

Structural Damping in Laminated Beams Due to Interfacial Slip

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ABSTRACT

Three closely related models for two-layered beams in which slip can occur at the interface are described. In the first, beam layers are modeled under the assumptions of Timoshenko beam theory. Along the interface, an adhesive layer of negligible thickness bonds the surfaces so that a small amount of slip is possible. Friction is assumed to be proportional to the rate of slip. The second is obtained from the first by letting the shear stiffness of each beam tend to infinity. The third is obtained from the second by assuming that the moment of inertia parameter is negligible. In the last case, an analog of the Euler-Bernoulli beam is obtained which exhibits frequency-proportional damping characteristics. The three models are compared, both numerically and analytically, and found to be in close agreement for low frequency motions. Optimal damping rates are calculated for these low frequency motions.

1. INTRODUCTION

In Hansen [1] a model for a two-layered plate in which slip can occur along the interface is derived. There, it is assumed that an adhesive layer of negligible thickness and mass bonds the two adjoining surfaces in such a way that the restoring force created by the adhesive is proportional to the amount of slip. The plate layers are modeled under the assumptions of Reissner-Mindlin plate theory, i.e., the straight filaments orthogonal to each center sheet at equilibrium remain straight during deformation and the transverse displacements are constant along each deformed filament. In the symmetric case, where both plates are identical, a system of bending equations can be decoupled from a set that describes the in-plane motions. These bending equations (see [1, (3.16)]) form a coupled hyperbolic system involving the transverse displacement, rotation angles and slips (i.e. the slip in two independent directions). This plate model reduces to the Reissner-Mindlin plate in the limit as the adhesive stiffness tends to infinity.

In this article we will be concerned with the beam analog, with strain-rate damping included, of the above described plate model [1, (3.16)]. The equations for this beam model in the absence of external forces are

$$\begin{cases} \text{(i)} & \rho\ddot{w} + (G(\psi - w_x))_x = 0 \\ \text{(ii)} & I_\rho(3\ddot{s} - \ddot{\psi}) - G(\psi - w_x) - (D(3s_x - \psi_x))_x = 0 \\ \text{(iii)} & I_\rho\ddot{s} + G(\psi - w_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta\dot{s} - (Ds_x)_x = 0, \end{cases} \quad (1)$$

where “ $\dot{\cdot}$ ” denotes differentiation with respect to time, the subscripted x denotes differentiation with respect to the longitudinal spatial variable, w denotes the transverse displacement, ψ represents the rotation angle, s is proportional to the amount of “slip” along the interface and the parameters ρ , G , I_ρ , D , γ , β are the density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness and adhesive damping parameter, respectively (see Section 2). The system (1) is an analog of the Timoshenko beam model in the sense that equations (i) and (ii) of (1) reduce exactly to the Timoshenko beam system if the slip s is assumed to be identically zero. The third equation describes the dynamics of the slip. A derivation of the system (1) with external forces included is given in Section 2.

Our main interest in this article is to compare the spectral characteristics of (1) with lower-order approximations of (1) in the case where the shear stiffness parameter G is large and the moment of inertia parameter I_ρ is small in comparison to D/ρ .

By letting $G \rightarrow \infty$ in (1) we obtain an analog of the Rayleigh beam, also involving an extra equation for the slip (see Section 3). Then by setting $I_\rho = 0$ in the latter equation we obtain the following:

$$\begin{cases} \text{(i)} & \rho \ddot{w} - (D \xi_x)_{xx} = 0 \\ \text{(ii)} & \beta(\dot{\xi} + \dot{w}_x) + \gamma(\xi + w_x) - 3(D \xi_x)_x - \frac{3}{4}(D w_{xx})_x = 0, \end{cases} \quad (2)$$

where $\xi = 3s - \psi$ is the *effective rotation angle*.

The spectral properties of (2) are analyzed in Section 4.

To describe some of our results, first consider (2) in the undamped case, i.e., where $\beta = 0$. If the adhesive stiffness γ tends to infinity (effectively causing a no-slip condition between the two layers), then (2)-(ii) suggests that $\xi \approx -w_x$, so that (2)-(i) reduces to the Euler-Bernoulli (EB) beam equation $\rho \ddot{w} + (D w_{xx})_{xx} = 0$. On the other hand, if γ tends to zero (so that the two layers effectively become delaminated) then (2) reduces to an EB beam with one fourth the stiffness of the case where $\gamma \rightarrow \infty$. (This is the same as the sum of the stiffnesses of two EB beams of thickness $h/2$.) For $\gamma > 0$ we find that (2) predicts a behavior which is intermediate between these two extremes, with high-frequency behavior approaching that of the case $\gamma = 0$.

If $\beta > 0$, the spectrum associated with (2) consists of a sequence of negative real eigenvalues and a sequence of complex conjugate pairs which exhibit *frequency-proportional damping*, that is, the real parts of the eigenvalues are approximately proportional to the imaginary parts. As the amount of damping in the adhesive layer is increased (i.e., as β increases), after some critical amount, the damping rates of the modes begin to decrease. Thus (2) predicts an optimal level of damping, after which, adding more damping is counterproductive. The value of β that gives the optimal damping is given in Proposition 4.1.

In Section 5 we compare the behavior of (1) and (2) both analytically and numerically. We find that (1) and (2) are in very close agreement provided that the frequency ω is small compared to G/ρ and ρ/I_ρ . In the example we consider (a steel two-layer beam a meter long and half a centimeter thick; see Section 5), the spectrum associated with the two models agree to within about one percent for the first twenty modes (up to about 15 KHz).

2. MODEL DESCRIPTION

The beam model we consider here coincides with a special case of the two-layer plate model [1, (3.16)] when the composite plate consists of two identical rectangular plates of uniform thickness. Thus refer to [1] for a precise derivation of the model. Here we outline only the main points needed to derive the beam analog of the plate model in [1].

The laminated beam we consider consists of two identical beams of uniform thickness $h/2$, width r and length ℓ (see Figure 1). Assume that at equilibrium the upper and lower beams occupy the respective regions

$$\begin{aligned} Q^+ &= \{(x, y, z) \in (0, \ell) \times (-r/2, r/2) \times (0, h/2)\}, \\ Q^- &= \{(x, y, z) \in (0, \ell) \times (-r/2, r/2) \times (-h/2, 0)\}. \end{aligned}$$

2.1 KINEMATIC ASSUMPTIONS.

Let $(U_1, U_2, U_3)(x, y, z)$ denote the displacement vector of the point which, when the beam is in equilibrium has coordinates (x, y, z) . (We suppress any time dependence in this notation). So that we may obtain a beam theory, all the displacements are assumed to be independent of the y coordinate, and the deformations are zero in the y direction:

$$\begin{cases} U_i(x, y, z) = U_i(x, 0, z) & i = 1, 3, & \forall (x, y, z) \in Q^+ \cup Q^-, \\ U_2(x, y, z) \equiv 0 & & \forall (x, y, z) \in Q^+ \cup Q^-. \end{cases} \quad (3)$$

Since this beam model allows slip to occur along the interface, there will usually be a discontinuity in the displacement along the interface. Define u^+ and u^- by

$$u_i^\pm(x) = \lim_{z \rightarrow 0^\pm} U_i(x, y, z), \quad i = 1, 3 \quad \forall x \in (0, \ell).$$

Both the upper and lower beams will be modeled as Timoshenko beams which allow longitudinal motions (i.e., stretching motions). Thus in each beam layer, planar cross sections that at equilibrium are orthogonal to the centerlines of each beam are assumed to remain planar, but may translate transversely or longitudinally and also may rotate parallel to the y axis. Furthermore, as is assumed in the Timoshenko theory, the transverse displacements are assumed to be constant throughout the deformed cross sections.

Thus the displacements are of the form

$$\begin{cases} U_1(x, y, z) = \begin{cases} u_1^+(x) - z\psi^+(x), & \text{if } (x, y, z) \in Q^+ \\ u_1^-(x) - z\psi^-(x), & \text{if } (x, y, z) \in Q^- \end{cases} \\ U_3(x, y, z) = \begin{cases} u_3^+(x), & \text{if } (x, y, z) \in Q^+ \\ u_3^-(x), & \text{if } (x, y, z) \in Q^-. \end{cases} \end{cases} \quad (4)$$

where $\psi^+(x)$ (respectively, $\psi^-(x)$) is the rotation angle, with positive orientation, through which the cross section of the upper (respectively, lower) beam has rotated relative to its equilibrium position (x, \cdot, \cdot) (see Figure 1).

So that the beams do not delaminate (in the transverse direction) we assume that the transverse displacements of both beam layers are the same (and hence may be expressed in terms of a single transverse displacement variable w):

$$w = u_3^+ = u_3^-. \quad (5)$$

Implicit in (5) is a “small slip” assumption, namely, for (5) to be valid to linear order, the *interfacial slip* $u_1^+ - u_1^-$ should be small in comparison to h (see Figure 1).

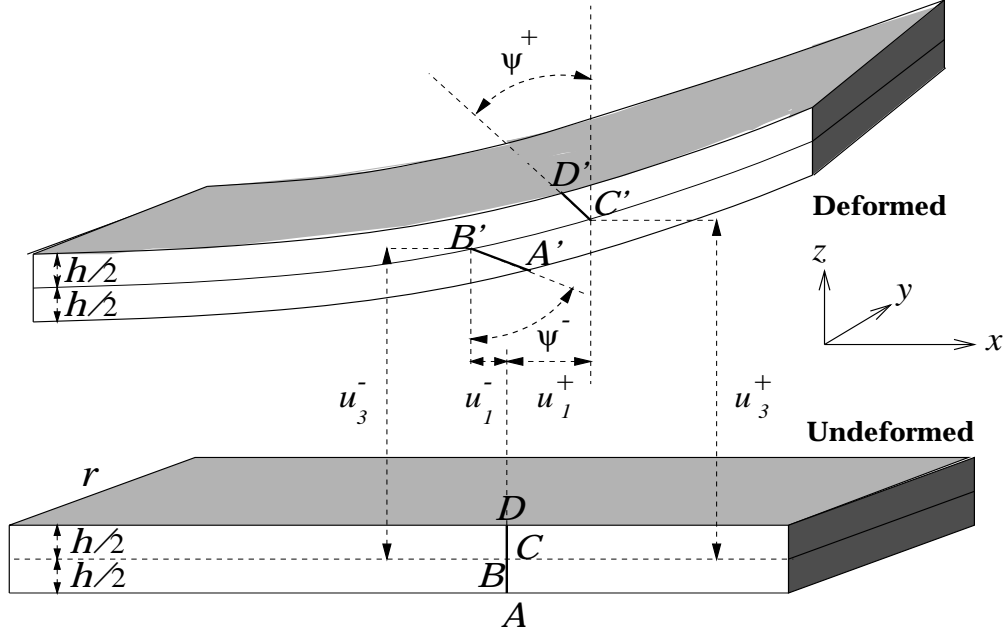


Figure 1: A cross section through the points A, B, C, D and parallel to the y - z plane deforms independently in each layer into cross sections through A', B' in the lower layer and C', D' in the upper layer. The variables ψ^\pm, u_1^\pm and u_3^\pm are indicated. (No symmetries are assumed in this figure).

