SEMIGROUP WELL-POSEDNESS OF MULTILAYER MEAD-MARKUS PLATE WITH SHEAR DAMPING

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Abstract A multilayer plate model consisting of $m+1$ relatively stiff plate layers bonded together by m compliant plate layers is described. In the case of only three layers, the model reduces to a Mead-Markus sandwich plate with viscous damping in the central layer. Existence and uniqueness of solutions are established using semigroup theory. In particular, the homogeneous problem may be written in the form $x'(t) = Ax(t)$ where A is the generator of a C_0 contraction semigroup.

Keywords: Multilayer plate, Mead-Markus plate, sandwich plate, shear damping

Introduction

The classical sandwich plate is a three-layer plate model consisting of two relatively stiff outer layers and a more compliant inner layer. Various sandwich plate and constrained layer models have been proposed and analyzed, see e.g., DiTaranto [2], Mead and Markus [9], Rao and Nakra [10]. For a review and some comparisons of these models see e.g., Sun and Lu [11]or Mead [8].

In Hansen [3], a multilayer generalization of the Mead-Markus model (and Rao-Nakra model) was derived. The multilayer models consist of alternating "stiff" and "compliant" layers. The stiff layers do not allow shear and are modelled as Kirchhoff plates, while the compliant layers allow shear and may be modelled in various ways. For the multilayer Mead-Markus model, the in-plane inertia is ignored in each layer and bending stiffness are ignored in the compliant layers. In the undamped

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case, existence and uniqueness of solutions for the multilayer systems in [3]was established on the natural energy spaces using the standard variational theory (e.g., Lions and Dautray [1]). However, in the case when shear damping is included in the multilayer Mead-Markus model, the variational approach approach in ([3]) could not be applied (at least directly) due a lack of coerciveness in the bilinear form associated with the highest order time-derivatives.

We overcome this problem in this article for the special case in which all of the stiff layers have the same Poisson's ratio. In this case it is possible to solve for the shear in each compliant layer and write the homogeneous problem in the standard form $y' = Ay$. We then show that A is the generator of a strongly continuous semigroup of contractions on the natural energy space.

Analogous models to the one of this article have been investigated in the case of a layered beam. For example, an analogous three-layer beam model was investigated in [4], where it was found (for either clamped or simply-supported boundary conditions) that the associated semigroup is in fact exponentially stable (moreover is analytic, if rotational moment of inertia is neglected). Similar results were found in the multilayer beam model in [5]for the case of simply supported boundary conditions.

1. Multilayer Mead-Markus Model

We first describe the "multilayer Mead-Markus" of [3]. (For a more careful derivation see [3].)

The multilayer sandwich plate is assumed to consist of $n = 2m + 1$ layers of alternating "stiff" and "compliant" plates, that occupy the region $\Omega \times (0, h)$ at equilibrium, where Ω is a smooth bounded domain in the plane. The layers are indexed from 1 to n, from bottom to top, with odd indices for stiff layers and even indices for compliant layers.

We use coordinates $x = \{x_1, x_2\}$ to denote points in Ω and use x_3 as the transverse independent variable. As is typical in plate theories, it is assumed that the transverse displacement is independent of x_3, i.e., we may use the scalar $w(x)$ to denote the transverse displacement at the point $x \in \Omega$. We let $v^i = \{v_1^i, v_2^i\}$ $i = 1, 2, \ldots n$ denote the in-plane displacements along the midplane of the ith layer.

It is assumed that all the layers are bonded to one another so that no slip occurs. The Kirchhoff hypothesis applies to the stiff layers, (i.e., no shear) while the compliant layers allow shear and deform linearly with respect to the transverse variable. Under these assumptions any displacement is completely determined by specification of the state variables v^i, i odd, and w.

If θ and ξ are matrices in $\mathbf{R}^{lm}$, by $\theta : \xi$ we mean the scalar product in $\mathbf{R}^{lm}$. We also denote

$$(\theta, \xi)_\Omega = \int_\Omega \theta : \xi \, dx, \qquad (\theta, \xi)_\Gamma = \int_\Gamma \theta : \xi \, d\Gamma.$$

Define the form ℓ for functions $\theta(x) = \{\theta_1(x), \theta_2(x)\}$ by

$$\begin{aligned} \ell(\theta; \hat\theta) \;=\; & \left(\frac{\partial\theta_1}{\partial x_1}, \frac{\partial\hat\theta_1}{\partial x_1}\right)_\Omega + \left(\frac{\partial\theta_2}{\partial x_2}, \frac{\partial\hat\theta_2}{\partial x_2}\right)_\Omega + \left(\nu\frac{\partial\theta_2}{\partial x_2}, \frac{\partial\hat\theta_1}{\partial x_1}\right)_\Omega \\ & + \left(\nu\frac{\partial\theta_1}{\partial x_1}, \frac{\partial\hat\theta_2}{\partial x_2}\right)_\Omega + \left(\left(\frac{1-\nu}{2}\right)\left(\frac{\partial\theta_1}{\partial x_2} + \frac{\partial\theta_2}{\partial x_1}\right), \left(\frac{\partial\hat\theta_1}{\partial x_2} + \frac{\partial\hat\theta_2}{\partial x_1}\right)\right)_\Omega . \end{aligned}$$

where ν is the Poisson's ratio ($0 < \nu < 1/2$).

