dots, \varphi^{(k-1)}(t-t_0))$$

But also $\psi^{(j)}(t) = \varphi^{(j)}(t-t_0)$ by the chain rule.

$$\therefore \quad \psi^{(j)}(t) = f(\psi(t), \psi^{(1)}(t), \dots, \psi^{(k-1)}(t))$$

where $t - t_0 \in I$

i.e. $t \in t_0 + I \equiv J$

1.12 (iii) $\dfrac{dx}{dt} = (\sin t) e^{x}$

$dx \cdot e^{-x} = \sin t \cdot dt$

$\Rightarrow \int e^{-x} dx = \int \sin t \, dt \Rightarrow -e^{-x} = -\cos t + c$

or $e^{-x} = \cos t + \tilde{c} \Rightarrow -x = \ln(\tilde{c} + \cos t)$

$x = -\ln(\tilde{c} + \cos t)$

If $x(t_0) = x_0$ then $x_0 = -\ln(\tilde{c} + \cos t_0)$

$\Rightarrow \tilde{c} = e^{-x_0} - \cos t_0$

$\Rightarrow x = -\ln(e^{-x_0} - \cos t_0 + \cos t)$

$= -\ln(e^{-x_0} - (\cos t_0 - \cos t))$

For bounded sol's, need $e^{-x_0} > \underbrace{\cos t_0 - \cos t}_{\text{periodic}}$

$\Rightarrow$ need $e^{-x_0} > 2$ then for any t_0 $\Rightarrow$ bdd sol.'s

If $e^{-x_0} > 1 + \cos t_0 \Rightarrow$ bdd sol. for this particular t_0

1.12 (iii) cont'

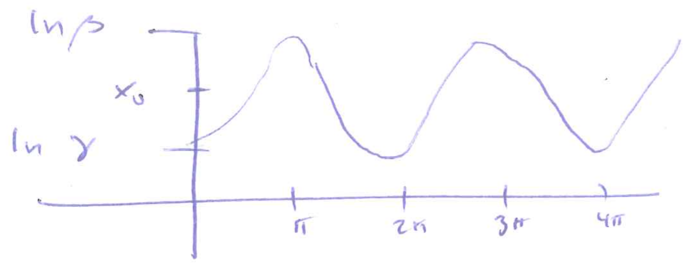
For $e^{-x_0} > 1 + \cos t_0$, if $\beta = e^{-x_0} - \cos t_0 - 1$

$\gamma = e^{-x_0} - \cos t_0 + 1$

$$x = -\ln(e^{-x_0} - \cos t_0 + \cos t)$$

satisfies $-\ln \gamma \le x \le \ln \beta$ and is 2π periodic

max at $\pi + 2k\pi$, min at $2k\pi$



$f(x) = x(1-x) \qquad x(0) = x_0$

$$F(x) = \int_{x_0}^{x} \frac{1}{y(1-y)}\,dy = \int_{x_0}^{x} \frac{1}{y} + \frac{1}{1-y}\,dy$$

$$= \ln|y| - \ln|1-y| \Big|_{x_0}^{x} = \ln\left|\frac{y}{1-y}\right| \Big|_{x_0}^{x} = \ln\left|\frac{\frac{x}{1-x}}{\frac{x_0}{1-x_0}}\right|$$

$$= \ln\left(\frac{x}{x_0}\right)\left(\frac{1-x_0}{1-x}\right)$$ (this is valid for each region: $(-\infty, 0)$, $(0,1)$, $(1,\infty)$)

If $0 < x_0 < 1$ $\quad t = F(x) \to \infty$ as $x \to 1^-$

$t \to -\infty$ as $x \to 0^+$

$\therefore \; (x_1, x_2) = (0,1) \qquad T_- = -\infty, \; T_+ = \infty$

If $x_0 > 1$ $\quad t = \ln\left(\left(\underbrace{\frac{x_0-1}{x_0}}_{>0}\right)\left(\underbrace{\frac{x}{x-1}}_{>0}\right)\right) \to \infty$ as $x \to 1^+$

$t = 0$ when $x = x_0 > 1 \qquad (x_1, x_2) = (1, \infty)$

as $x \to \infty \quad t \to \ln\left(\frac{x_0-1}{x_0} \cdot 1\right)$

so

$x = 1$ corresponds to $T_+ = \infty$

$x = \infty$ " " $T_- = \ln \frac{x_0 - 1}{x_0} < 0$

If $x_0 < 0$ $\quad t = \ln\left(\underbrace{\frac{x}{x-1}}_{>0}\right)\left(\underbrace{\frac{x_0-1}{x_0}}_{>0}\right) \qquad (x_1, x_2) = (-\infty, 0)$

as $x \to 0^- \quad t \to -\infty \qquad T_- = -\infty$

as $x \to -\infty$, $t \to T_+ = \ln\left(\frac{x_0 - 1}{x_0}\right) > 0$

If $x_0 = 0$, or $x_0 = 1 \Rightarrow x \equiv x_0 \qquad (T_-, T_+) = (-\infty, \infty)$

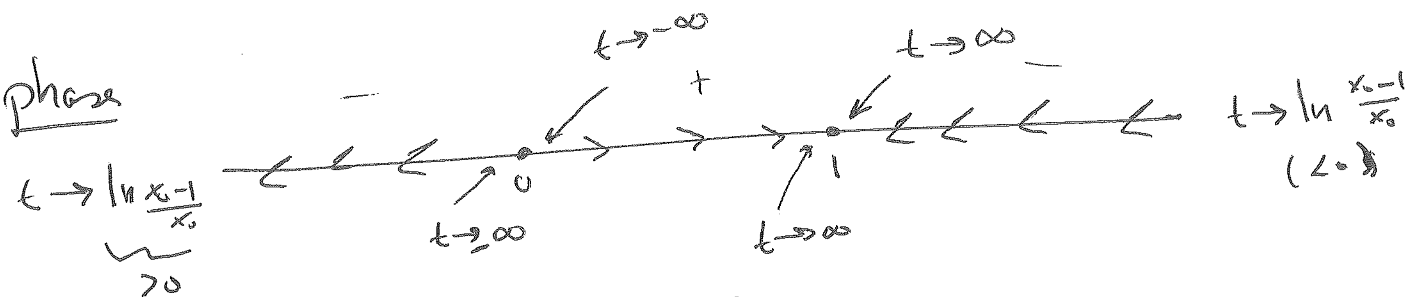
Sol. $$t = \ln\left(\frac{x}{x_0}\right)\left(\frac{1-x_0}{1-x}\right) = \ln\left(\frac{x}{1-x}\right)\underbrace{\left(\frac{1-x_0}{x_0}\right)}_{\alpha_0} = \ln\left(\frac{x}{1-x}\right)\alpha_0$$

$$e^t = \frac{x}{1-x}\alpha_0 \qquad (1-x)e^t = x\alpha_0$$

$$x(\alpha_0 + e^t) = e^t \qquad x = \frac{e^t}{\alpha_0 + e^t} = \frac{1}{\alpha_0 e^{-t} + 1}$$

$$x = \frac{1}{\left(\frac{1-x_0}{x_0}\right)e^{-t} + 1}$$

phase



$$\dot{x} = x(1-x)$$