Since the two beams are identical, as a consequence of conservation of momentum, the *bending motions* decouple from the *longitudinal motions* and possess the symmetry

$$U_1(x, y, z) = -U_1(x, y, -z) \quad \forall (x, y, z) \in Q^+ \cup Q^-. \quad (6)$$

(The longitudinal motions satisfy $U_1(x, y, z) = U_1(x, y, -z)$ and $U_3 \equiv 0$). Thus, for the purpose of studying bending, (6) may be assumed *a-priori*, and does not constitute a

kinematic restriction. As a consequence of (6) we have that

$$u_1^+ + u_1^- = 0, \quad \psi^+ = \psi^-.$$

Let us denote

$$s = \frac{u_1^+ - u_1^-}{2h}, \quad \psi = \psi^+.$$

The *dimensionless slip* s is proportional to the interfacial slip $u_1^+ - u_1^-$. Note that $|s|$ should be small in comparison to 1 for consistency with the assumption (5).

It follows from (5), (6) that for $(x, y, z) \in Q^+$, (4) may be rewritten as

$$\begin{cases} U_1(x, y, z) = s(x)h - z\psi(x), \\ U_3(x, y, z) = w(x). \end{cases} \quad (7)$$

Thus the displacement is uniquely determined in terms of the transverse displacement w , the *dimensionless slip* s and the *average rotation angle* ψ by (3), (5)–(7).

2.2 KINETIC AND POTENTIAL ENERGY OF BEAM LAYERS.

Since we have assumed that the mass of the adhesive layer is negligible, the kinetic energy is the sum of the kinetic energy of each beam:

$$\mathcal{K} = \frac{1}{2} \int_{Q^+ \cup Q^-} \rho(\dot{U}_1^2 + \dot{U}_2^2 + \dot{U}_3^2) dQ,$$

where “ $\dot{\cdot}$ ” = $\partial/\partial t$ and $\rho = \rho(x) > 0$ is the volume density. Using (3), (5)–(7), we obtain

$$\mathcal{K} = \frac{hr}{2} \int_0^\ell \left[\rho(\dot{w})^2 + I_\rho \left((\dot{\psi} - 3\dot{s})^2 + 3\dot{s}^2 \right) \right] dx, \quad (8)$$

where $I_\rho = \rho h^2/12$.

The potential energy consists of the contribution from each beam and the adhesive. The potential energy for the beams is given in terms of the stresses σ_{ij} and strains ϵ_{ij} by

$$\mathcal{P}_b = \frac{1}{2} \int_{Q^+ \cup Q^-} \sum_{i,j=1}^3 \sigma_{ij} \epsilon_{ij} dQ.$$

Using (7) and small strain assumptions we obtain

$$\epsilon_{11} = \frac{\partial U_1}{\partial x} = s_x h - z\psi_x, \quad \epsilon_{31} = \epsilon_{13} = \frac{1}{2} \left(\frac{\partial U_1}{\partial z} + \frac{\partial U_3}{\partial x} \right) = \frac{1}{2}(w_x - \psi).$$

(The other strains vanish due to (3) and the assumption that the transverse displacements are constant throughout the thickness). If the beam layers are homogeneous and transversely isotropic then

$$\sigma_{11} = \frac{E}{1 - \nu^2} \epsilon_{11}, \quad \sigma_{31} = \sigma_{13} = 2G\epsilon_{13},$$

where E denotes the in-plane Young's modulus, ν represents the in-plane Poisson's ratio, G represents the transverse shear modulus. In the isotropic case $G = E/2(1 + \nu)$. We may allow E , G , ν to be x -dependent as long as they are assumed to be strictly positive and bounded with $\nu < \bar{\nu} < 1/2$.

Due to the symmetries in (5), (6) the strain energy of the upper and lower beams are the same, and thus from (7) we obtain

$$\begin{aligned} \mathcal{P}_b &= \int_{Q^+} \left[\frac{E}{1 - \nu^2} (s_x h - z\psi_x)^2 + G(\psi + w_x)^2 \right] dz dy dx \\ &= \frac{rh}{2} \int_0^\ell \{ D [(\psi_x - 3s_x)^2 + 3s_x^2] + G(\psi - w_x)^2 \} dx, \end{aligned} \quad (9)$$

where $D = Eh^2/12(1 - \nu^2)$ is the modulus of flexural rigidity (for a Timoshenko beam of thickness h , Young's modulus E and Poisson ratio ν).

2.3 POTENTIAL ENERGY OF ADHESIVE LAYER.

We assume that the thickness of the adhesive bonding the two plate layers is small enough so that the contribution of its mass to the kinetic energy of the entire beam may be ignored (or included in the beam layers).

To derive an expression for the potential energy carried by the adhesive, we *initially* assume that the adhesive layer may be modeled as a Timoshenko beam of thickness d that is perfectly bonded to the beam layers (at $z = \pm d/2$). Assume that within this adhesive layer the following stress-strain law holds:

$$\sigma_{11} = E_a \epsilon_{11}, \quad \sigma_{13} = 2G_a \epsilon_{13}. \quad (10)$$

By (6), the displacements within this thin layer are of the form

$$U_1(x, y, z) = z\theta(x), \quad U_3(x, y, z) = w(x)$$

where θ is the rotation of a beam element. Since the total slip between the the beam surfaces is $2sh$, for perfect bonding of the adhesive layer to the beam layers (at $z = \pm d/2$) we have

$$\theta = \frac{2sh}{d}.$$

Therefore from small strain assumptions we obtain

$$\epsilon_{11} = \frac{2s_x h z}{d} \quad \epsilon_{13} = \epsilon_{31} = \frac{1}{2} \left(\frac{2sh}{d} + w_x \right).$$

The potential energy \mathcal{P}_a of the adhesive layer is given by

$$\mathcal{P}_a = \frac{1}{2} \int_0^\ell \int_{-r/2}^{r/2} \int_{-d/2}^{d/2} \sum_{i,j=1,2,3} \sigma_{ij} \epsilon_{ij} dz dy dx.$$

Performing the required integrations we obtain

$$\mathcal{P}_a = \frac{r}{2} \int_0^\ell \frac{dE_a h^2 s_x^2}{3} + dG_a \left(\frac{2sh}{d} + w_x \right)^2 dx.$$

Now we use the assumption that the adhesive layer is thin and compliant: we pass to the limit as $d \rightarrow 0$, $G_a \rightarrow 0$ with

$$\gamma := \frac{G_a h}{d} \text{ fixed.} \quad (11)$$

In the limit we obtain

$$\mathcal{P}_a \rightarrow \frac{rh}{2} \int_0^\ell 4\gamma s^2 dx. \quad (12)$$

The limiting expression (12) for the potential energy of the adhesive layer can be viewed most simply as the potential energy obtained under the assumption that the adhesive supplies a restoring force proportion to the interfacial slip (i.e., Hooke's Law).

2.4 WORK.

To fix ideas, let us assume that the beam is clamped at the end $x = 0$, subject to the volume distribution of body forces $(\tilde{f}_1(x, z), 0, \tilde{f}_3(x, z))$, and at the end $x = \ell$ is subject to the (area) distribution of forces $(\tilde{g}_1(z), 0, \tilde{g}_3(z))$. (The forces are assumed to be independent of y , with no forces acting in the y direction). Let us define the resultant force and moment density functions

$$\begin{cases} f_i(x) = \frac{1}{rh} \int_A \tilde{f}_i dS & g_i = \frac{1}{rh} \int_A \tilde{g}_i dS, & i = 1, 3 \\ M_i(x) = \frac{1}{rh} \int_A \tilde{f}_i z dS & m_i = \frac{1}{rh} \int_A \tilde{g}_i z dS, & i = 1, 3 \\ P_i(x) = \frac{1}{rh} \int_A \tilde{f}_i \operatorname{sgn}(z) dS & p_i = \frac{1}{rh} \int_A \tilde{g}_i \operatorname{sgn}(z) dS, & i = 1, 3 \end{cases}$$

where $A = (-r/2, r/2) \times (-h/2, h/2)$ and $\text{sgn}(z) = |z|/z$ denotes the sign function. The forces and moments f_1 , M_3 , P_3 , g_1 , m_1 and p_3 can produce no work due to the kinematic assumptions in (4), (5) and the decoupling in (6). Thus for consistency with these constraints one should assume that

$$f_1 = M_3 = P_3 = 0, \quad g_1 = m_3 = p_3 = 0.$$

The work done by the remaining forces and moments on the plate is

$$\begin{aligned} \mathcal{W} &= \int_{Q^+ \cup Q^-} \left(\tilde{f}_1 U_1 + \tilde{f}_3 U_3 \right) dQ + \int_{x=\ell} \left(\tilde{g}_1 U_1 + \tilde{g}_3 U_3 \right) dy dz \\ &= rh \int_0^\ell \left(hs P_1 - \psi M_1 + w f_3 \right) dx + rh \left[hs(x) p_1 - \psi(x) m_1 + w(x) g_3 \right] \Big|_{x=\ell}. \end{aligned}$$

2.5 EQUATIONS OF MOTION.

The Lagrangian \mathcal{L} on the interval $(0, T)$ is given by

$$\mathcal{L} = \int_0^T [\mathcal{K}(t) + \mathcal{W}(t) - \mathcal{P}_b(t) - \mathcal{P}_a(t)] dt.$$

By the principle of virtual work, the equations of motion can be found by setting the variation of the Lagrangian to zero, where the variation is taken with respect to all kinematically admissible motions.