It is assumed that the in-plane kinetic energy is negliglble and bending potential energy of the even layers are negligible in comparison to those of the surrounding odd layers. The energy is the sum of the kinetic and potential energies of each layer $\mathcal{E}_i = \mathcal{K}_i + \mathcal{P}_i$, where

$$\mathcal{K}_i = \begin{cases} \frac{1}{2}\int_\Omega \rho_i h_i (w')^2 + \alpha_i |\nabla w'|^2 \, dx & i \text{ odd} \\ \frac{1}{2}\int_\Omega \rho_i h_i (w')^2 \, dx & i \text{ even} \end{cases}$$

$$\mathcal{P}_i = \begin{cases} \frac{1}{2}\int_\Omega K\ell(h_i^3 D_i \nabla w; \nabla w) + 12\ell(h_i D_i v^i; v^i) \, dx & i \text{ odd} \\ \frac{1}{2}\int_\Omega G_i h_i |\varphi_i|^2 \, dx & i \text{ even} \end{cases}$$

In the above primes (e.g., w') denote differentiation with respect to time, $D_i = E_i/12(1-\nu^2)$, $\alpha_i = \rho_i h_i^3/12$ where $E_i > 0$ denotes the in-plane Young's modulus, h_i the thickness, ρ_i the volume density, all for the ith layer. In addition, G_i is the transverse shear modulus. The variable φ_i is the shear of the ith layer defined in (1) below.

Define the following n by n matrices:

$$\begin{aligned} \mathbf{h} &= \operatorname{diag}(h_1, h_2, \ldots, h_n) \qquad & \mathbf{D} &= \operatorname{diag}(D_1, D_2, \ldots, D_n) \\ \mathbf{p} &= \operatorname{diag}(\rho_1, \rho_2, \ldots, \rho_n) \qquad & \mathbf{G} &= \operatorname{diag}(G_1, G_2, \ldots, G_n). \end{aligned}$$

In addition, we let e.g., $\mathbf{h}_O$ and $\mathbf{h}_E$ denote the diagonal matrices of odd-indexed and even-indexed thicknesses h_i, respectively. Actually, we will only need to refer to $\mathbf{h}_O$, $\mathbf{h}_E$, $\mathbf{D}_O$, $\mathbf{G}_E$, $\mathbf{p}_O$. Also define $\vec{1}_E$ and $\vec{1}_O$ as the column vectors of m and $m+1$ ones, respectively.

Let v_O denote the $(m+1)\times 2$ matrix with rows v^i, $i = 1, 3, 5, \ldots 2m+1$. Likewise, let φ_O and φ_E denote the matrix with rows φ^i, i odd and even, respectively. Since no shear occurs in the odd layers, $\varphi_O = 0$. The shear φ_E can be expressed in terms of v_O, ∇w as follows:

$$\mathbf{h}_E \varphi_E = B v_O + \mathbf{h}_E N \nabla w; \quad N = \mathbf{h}_E^{-1} A \mathbf{h}_O \vec{1}_O + \vec{1}_E \tag{1}$$

where $A = (a_{ij})$, $B = (b_{ij})$ are the $m \times (m+1)$ matrices defined by

$$a_{ij} = \begin{cases} 1/2 & \text{if } j = i \text{ or } i+1 \\ 0 & \text{otherwise} \end{cases} \qquad b_{ij} = \begin{cases} (-1)^{i+j+1} & \text{if } j = i \text{ or } i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Also define

$$\ell_O(v_O, \hat{v}_O) = \sum_{i \ odd}^{n} \ell(v^i; \hat{v}^i)$$

Collecting the energies one finds that the total potential and kinetic energy may be expressed as

$$\mathcal{K}(t) = c(v_O', w'; v_O', w')/2 \qquad \mathcal{P}(t) = a(v_O, w; v_O, w)/2$$

where $c(\cdot\,;\cdot)$ and $a(\cdot\,;\cdot)$ denote the bilinear forms

$$\begin{aligned} c(v_O, w; \hat{v}_O, \hat{w}) &= (mw, \hat{w})_\Omega + (\alpha \nabla w, \nabla \hat{w})_\Omega \\ a(v_O, w; \hat{v}_O, \hat{w}) &= \ell_O(K\vec{1}_O \nabla w; \vec{1}_O \nabla \hat{w}) + 12\ell_O(\mathbf{h}_O \mathbf{D}_O v_O; \hat{v}_O) \\ &\quad + (\mathbf{G}_E \mathbf{h}_E \varphi_E, \hat{\varphi}_E)_\Omega \end{aligned}$$

where φ and $\hat{\varphi}$ satisfy (1), and

$$m = \sum_{i=1}^{n} h_i \rho_i, \qquad \alpha = \frac{1}{12} \sum_{i \, odd}^{n} \rho_i h_i^3, \qquad K = \sum_{i \ odd} D_i h_i^3.$$

Let us assume the plate is clamped on a portion Γ_0 of the boundary Γ and subject to applied forces on the complementary portion Γ_1 and distributed forces on Ω.

The equations of motion are determined by Hamilton's principal from the energy. The variational differential equation one obtains is the following:

$$\begin{aligned} &(mw'', \hat{w})_\Omega + (\alpha \nabla w'', \nabla \hat{w})_\Omega + \ell(K \nabla_O w, \nabla \hat{w}) \\ &+ 12\ell_O(\mathbf{h}_O \mathbf{D}_O v_O; \hat{v}_O) + (\mathbf{G}_E \mathbf{h}_E \varphi_E, \mathbf{h}_E^{-1} B \hat{v}_O + N \nabla \hat{w}) \\ &= W(\{\hat{v}_O, \hat{w}\}) = \textstyle\int_\Omega \hat{w} f_3 + \hat{v}_O f_O \, dx + \int_{\Gamma_1} \hat{w} g_3 + \hat{v}_O g_O - \hat{w}_n M_n \, ds. \end{aligned}$$

In the above, f_3 is the (scalar) transverse applied force in Ω and f_O is the net in-plane force acting on the odd layers in Ω. (f_O has rows $f^i = \{f_1^i, f_2^i\}$, $i = 1, 3, 5, \ldots 2m+1$.) The boundary forces acting on Γ_1 are: the transverse force g_3 (scalar valued), the in-plane force (dimensions matching f_O) and the bending moment M_n (scalar valued). (For precise description of forces see [3].) The test functions $\hat{w}$, $\hat{v}_O$ satisfy the clamped boundary conditions on Γ_0 and are sufficiently regular.

