Wan ~~[illegible]~~ Introduction (Ch. I)

The modeling cycle

- Identify physical problem, parameters, key variables
- Formulate initial math model (simple to start)
- Math, computational analysis
- Validation - data comparisons
- Improved formulation

repeat

Hope to describe
- evolution of system
- stability
- qualitative properties (limit cycles / bifurcations)
- range of validity

add ons:
- feed back
- control
- diffusion (pde)
- ~~limits~~ stochastic terms

population modeling (Wan Ch. 2)

Let $y(t)$ denote population time t

(could be in billions or by mass depending on problem)

- assume contin. var.

- start with a simple model (later adjust after comparisons with data)

$\frac{dy}{dt} = f(t,y)$ 1st order

one hopes that population should depend on y mainly (so to start, assume autonomous).

$\frac{dy}{dt} = f(y)$ predict world human population at a later time

properties if $y=0$ no growth $\Rightarrow f(0)=0$

taylor's series $f(y) = f'(0)y + \frac{1}{2}f''(0)y^2 + \dots + \frac{1}{n!}f^{(n)}(0)y^n$

$+ \frac{1}{(n+1)!}f^{(n+1)}(c)y^{n+1}$ ← error term

where $0 < c < y^{n+1}$

for small enough populations

$$f(y) = f'(0)y + \underbrace{\frac{1}{2}f''(0)y^2}_{\text{error}}$$

$$\approx \underbrace{f'(0)}_{a_0}y$$

if $a_0 > 0 \Rightarrow$ growing population

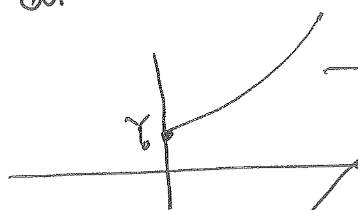
if $a_0 < 0 \Rightarrow$ dying population

$\underbrace{\frac{dy}{dt} = a_0 y}_{\text{linear growth rate model}} \Rightarrow y = y(0)e^{a_0 t} = Y_0 e^{a_0 t}$

a_0 = growth rate constant

math analysis of simple population model:

$$\frac{dy}{dt} = a_0 y \quad y(0) = Y_0 \quad \Rightarrow \quad y = Y_0 e^{a_0 t}$$



If $a_0 > 0$ population grows exponentially

① – might not be realistic due to limited space

② if y is very small, for some types of populations e.g. wolves, it does not match reality

– $a_0 < 0$ with y small enough

∴ prob. only valid for limited range of y

$f(y) = a_0 y$:

linear growth rate

a_0: growth rate const.

$$a_0 = \frac{dy/dt}{y} = \text{"relative" growth rate}$$

ex At 10% interest rate, 1000\$ becomes 1100 after 1 year

$$\frac{\Delta y / \Delta t}{y} = \frac{100/1}{1000} = .1$$

Model validation: see how $Y_0 e^{a_0 t}$ fits data

· Suppose 2 data points, y: world population at time t

$t = 0$ at 2011

date	t	y
2011	0	7 (billion)
1974	-37	4

$t=0$ $\quad Y_0 e^0 = 7 \quad\quad Y_0 = 7$

$t=-37$ $\quad Y_0 e^{-37 a_0} = 4$

$$e^{-37 a_0} = 4/7$$

$$a_0 = \frac{-1}{37} \ln\left(\frac{4}{7}\right)$$

Table 2.1 $\left(y = 7e^{a_0 t} \quad a_0 = \frac{-1}{37}\ln\left(\frac{4}{7}\right) \right)$

	1804	1927	1959	1974	1987	1999	2011
t_n	-207	-84	-52	-37	-24	-12	0
$y(t_n)$	.306	1.97	3.19	4	4.87	5.84	7
$\bar{y}_n$ = data	1	2	3	4	5	6	7

(see book)

~7% error back to 1927 (1804 was off)

$\therefore$ would guess it is accurate for 50 years forward

Model can predict: $y(9) =$ 2020 world pop

$y(39) =$ 2050 " "

when will p = 10 billion?

Improved parameter estimation

Suppose $y(t) = Y(t, A)$ is the model. $A = \{a_1, a_2, \ldots, a_m\}$ = parameter set

(in above example, $A = \{a_1, a_2\} = \{Y_0, a_0\}$)

Suppose we have data $\bar{y}_k$ at time t_k $k = 1, 2, \ldots, N$

and model values $y_k = Y(t_k, A)$

residual (error) $= E_k = Y(t_k, A) - \bar{y}_k$

Main case of interest: $N > m$

((i) $N = m \Rightarrow$ should be a unique sol.

$N < m \Rightarrow$ many sol's)

$N > m$: usually no exact sol.

