Optimal Damping in Multilayer Sandwich Beams

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ABSTRACT

We describe some possible models for a multilayer sandwich beam consisting of alternating stiff and compliant beam layers. The stiff layers are modeled under Euler-Bernoulli assumptions while the compliant layers essentially only carry the shear. We include viscous damping in the compliant layers and consider the optimization problem of choosing the damping parameters for each layer so that the maximal asymptotic damping angle in the system eigenvalues is obtained. The solution is obtained analytically as a closed-form function of the various material parameters.

Keywords: Sandwich beam, sandwich plate, multilayer plate, optimal design

1. INTRODUCTION

In this article we describe systems of partial differential equations that provide multilayer generalizations of the sandwich beam models of Rao-Nakra[1] and Mead-Markus,[2] DiTaranito.[3] These classical sandwich beam models consist of two outer layers of stiff material and a more flexible core layer. We consider a multiple layer situation in which alternate layers are assumed to be stiff and compliant. We first derive an "accurate model" (analogous to the Rao-Nakra model) that includes all the energy terms that exist under the following initial modeling assumptions: (i) layers are bonded together so that no slip occurrs along the interfaces; (ii) Euler-Bernoulli beam theory is used for the stiff beam layers; (iii) Timoshenko beam assumptions are used for the compliant layers. As long as the compliant layers are relatively thin and very flexible in comparison to the stiff layers, a good approximation results in neglecting all but the shear and transverse momentum of the compliant layers. This leads to a multilayer generalization of the Mead-Markus model.

We also solve an optimal design problem in which the eigenvalues of the multilayer Mead-Markus system with strain-rate damping in the compliant layers are optimally placed. We discover that the system exhibits "frequency-proportional" damping characteristics; that is, that at high frequencies, the real and imaginary parts of the eigenvalues are of a constant ratio. We find that (as is typical in layered beams) there is an optimal level of damping (among all possible damping constants for the compliant layers) that leads to an optimum asymptotic damping angle. In each case we obtain an explicit solution. This result extends to multiple layers earlier optimality results for three-layer beams.[4, 5]

2. DERIVATION OF MULTILAYER SANDWICH BEAM

The multilayer sandwich beam is assumed to consist of $n = 2m + 1$ layers that occupy the region $\Omega \times (0, h)$ at equilibrium, where $\Omega = (0, L) \times (0, R)$ is a rectangle of length L and width R. The sandwich beam is assumed to consist of alternating "stiff" and "compliant" beam layers, with stiff layers on the top and bottom. The layers are indexed from 1 to n, with odd indices for stiff layers and even indices for compliant layers.

Let

$$0 = z_0 < z_1 < \ldots < z_{n-1} < z_n = h, \qquad h_i = z_i - z_{i-1},\ i = 1, 2, \ldots, n.$$

We use the rectangular coordinates $\underline{x} = \{x_1, x_2\} = \{x, x_2\}$ to denote points in Ω and $\{\underline{x}, x_3\}$ to denote points in $Q = \cup_{i=1}^{n} Q_i$, where Q_i is the reference configuration of the ith layer given by $Q_i = \Omega \times (z_{i-1}, z_i)$.

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In addition to the above mentioned modeling assumptions (i), (ii), (iii), to obtain a beam theory, we assume (iv) that all motion occurs in the x_1-x_3 plane. Thus we may assume that the displacement vector U is of the form

$$U(\underline{x}, x_3) = \{U_1, 0, U_3\}(x_1, x_3).$$

Define $\hat{z}_i = (z_{i-1} + z_i)/2$. Under Timoshenko displacement assumptions, the displacement within the ith layer can be written as

$$\begin{aligned} U_1(x, x_3) &= v^i(x) + (x_3 - \hat{z}_i)\psi^i(x) \quad z_{i-1} < x_3 < z_i \\ U_3(x, x_3) &= w(x) \qquad\qquad\qquad\qquad z_{i-1} < x_3 < z_i \end{aligned} \tag{1}$$

where $v^i(x) = U_1(x, \hat{z}_i)$ is the longitudinal displacement of the ith layer at the point x and $\psi^i(x) = \frac{1}{h_i}(U_1(x, z_i) - U_1(x, z_{i-1}))$ is the rotation angle (with negative orientation) at x of the deformed filament within the i-th layer in the x_1-x_3 plane. In stiff layers the Euler-Bernoulli assumptions hold and (1) is adjusted by setting

$$\psi^i = -w_x \qquad i \text{ odd.} \tag{2}$$

We define the shear of the ith layer by

$$\varphi^i = \psi^i + w_x, \tag{3}$$

Hence on odd layers the shear vanishes.