Inclusion of shear damping. Damping may be introduced into any of the plate layers by replacing the standard stress-strain relation for transversely isotropic materials by an appropriate dissipative constitutive law. In the case of *strain-rate shear damping*, the stress-strain relation for transverse shear: $\sigma_{13} = 2G\epsilon_{13}$ is replaced by a dissipative stress-strain relation of the form: $\sigma_{13} = 2(G + \tilde{G}\frac{d}{dt})\epsilon_{13}$. The equations of motion are then modified by the correspondence $G \to (G + \tilde{G}\frac{\partial}{\partial t})$.

In our situation, shear damping occurs only in the even layers. Let $\tilde{\mathbf{G}}_E$ be the diagonal matrix with diagonal elements $\tilde{G}_i$, i even. We obtain the following variational differential equation when shear damping is included in the even layers:

$$\begin{aligned}(mw'', \hat{w})_\Omega + (\alpha\nabla w'', \nabla\hat{w})_\Omega + \ell(K\nabla w, \nabla\hat{w}) + 12\ell_O(\mathbf{h}_O\mathbf{D}_O v_O; \hat{v}_O) \\ +(\mathbf{G}_E\mathbf{h}_E\varphi_E + \tilde{\mathbf{G}}_E\mathbf{h}_E\varphi'_E, \mathbf{h}_E^{-1}B\hat{v}_O + N\nabla\hat{w}) = W(\{\hat{v}_O, \hat{w}\}). \quad (2)\end{aligned}$$

1.1 Boundary value problem

We first need to define some operators and boundary operators.

Define the operator L by $L\phi = L\{\phi_1, \phi_2\} = \{L_1(\phi), L_2(\phi)\}$ where

$$\begin{aligned} L_1\phi &= \tfrac{\partial}{\partial x_1}[(\tfrac{\partial\phi_1}{\partial x_1} + \nu\tfrac{\partial\phi_2}{\partial x_2})] + \tfrac{\partial}{\partial x_2}[(\tfrac{1-\nu}{2})(\tfrac{\partial\phi_1}{\partial x_2} + \tfrac{\partial\phi_2}{\partial x_1})] \\ L_2\phi &= \tfrac{\partial}{\partial x_2}[(\tfrac{\partial\phi_2}{\partial x_2} + \nu\tfrac{\partial\phi_1}{\partial x_1})] + \tfrac{\partial}{\partial x_1}[(\tfrac{1-\nu}{2})(\tfrac{\partial\phi_2}{\partial x_1} + \tfrac{\partial\phi_1}{\partial x_2})]. \end{aligned}$$

The associated boundary operator $\mathcal{B}\phi = \{\mathcal{B}_1(\phi_1, \phi_2), (\mathcal{B}_2(\phi_1, \phi_2)\}$ is defined by

$$\begin{aligned} \mathcal{B}_1(\phi_1, \phi_2) &= \left[\left(\tfrac{\partial\phi_1}{\partial x_1}n_1 + \nu\tfrac{\partial\phi_2}{\partial x_2}n_1\right) + \left(\tfrac{1-\nu}{2}\right)\left(\tfrac{\partial\phi_1}{\partial x_2} + \tfrac{\partial\phi_2}{\partial x_1}\right)n_2\right] \\ \mathcal{B}_2(\phi_1, \phi_2) &= \left[\left(\tfrac{\partial\phi_2}{\partial x_2}n_2 + \nu\tfrac{\partial\phi_1}{\partial x_1}n_2\right) + \left(\tfrac{1-\nu}{2}\right)\left(\tfrac{\partial\phi_2}{\partial x_1} + \tfrac{\partial\phi_1}{\partial x_2}\right)n_1\right]. \end{aligned}$$

where $\vec{n} = (n_1, n_2)$ denotes the outward unit normal to Γ. (For later reference: also define $\vec{\tau} = (-n_2, n_1)$ as the unit tangent vector to Γ.)

The following Green's formula is valid for all sufficiently smooth $\hat{\phi}$, ϕ:

$$\ell(\phi, \hat{\phi}) = (\mathcal{B}\phi, \hat{\phi})_\Gamma - (L\phi, \hat{\phi})_\Omega. \quad (3)$$

For $\xi = (\xi_j^i)$ $(i = 1, 2, \ldots, n,\ j = 1, 2)$ define the matrices $\mathbf{L}\xi$ and $\mathbf{B}\xi$ by

$$(\mathbf{L}\xi)_{ij} = (L_j\xi^i), \qquad (\mathbf{B}\xi)_{ij} = (\mathcal{B}_j\xi^i), \qquad i = 1, 2, \ldots, n,\ j = 1, 2.$$

Furthermore we define the operators $\mathbf{L}_O$, $\mathbf{L}_E$, $\mathbf{B}_O$, $\mathbf{B}_E$ from $\mathbf{L}$ and $\mathbf{B}$ based upon the convention that O and E subscripts refer to the parts of the operators that act upon odd and even rows respectively.