If we define the *effective rotation angle* ξ by

$$\xi = 3s - \psi,$$

we obtain (see [2] for a similar calculation)

$$\begin{cases} \text{(i)} & \rho \ddot{w} + (G\varphi)_x = f_3 \\ \text{(ii)} & I_\rho \ddot{\xi} - G\varphi - (D\xi_x)_x = M_1 \\ \text{(iii)} & I_\rho \ddot{s} + G\varphi + \frac{4}{3}\gamma s - (Ds_x)_x = \frac{1}{3}hP_1 - M_1, \\ & \text{where } \varphi = 3s - \xi - w_x, \quad 0 < x < \ell, \quad t > 0. \end{cases} \quad (13)$$

The cantilever boundary conditions (obtained from the principle of virtual work) are

$$\begin{cases} \text{(i)} & w(0) = \xi(0) = s(0) = 0 \\ \text{(ii)} & G(\ell)(3s(\ell) - \xi(\ell) - w_x(\ell)) = g_3 \\ \text{(iii)} & D(\ell)\xi_x(\ell) = m_1 \\ \text{(iv)} & D(\ell)s_x(\ell) = \frac{1}{3}hp_1 - m_1, \end{cases} \quad t > 0. \quad (14)$$

Initial conditions may be given as

$$\begin{cases} (\xi, s, w)|_{t=0} = (\xi^0, s^0, w^0) \\ (\dot{\xi}, \dot{s}, \dot{w})|_{t=0} = (\xi^1, s^1, w^1), \end{cases} \quad 0 < x < \ell. \quad (15)$$

Equations (13)–(15) are the beam analogs of the plate system [1, (3.16)–(3.18)].

A couple of comments are in order.

(i) If $\gamma \rightarrow \infty$ (so that the adhesive becomes infinitely stiff) one expects that $s \rightarrow 0$ (in some sense). Putting $s = 0$ in (13)-(i), (13)-(ii) results in the Timoshenko beam system. A precise statement of the convergence of solutions of the system (13)–(15) to those of the Timoshenko system is given in [1].

(ii) If $\gamma = 0$ there is no coupling between the two beams other than applied forces. Thus (with appropriate initial conditions, boundary conditions and external forces) (13)–(15) reduce to a system describing two uncoupled Timoshenko beams of thickness $h/2$. (Again, see [1] for a more detailed discussion).

2.6 DAMPING IN THE ADHESIVE LAYER.

We will also be concerned with the effect of damping in the adhesive layer.

If the adhesive is modeled as a linear viscoelastic material then one replaces the stress-strain law in (10) by an appropriate viscoelastic constitutive law. Since the influence of direct stresses in the adhesive layer is not present in the limiting potential energy (12), we only need to consider the stress-strain law for the transverse shear within the adhesive layer. The stress-strain law for the transverse shear (10) is modified to

$$\sigma_{13} = 2 \int_0^t \tilde{G}_a(t - \tau) \dot{\epsilon}_{13}(\tau) d\tau, \quad (16)$$

where \tilde{G}_a is the relaxation modulus of shear. Note that (16) and (10) are the same if \tilde{G}_a is constant. (It is assumed that the material is relaxed at time 0 in (16).)

According to the viscoelastic correspondence principle the equations of motion are modified in the same way that (10) is modified to (16) (see [2], for example). Therefore the equations of motion (13), (14) are modified by the substitution

$$\gamma s(x, t) \longrightarrow \int_0^t \tilde{\gamma}(t - \tau) s(x, \tau) d\tau, \quad (17)$$

where the relaxation function $\tilde{\gamma}(t)$ is defined in terms of the relaxation function \tilde{G}_a by (11) (with γ and G_a replaced by $\tilde{\gamma}$ and \tilde{G}_a).

To simplify the present analysis we will assume for this article that $\tilde{\gamma}$ is a may be approximated by a linear combination of a constant function and an impulse at the origin:

$$\tilde{\gamma}(t) = \beta\delta(t) + \gamma.$$

This corresponds to a constitutive law of the form

$$\sigma_{13} = 2(G_a\epsilon_{13} + H_a\dot{\epsilon}_{13}), \quad (18)$$

where G_a and γ are related by (11) and H_a is similarly related to β . The damping obtained from (18) is referred to as *strain-rate* damping or *Kelvin-Voigt* damping and states the adhesive supplies a restoring force proportional to the rate of strain. In the limit as the thickness d of the adhesive layer tends to zero, β can be viewed as a coefficient of sliding friction for the interface between the beam layers.

Applying the correspondence principle we find that the equations of motion (13) are modified to

$$\left\{ \begin{array}{l} \text{(i)} \quad \rho\ddot{w} + (G\varphi)_x = f_3 \\ \text{(ii)} \quad I_\rho\ddot{\xi} - G\varphi - (D\xi_x)_x = M_1 \\ \text{(iii)} \quad I_\rho\ddot{s} + G\varphi + \frac{4}{3}\gamma s + \frac{4}{3}\beta\dot{s} - (Ds_x)_x = \frac{1}{3}hP_1 - M_1, \\ \text{where } \varphi = 3s - \xi - w_x, \quad 0 < x < \ell, \quad t > 0. \end{array} \right. \quad (19)$$

The boundary conditions (14) and initial conditions (15) are unchanged. Note that (19) is the same as (1) in the absence of external forces.

It should be mentioned that if one is only interested in the forced response of the beam to harmonic inputs, the widely-used complex modulus approach (see [3] for a survey of this approach and many references) may be applied to model the viscoelastic adhesive layer. In this case the equations in (13) are modified by simply allowing γ to be complex. However this approach ignores any transient behavior and consequently can not be used to study free motions or solutions of initial value problems.

3. LOW FREQUENCY APPROXIMATIONS

In this section we discuss two approximations of the system (19) which are valid for stiff and moderately thin beams. The first is obtained by letting the shear stiffness G tend to infinity in (19), while the second is obtained from the first by letting the moment of inertia parameter I_ρ tend to 0.

3.1. LIMITING SYSTEM AS $G \rightarrow \infty$.

If the shear stiffness G of the beam layers is large enough, the shear motions will usually be minimal and consequently a reasonable beam model should be obtained without allowing shear deformation.

In this subsection we obtain such a beam model as a limit as $G \rightarrow \infty$ of the original (shear deformable) model (19). For now, we proceed in a formal way. Justification will be provided in the next subsection.

Let us first eliminate the term $(G\varphi)$ from two of the equations in (19). By taking appropriate linear combinations of these equations and their derivatives, we obtain

$$\left\{ \begin{array}{l} \text{(i)} \quad \rho\ddot{w} + \frac{1}{4}(I_\rho\ddot{\xi})_x - \frac{3}{4}(I_\rho\ddot{s})_x - (\gamma s)_x - (\beta\dot{s})_x - \frac{1}{4}(D\xi_x)_{xx} + \frac{3}{4}(Ds_x)_{xx} \\ \quad = f_3 + (M_1 - hP_1/4)_x \\ \text{(ii)} \quad I_\rho(\ddot{\xi} + \ddot{s}) + \frac{4}{3}\gamma s + \frac{4}{3}\beta\dot{s} - (D(\xi_x + s_x))_x = \frac{1}{3}hP_1 \\ \text{(iii)} \quad I_\rho(\ddot{\xi} - 3\ddot{s}) - 4G\varphi - 4\gamma s - 4\beta\dot{s} - (D(\xi_x - 3s_x))_x = 4M_1 - hP_1, \\ \quad \text{where } \varphi = 3s - \xi - w_x, \quad 0 < x < \ell, \quad t > 0. \end{array} \right. \quad (20)$$

Likewise, taking linear combinations of the boundary conditions in (14) and using (20)-(iii) at $x = \ell$ (to eliminate $G(\ell)\varphi(\ell)$ from the boundary condition (14)-(ii)) leads to

$$\left\{ \begin{array}{l} \text{(i)} \quad \xi(0) = w(0) = w_x(0) = 0 \\ \text{(ii)} \quad \left[\frac{1}{4}I_\rho(\ddot{\xi} - 3\ddot{s}) - \gamma s - \beta\dot{s} - \frac{1}{4}(D(\xi_x - 3s_x))_x \right]_{x=\ell} = g_3 + M_1(\ell) - \frac{h}{4}P_1(\ell) \\ \text{(iii)} \quad D(\ell)\xi_x(\ell) = m_1 \\ \text{(iv)} \quad D(\ell)(\xi_x(\ell) - 3s_x(\ell)) = 4m_1 - hp_1, \quad t > 0. \end{array} \right. \quad (21)$$

As $G \rightarrow \infty$, by (9), finite energy solutions can exist only if the shear φ tends to zero. Setting $\varphi = 0$ in (20)-(21) gives

$$\left\{ \begin{array}{l} \text{(i)} \quad \rho\ddot{w} - \frac{1}{4}(I_\rho\ddot{w}_x)_x - (\gamma s)_x - (\beta\dot{s})_x + \frac{1}{4}(Dw_{xx})_{xx} = f_3 + (M_1 - \frac{h}{4}P_1)_x \\ \text{(ii)} \quad I_\rho(4\ddot{\xi} + \ddot{w}_x) + 4\gamma s + 4\beta\dot{s} - (D(4\xi_x + w_{xx}))_x = hP_1, \\ \quad \text{where } s = \frac{1}{3}(\xi + w_x), \quad 0 < x < \ell, \quad t > 0, \end{array} \right. \quad (22)$$

with boundary conditions

$$\left\{ \begin{array}{l} \text{(i)} \quad \xi(0) = w(0) = w_x(0) = 0 \\ \text{(ii)} \quad -\frac{1}{4}I_\rho(\ell)\ddot{w}_x(\ell) - \gamma(\ell)s(\ell) - \beta(\ell)\dot{s}(\ell) + \frac{1}{4}(Dw_{xx})_x(\ell) \\ \quad = g_3 + M_1(\ell) - \frac{h}{4}P_1(\ell) \\ \text{(iii)} \quad D(\ell)\xi_x(\ell) = m_1 \\ \text{(iv)} \quad D(\ell)w_{xx}(\ell) = hp_1 - 4m_1, \quad t > 0. \end{array} \right. \quad (23)$$

Initial conditions for (22), (23) take the form

$$\begin{cases} \text{(i)} & (\xi, w)|_{t=0} = (\xi^0, w^0) \\ \text{(ii)} & (\dot{\xi}, \dot{w})|_{t=0} = (\xi^1, w^1). \end{cases} \quad (24)$$

The system (22)–(24) is (at least formally) the limiting form of the equations satisfied by (ξ, w) as $G \rightarrow \infty$. We discuss the convergence of the corresponding solutions in the next subsection.