Let ψ, v, φ denote the vectors having ith row ψ^i, v^i, φ^i, respectively. Furthermore let $\psi_{\mathcal{E}}$ denote the vector consisting of the even indexed rows of ψ, i.e, $\psi^i_{\mathcal{E}} = \psi^{2i}$, for $i = 1, 2, \ldots m$. Analogously define the quantities $\varphi_{\mathcal{E}}$, $v_{\mathcal{E}}$. Likewise define $\psi_{\mathcal{O}}$, $\varphi_{\mathcal{O}}$, $v_{\mathcal{O}}$, as the vector consisting of the odd-indexed rows of ψ, φ, v, respectively. As further notation is developed, we use the same system regarding the subscripts $\mathcal{O}$ and $\mathcal{E}$.

Equation (2) implies $\varphi = 0$ in odd layers and hence we have the following equivalent equations:

$$\varphi_{\mathcal{O}} = 0; \qquad \psi_{\mathcal{O}} = -\vec{1}_{\mathcal{O}} w_x, \tag{4}$$

where $\vec{1}_{\mathcal{O}}$ denotes the $(m+1)$-vector of ones. (Later we will also use $\vec{1}_{\mathcal{E}}$ for the m-vector of ones.)

Let $\sigma_{jk}, \epsilon_{jk}$ $(j, k = 1, 2, 3)$ denote the stress and strain tensors, respectively. For a beam theory, we assume all strains vanish except the ϵ_{11} and $\epsilon_{13} = \epsilon_{31}$ in the even layers and ϵ_{11} in the odd layers. The Timoshenko stress-strain assumptions are

$$\sigma_{11} = E_i \epsilon_{11} \qquad \sigma_{13} = 2G_i \epsilon_{13} \tag{5}$$

where $E_i > 0$ denotes the longitudinal Young's modulus, $G_i > 0$ denotes the transverse shear modulus. For the stiff layers we effectively have infinite transverse shear moduli, and hence the above is adjusted setting

$$\epsilon_{13} = \epsilon_{23} = 0 \text{ in stiff (odd indexed) layers.} \tag{6}$$

For a small displacement theory it is assumed that $\epsilon_{jk}(x) = \frac{1}{2}\left(\frac{\partial U_j(x)}{\partial x_k} + \frac{\partial U_k(x)}{\partial x_j}\right)$. Thus we obtain the strain within the i-th layer:

$$\epsilon_{11} = v^i_x + (x_3 - \hat{z}_i)\psi^i_x \quad \epsilon_{13} = \frac{1}{2}(\varphi^i) \tag{7}$$

In the stiff layers the above is adjusted by imposing (4).

2.1. Calculation of Lagrangian

The strain energy $\mathcal{P} = \sum_{i=1}^n \mathcal{P}_i$ and kinetic energy $\mathcal{K} = \sum_{i=1}^n \mathcal{K}_i$ for the composite beam are given by

$$\mathcal{P}_i = \frac{1}{2}\int_{Q_i} \sum_{j,k=1}^{3} \epsilon_{jk}\sigma_{jk} d\underline{x} dx_3, \qquad \mathcal{K}_i = \frac{1}{2}\int_{Q_i} \rho_i(\dot{U}_1^2 + \dot{U}_2^2 + \dot{U}_3^2) d\underline{x} dx_3,$$

where $\dot{} = d/dt$ and $\rho_i > 0$ denotes the mass density per unit volume within the i-th layer.

From (5)–(7) the strain energy of the ith layer can be written as

$$\mathcal{P}_i = \frac{RE_i h_i^3}{24}\int_0^L \left(\psi_x^i\right)^2 dx + \frac{Rh_i}{2}\int_0^L E_i\left(v_x^i\right)^2 + G_i(\varphi^i)^2 dx. \tag{8}$$

For a stiff (odd-indexed) layer, the shear vanishes and hence the last term in (8) is absent and $\psi_{\mathcal{O}} = -\vec{1}_{\mathcal{O}} w_x$.

Likewise the kinetic energy of the ith layer is

$$\mathcal{K}_i = \frac{R}{2}\int_0^L \rho_i h_i(\dot{w})^2 + \frac{\rho_i h_i^3}{12}\left(\dot{\psi}^i\right)^2 + \rho_i h_i\left(\dot{v}^i\right)^2 dx. \tag{9}$$

Define the following n by n matrices:

$$\begin{aligned} \mathbf{h} &= \operatorname{diag}(h_1, h_2, \ldots, h_n) \qquad & \mathbf{E} &= \operatorname{diag}(E_1, E_2, \ldots, E_n) \\ \mathbf{p} &= \operatorname{diag}(\rho_1, \rho_2, \ldots, \rho_n) \qquad & \mathbf{G} &= \operatorname{diag}(G_1, G_2, \ldots, G_n). \end{aligned}$$