Similar to (3) the following Green's formula is valid for all sufficiently smooth ξ, $\hat{\xi}$:

$$\ell_O(\xi,\hat{\xi}) = (\mathbf{B}_O\xi,\hat{\xi})_\Gamma - (\mathbf{L}_O\xi,\hat{\xi})_\Omega. \tag{4}$$

The equations of motion can be found from (2) using the Green's formulas (3), (4) and further integrations by parts:

$$\left.\begin{array}{rcl} mw'' - \alpha\Delta w'' + K\Delta^2 w - \operatorname{div}\vec{N}^T\mathbf{h}_E(\mathbf{G}_E\varphi_E + \tilde{\mathbf{G}}_E\varphi_E') & = & f_3 \\ -12\mathbf{h}_O\mathbf{D}_O\mathbf{L}_O v_O + B^T(\mathbf{G}_E\varphi_E + \tilde{\mathbf{G}}\varphi_E') & = & f_O \end{array}\right\} \tag{5}$$

$$\text{where } \varphi_E = \mathbf{h}_E^{-1}Bv_O + N\nabla w$$

$$\left.\begin{array}{rcl} \alpha w_n'' - K(\frac{\partial}{\partial\vec{\tau}}(\bar{\mathcal{B}}\nabla w)\cdot\vec{\tau}) - K(\Delta w)_n & & \\ +\vec{N}^T\mathbf{h}_E(\mathbf{G}_E\varphi_E + \tilde{\mathbf{G}}_E\varphi_E)\cdot\vec{n} & = & g_3 \\ K(\mathcal{B}\nabla w)\cdot\vec{n} & = & -M_n \\ 12\mathbf{D}_O\mathbf{B}_O\mathbf{h}_O v_O & = & g_O \end{array}\right\} \tag{6}$$

$$\left.\begin{array}{rcl} w = w_n & = & 0 \\ v_O & = & 0 \end{array}\right\} \tag{7}$$

where (5) holds on $Q = \Omega\times(0,\infty)$, (6) holds on $\Sigma_1 = \Gamma_1\times(0,\infty)$, and (7) holds on $\Sigma_0 = \Gamma_0\times(0,\infty)$.

Appropriate initial conditions are of the form

$$w(0) = w^0, \qquad w'(0) = w^1, \qquad \varphi_E(0) = \varphi_E^0 \tag{8}$$

2. State variable formulation

There is a difficulty in proving existence and uniqueness in the formulation above since it is not an easy task to solve for φ_E' in (5), which is needed to in order to solve for the generator of a semigroup. However, this can be accomplished in the present situation (when all the Poisson ratios in the stiff layers are same) due to the decomposition described in this section.

Let $W = \operatorname{span}\vec{1}_O$, and $V = \operatorname{span}\mathbf{D}_O\mathbf{h}_O\vec{1}_O$. Clearly these spaces are not orthogonal. Therefore there is a unique decomposition of $\mathbf{R}^{m+1}$ (or $\mathbf{C}^{m+1}$) into a part in $W^\perp$ and a part in V. Since the kernel of B is easily seen to be W, any vector in $W^\perp$ can be written as an element in the image of B^T. Thus at each point in Ω and Γ_1, respectively we have the decomposition

$$f_O = B^T f_E + \tilde{f}_O, \qquad g_O = B^T g_E + \tilde{g}_O$$

where $\tilde{f}_O$ and $\tilde{g}_O$ at each point belong to $V\times V$ and f_E and g_E at each point belong to $\mathbf{R}^m$.

We return to (5)–(7) but focus on the terms coupled to v_O. The relevant part of the system is

$$\begin{aligned} -12\mathbf{h}_O\mathbf{D}_O\mathbf{L}_O v_O + B^T(\mathbf{G}_E\varphi_E + \tilde{\mathbf{G}}_E\varphi_E') &= \tilde{f}_O + B^T f_E \text{ in } \Omega & (9)\\ 12\mathbf{D}_O\mathbf{h}_O\mathbf{B}_O v_O &= \tilde{g}_O + B^T g_E \text{ on } \Gamma_1 & (10)\\ v &= 0 \text{ on } \Gamma_0. & (11) \end{aligned}$$

We multiply (9) and (10) on the left by $\vec{1}_O^T$ and obtain

$$\begin{aligned} -12L(\sum_{i\,odd} D_i h_i v^i) &= \sum_{i\,odd} \tilde{f}_O^i \quad \text{in } \Omega & (12)\\ 12\mathcal{B}(\sum_{i\,odd} D_i h_i v^i) &= \sum_{i\,odd} \tilde{g}_O^i \quad \text{in } \Gamma_1. & (13) \end{aligned}$$

If we define

$$\bar{v}_O = \vec{C}^T v_O = \sum_{i\,odd} D_i h_i v^i, \quad \bar{f} = \sum_{i\,odd} \tilde{f}_O^i, \quad \bar{g} = \sum_{i\,odd} \tilde{g}_O^i \tag{14}$$

then (12)–(13) can be written as

$$\begin{aligned} -12L\bar{v}_O &= \bar{f} && \text{in } \Omega\\ 12\mathcal{B}\bar{v}_O &= \bar{g} && \text{on } \Gamma_1\\ \bar{v}_O &= 0 && \text{on } \Gamma_0. \end{aligned} \tag{15}$$

When the Poisson's ratios are constant we can write

$$B\mathbf{L}_O v_O = \mathbf{L}_E B v_O = \mathbf{L}_E \mathbf{h}_E(\varphi_E - \vec{N}\nabla w). \tag{16}$$

Define the matrix P by

$$P = \frac{1}{12} B\mathbf{D}_O^{-1}\mathbf{h}_O^{-1}B^T. \tag{17}$$

We multiply (9) and (10) by $\frac{1}{12}B\mathbf{h}_O^{-1}\mathbf{D}_O^{-1}$ and obtain

$$-B\mathbf{L}_O v_O + P\mathbf{G}_E\varphi_E \quad = \quad Pf_E \quad \text{in } \Omega. \tag{18}$$