Remark 3.1. The additional force terms in (22)-(i) and (23)-(ii) (besides f_3, g_3) are due to moments applied to the individual layers. If the resultant applied moment in each beam vanishes (this does not rule out an overall applied moment) then \tilde{f}_1 satisfies

$$\int_{-h/2}^0 \int_{-r/2}^{r/2} (z + h/4) \tilde{f}_1 dy dz = \int_0^{h/2} \int_{-r/2}^{r/2} (z - h/4) \tilde{f}_1 dy dx = 0, \quad 0 < x < \ell.$$

A similar condition also applies to the surface force \tilde{g}_1 . Under these conditions the moments M_1 and P_1 are no longer independent and are related by

$$hP_1 = 4M_1, \quad hp_1 = 4m_1,$$

and consequently the additional forcing terms appearing in (22)-(i), (23)-(ii) vanish.

Remark 3.2. Eq. (22) is closely related to the Rayleigh beam:

$$\rho \ddot{w} - (I_\rho \ddot{w}_x)_x + (Dw_{xx})_{xx} = 0. \quad (25)$$

If $\gamma \rightarrow \infty$ or $\beta \rightarrow \infty$ (so that the glue becomes infinitely stiff or viscous) it can be shown ([1]) that $s \rightarrow 0$ (weakly). Putting $s = 0$ in (20)-(iii) and combining this with (20)-(i) gives (25). On the other hand if γ and β are zero then (22)-(i) reduces to two uncoupled Rayleigh beams of thickness $h/2$ (as opposed to thickness h , as in (25)).

3.2. CONVERGENCE OF SOLUTIONS AS $G \rightarrow \infty$.

The previous equations (22)–(24) are *formally* the limiting system as $G \rightarrow \infty$ of the system (14), (15), (19). However, this does not necessarily imply any resemblance between corresponding solutions due to the presence of a boundary layer as $G \rightarrow \infty$. If (22)–(24) is meant to approximate a composite beam in which the shear moduli of the beam layers are large, then one would hope that solutions of (14), (15), (19) become close (in some sense) to solutions of (22)–(24) as $G \rightarrow \infty$.

In this subsection we describe in a precise way the sense in which solutions of (14), (15), (19) converge to solutions of the limiting system (22)-(24). In order to do so we will need a few mathematical definitions and theorems, which will be used solely in this subsection.

For simplicity we limit our discussion to the unforced problem with constant coefficients. (Similar results also hold for the forced problem with varying coefficients, but are more complicated to describe).

We first need to describe the spaces of existence and uniqueness of finite energy solutions for the original system (14),(15), (19) and the limiting system (22)–(24).

Let $L^2(0, \ell)$ denote the set of all square-integrable functions on the interval $(0, \ell)$ and denote $H^n(0, \ell) = \{\varphi: \varphi, \varphi' \dots, \varphi^{(n)} \in L^2(0, \ell)\}$. In the case of the cantilever boundary conditions (14) we let

$$H_*^1 = \{\varphi: \varphi \in H^1(0, \ell), \varphi(0) = 0\}$$

and define the spaces

$$\mathcal{V} = (H_*^1)^3, \quad \mathcal{H} = (L^2(0, \ell))^3.$$

Using [1, Theorem 4.1], one can prove the following result.

Theorem 3.1. *Given any $(\xi^0, s^0, w^0) \in \mathcal{V}$ and any $(\xi^1, s^1, w^1) \in \mathcal{H}$ there is a uniquely defined solution (ξ, s, w) to the unforced problem (14), (15), (19) for which*

$$(\xi, s, w) \in C([0, \infty); \mathcal{V}) \cap C^1([0, \infty); \mathcal{H}). \quad (26)$$

Equation (26) states that the position variables vary continuously in time in the space \mathcal{V} while the velocity variables vary continuously in time in the space \mathcal{H} . Since these spaces correspond to finite potential and kinetic energy, $\mathcal{V} \times \mathcal{H}$ is referred to as the *finite energy space* for (14), (15), (19).

Similar results are valid for the nonhomogeneous problem provided the applied forces in (15), (19) satisfy standard regularity conditions; see [2].

Now define $H_*^2 = \{\varphi \in H^2(0, \ell): \varphi(0) = \varphi_x(0) = 0\}$ and denote

$$W = H_*^1 \times H_*^2, \quad V = L^2(0, \ell) \times H_*^1.$$

The space $W \times V$ is the finite energy space for (22)–(24). More precisely, one can prove (similar to [1, Theorem 5.3]) the following result.

Theorem 3.2. *For the unforced problem (22)–(24), given any $(\xi^0, w^0) \in W$ and any $(\xi^1, w^1) \in V$ there is a unique solution (ξ, w) to (22)–(24) for which*

$$(\xi, w) \in C([0, \infty), W) \cap C^1([0, \infty), V).$$

Now consider a family of solutions $(\xi(G), s(G), w(G))$ of (14), (15), (19) corresponding to shear modulus G and a fixed set of initial data $(\xi^0, s^0, w^0, \xi^1, s^1, w^1)$ of finite energy. By Theorem 3.1 there is a uniquely defined solution for every G . Our plan is to show that $(\xi(G), s(G), w(G))$ converges weakly to a limit $(\xi(\infty), s(\infty), w(\infty))$ and furthermore that $(\xi(\infty), s(\infty), w(\infty))$ is a solution of (22)–(24).

Since this program has been carried out for the analogous plate system in [1], here we only describe enough to understand the result.

In order that the initial data have finite energy as $G \rightarrow \infty$ we need to impose that $w^0 \in H_*^2$ and $w^1 \in H_*^1$ and furthermore that within each beam initially there is no shear, i.e.

$$3s^0 = \xi^0 + w_x^0, \quad 3s^1 = \xi^1 + w_x^1. \quad (27)$$

It follows from the fact that the unforced problem is dissipative that the energy $\mathcal{E}(t)$ remains bounded for all t in $[0, T]$ and for all G . It thus follows from (9) that the shear φ must tend to zero uniformly in t as $G \rightarrow \infty$. Furthermore, since the initial energy is independent of G (since (27) is imposed) one has (again, by using that the energy is dissipative) that the $\mathcal{V} \times \mathcal{H}$ norms of the solutions are uniformly bounded independent of t and G . This implies the existence of a weak limit point as $G \rightarrow \infty$. It is then possible to show that this limit point is unique and is a solution of the limiting system (22)–(24).

More precisely, from [1] we have the following result.

Theorem 3.3. *Let $(\xi^0, w^0), (\xi^1, w^1)$ be given in $W \times V$ and moreover assume that s^0 and s^1 satisfy (27). Then for any $T > 0$ the solution $(\xi(G), s(G), w(G))$ to the unforced problem (14), (15), (19) tends to (ξ, s, w) as $G \rightarrow \infty$ in the sense that*

$$\begin{cases} (\xi(G), s(G), w(G)) \rightarrow (\xi, s, w) & \text{in } L^\infty(0, T, \mathcal{V}) \text{ weak-star} \\ \left(\dot{\xi}(G), \dot{s}(G), \dot{w}(G) \right) \rightarrow (\dot{\xi}, \dot{s}, \dot{w}) & \text{in } L^\infty(0, T, \mathcal{H}) \text{ weak-star} \end{cases}$$

where (ξ, w) is the unique solution of (22)–(24) (with no external forces) and $3s = \xi + w_x$.

The weak convergence in Theorem 3.3 is probably the best that can be expected since for each $G > 0$ “shear modes” will be present, and as the shear stiffness increases, these

oscillations increase in frequency, but may not necessarily go to zero pointwise. In the limiting system these oscillations are eliminated through “averaging effects”; i.e., in much the same way that $\sin(Kx) \rightarrow 0$ (weakly) as $K \rightarrow \infty$.

3.3. LIMIT AS $I_\rho \rightarrow 0$.

For the moment consider the Rayleigh beam (25). If I_ρ/ρ is very small then the EB beam obtained by setting I_ρ equal to 0 will provide a close approximation to (25) for the low frequency range of the spectrum. Of course this is not true for the higher frequencies since the principal symbols associated with (25) and the EB beam are different. In the same way, we will see that the system obtained by setting $I_\rho = 0$ in (22)–(24) also provides a close approximation to (22)–(24) for the low frequencies.