If $\mathbf{K}$ represents one of the above diagonal matrices, $\mathbf{K}_{\mathcal{O}}$ and $\mathbf{K}_{\mathcal{E}}$ represent the associated diagonal matrix of odd-indexed and even-indexed diagonal elements, respectively. If θ and ξ are matrices in $\mathbf{R}^{lm}$, $\theta : \xi$ denotes the Hadamard product (the scalar product in $\mathbf{R}^{lm}$) and if they are vector functions defined on $(0, L)$ we denote

$$(\theta, \xi) = \int_0^L \theta \cdot \xi \, dx.$$

The expressions for the kinetic and potential energy can be rewritten as

$$\mathcal{K}(t) = R\tilde{c}(\dot{v}, \dot{\psi}, \dot{w})/2 \qquad \mathcal{P}(t) = R\tilde{a}(v, \psi, \varphi)/2$$

where $\tilde{c}$ and $\tilde{a}$ are defined by

$$\begin{aligned} \tilde{c}(\psi, v, w) &= ((\mathbf{h} : \mathbf{p})w, w) + ((\mathbf{p}\mathbf{h}^3/12)\psi, \psi) + (\mathbf{hp}v, v) \\ \tilde{a}(\psi, v, \varphi, \psi, v, \varphi) &= (\mathbf{h}^3\mathbf{E}\psi_x, \psi_x)/12 + (\mathbf{hE}v_x; v_x) + (\mathbf{Gh}\varphi, \varphi). \end{aligned} \tag{10}$$

We can further decompose the energy as follows:

$$\begin{aligned} \tilde{c}(\psi, v, w) &= ((\mathbf{h} : \mathbf{p})w, w) + c_E(\psi_{\mathcal{E}}, v_{\mathcal{E}}) + c_O(\psi_{\mathcal{O}}, v_{\mathcal{O}}) \\ \tilde{a}(\psi, v, \varphi) &= a_E(\psi_{\mathcal{E}}, v_{\mathcal{E}}) + a_O(\psi_{\mathcal{O}}, v_{\mathcal{O}}) + (G_{\mathcal{E}}\mathbf{h}_{\mathcal{E}}\varphi_{\mathcal{E}}, \varphi_{\mathcal{E}}) \end{aligned}$$

where

$$\begin{aligned} c_E(\psi_{\mathcal{E}}, v_{\mathcal{E}}) &= (\mathbf{p}_{\mathcal{E}}\mathbf{h}_{\mathcal{E}}^3\psi_{\mathcal{E}}, \psi_{\mathcal{E}})/12 + (\mathbf{h}_{\mathcal{E}}\mathbf{p}_{\mathcal{E}}v_{\mathcal{E}}, v_{\mathcal{E}}) \\ c_O(\psi_{\mathcal{O}}, v_{\mathcal{O}}) &= (\mathbf{p}_{\mathcal{O}}\mathbf{h}_{\mathcal{O}}^3\psi_{\mathcal{O}}, \psi_{\mathcal{O}})/12 + (\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}v_{\mathcal{O}}, v_{\mathcal{O}}) \\ a_O(\psi_{\mathcal{O}}, v_{\mathcal{O}}) &= (\mathbf{h}_{\mathcal{O}}^3\mathbf{E}_{\mathcal{O}}\psi_{\mathcal{O}x}, \psi_{\mathcal{O}x})/12 + (\mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}v_{\mathcal{O}x}, v_{\mathcal{O}x}) \\ a_E(\psi_{\mathcal{E}}, v_{\mathcal{E}}) &= (\mathbf{h}_{\mathcal{E}}^3\mathbf{E}_{\mathcal{E}}\psi_{\mathcal{E}x}, \psi_{\mathcal{E}x})/12 + (\mathbf{h}_{\mathcal{E}}\mathbf{E}_{\mathcal{E}}v_{\mathcal{E}x}, v_{\mathcal{E}x}) \end{aligned}$$