Thus, after substitution of (16) into the above, we obtain the system

$$\begin{aligned} -\mathbf{L}_E\mathbf{h}_E(\varphi_E - \vec{N}\nabla w) + P(\mathbf{G}_E\varphi_E + \tilde{\mathbf{G}}_E\varphi_E') &= Pf_E \quad \text{in } \Omega & (19)\\ \mathbf{B}_E\mathbf{h}_E(\varphi_E - \vec{N}\nabla w) &= Pg_E \quad \text{on } \Gamma_1 & (20)\\ v_O &= 0 \quad \text{on } \Gamma_O. & (21) \end{aligned}$$

Lemma 1 *The matrix P in (17) is positive, i.e., $z^T P z > 0$ for all $z \in \mathbf{R}^m$. Moreover, every element p_{ij} of P^{-1} is positive.*

Proof: We can write $P = B\Lambda B^T$ where $\Lambda = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_{m+1})$, with $\lambda_i > 0$, all i. Moreover B can be written as the sum of partitioned matrices $B = (\vec{0} \vdots I_m) - (I_m \vdots \vec{0})$, where I_m is the identity on $\mathbf{R}^m$, and $\vec{0}$ is a column vector of zeros. Performing the block multiplications in (17), one obtains

$$P = \Lambda_{11} + \Lambda_{m+1,m+1} - \Lambda_{1,m+1} - \Lambda_{m+1,1}$$

where Λ_{ij} is the minor for the ij-th spot in Λ. Thus the diagonal of P has the elements $p_{ii} = \lambda_i + \lambda_{i+1}$, for $i = 1, 2, \ldots m$. The superdiagonal and subdiagonal sequence (beginning in the upper left) are each $-\lambda_2, -\lambda_3, \ldots, -\lambda_m$. All other elements are zero. We therefore have

$$\begin{aligned} p_{11} = \lambda_1 + \lambda_2 &> \sum_{k=2}^{m} |p_{1,k}| = \lambda_2 \\ p_{mm} = \lambda_m + \lambda_{m+1} &> \sum_{k=1}^{m-1} |p_{m,k}| = \lambda_m \\ p_{kk} = \lambda_k + \lambda_{k+1} &\geq \sum_{j \neq k} |p_{1,j}| = \lambda_k + \lambda_{k+1}. \end{aligned}$$

Thus P is diagonally dominant (with strict inequality in the first and last rows). Furthermore it is easily verified that P is irreducible. It follows from the theory of M-matrices (see e.g., [7],Theorem 3, p. 531) that P^{-1} is a nonnegative matrix (every element nonnegative). This completes the proof.

Since P is invertible, one can solve for $\mathbf{G}_E\varphi_E + \tilde{\mathbf{G}}_E\varphi_E'$ in (19)–(21) and substitute into (5)–(7) to obtain the following system:

$$\left.\begin{aligned} mw'' - \alpha\Delta w'' + K\Delta^2 w & \\ -\mathrm{div}\,\vec{N}^T\{\mathbf{h}_E P^{-1}\mathbf{L}_E B v_O\} &= f_3 - \mathrm{div}\,\vec{N}^T\mathbf{h}_E f_E \\ \tilde{\mathbf{G}}_E\varphi'_E + \mathbf{G}_E\varphi_E - P^{-1}\mathbf{L}_E B v_O &= f_E \end{aligned}\right\} \text{ in } Q \quad (22)$$

$$\left.\begin{aligned} \alpha(w_n)'' - \tfrac{\partial}{\partial\tau}(\vec{\tau}\cdot K\mathcal{B}\nabla w) + K(\Delta w)_n & \\ +\vec{N}^T\mathbf{h}_E(\mathbf{G}_E\varphi_E + \tilde{\mathbf{G}}_E\dot{\varphi}_E)\cdot\vec{n} &= g_3 \\ -K\mathcal{B}\nabla w\cdot\vec{n} &= M_n \\ P^{-1}\mathbf{B}_E B v_O &= g_E \end{aligned}\right\} \text{ on } \Sigma_1 \quad (23)$$

$$\varphi_E = 0, \qquad w = 0, \qquad w_n = 0 \qquad \text{on } \Sigma_0 \quad (24)$$

Where

$$B v_O = \mathbf{h}_E(\varphi_E - \vec{N}\nabla w), \qquad \vec{C}^T v_O = \bar{v}_O, \qquad (25)$$

where $\bar{v}_O$ is the solution to the stationary Lamé system (15).

Initial conditions are specified for w, w' and φ_E.

Suppose for the moment that the system (22)–(24) above together with appropriate initial conditions uniquely determines φ_E and $\bar{v}_O$. Then v_O is also determined: Indeed, define B_C to be the partitioned (square) matrix defined by $B_C^T = (B^T \vdots \vec{C})$, ($\vec{C}$ is defined in (14)). Then (since B_C is invertible from our earlier discussion) v_O can be obtained from the following:

$$B_C v_O = \begin{pmatrix} \mathbf{h}_E \varphi_E - \mathbf{h}_E \vec{N} \nabla w \\ \bar{v}_O \end{pmatrix}. \tag{26}$$

3. Semigroup formulation of homogeneous problem

We consider the homogeneous problem

$$\left.\begin{array}{rcl} mw'' - \alpha \Delta w'' + K\Delta^2 w - \operatorname{div} \vec{N}^T \{\mathbf{h}_E P^{-1} \mathbf{L}_E B v_O\} & = & 0 \text{ in } Q \\ \tilde{\mathbf{G}}_E \varphi_E' + \mathbf{G}_E \varphi_E - P^{-1} \mathbf{L}_E B v_O & = & 0 \text{ in } Q \end{array}\right\} \tag{27}$$

For simplicity, we restrict our interest here to the case of case of hinged boundary conditions:

$$\left.\begin{array}{rcl} \mathcal{B} \nabla w \cdot \vec{n} & = & 0 \\ \mathbf{B}_E B v_O & = & 0 \end{array}\right\} \quad \text{on } \Sigma \tag{28}$$

$$w = 0 \qquad \text{on } \Sigma. \tag{29}$$

Here $\Sigma = \partial\Omega \times (0, \infty)$.