Putting $I_\rho = 0$ in (22)–(24) gives

$$\left\{ \begin{array}{l} \text{(i)} \quad \rho \ddot{w} - (\gamma s)_x - (\beta \dot{s})_x + \frac{1}{4}(Dw_{xx})_{xx} = f_3 + (M_1 - \frac{h}{4}P_1)_x \\ \text{(ii)} \quad \beta \dot{s} + \gamma s - (D\xi_x + \frac{1}{4}Dw_{xx})_x = \frac{h}{4}P_1, \\ \text{where } 3s = \xi + w_x, \quad 0 < x < \ell, \quad t > 0, \end{array} \right. \quad (28)$$

with boundary conditions

$$\left\{ \begin{array}{l} \text{(i)} \quad \xi(0) = w(0) = w_x(0) = 0 \\ \text{(ii)} \quad -\frac{\gamma(\ell)}{3}(\xi(\ell) + w_x(\ell)) - \frac{\beta(\ell)}{3}(\dot{\xi}(\ell) + \dot{w}_x(\ell)) \\ \quad \quad \quad + \frac{1}{4}(Dw_{xx})_x(\ell) = g_3 + M_1(\ell) - \frac{h}{4}P_1(\ell) \\ \text{(iii)} \quad D(\ell)\xi_x(\ell) = m_1 \\ \text{(iv)} \quad D(\ell)w_{xx}(\ell) = hp_1 - 4m_1, \quad t > 0. \end{array} \right. \quad (29)$$

The initial conditions may be specified as

$$w|_{t=0} = w^0, \quad \dot{w}|_{t=0} = w^1, \quad \xi|_{t=0} = \xi^0. \quad (30)$$

If $\beta = 0$ it is necessary to specify the compatibility condition

$$\frac{\gamma}{3}(\xi^0 + w^0) - \left[D \left(\xi_x^0 + \frac{1}{4}w_{xx}^0 \right) \right]_x = M_1|_{t=0}. \quad (31)$$

A thorough investigation of the existence and uniqueness properties of (28)–(30) is beyond the scope of the present article. However, in the next section we examine the spectrum of (28)–(30) in the case of constant coefficients and simply supported boundary conditions. For that case a unique solution is well defined through separation of variables.

The system (28)–(29) can often be written in a simpler form. In particular, Remark 3.1 concerning the terms $(M_1 - \frac{h}{4}P_1)_x$ in (28)-(i) and similar terms in (29)-(ii) and (29)-(iv) still applies.

In the absence of all external forces (28) can be rewritten as

$$\begin{cases} \text{(i)} & \rho\ddot{w} - (D\xi_x)_{xx} = 0 \\ \text{(ii)} & \frac{\beta}{3}(\dot{\xi} + \dot{w}_x) + \frac{\gamma}{3}(\xi + w_x) - (D(\xi_x + \frac{1}{4}w_{xx}))_x = 0, \end{cases} \quad 0 < x < \ell, \quad t > 0, \quad (32)$$

which is the same as (2). Thus, as mentioned in the introduction, (28) is closely related to the EB beam. We describe this relationship in detail in the next section.

Other boundary conditions besides (29) are easily deduced from (29). For example, in the simply supported case there are no applied moments at the ends and the ends are fixed. If there are no applied moments on the interior then the simply supported boundary conditions take the form

$$w(i) = w_{xx}(i) = \xi_x(i) = 0 \quad i = 0, \ell, \quad t > 0, \quad (33)$$

Remark 3.3. In the constant coefficient case (32) (or (28)) is also closely related to the sixth-order beam model of DiTaranto [4] and Mead and Markus [5]. (These are both the same model; see [6].) If one begins with the three-layer model in [6], in the symmetric case (with the outer layers identical) and a center layer of thickness d and shear modulus K , then passes to the limit as $d \rightarrow 0$, $K \rightarrow 0$ with K/d fixed, one obtains a sixth order equation which is, under certain conditions, equivalent to (32). This only applies to the constant coefficient case with special boundary conditions and initial conditions that admit sinusoidal solutions.

4. SPECTRAL ANALYSIS

In this section we analyze the spectral characteristics of (32)–(33) in the case of constant coefficients. In this case, the eigenfunctions associated with (32)–(33) are purely sinusoidal. Thus, to analyze the spectrum we assume solutions to (32)–(33) of the form

$$w = e^{s_k t} \sin(\alpha_k x), \quad \xi = B e^{s_k t} \cos(\alpha_k x)$$

where $\alpha_k = k\pi/\ell$. Then the boundary conditions (33) are satisfied and (32) holds provided

$$\begin{cases} \text{(i)} & \rho s_k^2 - BD\alpha_k^3 = 0 \\ \text{(ii)} & (\beta s_k + \gamma)(B + \alpha_k) + 3D\alpha_k^2(B + \alpha_k/4) = 0. \end{cases} \quad (34)$$

Solving for B in (34)-(i) and substituting this into (34)-(ii) gives

$$\beta \rho s_k^3 + (3D\alpha_k^2 + \gamma)\rho s_k^2 + \beta D\alpha_k^4 s_k + D\alpha_k^4(\gamma + 3D\alpha_k^2/4) = 0. \quad (35)$$

4.1 THE CASE $\beta = 0$.

When $\beta = 0$ there are two roots $s_k = \sigma_k$ and $s_k = \bar{\sigma}_k$ to (35), where

$$\sigma_k = i \sqrt{\frac{D\alpha_k^4(\gamma + 3D\alpha_k^2/4)}{\rho(\gamma + 3D\alpha_k^2)}}. \quad (36)$$

The *flexural wave speeds* are defined by $\nu_k = \sigma_k/(i\alpha_k)$. If $\gamma \rightarrow \infty$ one sees from (36) that $\nu_k \rightarrow \alpha_k \sqrt{D/\rho}$, the same as in the EB beam $\rho \ddot{w} + Dw_{xxxx} = 0$. If $\gamma \rightarrow 0$ the wave speeds tend to half this value, the same as an EB beam with stiffness $D/4$. For fixed $\gamma > 0$ the wave speeds are between these two extremes, asymptotically approaching the values of the case $\gamma = 0$ as the frequency increases. Similar behavior of flexural wave speeds has been seen in sandwich plate theories [6]. Furthermore, the decrease in flexural wave speeds in comparison to what is predicted by the EB theory for composite materials is also supported by experiments ([7]).

4.2 THE CASE $\beta \neq 0, \gamma = 0$.

When $\beta > 0, \gamma = 0$, letting $v_k = -\alpha_k^2/s_k$, (35) becomes

$$\frac{3D^2}{4}v_k^3 - D\beta v_k^2 + 3D\rho v_k - \beta\rho = 0, \quad (37)$$

or, equivalently $F(v_k) = 3D/\beta$, where

$$F(v) \doteq \frac{\frac{D}{\rho}v^2 + 1}{v(\frac{D}{4\rho}v^2 + 1)}.$$

By noting that F is an odd function with the properties: $F(v) \rightarrow +\infty$ as $v \rightarrow 0^+$, $F(v) \rightarrow 0$ as $v \rightarrow +\infty$, F is decreasing on $(0, \infty)$, it follows that (37) has only one real root $v = p$ which satisfies

$$\frac{\beta}{3D} < p < \frac{4}{3} \frac{\beta}{D} \quad (38)$$

and a pair of complex conjugate roots c, \bar{c} . By factoring out p we find that

$$|c|^2 = \frac{4\beta\rho}{3pD^2}, \quad \sqrt{\frac{\rho}{D}} < \text{Im } c < 2\sqrt{\frac{\rho}{D}}, \quad \text{Re } c = \frac{1}{2} \left(\frac{4}{3} \frac{\beta}{D} - p \right). \quad (39)$$

Furthermore,

$$\frac{p}{\beta} \rightarrow \frac{1}{3D}, \quad \text{Im } c \rightarrow 2\sqrt{\frac{\rho}{D}}, \quad \text{Re } c \rightarrow 0 \quad \text{as } \beta \rightarrow 0 \quad (40)$$

$$\frac{p}{\beta} \rightarrow \frac{4}{3D}, \quad \text{Im } c \rightarrow \sqrt{\frac{\rho}{D}}, \quad \text{Re } c \rightarrow 0 \quad \text{as } \beta \rightarrow \infty. \quad (41)$$

We thus find that $s_k = -\alpha_k^2/v_k$, ($k = 1, 2, \dots$), and since the roots of (37) consist of c , \bar{c} and p (independent of k), the eigenvalues associated with (32)–(33) for the case $\gamma = 0$ consist of $\{\lambda_k(\beta)\} = \{\omega_k(\beta)\} \cup \{\sigma_k(\beta)\} \cup \{\bar{\sigma}_k(\beta)\}$ where

$$\omega_k(\beta) = \frac{-k^2\pi^2}{\ell^2 p(\beta)} \quad k = 1, 2, \dots \quad (42)$$

$$\sigma_k(\beta) = \frac{-k^2\pi^2}{\ell^2 c(\beta)} \quad k = 1, 2, \dots \quad (43)$$

where $p(\beta)$ and $c(\beta)$ satisfy (37)–(41).

There are several interesting things to note.

(i) For fixed $\beta > 0$ the branch $\{\sigma_k(\beta)\}$ tends to ∞ (as $k \rightarrow \infty$) along a ray (from the origin) in the left half of the complex plane. Thus the real and imaginary parts of $\sigma_k(\beta)$ are proportional, i.e., the damping is *frequency-proportional* (see Figure 2). Previously known frequency proportional damping models include the *square-root* model ([8]) and the *spatial-hysteresis* model ([9]). This type of damping is in agreement with experimental measurement in composite beams ([7]).

(ii) The branch of eigenvalues $\{\omega_k(\beta)\}$ is due to the parabolic component in (32)–(33). As $\beta \rightarrow 0$, each of these eigenvalues tends to ∞ along the negative real axis, while as $\beta \rightarrow \infty$ they tend toward the origin. (Again, see Figure 2). This type of pattern, where there is a branch of negative real eigenvalues along with the vibrational eigenvalues is typical of thermoelastic systems as well as some viscoelastic beam models.

(iii) By (40), (41), $c(\beta)$ tends to the imaginary axis in both extremes: as $\beta \rightarrow 0$ and as $\beta \rightarrow \infty$. Thus (43) indicates that all the eigenvalues tend to the imaginary axis as $\beta \rightarrow 0$ and also as $\beta \rightarrow \infty$. Thus the model (32)–(33) suggests that there is an optimal level of damping achievable and beyond that point, adding more damping to the adhesive layer is counterproductive. This phenomenon has long been known to occur in composite beams involving damping layers ([10]). Figure 3 illustrates the trajectories of the eigenvalues $\sigma_k(\beta)$. The value of β which gives the optimal damping is given in Proposition 4.1.

