Classification of DE's

DE: $F(t, x, x^{(1)}, \dots, x^{(k)}) = 0$

$$x^{(j)}(t) = \frac{d^j x}{dt^j}(t) \qquad j \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$$

t indep var   solve for $x \in C^k(J)$ $J \subseteq \mathbb{R}$

x dep. var

k order

~~Solution~~: F a function of $k+2$ vars on some set U

$$U \subset \mathbb{R}^{k+2}$$

Sol. on interval $I \subset J$:

A function $\varphi(t)$ so

$$F(t, \varphi(t), \varphi'(t), \dots, \varphi^{(k)}(t)) = 0$$

$$\forall\, t \in I$$

std form "solve" for $x^{(k)}$

$$x^{(k)} = f(t, x, x^{(1)}, \dots x^{(k-1)})$$

(solving for $(t, y) \in U$ can be done locally

if $\frac{\partial F}{\partial y_k} \neq 0$ at (t, y))

by implicit function theorem

DE is linear if

$$f(t, \underbrace{y_0, y_1, \dots y_{k-1}}) = f(t,y)$$

is of the form $g(t) + l(t,y)$

where l is linear in y

i.e. $\cancel{l(t,\alpha y+\beta)}$ $l(t, \alpha x + \beta y)$

$= \alpha l(t,x) + \beta l(t,y)$

i.e. f is of the form

$$f(t,y) = a_0(t)y_0 + a_1(t)y_1 + \dots + a_k(t)y_k$$

linear DE is <u>autonomous</u> if $g(t) \equiv 0$

(non lin: $f(t,y)$ is a function of y alone)

<u>ex</u>

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \sin x = 2$$

$$\frac{d^2y}{dx^2} + \sin^2 x = 0$$

$$\frac{d^2y}{dx^2} + \sin^2 y = 0$$

Systems

ex $\quad y''' = f(t,y)$

let $v_1 = y \quad v_2 = y' \quad v_3 = y''$

$$\Rightarrow \frac{dv_3}{dt} = f(t,y) = f(t,v_1)$$

$$\frac{dv_2}{dt} = v_3$$

$$\frac{dv_1}{dt} = v_2$$

$$\frac{dv}{dt} = \begin{pmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{dv_3}{dt} \end{pmatrix} = \begin{pmatrix} \cancel{f(t,v_1)}\ v_2 \\ v_3 \\ f(t,y) \end{pmatrix}$$

General system:

$$\frac{dv}{dt} = \begin{pmatrix} f_1(t,v) \\ f_2(t,v) \\ \vdots \\ f_n(t,v) \end{pmatrix}$$

is <u>linear</u> if $\quad f_i(t,v) = g_i(t) + l_i(t,v)$

where l_i is linear in v

(std form)

Any nth order DE can be ~~[illegible]~~ converted to 1st ~~[illegible]~~ order form

~~ex~~ $\frac{d^n x}{dt^n} = f(t, x, x^{(1)}, \ldots, x^{(n-1)})$

$y_1 = x \quad y_2 = x' \ldots \quad y_n = x^{(n-1)}$

$y_1' = y_2 \quad y_2' = y_3 \ldots \quad y_{n-1}' = y_n \qquad y_n' = f(t, y_1, y_2, \ldots y_n) = f(t, y)$

$$\frac{dy}{dt} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ f(t,y) \end{pmatrix} = F(t, y)$$