Using the relations (1), (2) together with the no-slip condition it is possible to solve for $v_{\mathcal{E}}$, $\psi_{\mathcal{E}}$ in terms of $v_{\mathcal{O}}$, w_x:

$$\begin{aligned} v_{\mathcal{E}} &= Av_{\mathcal{O}} + \tfrac{1}{4}B\mathbf{h}_{\mathcal{O}}\vec{1}_{\mathcal{O}}w_x \\ \mathbf{h}_{\mathcal{E}}\psi_{\mathcal{E}} &= Bv_{\mathcal{O}} + A\mathbf{h}_{\mathcal{O}}\vec{1}_{\mathcal{O}}w_x \end{aligned} \tag{11}$$

where $A = (a_{ij})$, $B = (b_{ij})$ are the $m \times (m+1)$ matrices defined by

$$a_{ij} = \begin{cases} 1/2 & \text{if } j = i \text{ or } j = i+1 \\ 0 & \text{otherwise} \end{cases} \qquad b_{ij} = \begin{cases} (-1)^{i+j+1} & \text{if } j = i \text{ or } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Now define the quadratic forms a and c by

$$\begin{aligned} a(v_{\mathcal{O}}, w) &= \tilde{a}(\psi, v, \varphi) \\ c(v_{\mathcal{O}}, w) &= \tilde{c}(\psi, v, w) \end{aligned}$$

where the variables on the right hand side are expressed in terms of $v_{\mathcal{O}}$, w using (11). Also let $a(v_{\mathcal{O}}, w; \hat{v}_{\mathcal{O}}, \hat{w})$ and $c(v_{\mathcal{O}}, w; \hat{v}_{\mathcal{O}}, \hat{w})$ denote the bilinear form that coincide with $a(v_{\mathcal{O}}, w)$ and $c(v_{\mathcal{O}}, w)$ when $\{vo, w\} = \{\hat{v}o, \hat{w}\}$.

According to the *principle of virtual work*, the solution trajectory is the trajectory which renders the Lagrangian stationary under all kinematically admissible displacements. The Lagrangian $\mathcal{L}$ on $(0, T)$ is defined by

$$\mathcal{L} = \int_0^T \mathcal{K}(t) - \mathcal{P}(t) dt.$$

By choosing appropriate test functions $\{\hat{v}_{\mathcal{O}}, \hat{w}\}$ and calculating the variation we obtain the weak form of the equations of motion. In our case we will consider simply-supported boundary conditions and hence we impose that the test functions $\hat{v_{\mathcal{O}}}$ and $\hat{w}$ are sufficiently smooth and vanish at $x = 0$ and $x = L$. We then obtain the following weak form of the equations of motion:

$$c(\ddot{v}_{\mathcal{O}}, w; \hat{v}_{\mathcal{O}}, \hat{w}) + a(\ddot{v}_{\mathcal{O}}, w; \hat{v}_{\mathcal{O}}, \hat{w}) = 0 \qquad \text{for all test functions } \{\hat{v}_{\mathcal{O}}, \hat{w}\}. \tag{12}$$

The strong form of the equations of motion can be obtained from integration by parts of (12). In the case of three layers, one obtains the model of Rao and Nakra.[1] Even for only three layers the resulting system and associated boundary conditions are quite lengthy to write out and we omit this calculation here for our n-layer model. However, the situation improves significantly if we are allowed to make some very modest approximations.

Namely, we consider the approximation of the full system obtained by dropping the energy terms $c_{\mathcal{E}}(\psi_{\mathcal{E}}, v_{\mathcal{E}})$ and $a_{\mathcal{E}}(\psi_{\mathcal{E}}, v_{\mathcal{E}})$. This is equivalent to dropping the parameters $\mathbf{E}_{\mathcal{E}}$ and $\rho_{\mathcal{E}}$ in the expressions for $a_{\mathcal{E}}$ and $c_{\mathcal{E}}$. It can be shown that the perturbation resulting in letting these parameters limit toward zero is *regular*. This guarantees that if the longitudinal inertia and stiffness of the compliant (even indexed) layers are small in relation to those of the odd layers, then the solutions of the approximate system are indeed close to the solutions of the full system; i.e., this limit does not have a boundary layer. We refer to the resulting beam system as the "Multilalyer Rao-Nakra Approximation".

2.2. Multilayer Rao-Nakra Approximation

When $\mathbf{p}_{\mathcal{E}}$ is dropped we have

$$\begin{aligned} c(v_{\mathcal{O}}, w_{\mathcal{O}}; v_{\mathcal{O}}, w) &= (\mathbf{h} : \mathbf{p} w, w) + \frac{1}{12}(\mathbf{p}_{\mathcal{O}} \mathbf{h}_{\mathcal{O}}^3 \vec{1}_{\mathcal{O}} w_x, \vec{1}_{\mathcal{O}} w_x) + (\mathbf{h}_{\mathcal{O}} \mathbf{p}_{\mathcal{O}} v_{\mathcal{O}}, v_{\mathcal{O}}) \\ &= (mw, w)_{\Omega} + \alpha(w_x, w_x) + (\mathbf{h}_{\mathcal{O}} \mathbf{p}_{\mathcal{O}} v_{\mathcal{O}}, v_{\mathcal{O}}) \end{aligned}$$

where

$$m = \mathbf{h} : \mathbf{p} = \sum_{i=1}^n h_i \rho_i, \qquad \alpha = \frac{1}{12} \vec{1}_{\mathcal{O}}^T \mathbf{p}_{\mathcal{O}} \mathbf{h}_{\mathcal{O}}^3 \vec{1}_{\mathcal{O}} = \frac{1}{12} \sum_{i\, odd}^n \rho_i h_i^3. \tag{13}$$