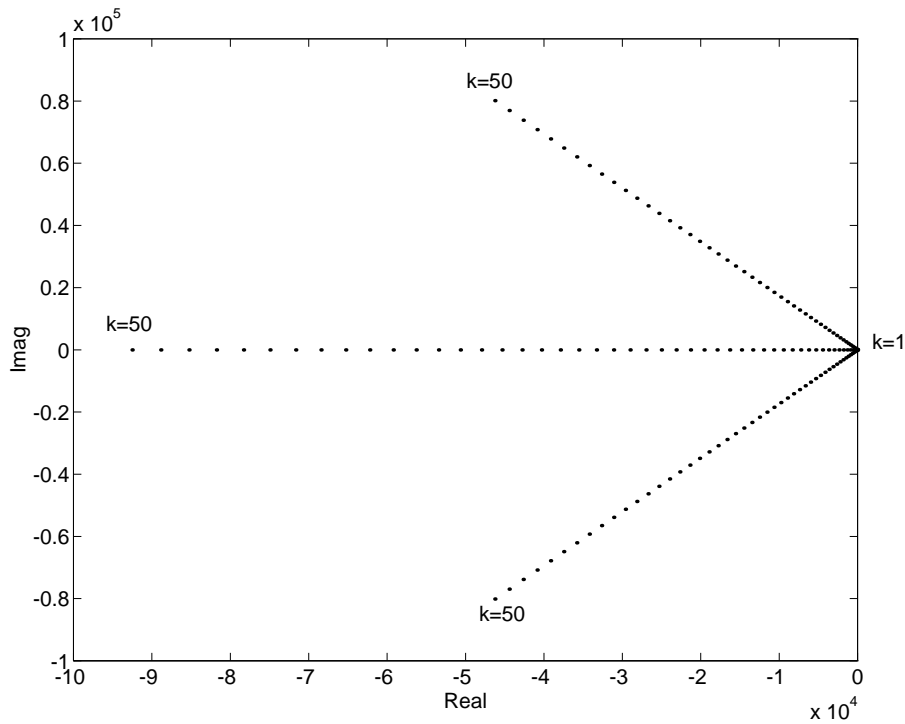


Figure 2: The three branches of eigenvalues for the model (32), (33) with $\gamma = 0$ (solutions of equation (35)).

Figure 2 shows the first 50 eigenvalues of (32)–(33) for the case $\gamma = 0$, i.e. the roots of (35) for $k = 1, 2, \dots, 50$. The parameters used are (all in MKS units) $\rho = 2000$, $D = 56250$, $\ell = 1$, $\beta = 22500$. The value of D was obtained by using $E = 2.4 \times 10^{10}$ (a typical value

for steel), $h = 0.005$ and $\nu = 1/3$. The value of β was picked to give optimal damping (see Proposition 4.1).

We can prove the following result concerning the problem of optimal damping of (32)-(33) for the case $\gamma = 0$.

Proposition 4.1. *Let $\sigma_k = \sigma_k(\beta)$, $k = 1, 2, \dots$ be the eigenvalues as defined by (43). Then $\arg \sigma_k$ is independent of k and*

$$\max_{0 \leq \beta < \infty} \arg \sigma_k(\beta) = \frac{2\pi}{3}. \quad (44)$$

Furthermore, $\beta = 3\sqrt{D\rho/2}$ is the unique value of β for which $\arg \sigma_k(\beta) = 2\pi/3$.

Proof. We know by (43) that the argument of the eigenvalues σ_k , $k = 1, 2, \dots$ are all the same and satisfy

$$\arg \sigma_k = -\arg(-c). \quad (45)$$

Note also that the values of $c(\beta)$, $0 \leq \beta < \infty$ together with the segment from $i\sqrt{\rho/D}$ to $2i\sqrt{\rho/D}$ form a continuous closed curve Σ in the closed right-half complex plane. Let us find the points of intersection (if any) of Σ with the ray $R \doteq \{re^{i\frac{\pi}{3}} : 0 \leq r < \infty\}$. If $v = re^{i\frac{\pi}{3}} \in \Sigma \cap R$ then v satisfies (37), i.e.

$$-\frac{3D^2r^3}{4} - D\beta r^2 e^{i\frac{2\pi}{3}} + 3D\rho r e^{i\frac{\pi}{3}} - \rho\beta = 0$$

which implies that $\text{Im} \left(D\beta r^2 e^{i\frac{2\pi}{3}} \right) = \text{Im} \left(3D\rho r e^{i\frac{\pi}{3}} \right)$ and therefore $r = 3\rho/\beta$. Plugging $v = \frac{3\rho}{\beta} e^{i\frac{\pi}{3}}$ into (37) we immediately obtain $\left(\frac{9D\rho}{2\beta} - \beta \right)^2 = 0$ which implies $\beta = \beta^* = 3\sqrt{D\rho/2}$. Hence the ray R intersects the curve Σ only once at the point $c = c^*(\beta) = \frac{3\rho}{\beta^*} e^{i\frac{\pi}{3}} = \sqrt{2\rho/D} e^{i\frac{\pi}{3}}$. Since Σ encloses a convex region of the complex plane, uniqueness of this intersection implies optimality; i.e., $\min_{0 \leq \beta < \infty} \arg c(\beta) = \arg c(\beta^*) = \pi/3$. Thus (44) follows from (45). \square

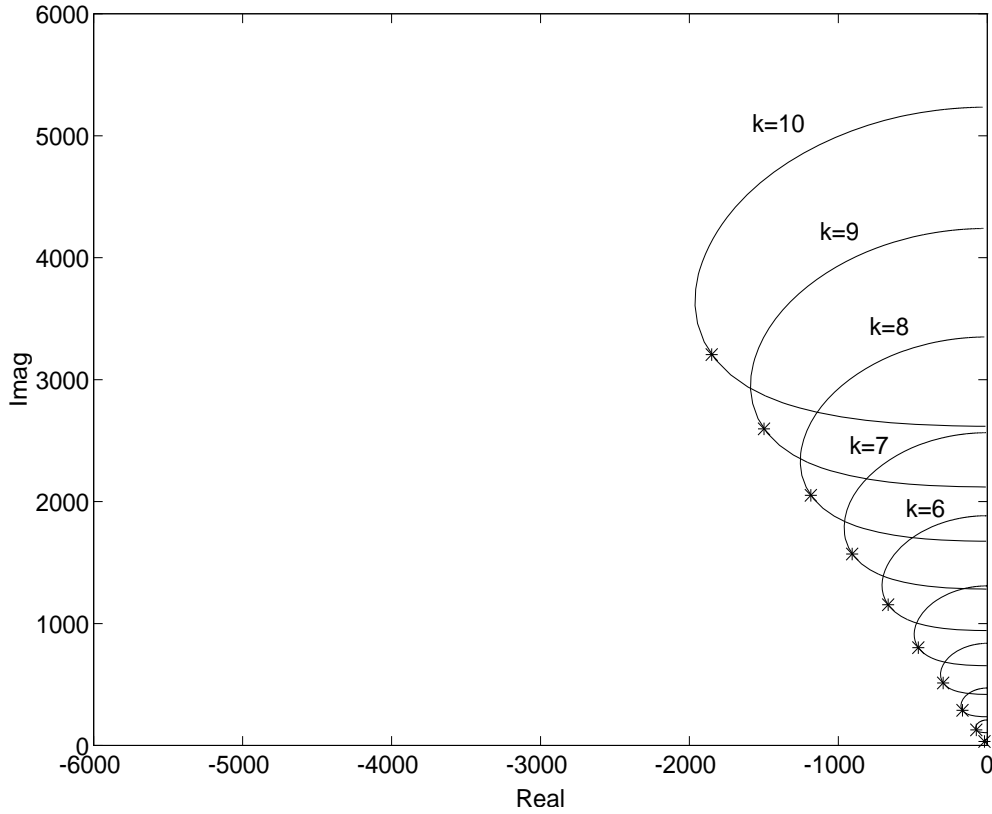


Figure 3: Trajectories of $\sigma_k(\beta)$ for $0 \leq \beta < \infty$, $1 \leq k \leq 10$, $\gamma = 0$ (equation (35)). Points marked “ * ” correspond to the optimal damping $\beta = 3\sqrt{D\rho/2}$.

Figure 3 shows the trajectories of the eigenvalues $\sigma_k(\beta)$, $1 \leq k \leq 10$, for $0 \leq \beta < \infty$. As β goes from zero to infinity the corresponding frequencies of each σ_k doubles. The values of ρ , D and ℓ are the same as those used in Figure 2.

4.3 THE CASE $\gamma > 0$, $\beta > 0$.

Let us briefly discuss the spectrum for the case $\beta > 0$, $\gamma > 0$. From Rouché’s Theorem applied to the polynomial in (35) one can deduce the following: If $\sigma_k(\gamma, \beta)$, $\bar{\sigma}_k(\gamma, \beta)$, $\omega_k(\gamma, \beta)$ denote the roots of the polynomial in (35), then

$$\frac{|\sigma_k(0, \beta) - \sigma_k(\gamma, \beta)|}{|\sigma_k(0, \beta)|} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

$$\frac{|\omega_k(0, \beta) - \omega_k(\gamma, \beta)|}{|\omega_k(0, \beta)|} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

that is, the stiffness of the adhesive does not significantly affect the position of the high

frequency eigenvalues. It follows that all the conclusions reached for the case $\gamma = 0$ also hold (asymptotically) for the case $\gamma > 0$.

Figure 4 shows a comparison of the eigenvalues for the cases $\gamma = 0$ (same as in Figure 2) and the case $\gamma = 2.0 \times 10^6$. This value of γ was picked using (11) with a typical value for a Young's modulus of a glue. Here we used $\ell = 10$ since with $\ell = 1$ the difference between the two branches was insignificant.