$\frac{dy}{dt} = F(t, y)$ <u>linear</u> if

$$F(t,y) = \underbrace{G(t)}_{n \times 1} = \underbrace{A(t)}_{n \times n} y$$

Autonomous if F has no t dependence

<u>Simplest case $n = 1$</u>

$$\frac{dy}{dt} = F(t, y)$$

<u>Autonomous nonlin</u> $\frac{dy}{dt} = F(y)$

<u>linear</u> $\frac{dy}{dt} = a(t) y + g(t)$

<u>linear auton</u> $\frac{dy}{dt} = a y$

convert non auton. DE to auton. system: ex. $\frac{d^2x}{dt^2} = t^2 \sin x$

$y_1 = x$
$y_2 = x'$
$y_3 = t$

$$\frac{dy}{dt} = \begin{pmatrix} y_2 \\ y_3^2 \sin y_1 \\ 1 \end{pmatrix}$$

ex $\ddot{x} + te^{\dot{x}} = x$

ex $\ddot{x} = y \qquad \dot{y} = -x$

~~Defn~~ linear 1st order autonomous

$\dot{x} = ax$

system $\dfrac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$

$\vdots$

$\dfrac{dx_n}{dt} = a_{n1}x_1 + \cdots \qquad + a_{nn}x_n$

or $\dfrac{dx}{dt} = Ax$

$\boxed{\dot{x} = ax}$: $\dot{x} - ax = 0$

$e^{-at}\dot{x} - e^{-at}x = 0$

$\dfrac{d}{dt}\left(e^{-at}x\right) = 0 \qquad e^{-at}x = c$

$\vdots$

~~B~~ 1.5

Review DE: $F(t, x, x^{(1)}, \dots, x^{(n)}) = 0$ nth order DE

std form $x^{(n)}(t) = f(t, x, x^{(1)}, \dots, x^{(n-1)})$

ex $F(t, a, b) = t^2 + a^2 - b^2$ $\qquad \frac{\partial F}{\partial b} = -2b$

$\Rightarrow t^2 + x^2 - (x')^2$

if $(x') \neq 0$ can invert: $(x')^2 = t^2 + x^2$

$x' = \pm\sqrt{x^2 + t^2}$

~~it~~ if $(t_0, x_0, y_0) = (1, 0, 1)$

local inverse exists

$x' = \sqrt{x^2 + t^2}$

near $(1, 0, -1)$ would be

$x' = -\sqrt{x^2 + t^2}$

Sol. methods $\qquad$ sep. of var's $\qquad$ e.g. $x' = x^2$

$\Rightarrow \frac{dx}{dt} = x^2 \Rightarrow \frac{dx}{x^2} = dt$

$-\frac{1}{x^2} = t + c$

~~integ~~ ex $x' = ax$

~~int factor~~ linear 1st order

1st order autonomous

$\dot{x} = f(x) \quad x(0) = x_0 \qquad f \in C(\mathbb{R})$

Assume $f(x_0) \neq 0$ so $f(x)$ is nonzero for $|x - x_0|$ small enough

$\frac{dx}{f(x)} = dt \quad \Rightarrow \quad \int_0^t \frac{\frac{dx}{ds}}{f(x(s))} ds = t$

~~Let $F(x) = \int$~~

$$= \underbrace{\int_{x_0}^{x} \frac{dx}{f(x)}}_{F(x)} = t \qquad \left(F'(x) = \frac{1}{f(x)}\right)$$

$\Rightarrow$ sol'n $x(t)$ satisfies

$F(x(t)) = t. \qquad f(x_0) \neq 0 \Rightarrow F(x)$ monotone near x_0

there is a unique inverse $\qquad (F(x_0) = 0 \Rightarrow \;)$

$x(t) = F^{-1}(t) \qquad x(0) = F^{-1}(0) = x_0$

If $f(x_0) > 0 \Rightarrow f(x) > 0$ for $x \in (x_1, x_2)$

(interval contains x_0)

– pick this as big as possible

(maybe $x_1 = -\infty$, $x_2 = \infty$)

define $T_+ = \lim_{x \nearrow x_2} F(x) \in (0, \infty]$ (monotone, bdd seq's converge)

$T_- = \lim_{x \searrow x_1} F(x) \in [-\infty, 0)$

then $x \in C^1(T_-, T_+)$ and

$\lim_{t \nearrow T_+} x(t) = x_2, \quad \lim_{t \searrow T_-} x(t) = x_1$

~~If $T_+ < \infty$ either $x_2 = \infty$ or $x_2 < \infty$~~

In particular, x is defined for all $t>0$ iff

$$T_+ = \int_{x_0}^{x_2} \frac{dy}{f(y)} = +\infty$$

i.e. $\frac{1}{f(x)}$ not integrable near x_2

likewise ~~TO~~ x defined for all $t<0$ iff

$\frac{1}{f(x)}$ not integrable near x_1

Two cases if $T_+ < \infty$

1) $x_2 = \infty \Longrightarrow$ x went to ∞ as $t \to T_+$

2) $x_2 < \infty$

$$\lim_{t \nearrow T_+} x(t) = x_2 < \infty$$

if $f(x_2) > 0$ then x_2 was not chose maximally

if $f(x_2) = 0$, ~~$x = \varphi(t)$~~ extend ~~φ by setting~~ $x(t)$ as a constant for $t \geq T_+$