Define the rigidity K by

$$K = \vec{1}_{\mathcal{O}}^T E_{\mathcal{O}} \mathbf{h}_{\mathcal{O}}^3 \vec{1}_{\mathcal{O}}/12 = \sum_{i\ odd} E_i h_i^3/12.$$

The form a becomes:

$$a(v_{\mathcal{O}}, w) = K(w_{xx}, w_{xx}) + (\mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} v_{\mathcal{O}x}, v_{\mathcal{O}x}) + (\mathbf{G}_{\mathcal{E}} \mathbf{h}_{\mathcal{E}} \phi_{\mathcal{E}}, \phi_{\mathcal{E}})$$

Note also that with $\mathbf{E}_{\mathcal{E}}$ set to zero, the only potential energy remaining associated with the even layers is the shear energy term $(G_{\mathcal{E}} \mathbf{h}_{\mathcal{E}} \phi_{\mathcal{E}}, \phi_{\mathcal{E}})$.

From (3), (11) we can write

$$\varphi_{\mathcal{E}} = \mathbf{h}_{\mathcal{E}}^{-1} B v_{\mathcal{O}} + N w_x; \qquad N = \mathbf{h}_{\mathcal{E}}^{-1} A \mathbf{h}_{\mathcal{O}} \vec{1}_{\mathcal{O}} + \vec{1}_{\mathcal{E}}. \tag{14}$$

The explicit formulation of the variational differential equation in (12) is

$$(m\ddot{w}, \hat{w})_\Omega + \alpha(\ddot{w}_x, \hat{w}_x) + (\mathbf{h}_{\mathcal{O}} \mathbf{p}_{\mathcal{O}} \ddot{v}_{\mathcal{O}}, \hat{v}_{\mathcal{O}}) + K(w_{xx}, \hat{w}_{xx}) + (\mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} v_{\mathcal{O}}, \hat{v}_{\mathcal{O}}) + (\mathbf{G}_{\mathcal{E}} \varphi_{\mathcal{E}}, B v_{\mathcal{O}} + \mathbf{h}_{\mathcal{E}} N w_x) = 0 \tag{15}$$

Using integrations by parts of (15) one obtains

$$(F_1, \hat{w}) + (F_2, \hat{v}_{\mathcal{O}}) + (F_3, \hat{w}) \Big|_0^L + (F_4, \hat{w}_x) \Big|_0^L + (F_5, \hat{v}_{\mathcal{O}}) \Big|_0^L = 0 \tag{16}$$

where (with $D_x^k = \frac{\partial^k}{\partial x^k}$)

$$\begin{aligned}
F_1 &= m\ddot{w} - \alpha D_x^2 \ddot{w} + K D_x^4 w - D_x N^T \mathbf{G}_{\mathcal{E}} \mathbf{h}_{\mathcal{E}} \varphi_{\mathcal{E}} \\
F_2 &= \mathbf{h}_{\mathcal{O}} \mathbf{p}_{\mathcal{O}} \ddot{v}_{\mathcal{O}} - \mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} D_x^2 v_{\mathcal{O}} + B^T \mathbf{G}_{\mathcal{E}} \varphi_{\mathcal{E}} \\
F_3 &= \alpha \ddot{w}_n - K(D_x^3 w) + N^T \mathbf{G}_{\mathcal{E}} \mathbf{h}_{\mathcal{E}} \varphi_{\mathcal{E}} \\
F_4 &= K w_{xx} \\
F_5 &= \mathbf{E}_{\mathcal{O}} \mathbf{h}_{\mathcal{O}} v_{\mathcal{O}x}.
\end{aligned}$$

The appropriate boundary conditions are easily found from (16). In particular, the boundary value problem associated with simply-supported boundary conditions is

$$\begin{aligned}
m\ddot{w} - \alpha D_x^2 \ddot{w} + K D_x^4 w - D_x N^T \mathbf{G}_{\mathcal{E}} \mathbf{h}_{\mathcal{E}} \varphi_{\mathcal{E}} &= 0 \quad \text{on } (0, L) \times (0, \infty) \\
\mathbf{h}_{\mathcal{O}} \mathbf{p}_{\mathcal{O}} \ddot{v}_{\mathcal{O}} - \mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} D_x^2 v_{\mathcal{O}} + B^T \mathbf{G}_{\mathcal{E}} \varphi_{\mathcal{E}} &= 0 \quad \text{on } (0, L) \times (0, \infty) \\
w = D_x^2 w = D_x v_{\mathcal{O}} &= 0 \quad x = 0, L, \; t > 0.
\end{aligned} \tag{17}$$