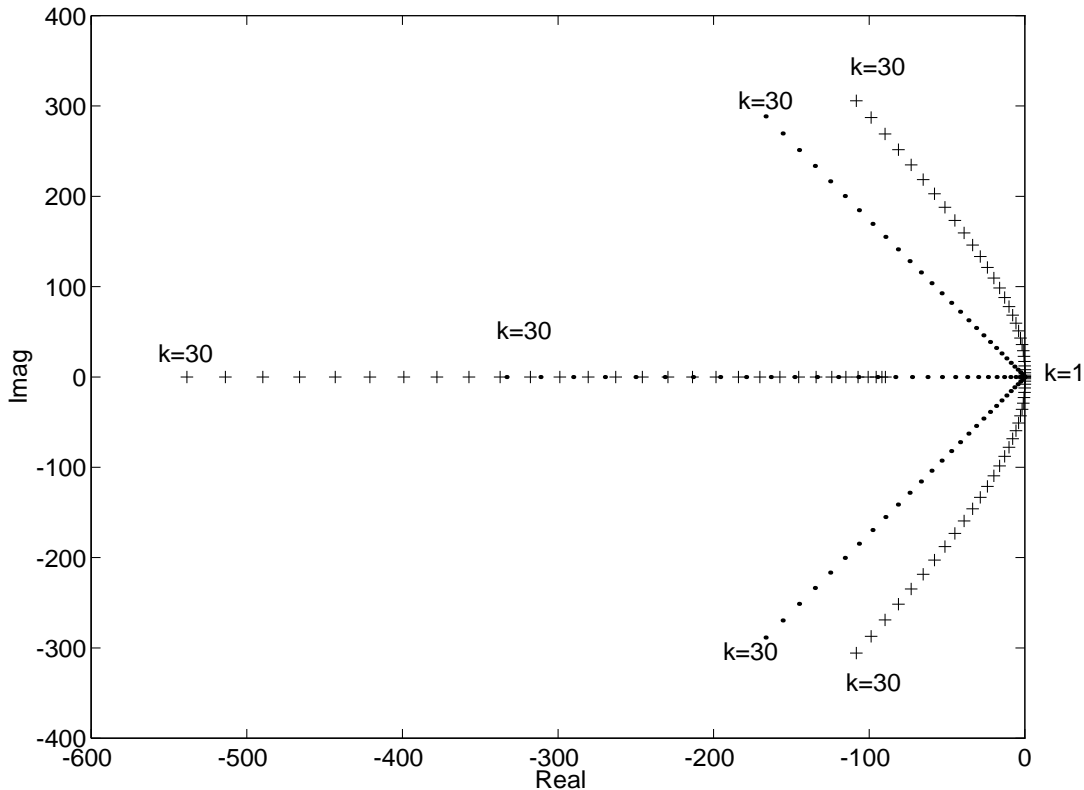


Figure 4: Comparison of the first 30 eigenvalues for the case $\gamma = 0$ (points marked “.”) and $\gamma = 2 \times 10^6$ (points marked “+”) in the model (32)–(33).

5. COMPARISON WITH THE GENERAL CASE $I_\rho > 0$, $0 < G < \infty$

In this section we compare the spectrum of (19) with the spectrum of (32) both analytically and numerically.

For the general unforced problem with simply supported boundary conditions and γ , D

constant, system (19) takes the form

$$\begin{cases} \text{(i)} & \rho\ddot{w} + G\varphi_x = 0 \\ \text{(ii)} & I_\rho\ddot{\xi} - G\varphi - D\xi_{xx} = 0 \\ \text{(iii)} & I_\rho\ddot{s} + G\varphi + \frac{4}{3}\gamma s + \frac{4}{3}\beta\dot{s} - Ds_{xx} = 0 \\ \text{(iv)} & \varphi = 3s - \xi - w_x, \end{cases} \quad 0 < x < \ell, \quad t > 0. \quad (46)$$

The simply supported boundary conditions take the form

$$w(i) = \xi_x(i) = s_x(i) = 0, \quad i = 0, \ell. \quad (47)$$

By letting

$$w = e^{s_k t} \sin(\alpha_k x), \quad \xi = B \cos(\alpha_k x) e^{s_k t} \quad \text{and} \quad s = C \cos(\alpha_k x) e^{s_k t},$$

where $\alpha_k = k\pi/\ell$, we obtain $\varphi = \left(3C - B - \frac{k\pi}{\ell}\right) \cos\left(\frac{k\pi}{\ell} x\right) e^{s_k t}$ and equations (i)–(iii) of (46) yield

$$\begin{cases} \text{(i)} & \rho s_k^2 + G \frac{k\pi}{\ell} \left(\frac{k\pi}{\ell} + B - 3C\right) = 0 \\ \text{(ii)} & I_\rho B s_k^2 - G \left(3C - B - \frac{k\pi}{\ell}\right) + DB \frac{k^2 \pi^2}{\ell^2} = 0 \\ \text{(iii)} & I_\rho C s_k^2 + G \left(3C - B - \frac{k\pi}{\ell}\right) + \frac{4}{3}\gamma C + \frac{4}{3}\beta C s_k + DC \frac{k^2 \pi^2}{\ell^2} = 0. \end{cases} \quad (48)$$

From (48)(i) and (48)(ii) we obtain

$$B = \frac{\rho s_k^2}{\alpha_k (I_\rho s_k^2 + D\alpha_k^2)}, \quad C = \frac{\alpha_k}{3} + \frac{\rho s_k^2}{3\alpha_k} \left(\frac{1}{I_\rho s_k^2 + D\alpha_k^2} + \frac{1}{G} \right).$$

Replacing these values of B and C into (48)(iii) we obtain

$$\sum_{i=0}^6 C_i s_k^i = 0 \quad (49)$$

where the coefficients C_i , $i = 1, \dots, 6$ are given by

$$\begin{cases} C_6 = \frac{3\rho I_\rho^2}{4G}, & C_5 = \frac{\beta\rho I_\rho}{G} \\ C_4 = I_\rho \left[\rho \left(3 + \frac{\gamma}{G} \right) + \frac{3\alpha_k^2}{4} \left(I_\rho + \frac{2\rho D}{G} \right) \right] \\ C_3 = \beta \left(\rho + I_\rho \alpha_k^2 + \frac{\rho D \alpha_k^2}{G} \right) \\ C_2 = \left[\rho \left(\gamma + 3D\alpha_k^2 \right) + I_\rho \left(\gamma \alpha_k^2 + \frac{3}{2} D \alpha_k^4 \right) + \frac{\rho D \alpha_k^2}{G} \left(\gamma + \frac{3D\alpha_k^2}{4} \right) \right] \\ C_1 = D\beta\alpha_k^4, & C_0 = \alpha_k^4 D \left(\gamma + \frac{3D\alpha_k^2}{4} \right). \end{cases} \quad (50)$$

Note that formally setting $I_\rho = 0$ and $\frac{1}{G} = 0$ in (49)–(50) results in precisely (35).

Since (49) is sixth order (for each k), there are three additional roots compared to the third degree polynomial in (35). These are due to the presence of rotational modes of vibration that are not present in the simplified model (32)–(33). However, under the assumption that G is very large and I_ρ is very small, the size of the eigenvalues corresponding to these rotational modes are quite large. We show in section 5.1 that the low frequency eigenvalues associated with (46)–(47) are closely approximated by corresponding eigenvalues associated with (32)–(33).

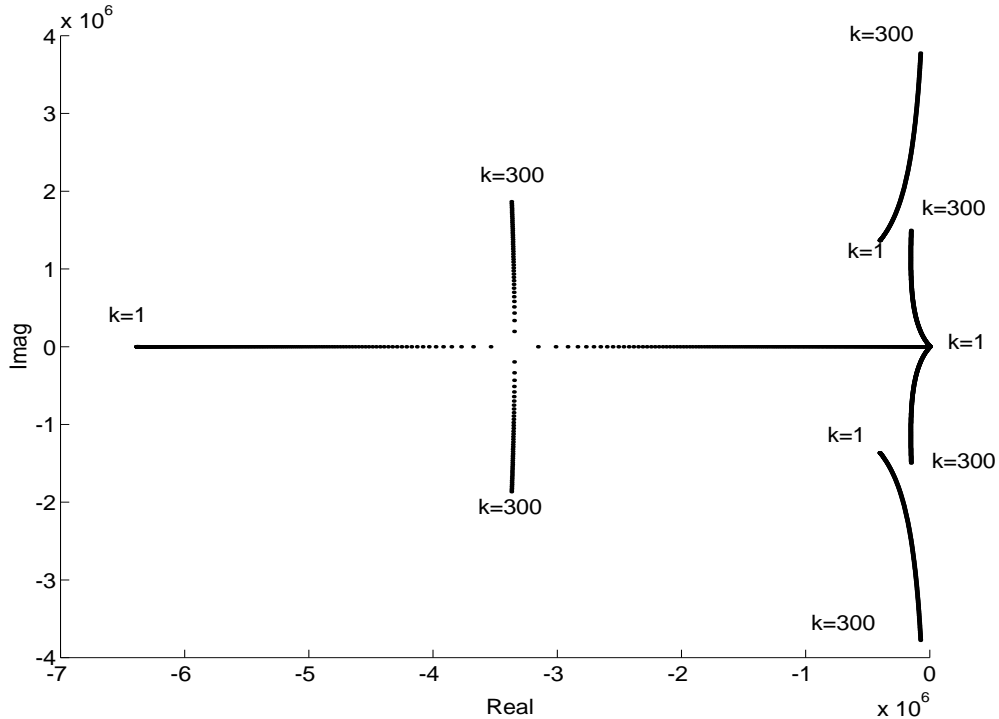


Figure 5: The six branches of eigenvalues obtained with the full model (46)–(47) (solutions of (49)–(50)).

Figure 5 shows the first 300 eigenvalues of (46)–(47). Here we used $I_\rho = .004166$, $G = 7.5 \times 10^9$ and $\gamma = 2 \times 10^6$. All the other parameters remain the same as those used in Figure 2. These values are consistent with those used in Figure 2 and $G = \kappa E/2(1 + \nu)$, where κ denotes a shear correction coefficient that is taken to be $5/6$. Note that the first of each of these “rotational” eigenvalues (marked “ $k = 1$ ”, but away from the origin) are of the order of the 300th eigenvalue corresponding to modes that are present in the model (32)–(33).

5.1 ASYMPTOTIC ANALYSIS OF (46)–(47) AS $I_\rho \rightarrow 0, G \rightarrow \infty$.

Let us suppose that $I_\rho = \epsilon \hat{I}_\rho$, $G^{-1} = \epsilon \hat{G}^{-1}$ where ϵ is a small parameter and $\hat{I}_\rho, \hat{G}, \gamma, \beta, \rho, D$ are all $\mathcal{O}(1)$, that is, much larger than ϵ , but much smaller than $1/\epsilon$ (some renormalization may be required for this hypothesis to apply).