For the variable $\bar{v}_O = \vec{C}^T v_O$ we assumme fixed boundary conditions, so that $\bar{v}_O$ is the solution of

$$\begin{array}{rll} -12 L \bar{v}_O & = 0 & \text{in } \Omega \\ \bar{v}_O & = 0 & \text{on } \Gamma_0. \end{array} \tag{30}$$

The system (30) has the unique solution $\bar{v}_O = 0$ since the form ℓ is known to be coercive on $H_0^1(\Omega)$ ([6]). Therefore we eliminate v_O and henceforth may assume $\vec{C}^T v_O = 0$.

Define the spaces

$$\begin{array}{rcl} L_O^2(\Omega) & = & \{v_O = (v_j^i), i = 1, 3, 5, \ldots n,\ j = 1, 2 : v_j^i \in L^2(\Omega)\} \\ H_{O,\Gamma}^1 & = & \{v_O \in L_O^2(\Omega) : v_j^i \in H_0^1(\Omega)\} \\ L_E^2(\Omega) & = & \{\phi_E = (\phi_j^i), i = 2, 4, \ldots 2m,\ j = 1, 2 : \phi_j^i \in L^2(\Omega)\} \\ H_E^1 & = & \{\phi_E \in L_E^2(\Omega) : \phi_j^i \in H^1(\Omega)\}. \end{array}$$

The energy space is $\{w, w', v_O \in \mathcal{V} : \vec{C}^T v_O = 0\}$ where

$$\mathcal{V} = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \times H^1_{O,\Gamma}.$$

Let $\phi = Ry$ denote the solution to the following elliptic problem

$$m\phi - \alpha\Delta\phi = y, \qquad \phi = 0 \text{ on } \Gamma. \tag{31}$$

Then R maps $H^{-1}(\Omega)$ into $H_0^1(\Omega)$ continuously.

Let $y^T = (y_1, y_2, y_3) = (w, w', \varphi_E)$ then

$$y' = Ay = \begin{pmatrix} y_2 \\ R(-K\Delta^2 y_1 + \operatorname{div} \vec{N}^T\{\mathbf{h}_E P^{-1}\mathbf{L}_E\mathbf{h}_E(y_3 - \vec{N}\nabla y_1)\} \\ -\tilde{\mathbf{G}}_E^{-1}\mathbf{G}_E y_3 + \tilde{\mathbf{G}}_E^{-1}P^{-1}L_E\mathbf{h}_E(y_3 - \vec{N}\nabla y_1) \end{pmatrix} \tag{32}$$

Define matrix S by

$$B_C\vec{y} = \begin{pmatrix} \vec{a} \\ \cdots \\ 0 \end{pmatrix} \Leftrightarrow \vec{y} = S\vec{a} \tag{33}$$

It is simple to verify that

$$BSy = y \qquad \forall\, y \in \mathbf{R}^m \tag{34}$$

$$SBu = u \qquad \forall\, u \in \mathbf{R}^{m+1} : \vec{C}^T u = 0 \tag{35}$$

Along with the state variables y, define $v_y = S\mathbf{h}_E(y_3 - \vec{N}\nabla y_1)$.

The energy inner product is defined by

$$\begin{aligned}(y, z)_{\mathcal{E}} &= (my_2, z_2)_\Omega + \alpha(\nabla y_2, \nabla z_2)_\Omega + K\ell_O(\vec{1}_0\nabla y_1, \vec{1}_0\nabla z_1) \\ &\quad + (\mathbf{G}_E\mathbf{h}_E y_3, z_3)_\Omega + 12\ell_0(\mathbf{h}_0\mathbf{D}_O v_y, v_z).\end{aligned}$$

Define $\mathcal{H}$ by

$$\mathcal{H} = \{w, w', \varphi_E : w \in H^2(\Omega) \cap H_0^1(\Omega),\ w' \in H_0^1(\Omega),\ \varphi_E \in H_E^1\}. \tag{36}$$

Then one can verify that $A : \mathcal{D}(A) \to \mathcal{H}$ where

$$\mathcal{D}(A) = \{y \in \mathcal{H} : y_1 \in H^3(\Omega), y_2 \in H^2(\Omega), y_3 \in H_E^3, + \text{BC's}\} \tag{37}$$

where "+BC's" means the boundary conditions $(\mathcal{B}\nabla y_1)\cdot\vec{n} = 0$, $\mathbf{B}_E(Bv_y) = 0$ are imposed on Γ.

Theorem 2 *The operator A in (32) is the generator of strongly continuous semigroup of contractions on $\mathcal{H}$. Moreover for any $y \in \mathcal{D}(A)$,*

$$Re\,(y, Ay)_{\mathcal{E}} = -(\tilde{\mathbf{G}}_E^{-1}\mathbf{h}_E U, U)_\Omega; \qquad U = P^{-1}B\mathbf{L}_O v_y - \mathbf{G}_E y_3. \tag{38}$$

Proof: To show that A is the generator of a contraction semigroup, By the Lumer-Phillips theorem it is enough to demonstrate that A is dissipative and satisfies the range condition: i.e., $(I-A)^{-1}$ maps $D(A)$ onto $\mathcal{H}$.