2.2.1. Damped sandwich beams

Damping may be introduced into any of the beam layers by replacing the stress-strain relation with an appropriate dissipative constitutive law. In the case of *strain-rate damping*, the equations of motion are given by simply replacing E and G by $E + \tilde{E} d/dt$ and $G + \tilde{G} d/dt$, respectively.

The boundary value problem (17) becomes

$$\begin{aligned}
m\ddot{w} - \alpha D_x^2 \ddot{w} + K D_x^4 w - D_x N^T \mathbf{h}_{\mathcal{E}} (\mathbf{G}_{\mathcal{E}} \varphi_{\mathcal{E}} + \tilde{\mathbf{G}}_{\mathcal{E}} \dot{\varphi}_{\mathcal{E}}) &= 0 \quad \text{on } (0, L) \times (0, \infty) \\
\mathbf{h}_{\mathcal{O}} \mathbf{p}_{\mathcal{O}} \ddot{v}_{\mathcal{O}} - \mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} D_x^2 v_{\mathcal{O}} + B^T (\mathbf{G}_{\mathcal{E}} \varphi_{\mathcal{E}} + \tilde{\mathbf{G}}_{\mathcal{E}} \dot{\varphi}_{\mathcal{E}}) &= 0 \quad \text{on } (0, L) \times (0, \infty) \\
w = D_x^2 w = D_x v_{\mathcal{O}} &= 0 \quad x = 0, L, \; t > 0
\end{aligned} \tag{18}$$

For a discussion of general viscoelastic damping within the sandwich beam framework see e.g., Sun and Lu.[6]

3. MULTILAYER MEAD-MARKUS MODEL

In the 3-layer Mead-Markus model, the energy contributed by the longitudinal and rotational momentum is ignored. We consider the same approximation of the multilayer system. In this case the terms $\mathbf{h}_{\mathcal{O}} \mathbf{p}_{\mathcal{O}} \ddot{v}_{\mathcal{O}}$ and $\alpha D_x^2 \ddot{w}$ in the previous system are dropped.

The boundary value problem (18) becomes

$$\begin{aligned}
m\ddot{w} + K D_x^4 w - D_x N^T \mathbf{h}_{\mathcal{E}} (\mathbf{G}_{\mathcal{E}} \varphi_{\mathcal{E}} + \tilde{\mathbf{G}}_{\mathcal{E}} \dot{\varphi}_{\mathcal{E}}) &= 0 \quad \text{on } (0, L) \times (0, \infty) \\
-\mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} D_x^2 v_{\mathcal{O}} + B^T (\mathbf{G}_{\mathcal{E}} \varphi_{\mathcal{E}} + \tilde{\mathbf{G}}_{\mathcal{E}} \dot{\varphi}_{\mathcal{E}}) &= 0 \quad \text{on } (0, L) \times (0, \infty) \\
w = D_x^2 w = D_x v_{\mathcal{O}} &= 0 \quad x = 0, L, \; t > 0
\end{aligned} \tag{19}$$

It is possible to solve for the shears in the second equation above by multiplying on the left by B and using the identity

$$\mathbf{h}_\mathcal{E}\varphi_\mathcal{E} = Bv_\mathcal{O} + \mathbf{h}_\mathcal{E}Nw_x.$$

We obtain

$$-D_x^2(\mathbf{h}_\mathcal{E}\varphi_\mathcal{E} - \mathbf{h}_\mathcal{E}Nw_x) + P(\mathbf{G}_\mathcal{E}\varphi_\mathcal{E} + \tilde{\mathbf{G}}_\mathcal{E}\dot{\varphi}_\mathcal{E}) = 0$$

where

$$P = B(\mathbf{h}_\mathcal{O}\mathbf{E}_\mathcal{O})^{-1}B^T.$$

It can be shown that i) the matrix P is an M-matrix, ii) P^{-1} exists and has only positive elements, iii) P is positive definite.