Therefore minimize errors

Least squares minim. $S_N^2 = \sum_{k=1}^{N} (y_k - \bar{y}_k)^2$

i.e., minimize $\| \{E_k\} \|_{\ell^2}$

$\| \{x_1, \ldots, x_n\} \|_{\ell^2} = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$

Here, $E_k^2 = (y_k - \bar{y}_k)^2 = (Y_0 \underbrace{e^{a_0 t_k}}_{A_k} - \bar{y}_k)^2$

$S_N^2 = \sum_{k=1}^{N} E_k^2$ $\quad S_N^2$ minimized when $\nabla_A S_N^2 = 0$

$$\frac{\partial S_N^2}{\partial Y_0} = \sum_{k=1}^{N} \frac{\partial E_k^2}{\partial Y_0} = \sum_{k=1}^{N} 2(Y_0 A_k - \bar{y}_k) A_k = 0$$

$$\Rightarrow Y_0 \sum_{k=1}^{N} A_k^2 = \sum_{k=1}^{N} \bar{y}_k A_k \Rightarrow \boxed{Y_0 = \frac{\sum_{k=1}^{N} \bar{y}_k A_k}{\sum_{k=1}^{N} A_k^2}} \quad (1)$$

$$\frac{\partial S_N}{\partial a_0} = \sum_{k=1} \frac{\partial E_k^2}{\partial a_0} = \sum 2(Y_0 A_k - \bar{y}_k) Y_0 t_k A_k$$

$$\Rightarrow \boxed{\sum_{k=1}^{N} (Y_0 A_k - \bar{y}_k) t_k A_k = 0} \quad (2)$$

2 eqn's, 2 unknowns. Plug (1) into (2)

(2) of the form $f(a_0) = 0$

– e.g., use bisection method.

<u>Review</u> $A_k = e^{a_0 t_k}$ $\qquad Y_0 = \dfrac{\sum_{k=1}^{N} \bar{y}_k A_k}{\sum_{k=1}^{N} A_k^2}$

$$\sum_{k=1}^{N} \left(Y_0 A_k - \bar{y}_k \right) t_k A_k = 0$$

(of the form $f(a_0) = 0$
- find root on Mathematica or MATLAB)

disregard

Thus $\dfrac{\sum_{k=1}^{N} \bar{y}_k A_k}{\sum_{k=1}^{N} A_k^2} \sum_{k=1}^{N} t_k A_k^2 = \sum_{k=1}^{N} t_k \bar{y}_k A_k$

$$\Rightarrow \sum_{k=1}^{N} \bar{y}_k A_k \sum_{k=1}^{N} t_k A_k^2 = \left(\sum_{k=1}^{N} t_k \bar{y}_k A_k \right) \sum_{k=1}^{N} A_k^2$$

Let $V_k = \bar{y}_k A_k$ $\quad P_k = A_k^2$ $\quad$ so $V_k > 0$, $P_k > 0$

$$\sum_{k=1}^{N} V_k \sum_{k=1}^{N} t_k P_k = \sum_{k=1}^{N} t_k V_k \sum_{k=1}^{N} P_k$$

... — seems tough to solve analytically —

Wan gets (using data points: 1974, 1987, 1999, 2011)
$(N = 4)$

$a_0 \simeq .0148186\ldots$

$Y_0 = 7.0624\ldots$

data fit is better for 1927 — 2011

— see table 2.3

Linear least squares

IF Model is of the form

$$y(t) = Y(t, A) = \sum_{k=1}^{m} a_k \varphi_k(t)$$

with $\{\varphi_k\}$ linearly indep. functions, ~ as vectors at data pts

can find least squares sol. analytically

To solve exactly:

$$\sum_{k=1}^{M} a_k \varphi_k(t_j) = \bar{y}_j \qquad j = 1, 2, \cdots N$$

$$N > m$$

$$\varphi_1(t_1) a_1 + \varphi_2(t_1) a_2 \cdots + \varphi_m(t_1) a_m = \bar{y}_1$$

$$\vdots$$

$$\varphi_1(t_N) a_1 + \varphi_2(t_N) a_2 + \cdots + \varphi_m(t_N) a_m = \bar{y}_N$$

$$\underbrace{\begin{bmatrix} \varphi_1(t_1) & \varphi_2(t_1) & \cdots & \varphi_m(t_1) \\ \vdots & & & \\ \varphi_1(t_N) & \varphi_2(t_N) & \cdots & \varphi_m(t_N) \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_N \end{bmatrix}}_{\bar{y}}$$

$$M x = \bar{y}$$

Normal eq's $\quad M^T M x = M^T \bar{y}$

linear indep $\Rightarrow$ $M^T M$ invertible

$$x = (M^T M)^{-1} M^T \bar{y}$$

<u>Example</u> $y = Y_0 e^{a_0 t}$

$$\ln y = \underbrace{\ln Y_0}_{\alpha_0} + a_0 t$$

Now apply linear least squares;

let $z_k = \alpha_0 + a_0 t_k \qquad \bar{z}_k = \ln \bar{y}_k$

$$\varphi_1(t) = 1 \quad \varphi_2(t) = t$$

$$\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_N \end{bmatrix} \begin{bmatrix} \alpha_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}$$

solve for $\hat{\alpha}_0$, $\hat{a}_0$ (least squares fit)

then $\hat{Y}_0 = e^{\hat{\alpha}_0} \qquad y = \hat{Y}_0 e^{\hat{a}_0 t}$

<u>Alternative</u> instead of $y' = a_0 y$, try $y' = b$

(constant growth)

$\Longrightarrow y = c + bt$

– apply linear least squares

or similarly try $y' = p(t)$ $\quad p$ degree $n-1$

$\Rightarrow y = b_0 + b_1 t + \ldots + b_n t^n$

apply linear least squares