Let $Q_k(s)$ denote the polynomial in (49) and let $P_k(s)$ denote the polynomial in (35) (both with s_k replaced by s). Let $\Gamma(z, R)$ denote the circle of radius R about the complex number z .

Proposition 5.1. *Assume there are numbers ϵ, δ ; $0 < \epsilon < \delta < 1/2$ such that*

$$|\lambda_k| < \frac{\delta}{\epsilon}, \quad (51)$$

where λ_k denotes a root of P_k . Then there exists $M > 0$, independent of k, δ, ϵ , such that Q_k has exactly one root inside $\Gamma(\lambda_k, M\delta|\lambda_k|)$.

The proof of Proposition 5.1 involves applying Rouché's theorem to the polynomials P_k and Q_k and is given in the Appendix.

Proposition 5.1 states that if λ_k is an eigenvalue corresponding to the model (2) (so that λ_k is one of the numbers $\sigma_k, \bar{\sigma}_k$ or ω_k in (42)–(43)), and k is small enough so that (51) holds, then there is a unique eigenvalue $\hat{\lambda}_k$ corresponding to the model (1) for which $|\lambda_k - \hat{\lambda}_k| < M\delta|\lambda_k|$. Thus (since M is independent of δ) the relative error in approximating $\hat{\lambda}_k$ by λ_k is of the order of δ . The constant M in Proposition 5.1 can be computed explicitly and is a maximum of sums, products and ratios (but not differences) of the parameters $\hat{I}_\rho, \hat{G}, \rho, D, \gamma, \beta$.

Figure 6 shows a comparison of the first 30 eigenvalues of the models (46)–(47) and (32)–(33). The values of the parameters used here are the same as those used in Figure 5.

We may apply Proposition 5.1 to the parameter values used in Figure 6 as follows. We first renormalize each parameter in (46) by division by ρ . This results in new parameter values: $\rho = 1$, $D = 28.125$, $G = 3.75 \times 10^6$, $\gamma = 1000$, $I_\rho = 2.08 \times 10^{-6}$, $\beta = 11.25$. Observe that by taking $\epsilon = 10^{-6}$, $\delta = .01$, Proposition 5.1 predicts a relative error of the order of 1% for $|\lambda_k| < 10^4$, i.e. up to 10 KHz, which is approximately the same as the actual relative error.

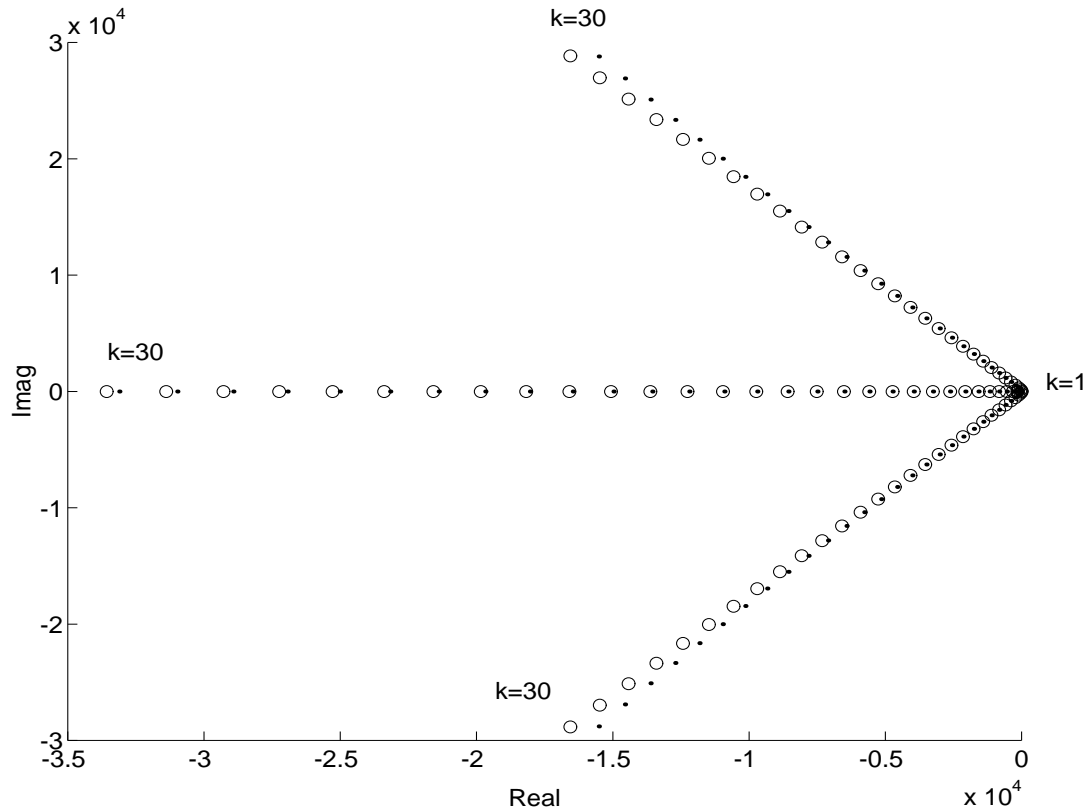


Figure 6: Comparison of the first 30 eigenvalues obtained with the full model (equations (49)-(50)) and the limit case $I_\rho = 1/G = 0$ (equations (35), (42)-(43)). Points marked with a “.” correspond to the full model while points marked “o” are the roots of $P_k(s_k)$.

6. SUMMARY AND CONCLUSIONS

In this paper a model for a symmetric, two-layer beam that allows interfacial slip was derived which includes the effects of shear deformation and rotary inertia in each beam layer. The two layers were assumed to be bonded by a viscous adhesive of negligible mass and thickness. By letting the shear stiffness parameter G tend to infinity and dropping the terms involving the rotational inertia I_ρ , a much simpler model (28) is produced. If G is large and I_ρ is small then this simpler model provides a close approximation to the original model for harmonic motions with frequencies that are small compared to G and $1/I_\rho$. For these frequencies, the discussion in Section 4 concerning the behavior of the spectrum and optimal damping also applies to the original model. We found that the frictional damping resulted in a frequency-proportional damping pattern in the spectrum and the maximal damping angle was described in Proposition 4.1.

This investigation was motivated towards understanding the damping characteristics in

a multi-laminated or possibly fiber-composite beams. Here we assumed a single location where slip was possible, since this simplified the eigenvalue calculations. Future research will hopefully extend much of this analysis to beam models with many adhesive layers.

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APPENDIX

Proof of Proposition 5.1. The polynomial Q_k may be written as

$$Q_k(s) = P_k(s) + \epsilon F_k(s) + \epsilon^2 G_k(s) + \epsilon^3 H_k(s).$$

The result follows from Rouché's Theorem if the following inequality can be proved:

$$|P_k(s)| > \epsilon |F_k(s)| + \epsilon^2 |G_k(s)| + \epsilon^3 |H_k(s)| \quad \forall s \text{ on } \Gamma(\lambda_k, M\delta|\lambda_k|). \quad (\text{A1})$$

There are two possibilities, either λ_k is a real root ω_k of (33) or it is one of the complex roots $\sigma_k, \bar{\sigma}_k$. Both cases are handled in the same way, so let us consider only the case $\lambda_k = \sigma_k$.

Let $R < |\sigma_k|/2$. Since $P_k(s) = \beta\rho(s - \sigma_k)(s - \bar{\sigma}_k)(s - \omega_k)$ and $\arg \sigma_k \leq 2\pi/3$ (see Proposition 4.1) it follows from simple geometry that

$$|P_k(s)| \geq \beta\rho R \frac{\sqrt{3}}{2} |\sigma_k| \frac{\sqrt{3}-1}{2} |\sigma_k| > .3\beta\rho |\sigma_k|^2 R \quad \text{on } \Gamma(\sigma_k, R). \quad (\text{A2})$$

To obtain an appropriate bound for the right hand side of (A1) we treat each term separately. For example, using (51) we have

$$\epsilon^3 |H_k(s)| = \left| \frac{3\rho \hat{I}_\rho^2}{4\hat{G}} \epsilon^3 s^6 \right| \leq M_1 \delta^3 |\sigma_k|^3 \quad \text{on } \Gamma(\sigma_k, R),$$

where $M_1 = (3\rho \hat{I}_\rho^2 / 4\hat{G})(3/2)^3$. The term $F_k(s)$ in (A1) is of the form

$$F_k(s) = \tilde{c}_2 s^2 + \tilde{c}_3 s^3 + \tilde{c}_4 s^4.$$

The term $\tilde{c}_2 s^2$ is the most troubling since $\tilde{c}_2 = 3\hat{I}_\rho D\alpha_k^4/2 + \text{l.o.t.}$, where l.o.t. denotes lower order terms in k . One has

$$|\tilde{c}_2 s^2| \leq M_2 \alpha_k^4 |\sigma_k|^2 \leq M_3 |\sigma_k|^4 \quad \text{on } \Gamma(\sigma_k, R).$$

Handling the other terms in the same way,

$$\epsilon |F_k(s)| \leq M_4 \delta |\sigma_k|^3 \quad \text{on } \Gamma(\sigma_k, R).$$

Likewise one can obtain a similar bound for $\epsilon^2 |G_k(s)|$ on $\Gamma(\sigma_k, R)$.

Thus we obtain

$$\epsilon |F_k(s)| + \epsilon^2 |G_k(s)| + \epsilon^3 |H_k(s)| < M_5 \delta |\sigma_k|^3 \quad \text{on } \Gamma(\sigma_k, R), \quad (\text{A3})$$

for any $R \leq |\sigma_k|/2$. Thus (A1) follows from (A2), (A3) provided $M_5 \delta |\sigma_k| / (.3\beta\rho) < R \leq |\sigma_k|/2$. Thus by choosing δ sufficiently small, (A1) follows. On the other hand, Proposition 5.1 becomes trivial if we restrict δ to any interval of the form $[\delta_0, 1/2]$, by taking $M = 2/\delta_0$.

□