The system (19) can be written as

$$\begin{aligned} m\ddot{w} + KD_x^4 w - D_x N^T \mathbf{h}_\mathcal{E}(\mathbf{G}_\mathcal{E}\varphi_\mathcal{E} + \tilde{\mathbf{G}}_\mathcal{E}\dot{\varphi}_\mathcal{E}) &= 0 \quad \text{on } \Omega\times(0,\infty) \\ -D_x^2\mathbf{h}_\mathcal{E}\varphi_\mathcal{E} + P(\mathbf{G}_\mathcal{E}\varphi_\mathcal{E} + \tilde{\mathbf{G}}_\mathcal{E}\dot{\varphi}_\mathcal{E}) &= -\mathbf{h}_\mathcal{E}ND_x^3 w \quad \text{on } \Omega\times(0,\infty) \end{aligned} \tag{20}$$

3.1. Spectral analysis of Multilayer Mead-Markus system

The method of separation of variables applied to system (20) leads to modal solutions of the form

$$w = e^{s_k t}\sin\sigma_k x, \qquad \varphi_\mathcal{E} = \vec{C}\cos e^{s_k t}\sigma_k x; \qquad \sigma_k = k\pi/L, \quad \vec{C}_k = (c_2, c_4, \dots\ c_{2m})^T \tag{21}$$

The modal solutions satisfy the boundary conditions. We seek s_k and $\vec{C}_k$ so that the PDE (20) is satisfied. For simplicity we omit the subscript k. Upon substitution of the modal solutions into (20) we obtain

$$ms^2 + K\sigma^4 + \sigma N^T\mathbf{h}_\mathcal{E}(\mathbf{G}_\mathcal{E} + s\tilde{\mathbf{G}}_\mathcal{E})\vec{C} = 0 \tag{22}$$

$$\sigma^2\mathbf{h}_\mathcal{E}\vec{C} + P(\mathbf{G}_\mathcal{E} + s\tilde{\mathbf{G}}_\mathcal{E})\vec{C} = \sigma^3\mathbf{h}_\mathcal{E}N \tag{23}$$

Solving for $\vec{C}$ in (23) and substituting the result into (22) gives

$$ms^2 + K\sigma^4 + \sigma N^T\mathbf{h}_\mathcal{E}(\mathbf{G}_\mathcal{E} + s\tilde{\mathbf{G}}_\mathcal{E})(\sigma^2\mathbf{h}_\mathcal{E} + P(\mathbf{G}_\mathcal{E} + s\tilde{\mathbf{G}}_\mathcal{E}))^{-1}\sigma^3\mathbf{h}_\mathcal{E}N = 0.$$

Let

$$H = \mathbf{h}_\mathcal{E}N, \qquad \Psi = \mathbf{h}_\mathcal{E}^{-1}\mathbf{G}_\mathcal{E}, \quad \Gamma = \mathbf{h}_\mathcal{E}^{-1}\tilde{\mathbf{G}}_\mathcal{E}.$$

Then we obtain

$$ms^2 + K\sigma^4 + \sigma^4 H^T(\Psi + s\Gamma)(\sigma^2 I + P(\Psi + s\Gamma))^{-1}H = 0. \tag{24}$$

Let

$$y = \frac{s}{\sigma^2}.$$

Then the characteristic equation becomes

$$my^2 + K + H^T(\Psi/\sigma^2 + y\Gamma)(I + P(\Psi/\sigma^2 + y\Gamma))^{-1}H = 0. \tag{25}$$

With no damping the system is conservative and hence y is on the imaginary axis. As the damping becomes infinite one again sees that the resulting characteristic equation again has imaginary roots. For nonzero damping the system is dissipative and hence y is in the left half-plane. As $\sigma_k \to \infty$ (25) becomes the following limiting characteristic equation:

$$my^2 + K + H^T y\Gamma(I + P(y\Gamma))^{-1}H = 0 \tag{26}$$

Hence the eigenvalues s_k tend toward constant (complex) multiples of σ_k^2. This means that at high frequencies, the roots s_k of (24) satisfy

$$s_k \sim y(k\pi/L)^2, \qquad \arg s_k \sim \arg y, \qquad \text{as } k \to \infty. \tag{27}$$

This type of eigenvalue behavior has been referred to as *frequency-proportional damping* and has been seen in other models of damped beams; see e.g., Chen and Russell.[7]

4. OPTIMAL DAMPING ANGLE

The complex number y in (27) can be viewed as a function of the damping vector Γ. Typically, in composite beam models, one finds that there is an optimal level of damping, beyond which, additional damping becomes counterproductive. We pose the optimization problem:

$$\text{Maximize } \lim_{k\to\infty} \arg s_k : \Gamma \in \{ \text{ diagonal matrices in } \mathbf{R}^{m\times m}\} \tag{28}$$