$\varphi(t) = x_2$ for $t \geq T_+$

(since $\dot{x} = 0$ for $t \geq T_+$)

i.e. $\varphi(t) = \begin{cases} x(t) & x \leq T_+ \\ x_2 & t \geq T_+ \end{cases}$

— but extension could be non unique

Ex $\frac{dx}{dt} = 2\sqrt{|x|}$ $x(0) = 0$

$\frac{dx}{\sqrt{|x|}} = 2\,dt$

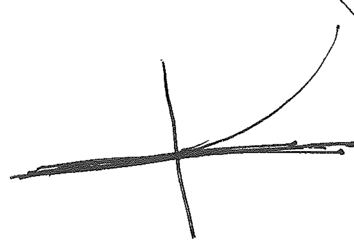
for $x>0$ $2x^{1/2}$

$2x^{1/2} = 2t + c$

$x^{1/2} = t + K$

$x = (t+K)^2$

$x(0) = 0 \Rightarrow K = 0$



next ~~page~~ do as example

What if $f(x_0) = 0$?

$\dot{x} = f(x) \quad x(0) = x_0$

(1) Note $\varphi(x) \equiv x_0$ is a solution:

$$\frac{d\varphi}{dt} \equiv 0, \quad f(x_0) = 0 \Rightarrow f(\varphi(t)) = 0$$

$$\varphi(0) = x_0 \checkmark$$

If $\left| \int_{x_0}^{x_0+\varepsilon} \frac{dy}{f(y)} \right| < \infty$ then there is another sol

$\varphi(t) = F^{-1}(t), \quad F(x) = \int_{x_0}^{x} \frac{dy}{f(y)}$

with $\varphi(0) = F^{-1}(0)$

$= x_0$ different from constant sol.

<u>HW7</u> 1.6, 1.7, 1.8, ~~1.9~~ 1.12 (iii)

For $\dot{x} = x(t-x)$ $x(0) = x_0$

Find (x_1, x_2) and (T_-, T_+) for all possible values of x_0.

~~Solve~~ Then solve by separation of vars ex $f(x) = x$ $\quad x_0 > 0$

$x_1 = 0, \; x_2 = \infty$

$$F(x) = \int_{x_0}^{x} \frac{dy}{y} = \ln|y|\Big|_{x_0}^{x} = \ln\left(\frac{x}{x_0}\right) = t$$

~~F positive on (x_0, ∞)~~

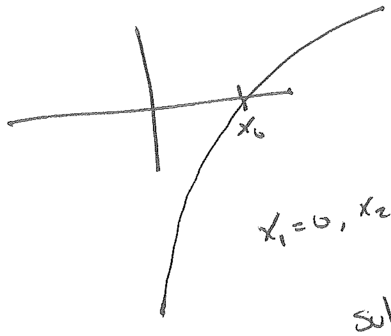
Note F monoton

$\cancel{\int_{x_0}^{x_1} F(y)\,dy \text{ finite}}$

as $x \to \infty$ $F(x) = t \to \infty$

as $x \to 0$ $F(x) = t \to -\infty$

so $T_+ = \infty$



$x_1 = 0, \; x_2 = \infty$

solve for x

$$x = x_0 e^t$$

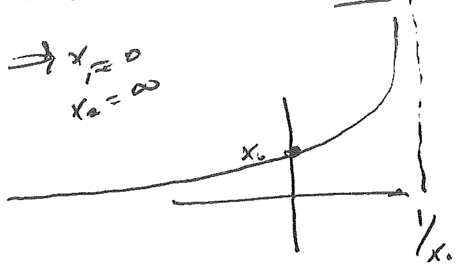
– can show is a sol. for all $x \in \mathbb{R}$

ex $f(x) = x^2$ $\quad x_0 > 0$

$\Rightarrow x_1 = 0$
$x_2 = \infty$

Sol. $\quad x(t) = \dfrac{x_0}{1 - x_0 t}$

$(T_-, T_+) = (-\infty, 1/x_0)$



$$F(x) = \int_{x_0}^{x} \frac{dy}{y^2} = \frac{1}{x_0} - \frac{1}{x}$$

$(x_1, x_2) = (0, \infty)$

$T_+ = 1/x_0 \quad T_- = -\infty$

$I = (-\infty, 1/x_0)$

as $x \to x_1 = 0$

$F(x) \to -\infty = T_-$

as $x \to x_2 = \infty$ $\quad F(x) \to \frac{1}{x_0}$

ex $f(x) = 2\sqrt{|x|}$

$x(0) = 0$