First we show dissipativity. Let $z = Ay$, then

$$\begin{aligned}
&(y, Ay)_{\mathcal{E}} = (my_2, z_2)_\Omega + \alpha(\nabla y_2, \nabla z_2)_\Omega + K\ell(\nabla y_1, \nabla z_1) \\
&\quad +(\mathbf{G}_E \mathbf{h}_E y_3, z_3)_\Omega + 12\ell_0(\mathbf{h}_0 \mathbf{D}_O v_y, v_z) \\
&= (my_2, R[-K\Delta^2 y_1 + \operatorname{div} \vec{N}^T\{\mathbf{h}_E P^{-1}\mathbf{L}_E \mathbf{h}_E(y_3 - \vec{N}\nabla y_1)\}])_\Omega \\
&\quad +\alpha(\nabla y_2, \nabla R[-K\Delta^2 y_1 + \operatorname{div} \vec{N}^T\{\mathbf{h}_E P^{-1}\mathbf{L}_E \mathbf{h}_E(y_3 - \vec{N}\nabla y_1)\}])_\Omega \\
&+K\ell(\nabla y_1, \nabla y_2) + (\mathbf{G}_E \mathbf{h}_E y_3, -\tilde{\mathbf{G}}_E^{-1}\mathbf{G}_E y_3 + \tilde{\mathbf{G}}_E^{-1} P^{-1} L_E \mathbf{h}_E(y_3 - \vec{N}\nabla y_1))_\Omega \\
&+12\ell_O(\mathbf{h}_O \mathbf{D}_O v_y, S\mathbf{h}_E(-\tilde{\mathbf{G}}_E^{-1}\mathbf{G}_E y_3 + \tilde{\mathbf{G}}_E^{-1} P^{-1} L_E \mathbf{h}_E(y_3 - \vec{N}\nabla y_1) - \vec{N}\nabla y_2)) \\
&= \{(y_2, -K\Delta^2 y_1)_\Omega + K\ell(\nabla y_1, \nabla y_2)\} + \{(y_2, \operatorname{div} \vec{N}^T \mathbf{h}_E P^{-1} L_E B v_y)_\Omega \\
&+12\ell_O(\mathbf{h}_O \mathbf{D}_O v_y, -S\mathbf{h}_e \vec{N}\nabla y_2)\} + \{\text{terms with } \tilde{\mathbf{G}}_E^{-1}\} \\
&=: \{T_1\} + \{T_2\} + \{T_3\}.
\end{aligned}$$

The real part of the first bracketed term T_1 is easily shown to vanish using integrations by parts. For the second term T_2 we integrate by parts, apply (16) and use the boundary conditions to obtain

$$\begin{aligned}
T_2 \quad &= \quad (y_2, \operatorname{div} \vec{N}^T \mathbf{h}_E P^{-1} \mathbf{L}_E B v_y)_\Omega + 12(\mathbf{h}_O \mathbf{D}_O \mathbf{B}_O v_y, -S\mathbf{h}_E \vec{N}\nabla y_2)_\Gamma \\
&\quad +12(\mathbf{h}_O \mathbf{D}_O \mathbf{L}_O v_y, S\mathbf{h}_E \vec{N}\nabla y_2)_\Omega = (y_2, \operatorname{div} \vec{N}^T \mathbf{h}_E P^{-1} B \mathbf{L}_O v_y)_\Omega \\
&\quad -12(\operatorname{div} \vec{N}^T \mathbf{h}_E S^T \mathbf{L}_O \mathbf{h}_O \mathbf{D}_O v_y, y_2)_\Omega + 12(S^T \mathbf{L}_O \mathbf{h}_O \mathbf{D}_O v_y \cdot \vec{n}, \mathbf{h}_E \vec{N} y_2)_\Gamma \\
&= \quad (y_2, \operatorname{div} \vec{N}^T \mathbf{h}_E P^{-1} B \mathbf{L}_O v_y)_\Omega \\
&\quad -(\operatorname{div} \vec{N}^T \mathbf{h}_E P^{-1} B \mathbf{D}_O^{-1} \mathbf{h}_O^{-1} B^T S^T \mathbf{h}_O \mathbf{D}_O \mathbf{L}_O v_y, y_2)_\Omega
\end{aligned}$$

One has that

$$(B - B\mathbf{D}_O^{-1}\mathbf{h}_O^{-1} B^T S^T \mathbf{h}_O \mathbf{D}_O) = B\mathbf{D}_O^{-1}\mathbf{h}_O^{-1}(I - B^T S^T)\mathbf{h}_O \mathbf{D}_O = 0$$

where we have applied the transposition of formula (35) to obtain the last equality. Thus the real part of T_2 vanishes.

Hence

$$\begin{aligned}
T_3 \quad &= (\mathbf{G}_E \mathbf{h}_E y_3, -\tilde{\mathbf{G}}_E^{-1}\mathbf{G}_E y_3)_\Omega + (\mathbf{G}_E \mathbf{h}_E y_3, \tilde{\mathbf{G}}_E^{-1} P^{-1} L_E B v_y)_\Omega \\
&\quad +12\ell_O(\mathbf{h}_O \mathbf{D}_O v_y, S\mathbf{h}_E(-\tilde{\mathbf{G}}_E^{-1}\mathbf{G}_E y_3 + \tilde{\mathbf{G}}_E^{-1} P^{-1} L_E B v_y)) \\
&= -(\mathbf{G}_E \mathbf{h}_E y_3, \tilde{\mathbf{G}}_E^{-1}\mathbf{G}_E y_3)_\Omega + (\mathbf{G}_E \mathbf{h}_E y_3, \tilde{\mathbf{G}}_E^{-1} P^{-1} B \mathbf{L}_O v_y)_\Omega \\
&\quad +12(\tilde{\mathbf{G}}_E^{-1} S^T \mathbf{h}_O \mathbf{D}_O \mathbf{L}_O v_y, \mathbf{h}_E \mathbf{G}_E y_3)_\Omega \\
&\quad -12(\mathbf{h}_O \mathbf{D}_O \mathbf{L}_O v_y, S\mathbf{h}_E \tilde{\mathbf{G}}_E^{-1} P^{-1} B \mathbf{L}_O v_y))_\Omega.
\end{aligned}$$