THEOREM 4.1. *Suppose $H^T P^{-1} H/K < 8$. Then $\lim_{k\to\infty} \arg s_k < \pi$. Let Γ^* denote a value of Γ that optimizes (28). Then Γ^* is unique and satisfies*

$$P\Gamma^* H = \lambda H, \qquad \textit{where } \lambda = \frac{m^{1/2}K^{1/4}}{(K + H^T P^{-1} H)^{3/4}}. \tag{29}$$

Sketch the proof: We rewrite the characteristic equation (26) as

$$my^2 + K + H^T Q^{-1} H = 0; \qquad Q = (\Gamma^{-1} y^{-1} + P). \tag{30}$$

We calculate implicitly the derivative y' of y at Γ^* in the direction of M where M is a real diagonal matrix. (Since Γ is positive and diagonal, all valid variations should be in a real, diagonal-matrix direction.) One finds

$$\begin{aligned} y' &= \frac{-H^T Q^{-1}(\Gamma^* y)^{-1} M y (\Gamma^* y)^{-1} Q^{-1} H}{2my + H^T Q^{-1}(\Gamma^* y)^{-1}\Gamma^*(\Gamma^* y)^{-1} H} \\ &= \frac{-H^T (I + \Gamma^* y P)^{-1} M y (I + P\Gamma^* y)^{-1} H}{2my + H^T (I + \Gamma^* y P)^{-1}\Gamma^*(I + P\Gamma^* y)^{-1} H}. \end{aligned} \tag{31}$$

The matrix P is symmetric so that the standard spectral theory applies to the matrix $P\Psi$. We apply the eigenvector condition (29) so that

$$(I + P\Gamma y)^{-1} H = \frac{1}{1 + y\lambda} H. \tag{32}$$

From (31) we obtain

$$\begin{aligned} \frac{y'}{y} &= -\frac{H^T M H/(1 + y\lambda)^2}{2my + H^T \Gamma^* H/(1 + y\lambda)^2} \\ &= -\frac{H^T M H}{2my(1 + y\lambda)^2 + H^T \Gamma^* H}. \end{aligned} \tag{33}$$

From elementary complex variable theory, the argument θ of y is optimized when $\theta' = 0$, i.e., when $(\log y)' = y'/y$ is real.

Thus to show that Γ^* is a critical point it is enough that (33) is real, independent of the direction of variation M. However, since M and Γ^* are real, it is enough to:

$$\text{show that } y(1 + y\lambda)^2 \quad \text{is real} \tag{34}$$

Using the eigenvector condition in (29) it is easy to show that $H^T \Gamma^* H = \lambda H^T P^{-1} H$. Thus (26) becomes

$$(my^2 + K)(1 + \lambda y) + \lambda y H^T P^{-1} H = 0.$$

This can be rewritten as

$$y^3 + \frac{1}{\lambda} y^2 + \frac{\tilde{K}}{m} y + \frac{K}{m\lambda} = 0; \qquad \tilde{K} = K + H^T P^{-1} H \tag{35}$$

We make the substitutions

$$y = u\sqrt{K/m}, \qquad t = (\lambda\sqrt{K/m})^{-1} \tag{36}$$

and using the value of λ in (29), the equation (35) becomes

$$u^3 + tu^2 + t^{4/3}u + t = 0. \tag{37}$$

For $1 \leq t \leq \sqrt{27}$ the roots of (37) are $-t^{1/3}$, $\frac{1}{2}t^{1/3}[(1 - t^{2/3} \pm i\sqrt{3 + 2t^{2/3} - t^{4/3}}]$. Furthermore, direct multiplication will verify that if u is a complex root of (37) then the quantity $u(t+u)^2$ is real. Upon application of the reverse substitutions in (36) we find that when y is a complex root of (26) that the quantity $y(1 + \lambda y)$ is real; i.e., the condition (34) has been verified. The condition that $t < \sqrt{27}$ translates to the condition that $H^T P^{-1} H/K < 8$ given in the hypothesis. The uniqueness of the critical point involves an addition argument which we omit. What remains is to justify that it is always possible to find a Γ^* that satisfies the condition in (29).

4.1. Calculation of Γ^*

We need to find Γ^* so that $P\Gamma^* H = \lambda H$. Equivalently, $\Gamma^* H = \lambda P^{-1} H$. Let $[H]$ be the diagonal matrix consisting of the elements of H and $\vec{\Gamma^*}$ the vector of diagonal elements of Γ^*. Then $\Gamma^* H = [H]\vec{\Gamma^*}$. Upon inversion of $[H]$ we obtain

$$\vec{\Gamma^*} = \lambda [H]^{-1} P^{-1} H. \tag{38}$$

As mentioned earlier, one can show that P^{-1} is positive definite, and consists *only of positive elements*. Consequently the value of Γ^* is guaranteed to consist of positive numbers.

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