One can easily show using (35) and the definition of P that

$$12S^T \mathbf{h}_O \mathbf{D}_O = P^{-1}B. \tag{39}$$

Applying (39) to the present calculation one finds that

$$\begin{aligned}\operatorname{Re}(y, Ay)_{\mathcal{E}} &= -(\tilde{\mathbf{G}}_E^{-1}\mathbf{h}_E\mathbf{G}_E y_3, \mathbf{G}_E y_3)_\Omega + \operatorname{Re}\{(\tilde{\mathbf{G}}_E^{-1}\mathbf{h}_E\mathbf{G}_E y_3, P^{-1}B\mathbf{L}_O v_y)_\Omega \\ &\quad +(\tilde{\mathbf{G}}_E^{-1}\mathbf{h}_E P^{-1}B\mathbf{L}_O v_y, \mathbf{G}_E y_3)_\Omega\} \\ &\quad -(\tilde{\mathbf{G}}_E^{-1}\mathbf{h}_E P^{-1}B\mathbf{L}_O v_y, P^{-1}B\mathbf{L}_O v_y)_\Omega \\ &= -(\tilde{\mathbf{G}}_E^{-1}\mathbf{h}_E(P^{-1}B\mathbf{L}_O v_y - \mathbf{G}_E y_3), (P^{-1}B\mathbf{L}_O v_y - \mathbf{G}_E y_3))_\Omega\end{aligned}$$

Since $\tilde{\mathbf{G}}^{-1}\mathbf{h}_E$ is positive definite, A is dissipative.

Next we check the range condition.

In what follows we use the notation

$$|w|_s = \|w\|_{H^s(\Omega)}, \qquad |\varphi_E|_{E,s} = \sum_{i\,\text{even}}^{m} \sum_{j=1,2} |\varphi_j^i|_s.$$

Assume that $-Ay + y = z$ where $z \in \mathcal{H}$. This is the same as

$$\begin{aligned} -y_2 + y_1 &= z_1 & (40)\\ -R(-K\Delta^2 y_1 + \operatorname{div}\vec{N}^T\mathbf{h}_E P^{-1}\mathbf{L}_E\mathbf{h}_E(y_3 - \vec{N}\nabla y_1)) + y_2 &= z_2 & (41)\\ \tilde{\mathbf{G}}_E^{-1}\mathbf{G}_E y_3 - \tilde{\mathbf{G}}_E^{-1}P^{-1}\mathbf{L}_E\mathbf{h}_E(y_3 - \vec{N}\nabla y_1) + y_3 &= z_3 & (42)\end{aligned}$$

Solving for $\Delta^2 y_1$ and $\mathbf{L}_E y_3$ in (41) and (42) gives

$$\begin{aligned} K\Delta^2 y_1 - \alpha\Delta y_1 + m y_1 &= \operatorname{div}\vec{N}^T\tilde{\mathbf{G}}_E\mathbf{h}_E(y_3 - z_3) + (m - \alpha\Delta)(z_1 + z_2) & (43)\\ \mathbf{G}_E y_3 - P^{-1}\mathbf{L}_E\mathbf{h}_E y_3 + \mathbf{G}_E y_3 &= -P^{-1}\mathbf{L}_E\mathbf{h}_E\vec{N}\nabla y_1 + \tilde{\mathbf{G}}_E z_3. & (44)\end{aligned}$$

The left hand side of (43) is associated with a coercive bilinear symmetric form on $H^2(\Omega) \cap H_0^1(\Omega)$ (see [3]) and hence for functions $y \in \mathcal{D}(A)$ one can show that

$$|y_1|_4 \le C|K\Delta^2 y_1 - \alpha\Delta y_1 + m y_1|_0.$$

Thus, considering the right hand side of (43),

$$|y_1|_4 \le C_1(|z_3|_{E,1} + |y_3|_{E,1} + |z_1|_2 + |z_2|_2). \tag{45}$$

Estimation of (43) with $R^{1/2}$ applied to both sides leads to

$$|y_1|_3 \le C_2(|z_3|_{E,1} + |y_3|_{E,1} + |z_1|_1 + |z_2|_1). \tag{46}$$

Using that P^{-1} is symmetric, positive definite, we also have that

$$|y_3|_{E,2} \le C|\mathbf{G}_E y_3 - P^{-1}\mathbf{L}_E\mathbf{h}_E y_3 + \tilde{\mathbf{G}}_E y_3|_0 \le C_1|z_3|_{E,0} + |y_1|_3. \tag{47}$$

Hence from (46)

$$|y_3|_{E,2} \leq C|z_3|_{E,1} + |y_3|_{E,1} + |z_1|_1 + |z_2|_1) \tag{48}$$

From (40) we also have

$$|y_2|_2 \leq C(|y_1|_2 + |z_1|_2). \tag{49}$$

Since A has is dissipative, $I - A$ is injective and hence any solutions to (40)–(42) are unique. Applying the usual compactness/uniqueness argument to the estimates (46), (48), (49) yields

$$|y_1|_3 + |y_2|_2 + |y_3|_{E,2} \leq C(|z_2|_1 + |z_3|_{E,1} + |z_1|_2). \tag{50}$$

Thus $y \in \mathcal{D}(A)$ and $I - A$ is surjective from $\mathcal{D}(A)$ to $\mathcal{H}$. This completes the proof.

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